Gabriel Uzquiano, *The Mereology of Classes*, Cambridge University Press, Cambridge 2024, pp. 80, $ 64.99, ISBN 9781009092241

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Mereology is the formal study of the relation "*x* is part of *y*", as referred to in sentences like "Arthur Rimbaud's right leg is part of the poet Arthur Rimbaud". While physical objects have been subjected to extensive mereological investigation, a key question is whether abstract entities, such as mathematical entities, have parts and, if so, whether mereology is equipped to shed light on some aspects of their part-whole structure. Uzquiano's work addresses these questions by focusing on classes, a specific type of mathematical entity. In this review, I will summarize the book's main contributions and will also raise two points of criticisms that may inspire further reflection on the mereology of classes.

The Mereology of Classes consists of six chapters. Chapters 1 and 2 discuss the philosophical motivations for a mereology of classes and present two prominent alternative theses concerning the characterization of the proper parts of classes: Main Thesis (MT), stating that the parts of a class are all and only its subclasses, and Hierarchical Composition (HC), roughly stating that the parts coincide with the members of the class, the members of those members, and so on, extending hierarchically. Chapter 3 delves into the formal background of some mereological theories and plural logic, and the formal study of their interactions (*plural mereology*). Chapter 4 critically examines MT, highlighting the dilemma it poses once combined with Classical Extensional Mereology (CEM) and discussing proposed solutions. Chapter 5 explores HC focusing on the theories of pseudoclasses and rigid embodiments. Since Chapter 6 summarizes the philosophical proposals on class mereology already presented in the book, it will not be covered in detail in this review.

In Chapter 1, Uzquiano explores whether mathematical objects have parts. He distinguishes between the thesis of compositional pluralism, which asserts that the part-whole relation differs across different domains (e.g., physical vs. abstract objects), and compositional monism, which claims that it operates the same way across these domains. The deeper issues are: (1) whether mereology applies to mathematical objects, (2) whether this application should be taken literally, and (3) what proper parts mathematical objects may have. While Uzquiano responds affirmatively to both (1) and (2), his main focus is on addressing (3) for classes. Over the course of the book he offers two plausible answers to (3): one is MT, which tends to align with monism insofar as it typically assumes Classical Extensional Mereology (CEM) to be true across categories, and the other is HC, which is compatible with both monism and pluralism.

In Chapter 2, Uzquiano begins to address the crucial questions of whether mathematical objects are composed of proper parts and, if so, which set-theoretic relations might be suitable for mereological analysis. Uzquiano delves into two main options to address (3) for classes: (MT) the parts of a class are all and only its subclasses; (HC) the immediate parts of a class are all and only its members, and the parts of a class include the class itself and all and only its ancestral immediate parts. Notably, HC does not imply that the subclasses of a class are amongst its parts. However, there is also a version of HC, called Liberal Hierarchical Composition (LHC) in which the proper parts of a class include both its subclasses and its ancestral immediate parts.

Lewis (1991) supports MT, but Hamkins and Kikuchi (2016) point out a limitation. If MT is true, then given a model of ZFC $\langle V; \in \rangle$, one can provide an automorphism τ of $\langle V; \subseteq \rangle$ which does not preserve the membership relation. Therefore, neither class theory nor set theory can be fully reduced to mereology.

HC states that the immediate parts of a class are its members. The parts of a class coincide with its immediate parts, along with the elements obtained through the transitive closure of the "membership" relation ($\in^{\infty}$) applied to these immediate parts. The emerging parthood relation does not reduce to set-theoretic membership (see Fine 1992), which limits the mereological analysis of classes in purely set-theoretic terms: given a model of ZFC $\langle V; \in \rangle$, one can provide an automorphism of $\langle V; \in^{\infty} \rangle$ that does not preserve the membership relation. Further explorations and ways to get around this limitation can be found in Forrest (2002) and

Caplan, Tillman, and Reeder (2010). Additionally, it can be shown that LHC suffers from the same limitation regarding the reduction of the membership relation to a full-fledged mereological analysis.

In Chapter 3, Uzquiano discusses CEM, as adopted in Lewis's version of MT, and contrasts it with a non-classical mereological theory, known as Heyting mereology. Unlike CEM, Heyting mereology does not include weak supplementation amongst its principles (if x is a proper part of y, then there must exist another part of y that is entirely disjoint from x). Therefore, Heyting mereology offers a suitable framework for proponents of HC, as it can be proved that HC is incompatible with weak supplementation.

At this point, Uzquiano outlines limitations for both MT and HC. Lewis's MT "entails that singletons are atoms, and a total injection of objects into singletons is, modulo the existence of more than one object, inconsistent with CEM" (p. 41). In accordance with MT, since identifying a class's parts with its subclasses amounts to treating classes as fusions of singletons this limits Lewis's MT.

Regarding HC, Uzquiano reviews attempts to define HC by means of the parthood relation within Heyting mereology (e.g., Simons 1987, Forrest 2002, Cotnoir and Varzi 2021). As a matter of fact, Heyting mereology is considered "a framework for a transitive relation of part to whole in line with Hierarchical Composition" (p. 22). However, the unsuitablities of these accounts leads to the opposite approach: treating the immediate-part relation as primitive and deriving the parthood relation from it. On this regard, Uzquiano presents Fine's (1992) theory of rigid embodiments, extended to second-order logic to ensure that a rigid embodiment can be composed of infinitely many immediate parts. This transition to second-order logic enables defining the part-whole relation in terms of the immediate-part relation, as shown by Jacinto and Cotnoir (2019). However, the plural Cantor theorem (see, e.g., Shapiro 1991) imposes a limit on HC: "[t]he plural formulation of the existence and identity postulates [of the theory of rigid embodiments] is inconsistent with the existence of more than one object" (p. 36, parentheses added).

In Chapter 4, Lewis's proposal is revisited, according to which the parts of a class coincide with its subclasses. In Chapter 3 Uzquiano outlined a limit of MT in terms of the inexistence of a total injection of all objects into mereological atoms. He now presents two proposals from the literature to overcome this issue. The first consists of restricting the scope of the singleton operation, a $\mapsto\{a\}$, in line with the structuralist approach adopted by Lewis. The idea underlying this approach is to exchange a philosophically opaque singleton operation for any operation that meets the formal requirements typically linked to the singleton operation. However, Uzquiano's concern regarding MT is that embracing a structuralist perspective might diminish its relevance. If structuralism is used to resolve the enigmatic relationship between objects and their singletons, it could similarly address the mystery of membership, thus undermining the necessity of MT itself.

The second proposal weakens CEM. While the proposal will not be discussed in detail, it should still be noted that this second approach doesn't fare any better than the first as it requires unintuitive limitations on the singleton operation.

In Chapter 5, Uzquiano presents two implementations of HC. The first is the theory of pseudoclasses. Forrest (2002) views the parthood relation as fundamental and seeks to recover the membership relation through a variant of the relation of immediate part as defined in Simons (1987) or Cotnoir and Varzi (2021), resulting in the relation of "pseudomembership", which is not extensional, unlike standard set-theroretic membership. Additionally, Forrest's system does not guarantee the existence of a class composed of all elements satisfying a certain property. However, he addresses this problem by associating a pseudoset with every "simply well-founded set", defined as any set whose membership relation's transitive closure forms a tree.

The second implementation is the theory of rigid embodiments. In fact, as Uzquiano notes: "however fruitful, the relation of pseudomember is at most inadequate for the more basic relation of member" (p. 59). This theory starts from the immediate part relation, considered primitive, and derives the notion of part from it. Each complex object consists of immediate parts unified by a certain form, understood as a principle of unity (Johnson 2006) or as an attribute (Fine 1992). Once a class is encoded as the rigid embodiment of certain objects, the notion of membership can be defined in terms of immediate parts. As already precised, however, the plural formulation of the existence and identity postulates for rigid embodiments is inconsistent with the existence of more than one object. At this point, Uzquiano presents several attempts to resolve this impasse (see Fine 2010 and Florio and Linnebo 2021). One can attempt to weaken the axioms of plural quantification, or consider that rigid entities are composed in stages of some cumulative hierarchy. However, there is not enough space here to discuss such proposals in depth.

I hope that this reconstruction demonstrates that Uzquiano's work is not merely a mereology of classes, as the title might suggest. It is a technically rigorous and philosophically engaging discussion of the issues that arise as soon as we begin to think of classes in terms of their part-whole structures. As such, it is accessible to those familiar with the standard concepts of mereology as well as the typical notions from set theory. The abstract states that "a clear picture of the mereology of classes may provide further insights into the foundations of set theory". However, I am under the impression that exploring the mereological behavior of classes does not provide a deeper understanding of foundation for set theory than the one set theory already has, which can be traced back to Cantor's precise analysis of the concept of set. I would have preferred greater emphasis on this point, as it seems central to the issue at hand. On a personal note, I retain that the study of the mereology of classes is valuable primarily for understanding the expressive power of mereology. That is, given the potential of mereological language and theories to be applied across various categories of entities, it is intriguing to explore how far mereology can be extended. Furthermore, considering the foundational nature of set theory, it is also interesting to understand set theory in mereological terms and vice versa, despite the fact that the two systems are distinct categorical frameworks.

An additional point on which I would have liked more clarification is the following: when employing plural logic, the relation "is one of", characteristic of plural logic, becomes closely akin to the relation "is a member of", with the exception that in the former case the second argument of the relation applies to a plurality of entities, while in the latter it applies to a single entity. To what extent, then, is the notion of membership a genuine result of the theory of rigid embodiment, and to what extent is it already inherently tied to the plural account of rigid embodiment?

A clarification of these two points, I believe, could broaden the debate on the mereology of classes and make explicit some of the objectives and assumptions underlying the discussion.

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