*Synthese*, Vol. 193, pp. 1-14, 2016.

# Paraconsistent Dynamics

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**Abstract**

It has been an open question whether or not we can define a belief revision operation that is distinct from simple belief expansion using paraconsistent logic. In this paper, we investigate the possibility of meeting the challenge of defining a belief revision operation using the resources made available by the study of dynamic epistemic logic in the presence of paraconsistent logic. We will show that it is possible to define dynamic operations of belief revision in a paraconsistent setting.

**Key Words**

Paraconsistent Logic, Dynamic Epistemic Logic, Public Announcement, Belief Revision

## 1 Introduction

What is the rational thing to do when new information *contradicts* old information? The standard assumption is that it is always rational to remove contradictions. This standard assumption has been challenged in recent years, most notably by Graham Priest.[1] For example, Bohr's theory of the atom contained both classical electrodynamics principles and quantum principles that were inconsistent, but reasoning with Bohr's theory was not thought of as irrational.[2] Moreover, even when we are rationally required to update inconsistent theories, the method

[1] Cf. Priest (1987) and Priest (2006) chaps. 7 and 8.

[2] Cf. Brown and Priest (2004) for a discussion of contradictions in the history of science and a paraconsistent logic dealing with some of those contradictions.

for the belief update cannot be trivial. The notion of rationality which incorporates information dynamics must thus integrate a method of sensibly dealing with contradictory information.

One way of sensibly dealing with contradictory information is to model information dynamics in terms of paraconsistent logic. In a paraconsistent logic, *ex contradictione quodlibet* ($\varphi, \neg\varphi \models \psi$ for any $\varphi, \psi$) is invalid. That is, inconsistency is not automatically treated as triviality. Armed with paraconsistent logic, we don't necessarily have to declare inconsistent information trivial. Thus, by assuming the underlying logic to be paraconsistent, we can separate inconsistent information from triviality. In the presence of paraconsistent logic, therefore, we can sensibly deal with contradictory information. We can then reject the assumption that it is always rational to avoid contradictory beliefs.[3]

Classically, if new information contradicts other beliefs, it is necessary to *revise*, rather than *expand*, beliefs in order to maintain non-triviality. This means that, classically, the need for revision is driven by consistency and consistency is maintained by defining revision as a different operation from (mere) expansion. If inconsistency does not imply triviality, however, the need for separating revision from expansion disappears. As Tanaka (2005) shows, AGM revision and expansion can be shown to be equivalent (with respect to Grove's sphere semantics) in most paraconsistent logics.

This does not mean that rationality *never* demands inconsistent beliefs or information to be revised, as Priest argues.[4] It may be the case that, if it is not rational to stipulate which of $p$ or $\neg p$ to reject when evidence is lacking for either case, the most rational thing to do is to expand one's beliefs by accepting both $p$ and $\neg p$ (at least for the time being). Nonetheless, when new information which carries sufficient evidential force becomes available, we may be rationally required to *revise* our beliefs by picking only one of $p$ or $\neg p$ and rejecting the other. Hence, even though paraconsistency does not force one to always resolve contradictory beliefs, it is important to distinguish between *expanding* one's beliefs by accepting contradictory beliefs and *revising* them by resolving contradictory beliefs. There is thus still a need for revision in paraconsistent logic.

We know that paraconsistency can provide a mechanism for accommodating contradictory beliefs (and information). Restall and Slaney (1995) used *First Degree Entailment* ($\mathsf{FDE}$) (a fragment of relevant logic containing only $\neg$, $\wedge$ and $\vee$) to show that a system of spheres based on $\mathsf{FDE}$ is sound with respect to all AGM postulates. Mares (2002) extended their analysis by using the (full) relevant logic $\mathsf{R}$ to show that the AGM operations can be satisfied paraconsistently. They did

[3]For introductions to paraconsistent logic, see, for example, Priest (2002) and Priest, Tanaka and Weber (2013).

[4]Cf. Priest (2006) chaps. 7 and 8.

not show, however, that the revision operation that they defined is distinct from expansion.[5] Moreover, Priest (2006 ch. 8) proposed a paraconsistent model of belief revision and showed that all of the AGM revision postulates fail in his system. Defining a distinct revision operation using paraconsistent logic, thus, remains an open question.[6]

In recent years, (classical) revision operations have been studied in the framework of *Dynamic Epistemic Logic* (DEL). DEL offers a general framework for analysing the dynamics of structured epistemic and doxastic states. Whereas the AGM theory treats beliefs as a set of sentences, DEL provides a framework for analysing the doxastic (and epistemic) *states*, rather than *sets*, by taking into consideration the doxastic (and epistemic) structure that represents beliefs (and knowledge).[7]

In this paper, we investigate the possibility of meeting the challenge of defining a revision operation that is not mere expansion using the resources made available by the study of DEL in the presence of paraconsistent logic. We will define two paraconsistent revision operations over belief states in appropriate extensions of the paraconsistent logic $\mathsf{LP}$.[8] We will expand $\mathsf{LP}$ in two ways. First, we introduce epistemic and doxastic operators. Second, we introduce a conditional (taken from D'Ottaviano and da Costa (1970)) necessary for the analysis of dynamics. We will then show that the operations of *public announcement* (Plaza (2007)) and *lexicographic upgrade* (van Benthem (2007)) can be defined in the expanded $\mathsf{LP}$-language. These two belief state revision operations are respectively called *radical* and *moderate* revision in Rott (2009). We thus have a possibility result: it is possible to define dynamic operations of belief revision in a paraconsistent setting. This solves the open problem of paraconsistently defining a revision operation that is distinct from expansion. Moreover, we provide an analysis of doxastic states, rather than doxastic sets, that may be contradictory by combining the studies of belief revision, DEL and paraconsistent logic for the very first time.

[5] Mares took expansion and contraction operations as primitive and defined them using the relevant logic $\mathsf{R}$. He then defined revision in terms of expansion and contraction via *Levi Identity*. As Tanaka (2005) shows, however, some of the AGM revision postulates fail if we define revision in terms of the *Levi Identity* in paraconsistent logics. It is thus not clear to what extent Mares has succeeded in defining revision paraconsistently.

[6] For a summary of paraconsistent approaches to AGM operations, see Wassermann (2011).

[7] Cf. van Benthem (2007), Baltag and Smets (2008), and Girard and Rott (2014). See also Rott (2009) for the difference between belief *sets* and belief *states*.

[8] $\mathsf{LP}$ was introduced by Priest (1979).

## 2 Many-Valued Epistemic and Doxastic Logic

We first investigate paraconsistent epistemic and doxastic logic. Following the presentation of paraconsistent logics in Priest (2008), we provide a general framework for many-valued epistemic and doxastic logic, and define epistemic and doxastic LP as a special case.

### 2.1 Many-Valued Epistemic Logic

Given a set of connectives $C$, a many-valued logic $L$ can be characterised by a structure $S_L = \langle \mathcal{V}, \mathcal{D}, \{f_c \mid c \in C\} \rangle$, where $\mathcal{V}$ is the set of truth-values which come with a complete ordering[9] $\leq$, $\mathcal{D} \subset \mathcal{V}$ is the set of designated values, and for each logical connective $c \in C$, $f_c$ is the truth function it denotes. A model for a many-valued propositional logic $L$ is a structure $M = \langle W, S_L, v \rangle$ where $W$ is a non-empty set of worlds, $S_L$ is a propositional many-valued structure for the logic $L$, and $v$ is a collection of propositional valuations $v_w$ for each world $w \in W$ such that $v_w(p) \in \mathcal{V}$. The truth values of complex formulas are computed using the function $f_c$:

$$v_w(c(\varphi_1, ..., \varphi_n)) = f_c(v_w(\varphi_1), ..., v_w(\varphi_n)).$$

For epistemic multi-valued logic, we expand the language with an epistemic operator $K$, and we add to models a corresponding epistemic equivalence relation $\sim$ on worlds.[10] A model for a many-valued epistemic logic is thus a structure $M = \langle W, \sim, S_L, v \rangle$ where $W$ is a non-empty set of worlds, $\sim$ is an equivalence relation on $W$, $S_L$ is a structure for the propositional many-valued logic $L$, and $v$ is a collection of propositional valuations $v_w$ for each world $w \in W$ such that $v_w(p) \in \mathcal{V}$. Truth values for epistemic formulas can be computed as follows:

$$v_{w_1}(K\varphi) = Glb\{v_{w_2}(\varphi) \mid w_1 \sim w_2\}$$

where $Glb$ is the greatest lower bound in the ordering $\leq$. As usual, we can define an operator $k\varphi$ dual to $K\varphi$, with truth values computed as follows:

$$v_{w_1}(k\varphi) = Lub\{v_{w_2}(\varphi) \mid w_1 \sim w_2\}$$

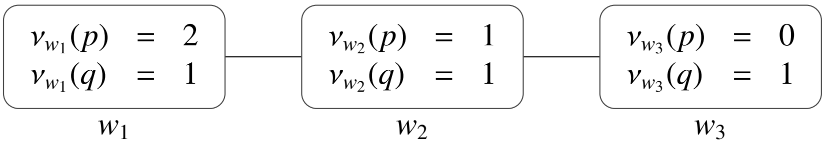
where $Lub$ is the least upper bound in the ordering $\leq$.[11]

[9] Complete so as to guarantee the existence of greatest and least bounds which we require for the semantics of modalities, as we will see below.

[10] Cf. Fagin *et al.* (1995).

[11] For special cases of many-valued epistemic logic, for instance LP, defining knowledge in this way allows $K\varphi$ to be evaluated as both true and false at some world. We acknowledge that there are other plausible definitions of knowledge in many-valued settings. For example, one can impose a classical definition of knowledge such that $K\varphi$ is 1 (the top value) if it is 1 in all epistemically equivalent worlds. But this would make the logic non-paraconsistent: $Kp$ would entail everything when $v_w(p) = b$ for all $w \in W$.

**Example 1.** *Let $\mathcal{V} = \{0, 1, 2\}$ and $\mathcal{D} = \{1, 2\}$. Consider the following model:*




*(Here and throughout the paper, we represent two worlds being epistemically equivalent by drawing a line between them. For readability, we omit transitive and reflexive relations between worlds.)*

*In this model, $Glb\{v_x(p) \mid w_1 \sim x\} = 0$, $Glb\{v_x(q) \mid w_1 \sim x\} = 1$, $Lub\{v_x(p) \mid w_1 \sim x\} = 2$ and $Lub\{v_x(q) \mid w_1 \sim x\} = 1$ for every $x \in W$. So $v_{w_1}(Kp) = 0$, $v_{w_1}(Kq) = 1$, $v_{w_1}(kp) = 2$ and $v_{w_1}(kq) = 1$.*

A valid consequence relation is defined in terms of the preservation of designated values.

> $\Sigma \models \varphi$ iff for every $M$ and every $w \in W$, if $v_w(\psi) \in \mathcal{D}$ for every $\psi \in \Sigma$ then $v_w(\varphi) \in \mathcal{D}$.

We say that a formula is valid iff $\varnothing \models \varphi$, which we abbreviate as $\models \varphi$.

## 2.2 Many-Valued Doxastic Logic

We now expand the language of many-valued epistemic logic by introducing a doxastic operator $B\varphi$.[12] We follow van Benthem (2007) and define belief in terms of maximal plausible worlds.

A model for a many-valued doxastic logic, $M$, is a structure $\langle W, \sim, \leq, S_L, v\rangle$ where $W$ is a non-empty set of worlds, $\sim$ is an equivalence relation on $W$, $\leq$ is a well-founded[13] preorder (reflexive and transitive relation) on $W$ with the restriction that $\leq \subseteq \sim$, $S_L$ is a structure for a many-valued logic $L$, and $v$ is a collection of propositional valuations $v_w$ for each world $w \in W$ such that $v_w(p) \in \mathcal{V}$. We define $u < v$ iff $u \leq v$ and $v \nleq u$ and say that a world $w$ is maximal (or most plausible) if there is no world $v$ such that $w < v$. Finally, we can define the semantics of the doxastic operator $B$ in terms of maximal worlds by adapting the classical definition; namely by taking the greatest lower bound of epistemically equivalent maximal worlds:

[12]The logic we develop in this section may best be described as an 'epistemic doxastic logic' since $B\varphi$ supplements rather than replaces knowledge operator. For simplicity, however, we refer to it as a 'doxastic logic' rather than 'epistemic doxastic logic'.

[13]For the concerned reader, well-foundedness guarantees the limit assumption which simplifies technical and conceptual details that are not critical in the context of this paper.

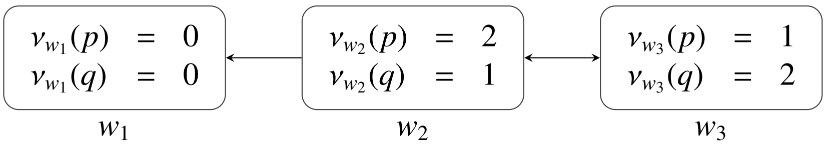
$$v_{w_1}(B\varphi) = Glb\{v_{w_2}(\varphi) \,|\, w_1 \sim w_2, w_2 \text{ is a maximal world}\}.$$

Notice that we added an epistemic restriction on the maximal worlds used for evaluating beliefs. This is to ensure doxastic introspection, so that agents know what they believe (if $B\varphi$, then $KB\varphi$). In the rest of the paper, we will assume that the epistemic relation is the universal relation for simplicity.

We can also define a corresponding existential belief operator $b$ with semantics given in terms of the least upper bounds:

$$v_{w_1}(b\varphi) = Lub\{v_{w_2}(\varphi) \,|\, w_1 \sim w_2, w_2 \text{ is a maximal world}\}.$$

**Example 2.** *As a simple example, consider the following model $M$:*




*(We assume that the epistemic relation is the universal relation and use arrows to represent plausibility orders. So, in this model, $x \sim y$ for every world $x, y$, and $w_1 < w_2$, $w_2 \leq w_3$ and $w_3 \leq w_2$. When we combine epistemic and doxastic relations, we will omit the epistemic relation altogether, since all worlds are epistemically equivalent in our examples. For readability, we also omit transitive and reflexive relations between worlds.)*

*According to our definition of maximality, $w_1$ is the maximal world in this model. So $Glb\{v_{w_1}(q)\} = 0$. Therefore, $v_x(Bq) = 0$ for every $x \in W$.*

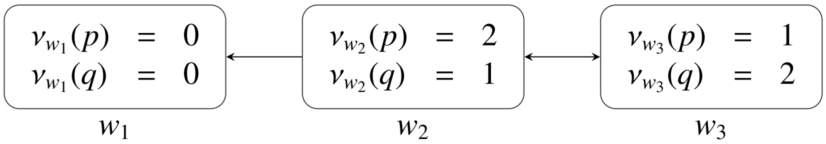
### 2.3 Conditional Belief

In a dynamic doxastic language, it is much easier to work with a conditional belief operator $B(\varphi \,|\, \psi)$, defined as the belief in $\varphi$ given $\psi$. The technical reason for this will become clear later.

To define the semantics of conditional belief, we adapt the notion of maximality to that of conditional maximality. We define a maximal $\psi$-world $w$ as a world such that $v_w(\psi) \in \mathcal{D}$ and there is no $x$ such that $w < x$ and $v_x(\psi) \in \mathcal{D}$. The truth-value of $B(\varphi \,|\, \psi)$ is then given by:

$$v_{w_1}(B(\varphi \,|\, \psi)) = Glb\{v_{w_2}(\varphi) \,|\, w_1 \sim w_2, w_2 \text{ is a maximal } \psi\text{-world}\}.$$

**Example 3.** *As a simple example, consider the following model $M$:*




*The maximal world in $M$ is $w_1$, but the maximal worlds conditional on $p$ are $w_2$ and $w_3$. Furthermore, $Glb\{v_{w_2}(q) = 2, v_{w_3}(q) = 1\} = 1$ (assuming that $0 \preceq 1 \preceq 2$). Therefore, $v_x(B(q \mid p)) = 1$ for every $x \in W$.*

## 2.4 Doxastic LP

Doxastic LP can be seen as a special case of many-valued doxastic logic with $\mathcal{V} = \{1, b, 0\}$, ordered as

$$0 \preceq b \preceq 1$$

or ordered by the following lattice:

$$\begin{array}{c} 1 \\ \uparrow \\ b \\ \uparrow \\ 0 \end{array}$$

For each propositional variable $p$ and $w \in W$, $v_w(p) = 1$, $v_w(p) = b$, or $v_w(p) = 0$. If we think of the value $b$ as 0.5, then the truth values of negated and disjoined formulas can be computed as follows:

$$v_w(\neg\varphi) = 1 - v_w(\varphi)$$
$$v_w(\varphi \vee \psi) = Max(v_w(\varphi), v_w(\psi)).$$

The functions that compute the truth values of negated and disjoined formulas give the following tables:

| $\neg$ | |
|---|---|
| 1 | 0 |
| $b$ | $b$ |
| 0 | 1 |

| $\vee$ | 1 | $b$ | 0 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| $b$ | 1 | $b$ | $b$ |
| 0 | 1 | $b$ | 0 |

Let $\mathcal{D} = \{1, b\}$. We will write doxastic LP models succinctly as $\langle W, \sim, \leq, v\rangle$, omitting the component $S_L$, which is now fixed for LP. A doxastic LP-valid consequence relation is then defined in terms of the preservation of designated values:

$\Sigma \models_{LP} \varphi$ iff for every doxastic LP-model, $\langle W, \sim, \leq, v\rangle$, and for every $w \in W$, if $v_w(\psi) \in \mathcal{D}$ for every $\psi \in \Sigma$ then $v_w(\varphi) \in \mathcal{D}$.

It is easy to show that $p, \neg p \vee q \not\models_{LP} q$, by taking a model with a single world $w$ such that $v_w(p) = b, v_w(q) = 0$. This means that $\vee$ is not a detachable connective in the sense that a disjunct cannot be isolated from a disjunction. In fact, as Beall, Forster and Seligman (2013) demonstrate, no non-trivial detachable connective can be defined in LP. This will become important for us below.

## 3 Dynamic Doxastic LP

Having treated (static) epistemic and doxastic paraconsistent logic, we now turn to the dynamics of belief revision. We will introduce two kinds of belief revision operations. The first is the operator known as public announcement (Plaza (2007)), but interpreted as an operation of *radical revision*, following the taxonomy of Rott (2009). Radical revision is an action which deletes all worlds in which $\varphi$ is not designated. The second is the lexicographic upgrade of van Benthem (2007), which Rott (2009) categorises as *moderate revision*. Moderate revision doesn't delete any world. Instead, it rearranges the plausibility order by putting the worlds in which $\varphi$ is not designated at the bottom of the order. From now on, we will refer to the two operations as radical and moderate revision, respectively defined in Sections 3.1 and 3.2.

### 3.1 Paraconsistent Radical Revision

Radically revising by $\varphi$ amounts to removing all worlds from a model in which $\varphi$ is not designated. To radically revise by $\varphi$ is thus to remove all worlds in which $\varphi$ is (strictly) false and preserve only worlds in which $\varphi$ is designated.

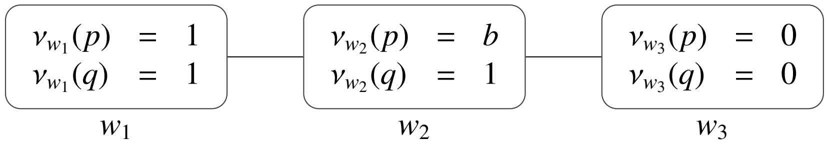
We extend our basic epistemic language with the radical revision operator $[!\varphi]$ and define the result of announcing $\varphi$, $[!\varphi]\psi$, in a model $M = \langle W, \sim, V\rangle$ in terms of the submodel $M^{\varphi} = \langle W^{\varphi}, \sim^{\varphi}, v^{\varphi}\rangle$ where:

- $W^{\varphi} = \{u \mid v_u(\varphi) \in \mathcal{D}\}$,
- $\sim^{\varphi} = \{\langle u, v\rangle \mid u, v \in W^{\varphi}, u \sim v\}$, and
- $v^{\varphi} = v - \{v_u \mid u \notin W^{\varphi}\}$.

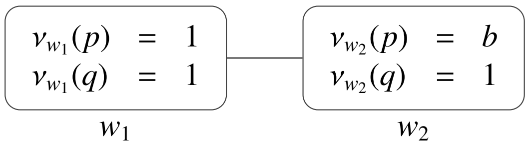
The semantic value of $[!\varphi]\psi$ is defined as:

$$v_w([!\varphi]\psi) = \begin{cases} v^{\varphi}_w(\psi) & \text{if } v_w(\varphi) \in \mathcal{D} \\ 1 & \text{otherwise} \end{cases}$$

**Example 4.** *Let $\mathcal{V} = \{0, b, 1\}$ and $\mathcal{D} = \{b, 1\}$, and consider the following model:*




*In this model, $v_{w_2}(Kq) = 0$, since $w_2 \sim w_3$ and $v_{w_3}(q) = 0$. Now, since $v_{w_3}(p) = 0$, the effect of announcing $p$ results in the following model:*




*In this new model, we have that $v^p_{w_2}(Kq) = 1$. Therefore, $v_{w_2}([!p]Kq) = 1$. (This example is specifically paraconsistent since the world $w_2$ in which $p$ is both true and false is preserved. A more classical operation would only preserve world $w_1$.)*

Following Rott (2009), if we think of a model as representing a doxastic (and epistemic) *state* and the change of models as representing the doxastic dynamics, Example 4 illustrates a revision operation where the information that $\neg q$ is lost by the announcement of $p$. Given that expansion is an operation where information is added and no information is removed, $[!\varphi]$ represents revision which is not (mere) expansion.

### 3.2 Paraconsistent Moderate Revision

For moderate belief revision, we further extend the language with the *lexicographic upgrade* operator $[\Uparrow \varphi]$ taken from van Benthem (2007). We will refer to the operator as a moderate revision operator from now on.[14] The main difference between radical and moderate revision is that the former deletes worlds whereas the latter preserves all worlds but re-orders them according to plausibility. Not deleting worlds allows them to be used for new revision in the future.

In modelling moderate revision, we modify the plausibility ordering by *upgrading* every world in which some formula $\varphi$ is designated, and we preserve the ordering otherwise. That is, given a model $M$, we define the action of moderate belief revision in terms of the model $M^{\Uparrow\varphi} = \langle W^{\Uparrow\varphi}, \sim^{\Uparrow\varphi}, \leq^{\Uparrow\varphi}, v^{\Uparrow\varphi}\rangle$, where

- $W^{\Uparrow\varphi} = W$,
- $\sim^{\Uparrow\varphi} = \sim$,

[14] This approach was further explored in a multi-agent system in Baltag and Smets (2008).

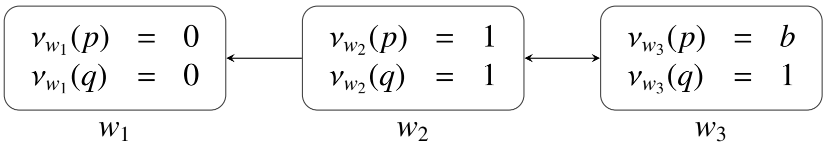
- $$\begin{aligned}\leq^{\Uparrow\varphi} \;=\; & \{\langle w_1, w_2\rangle \in \leq \;\mid\; v_{w_1}(\varphi), v_{w_2}(\varphi) \in \mathcal{D}\} \\ & \cup \{\langle w_1, w_2\rangle \in \leq \;\mid\; v_{w_1}(\varphi) = v_{w_2}(\varphi) = 0\} \\ & \cup \{\langle w_1, w_2\rangle \mid v_{w_1}(\varphi) = 0, v_{w_2}(\varphi) \in \mathcal{D}\}\end{aligned}$$

- $v^{\Uparrow\varphi} = v$.

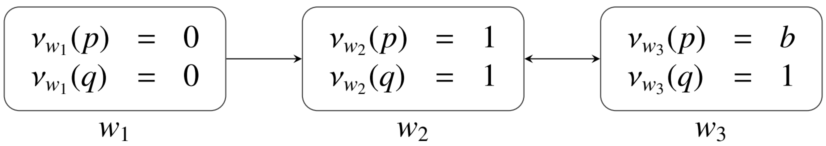
The semantics of $[\Uparrow \varphi]\psi$ is given by:

$$v_w([\Uparrow \varphi]\psi) = v_w^{\Uparrow\varphi}(\psi).$$

**Example 5.** *Consider the model M, given by:*




*In M, the maximal world is $w_1$ and $v_{w_1}(q) = 0$, so $v_x(Bq) = 0$ for every $x \in W$. The result $M^{\Uparrow p}$ of upgrading p in M is given by:*




*The maximal worlds in $M^{\Uparrow p}$ are $w_2$ and $w_3$, and $Glb\{v_{w_2}(q), v_{w_3}(q)\} = 1$. So $v_x(Bq) = 1$ for every $x \in W^{\Uparrow p}$. Therefore, $v_x([\Uparrow p]Bq) = 1$ for every $x \in W$.*

Example 5 illustrates a revision operation where the belief in $\neg q$ is lost as a result of upgrading $p$. As in radical revision, this is a case of revision and not expansion.

### 3.3 Compositional Analysis

Compositional analysis is a method of showing that the axiomatisation of dynamic logic can be reduced to that of the corresponding static logic via reduction principles (see for example van Benthem 2006 and Kooi 2007). The following two classical theorems can be found in van Benthem (2007).

**Theorem 1.** *The following are classically valid:*

1. $[!\varphi]p \equiv \varphi \to p$
2. $[!\varphi]\neg\psi \equiv \varphi \to \neg[!\varphi]\psi$
3. $[!\varphi](\psi \vee \xi) \equiv [!\varphi]\psi \vee [!\varphi]\xi$
4. $[!\varphi]K\psi \equiv \varphi \to K[!\varphi]\psi$
5. $[!\varphi]B(\psi \mid \xi) \equiv \varphi \to B([!\varphi]\psi \mid \varphi \wedge [!\varphi]\xi)$

**Theorem 2.** *The following are classically valid:*

1. $[\Uparrow\varphi]p \equiv p$
2. $[\Uparrow\varphi]\neg\psi \equiv \neg[\Uparrow\varphi]\psi$
3. $[\Uparrow\varphi](\psi\vee\xi) \equiv [\Uparrow\varphi]\psi\vee[\Uparrow\varphi]\xi$
4. $[\Uparrow\varphi]K\psi \equiv K[\Uparrow\varphi]\psi$
5. $[\Uparrow\varphi]B(\psi\,|\,\xi) \equiv (k(\varphi\wedge[\Uparrow\varphi]\xi)\wedge B([\Uparrow\varphi]\psi\,|\,(\varphi\wedge[\Uparrow\varphi]\xi))) \vee(\neg k(\varphi\wedge[\Uparrow\varphi]\xi)\wedge B([\Uparrow\varphi]\psi\,|\,[\Uparrow\varphi]\xi))$

Given an arbitrary formula containing revision operators, one proceeds inside-out, eliminating every announcement operator, using appropriate reduction principles from Theorem 1 or 2. The result is a logically equivalent formula which does not contain revision operators.[15]

Now, can we provide compositional analyses of our revision operators, defined in LP? As van Benthem (2007) demonstrates, the existence of compositional analysis marks the success of logical analysis of epistemic and doxastic dynamics. Offering a positive answer to the question, thus, allows us to meet the challenge of defining a revision operation separate from an expansion operation by using the resources made available by the study of DEL.

### 3.4 From LP to $\mathsf{LP}^{\to}$

In an LP environment, compositional analyses for radical and moderate revision are more complicated than in classical logic. An obvious problem is that LP doesn't have an implication like $p\to q$. The first attempt to get around this problem is to use the explicit definition of $\to$ as $\neg p\vee q$. But this will not do, since the two formulas $[!\varphi]p$ and $\neg\varphi\vee p$ are not logically equivalent in LP. Take $v_w(p)=b$ and $v_w(q)=0$, then $v_w([!p]q)=0$, but $v_w(\neg p\vee q)=b$. Hence the first reduction principle of Theorem1 doesn't hold.

The problem is an important one. Indeed, we can show that there are no reduction principles for the propositional case in LP. Consider the special case of propositional announcement $[!p]q$ whose truth value can be computed by the following table:

| $[!p]q$ | $q=1$ | $q=b$ | $q=0$ |
|---|---|---|---|
| $p=1$ | 1 | $b$ | 0 |
| $p=b$ | 1 | $b$ | 0 |
| $p=0$ | 1 | 1 | 1 |

[15] That the end-result is a logically equivalent formula follows from the fact that every step in the transformation produces an equivalent formula, pushing the dynamic modality inside until it is attached to a propositional variable and then eliminated. Take the following simple example to illustrate the point: $[!p](q\vee r)\equiv[!p]q\vee[!q]\equiv p\to q\vee[!p]r\equiv p\to q\vee p\to r$.

In order to satisfy the first reduction principle of radical revision in this special case, a connective with a matching truth-table definable in LP is required. But this is impossible, because such a connective satisfies detachment. To see this, consider adding $[!p]q$ to propositional LP. With $\varphi$ and $\psi$ purely propositional formulas, it's easy to prove that if $\varphi$ and $[!\varphi]\psi$ are both designated, then $\psi$ must also be designated. Given that no non-trivial detachable connective is definable in LP as we noted above, we cannot even get a reduction principle for the propositional case in LP.

However, what if we extend LP with a propositional conditional $\rightarrow$ with the following table?[16]

| $\rightarrow$ | 1 | $b$ | 0 |
|---|---|---|---|
| 1 | 1 | $b$ | 0 |
| $b$ | 1 | $b$ | 0 |
| 0 | 1 | 1 | 1 |

To our knowledge, this conditional was first introduced by D'Ottaviano and da Costa (1970) and axiomatised by Avron (1984). The logic is ideal in the sense of Arieli, Avron and Zamansky (2011); that is, it is maximally paraconsistent[17] and maximal relative to classical logic.[18] Furthermore, it satisfies both detachment and the deduction theorem, both of which are crucial for compositional analysis. Even though this conditional is well-studied, what we add to its history are new motivations stemming from considerations of dynamics.

Let's call the resulting logic $\mathsf{LP}^{\rightarrow}$. In the next section, we will show that compositional analyses of radical and moderate revision can be given in $\mathsf{LP}^{\rightarrow}$.

[16]Our choice of conditional is fully dictated by the semantics of $[!p]q$. For instance, the *RM*3 conditional would not suffice. The *RM*3 table for $\rightarrow$ is given by:

| $\rightarrow$ | 1 | $b$ | 0 |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| $b$ | 0 | $b$ | 0 |
| 0 | 1 | 1 | 1 |

Consider $[!p]q$ and take $p = 1$ and $q = b$. Then $[!p]q = b$, but $p \rightarrow q = 0$. Nevertheless, one could define a belief revision $[!\varphi]\psi$ that reduces to the *RM*3 arrow in the special case $[!p]q$. We leave this as an open problem.

[17]That is, extending $\mathsf{LP}^{\rightarrow}$ with any axiom or rule that produce new theorems yields a non-paraconsistent logic.

[18]That is, adding any non-provable classical tautologies as axioms yields a non-paraconsistent logic.

### 3.5 Compositional Analysis for Paraconsistent Doxastic Logic

In order to provide a full compositional analyses for radical and moderate revision, we define an equivalence connective $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, with the following derived table:

| $\leftrightarrow$ | 1 | $b$ | 0 |
|---|---|---|---|
| 1 | 1 | $b$ | 0 |
| $b$ | b | $b$ | 0 |
| 0 | 0 | 0 | 1 |

Radical and moderate revision operators are not conservative over LP, since they can define a detachable propositional connective. However, they are conservative over $\mathsf{LP}^{\to}$ as the following theorems show:

**Theorem 3.** *The following are valid:*

1. $[!\varphi]p \quad\leftrightarrow\quad \varphi \to p$
2. $[!\varphi]\neg\psi \quad\leftrightarrow\quad \varphi \to \neg[!\varphi]\psi$
3. $[!\varphi](\psi \vee \xi) \quad\leftrightarrow\quad [!\varphi]\psi \vee [!\varphi]\xi$
4. $[!\varphi]K\psi \quad\leftrightarrow\quad \varphi \to K[!\varphi]\psi$
5. $[!\varphi]B(\psi\,|\,\xi) \quad\leftrightarrow\quad \varphi \to B([!\varphi]\psi\,|\,\varphi \wedge [!\varphi]\xi)$

*Proof.* In each case, if $v_w(\varphi) \notin \mathcal{D}$, then both sides of the equations are designated. So assume that $v_w(\varphi) \in \mathcal{D}$.

1. This is immediate for the propositional case, by the choice of the semantics for the new connective.

2. 
$$\begin{array}{llll} v_w([!\varphi]\neg\psi) \in \mathcal{D} & \text{iff} & v_w^{\varphi}(\neg\psi) \in \mathcal{D} & \text{(definition)} \\ & \text{iff} & v_w(\psi) = b \text{ or } v_w(\psi) = 0 & \text{(definition)} \\ & \text{iff} & v_w([!\varphi]\psi) = b \text{ or } v_w([!\varphi]\psi) = 0 & (v_w(\varphi) \in \mathcal{D}) \\ & \text{iff} & v_w(\neg[!\varphi]\psi) \in \mathcal{D} & \text{(definition)} \\ & \text{iff} & v_w(\varphi \to \neg[!\varphi]\psi) \in \mathcal{D} & \text{(weakening)} \end{array}$$

3. Observe that $v_w^{\varphi}(\psi \vee \xi) \in \mathcal{D}$ iff $v_w^{\varphi}(\psi) \in \mathcal{D}$ or $v_w^{\varphi}(\xi) \in \mathcal{D}$, from which the result follows immediately.

4. Observe that, under the assumption that $v_w(\varphi) \in \mathcal{D}$, $Glb\{v_x^{\varphi}(\psi) \mid w \sim^{\varphi} x\} \in \mathcal{D}$ iff $Glb\{v_x([!\varphi]\psi) \mid w \sim x\} \in \mathcal{D}$, as can be seen in the truth table provided earlier for the propositional announcement $[!\varphi]\psi$. Thus,
$$\begin{array}{llll} v_w([!\varphi]K\psi) \in \mathcal{D} & \text{iff} & v_w^{\varphi}(K\psi) \in \mathcal{D} & \text{(definition)} \\ & \text{iff} & Glb\{v_x^{\varphi}(\psi) \mid w \sim^{\varphi} x\} \in \mathcal{D} & \text{(definition)} \\ & \text{iff} & Glb\{v_x([!\varphi]\psi) \mid w \sim x\} \in \mathcal{D} & \text{(observation above)} \\ & \text{iff} & v_w(K[!\varphi]\psi) \in \mathcal{D} & \text{(definition)} \\ & \text{iff} & v_w(\varphi \to K[!\varphi]\psi) \in \mathcal{D} & \text{(weakening)} \end{array}$$

5. This again follows directly from the observation that $Glb\{v_x^\varphi(\psi) \mid w \sim^\varphi x, x \text{ is a maximal } \xi\text{-world}\} \in \mathcal{D}$ iff $Glb\{v_x([!\varphi]\psi) \mid w \sim x, x \text{ is a maximal } (\varphi \wedge [!\varphi]\xi) - \text{world}\} \in \mathcal{D}$. □

**Theorem 4.** *The following are valid:*

$$\begin{array}{lcl} [\Uparrow\varphi]p & \leftrightarrow & p \\ [\Uparrow\varphi]\neg\psi & \leftrightarrow & \neg[\Uparrow\varphi]\psi \\ [\Uparrow\varphi](\psi\vee\xi) & \leftrightarrow & [\Uparrow\varphi]\psi\vee[\Uparrow\varphi]\xi \\ [\Uparrow\varphi]K\psi & \leftrightarrow & K[\Uparrow\varphi]\psi \\ [\Uparrow\varphi]B(\psi\,|\,\xi) & \leftrightarrow & (k(\varphi\wedge[\Uparrow\varphi]\xi)\wedge B([\Uparrow\varphi]\psi\,|\,(\varphi\wedge[\Uparrow\varphi]\xi))) \\ & & \vee(\neg k(\varphi\wedge[\Uparrow\varphi]\xi)\wedge\ B([\Uparrow\varphi]\psi\,|\,[\Uparrow\varphi]\xi)) \end{array}$$

*Proof.* Since the propositional valuation is not affected by doxastic dynamic, nor is the epistemic relation, the proof for the four first principles is immediate.

For the last principle, the right-to-left direction is a matter of unpacking the definitions. For the (harder) left-to-right direction, assume that $v_w([\Uparrow\varphi]B(\psi\,|\,\xi)) \in \mathcal{D}$ for some $w$. (In what follows, we simply sketch a proof.) We distinguish two cases, which correspond to the two disjuncts of the right-hand side.

Case 1: There are some worlds $x$ such that the following hold: 1) $v_x(\varphi) \in \mathcal{D}$, 2) $x \sim w$ and 3) $v_x^{\Uparrow^a\varphi}(\xi) \in \mathcal{D}$. That is, $v_w(k(\varphi \wedge [\Uparrow\varphi]\xi)) \in \mathcal{D}$.

Now, a simple argument establishes that the two following sets are identical:

$\alpha = \{x \in W^{\Uparrow^a\varphi} \mid w \sim x, x \text{ is a maximal } \xi\text{-world}\}$

$\beta = \{x \in W \mid w \sim x, x \text{ is a maximal } [\Uparrow\varphi]\xi\text{-world}, v_x(\varphi) \in \mathcal{D}\}$

For every $x \in \alpha, v_x^{\Uparrow^a\varphi}(\psi) \in \mathcal{D}$. So $v_x([\Uparrow\varphi]\psi \in \mathcal{D})$. Hence, for every $x \in \beta, v_x([\Uparrow\varphi]\psi) \in \mathcal{D}$. Thus, $v_w(B([\Uparrow\varphi]\psi\,|\,(\varphi \wedge [\Uparrow\varphi]\xi))) \in \mathcal{D}$.

Case 2: There are no worlds $x$ such that the following hold: 1) $v_x(\varphi) \in \mathcal{D}$, 2) $x \sim w$ and 3) $v_w^{\Uparrow^a\varphi}(\xi) \in \mathcal{D}$. That is, $v_w(\neg k(\varphi \wedge [\Uparrow\varphi]\xi)) \in \mathcal{D}$.

Again, a simple argument shows that the following two sets are identical:

$\alpha' = \{x \in W^{\Uparrow^a\varphi} \mid w \sim x, x \text{ is a maximal } \xi\text{-world}\}$

$\beta' = \{x \in W \mid w \sim x, x \text{ is a maximal } [\Uparrow\varphi]\xi\text{-world}\}$

Since for every $x \in \alpha', v_x^{\Uparrow^a\varphi}(\psi) \in \mathcal{D}$, we have that $v_x([\Uparrow\varphi]\psi) \in \mathcal{D}$, so for every $x \in \beta', v_x([\Uparrow\varphi]\psi) \in \mathcal{D}$. Thus, $v_w(B([\Uparrow\varphi]\psi\,|\,[\Uparrow\varphi]\xi)) \in \mathcal{D}$. □

## 4 Conclusion

We have now established a possibility result: it is possible to define dynamic operations of belief revision over belief states in the DEL style in the presence of paraconsistent logic. This result opens the door for further investigation into paraconsistent (and relevant) dynamic logic. In the literature on DEL, the notion of common knowledge plays a central role. How should common knowledge be defined in paraconsistent logic? How would it be affected by radical and moderate revision? Interesting issues akin to those arising for conditional belief also become prominent for conditional common knowledge with a paraconsistent logic in the background. Finally, one could define many more belief change operations with the propositional dynamic logic of Pratt (1976), as in Girard and Rott (2014). In this paper, we have provided the groundwork for further research in paraconsistent (and relevant) dynamic logic.

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