Check for updates

DOI: 10.1111/phpr.13115

ORIGINAL ARTICLE

Philosophy and Phenomenological Research

Bilateralism, coherence, and incoherence

Rea Golan

Ben Gurion University of the Negev, Beer Sheva, Israel

Correspondence
Rea Golan, Ben Gurion University of the Negev, Beer Sheva, Israel.
Email: reagolan@gmail.com

Abstract

Bilateralism is the view that the speech act of denial is as primitive as that of assertion. Bilateralism has proved helpful in providing an intuitive interpretation of formalisms that, *prima facie*, look counterintuitive, namely, multiple-conclusion sequent calculi. Under this interpretation, a sequent of the form $\Gamma \vdash \Delta$ is regarded as the statement that it is incoherent, according to our conversational norms, to occupy the position of asserting all the sentences in Γ and denying all the sentences in Δ. Some have argued, based on this interpretation, that the notion of invalidity is as conceptually primitive and important as the notion of validity: whereas the latter is couched in terms of what positions are incoherent and hence untenable, the former is couched in terms of what positions are coherent and hence tenable. My ultimate goal in this paper is to contest this view. Based on a novel technical account of the two notions—one that I find more accurate than the existing accounts in the literature—I shall argue that the notion of incoherence takes conceptual priority over the notion of coherence.

1 | INTRODUCTION

Bilateralism is the view that the speech act of denial is as primitive as that of assertion. Bilateralists thus hold that we cannot explain what it is to deny a given sentence in terms of what it is to

This is an open access article under the terms of the Creative Commons Attribution-NonCommercial-NoDerivs License, which permits use and distribution in any medium, provided the original work is properly cited, the use is non-commercial and no modifications or adaptations are made.
© 2024 The Author(s). *Philosophy and Phenomenological Research* published by Wiley Periodicals LLC on behalf of Philosophy and Phenonmenological Research Inc.

assert some other sentence.[1] In particular, contrary to philosophers such as Frege (1960), Geach (1965), and Dummett (1973, pp. 295–363), bilateralists hold that denying a sentence p should not be understood in terms of asserting the sentence "not-p." If anything, the order of explanation should run the other way around, and we should explain what "not-p" is in terms of denying p.

This view is argued for on various grounds, and couched technically in several ways (see, e.g., Price, 1990; Restall, 2005, 2020; Ripley, 2011, 2013, 2015; Rosenblatt, 2019; Rumfitt, 2000; Smiley, 1996). The one that will be at stake here, whose prominent advocates are Restall (e.g., in Restall, 2005, 2020) and Ripley (e.g., in Ripley, 2011, 2013, 2015), aims light on formalisms that, *prima facie*, look counterintuitive, namely, multiple-conclusion sequent calculi. Intuitively, a proof may have several premises, but only one conclusion. In multiple-conclusion sequent calculi, by contrast, a bunch of conclusions appear together on the right-hand side of the turnstile, and the sequent $\gamma_1, \ldots, \gamma_n \vdash \delta_1, \ldots, \gamma_m$ is regarded as equivalent to the statement $(\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow (\delta_1 \vee \ldots \vee \delta_m)$.

This is hard to explain, yet Restall and Ripley managed to provide here a neat bilateralist explanation.[2] To begin with, they point out that for an interlocutor to engage in a rational discourse, she has to occupy a *position* in it, by way of committing herself to some assertions and denials. Let us associate each such position with an ordered pair of the form $< \Gamma, \Delta >$, where Γ is the position's set of assertions, and Δ its set of denials.

Next, Restall and Ripley point out that positions are by nature subject to normative assessment. In particular, we hold interlocutors to conversational norms of *coherence* and so each position is either *coherent* or *incoherent* according to these norms. For example, the position that consists of the assertion "Today is Friday" and the denial "Tomorrow is Saturday" is incoherent, whereas the position that consists of the assertion "Today is Friday" and the denial "Tomorrow is Monday" is coherent.

With these notions in mind,[3] Restall and Ripley suggest reading the sequent $\Gamma \vdash \Delta$ as expressing the statement that the position $< \Gamma, \Delta >$ is incoherent. Hence, the reading of $\gamma_1, \ldots, \gamma_n \vdash \delta_1, \ldots, \gamma_m$ as equivalent to $(\gamma_1 \wedge \ldots \wedge \gamma_n) \rightarrow (\delta_1 \vee \ldots \vee \delta_m)$: provided that an interlocutor coherently asserts all of $\gamma_1, \ldots, \gamma_n$, she cannot coherently deny, in addition, all of $\delta_1, \ldots, \delta_m$; i.e., for her position to remain coherent, she must refrain from denying at least one such δ_i $(1 \leq i \leq m)$. Notice that this interpretation generalizes a familiar understanding of the notion of validity, namely, that an inference from A to B is valid just in case it is incoherent to assert A and deny B.

Restall and Ripley's interpretation is neat, intuitive, and smoothly extends to sequent rules. Take for example the sequent rules for classical negation, which might strike one as counterintuitive at first glance:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \neg L \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \neg R$$

[1] I take it that the contents of assertions and denials are *sentences* in a given (interpreted) language; we certainly cannot assert or deny formulas or sentences in uninterpreted languages. Alternatively, one may suggest that the contents of assertions and denials be understood as *propositions* rather than sentences. Yet, the bilateralist order of explanation seems committed to understanding propositions in terms of (the norms governing) assertions and denials, and not the other way around. At any rate, for my purposes in the present paper there is no need to take a firm stand on this issue.

[2] The above explanation focuses on conversational norms. Restall (2005) also mentions a more mental-oriented sort of explanation that will not be under consideration here.

[3] I take it that "coherent" and "incoherent" are *concepts* that apply to positions. Alternatively, one may view them as *relations* between collections of assertions and denials. Nothing essential in what follows depends on this distinction, and so I shall refer to them as *notions*, for the sake of neutrality.

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

In the proposed reading, $\neg L$ says that if rejecting A is incoherent (in the context where one asserts all the sentences in Γ and denies all the sentences in Δ), then so is asserting $\neg A$ (in the same context). Analogously, $\neg R$ says that if asserting A (in some context) is incoherent, then so is rejecting $\neg A$ (in the same context). By way of contraposition, Ripley makes things even simpler, justifying $\neg L$ and $\neg R$ on the grounds that "a negation $\neg A$ is assertible just when its negatum A is deniable, and $\neg A$ is deniable just when A is assertible" (Ripley, 2013, p. 142).

Now, if validity is couched in terms of incoherence, then its complementary notion of invalidity—couched in terms of coherence—seems just as conceptually primitive and important: whereas the former notion concerns what positions are incoherent and hence untenable according to our conversational norms, the latter notion concerns what positions are coherent and hence tenable. Indeed, even though the bilateralist literal reading of $\neg L$ and $\neg R$ is in terms of the *incoherence conditions* of certain positions, Ripley explains these rules by corresponding *coherence conditions*. For the latter explanation to go through, the very notion of coherence has to be no less explanatory, or conceptually primitive, than the notion of incoherence. Indeed, we have *prima facie* good reasons for considering the two complementary notions equally primitive and important. After all, it is equally important for our understanding of rational discourse to know which positions are coherent and which positions are incoherent.

This view—namely, that the two notions are equally primitive and important—has been argued for explicitly as well as comprehensively in a recent work by Rosenblatt (2019). In brief, his claim is that under the bilateralist interpretation, all the reasons to privilege a treatment of validities over one of invalidities lose much of their force: validity and invalidity are on a par, and there is no reason to treat validities primitively and to define invalidities in terms of them.

My ultimate goal in the present paper is to contest this view. I shall argue that, under the above bilateralist interpretation, the notion of incoherence (validity) takes conceptual priority over the notion of coherence (invalidity), and decisively so. My argument for this claim is based on a novel technical account of the two notions.

The paper proceeds as follows. In Section 2, I discuss bilateralism in more detail. In Section 3, I critically examine perhaps the only unified account of coherence and incoherence that is available in the literature (Rosenblatt, 2019) (based on the calculus of invalidities introduced by Tiomkin, 1988 and further elaborated, e.g., in Carnielli & Pulcini, 2017), in the form of a "hybrid calculus" that contains both sequent rules governing the notion of incoherence, and "anti-sequent" rules governing the notion of coherence. I argue that this account is flawed both conceptually and technically. In Section 4, I present an alternative account, based on an alternative calculus that I develop. The alternative calculus results from adding only two structural rules to the regular sequent rules; once these rules are added, it becomes possible to derive all the anti-sequent rules. Hence, we can derive from the notion of incoherence all the principles governing the notion of coherence. In Section 5, I show that the opposite move cannot be carried out in full. That is, we cannot derive from the notion of coherence all the principles governing the notion of incoherence. I therefore conclude that the notion of incoherence takes conceptual priority over the notion of coherence.

2 | MORE ON BILATERALISM

Bilateralism is an *inferentialist* theory, i.e., a theory that specifies the *meanings* of linguistic expressions in terms of the rules governing the *use* of these expressions, and in particular, their use

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

in inference making. As they understand 'inference' in terms of positions and whether they are coherent or incoherent, bilateralists are committed to specifying the meanings of linguistic expressions in terms of their assertibility and deniability conditions, i.e., the conditions under which asserting or denying a sentence whose principal operator is the expression under consideration makes one's position coherent or incoherent.

This point is nicely illustrated by the above-discussed interpretation of the sequent rules for negation. As we saw, the bilateralist reading of these rules amounts to the following conditions: $\neg A$ is coherently assertible just when A is coherently deniable, and $\neg A$ is coherently deniable just when A is coherently assertible. Conceived of from the perspective of a theory of meaning, these conditions specify the meaning of negation, as they specify the circumstances under which it is coherent (incoherent) to assert or deny sentences of the form $\neg A$.

We may think of the other sequent rules along the same lines. Consider the sequent rules for classical disjunction[4]:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\,\vee L \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\,\vee R$$

The bilateralist reading of $\vee L$ amounts to the condition that a disjunction $A \vee B$ is coherently assertible just when at least one disjunct—either A or B—is coherently assertible. The reading of $\vee R$ amounts to the condition that $A \vee B$ is coherently deniable just when both disjuncts are coherently deniable.

The structural rules are given a similar justification. To start off, assertion and denial are opposite speech acts by their very nature. Which is to say, it is incoherent for one to assert and deny the same sentence A, regardless of one's other assertions and denials. Hence, the axiom of identity ($\Gamma, A \vdash A, \Delta$). For technical reasons that will become important only later, we shall restrict the axioms to sequents that contain only atoms, namely, $\Gamma, p \vdash p, \Delta$ where the members of $\Gamma \cup \Delta$ are all atoms. Given all the rules discussed in this section, it is possible to prove by structural induction that any sequent of the form $\Gamma, A \vdash A, \Delta$ is derivable, regardless of the complexity of the relevant sentences.

Next, it is plausible to assume that if a position $< \Gamma, \Delta >$ is incoherent, then adding more assertions and denials to the position would not make it coherent. Hence, the structural rule of weakening:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Sigma \vdash \Delta, \Pi}\,Weak$$

Finally, the rule of cut

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}\,Cut$$

[4] For the sake of brevity, I confine the discussion in this paper to a language with only two connectives, namely, negation and disjunction. Nothing hangs on this, as the other connectives and their corresponding operational rules may clearly be given analogous justifications.

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

TABLE 1 The sequent rules.

Axioms (where $\Gamma \cup \Delta$ consists of atoms):

$$\Gamma, p \vdash p, \Delta$$

Structural Rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Sigma \vdash \Delta, \Pi}\ \mathit{Weak}$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}\ \mathit{Cut}$$

Operational Rules:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ {+\neg L} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ {+\neg R}$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ {+\vee L} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ {+\vee R}$$

is justified on the grounds that a coherent position can be coherently extended with any sentence A, either by asserting that sentence, or by denying it; otherwise, the position is incoherent to begin with.[5]

To sum up, I have laid down the bilateralist justifications for a bunch of rules, that form the sequent calculus for classical logic that appears in Table 1. Notice that I have labelled the operational sequent rules with the additional symbol $+$. This notation indicates their nature as *sequent rules*, as opposed to *anti-sequent rules* that will be introduced in Section 3. It is intrinsic to the bilateralist interpretation that the contexts are read here as *sets* rather than *multisets*. Consequently, further structural rules such as contraction and permutation are regarded here as built-in properties of the system. It is also worth pointing out that the operational rules are all invertible in this calculus: if the conclusion-sequent of each such rule is derivable, so is (are) the premise-sequent(s).[6] This fact—as well as its proof—will prove significant in what follows, and so it is discussed in greater detail in Sections 4 and 5.

Having introduced this system, it is time to go to the heart of the matter, which is the two complementary notions of coherence and incoherence underlying our understanding of the above rules. First, it is vital to point out that coherence and incoherence are regarded by bilateralists as *material* notions rather than *formal* ones. That is, whether a given position is coherent or incoherent is determined by matters that go beyond the logical structure of sentences.[7] To illustrate this point, recall the example given above: the position that consists of the assertion "Today is Friday" and the denial "Tomorrow is Saturday" is incoherent; clearly, such incoherence has nothing to

[5] Some bilateralists, notably Ripley (2013), object to the cut rule on the grounds that certain paradoxical sentences can be neither coherently asserted nor coherently denied. As I do not deal in this paper with paradoxes of any kind—in effect, I confine the discussion to a propositional language where such paradoxes do not show up—I shall assume that the cut rule does hold. For similar reasons, I do not consider here other ways of going substructural, e.g., by giving up the axioms or the rule of weakening, or even rules implicit in the bilateralist reading of sequents such as contraction.

[6] For a proof of this and further discussion, see (Negri and von Plato, 2001, pp. 49–60).

[7] See, e.g., (Rosenblatt, 2019, pp. 3-4) and the references therein. In this regard, bilateralists clearly follow Brandom (2000, pp. 52–55)

do with logical structure, as the logical constants do not even figure in these sentences. Consequently, the bilateralist account of *logical* expressions rests on the *material* notions of coherence and incoherence, and not the other way around.

As an immediate implication of such materiality, we may admit of "material axioms," in addition to the instances of identity. If we admit, e.g., that it is incoherent to assert "Today is Friday" while denying "Tomorrow is Saturday," then we have to add an axiom of the form $p \vdash q$, where p and q are atoms. Nonetheless, to keep things as simple as possible, I shall ignore the possibility of such material axioms here, and discuss only the logical aspect of the system under consideration.

Second, more needs to be said about the *extensional* as well as the the *conceptual* relations between the notions of coherence and incoherence. As for extensionality, it is common to make the following assumption:

> *The extensional assumption*: The two notions of coherence and incoherence are mutually exclusive and jointly exhaustive; namely, each position is either coherent or incoherent, and no position can be both.

This assumption is quite intuitive, and underlies much of the bilateralist literature. Indeed, Ripley's explanation of the negation sequent rules in terms of coherence conditions seems to depend on it.

It is worth mentioning, though, that the extensional assumption has been called into question recently. Confronted with a semantic paradox of incoherence, Rosenblatt (2019, 2021) suggests that certain positions constitute counterexamples to the assumption, violating either mutual exclusivity or joint exhaustivity.[8] Indeed, we may understand "incoherence" not as "not coherent," but positively as some sort of clashing, whereas "coherence" may be roughly be understood as some sort of safety. Such positive understanding may leave conceptual room for rejecting the extensional assumption.[9]

Be that as it may, the extensional assumption will be a working hypothesis of mine in this paper. For, as Rosenblatt himself acknowledges, such paradoxical positions are "very rare and infrequent" (Rosenblatt, 2019, p. 16). Indeed, semantic paradoxes—such as the one considered by Rosenblatt—show up only in the context of expressively rich languages. On the other hand, a "core theory" of coherence and incoherence—such as the one I aim to develop here—has to predominantly address the central characteristics of these notions, which are expressed in a material vocabulary that does not in and of itself require the sophistication of expressively rich languages.[10] If anything, we should go the other way around here: first, develop a "core theory" of coherence and incoherence, and then turn to see whether and how such a theory may deal with self-referential paradoxes. After developing such a theory in the main body of the paper, I will point out towards the end some implications the theory might have for dealing with relevant paradoxes.

In addition to the extensional assumption, one can find in the literature a *conceptual* assumption that will be at the center of my discussion. Given two mutually exclusive and jointly exhaustive notions, it is evidently possible to specify the extension of each notion by the extension

[8] As explained above, dealing with such paradoxes exceeds the scope of the present paper. Thus, the exact details of Rosenblatt's paradox are not so important. See (Rosenblatt, 2019, pp. 15–19) for an extensive discussion.

[9] I would like to thank an anonymous referee for making this point.

[10] Moreover, it is not even clear that such a theory can be formulated in first-order logic—which is what is required for generating self-reference—as the invalidities of first-order logic are not axiomatizable (as Rosenblatt himself mentions in Rosenblatt, 2019, pp. 7–9).

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

of the other. Hence, one cannot help but wonder whether such specification is sufficient for *defining* each concept, in which case the two notions are to be regarded as equally primitive. This case seems to be the default one: the two notions are to be regarded as equally primitive unless and until an argument is provided to the effect that one of them is conceptually more primitive, and the other notion has to be defined in terms of it. Such an argument may take various forms. To give a famous historical anecdote demonstrating this point, Descartes argues in the third of his Meditations on *cognitive* grounds that the infinite cannot be defined in terms of the finite: "[T]he idea of an infinite substance would not [...] be in me, since I am finite, unless it derived from some substance that is really infinite" (Descartes, 2008, p. 32).

Yet, on the face of it, no such argument is available in regard to the notions of coherence and incoherence. It is thus natural to make the following assumption, which I call the conceptual assumption:

> *The conceptual assumption*: The two notions of coherence and incoherence are equally primitive, as neither notion takes conceptual priority over the other.

Reading through the literature, one almost gets the feeling that the conceptual assumption has an unquestionable consensus.[11] Indeed, we already saw how Ripley makes implicit use of it, and that Rosenblatt defends it explicitly.[12]

As I said, my ultimate goal in the present paper is to contest the conceptual assumption. I shall argue that—despite appearances to the contrary—there is a more conceptually primitive notion here, and that is the notion of incoherence. But first things first, it is clear that to discuss these ideas we need a detailed account of the two notions. Section 3 is dedicated to a critical examination of the account that is already available in the literature. In Section 4, I develop my own account.

3 | THE COHERENCE RULES

To the best of my knowledge, one can find in the literature basically one calculus of invalidities introduced by Tiomkin (1988).[13] Based on this calculus, Rosenblatt (2019, 2021) introduces a hybrid calculus of validities and invalidities, of which he makes use for a bilateralist account of coherence and incoherence in Rosenblatt (2019). The basic idea behind this calculus is that we may derive not only validity statements of the form $\Gamma \vdash \Delta$, but also invalidity statements of the form $\Gamma \nvdash \Delta$. To put it bilateralistically, we denote by $\Gamma \nvdash \Delta$ the statement that the position $< \Gamma, \Delta >$ is coherent, i.e., that it is permissible to occupy that position according to our conversational norms. Following the literature, I shall sometimes refer to such coherence statements as "anti-sequents."

[11] One might think that the conceptual assumption is a natural analogue of the bilateralist's position for the conceptual parity between assertion and denial. But that does not seem parallel, since it is very plausible, on the face of it, that incoherence can directly be given as the failure of coherence—after all, that's what the word says. Denial is not the failure of assertion. I thank an anonymous referee for making this point to me.

[12] Likewise, Restall in his foundational work on bilateralism (Restall, 2005) justifies sequent rules in terms of coherence and incoherence conditions alternately, a move that seems committed both to the extensional assumption and to the conceptual one.

[13] See also Carnielli and Pulcini (2017) and the references therein for some variations of this calculus. Notice that I am focusing here on a *sequent calculus* of invalidities, rather than axiomatic or natural deduction systems.

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

To develop such a calculus, the first thing we have to do is specify what anti-sequents count as axioms. Given the extensional assumption, this matter has already been settled. If, as assumed above, the basic incoherent positions—associated with sequents that contain only atoms—are all of the form $\Gamma, p \vdash p, \Delta$ (where the members of $\Gamma \cup \Delta$ are all atoms), then the basic coherent positions—associated with anti-sequents that contain only atoms—are all of the form $\Gamma \nvdash \Delta$ where $\Gamma \cap \Delta = \emptyset$ (and where the members of $\Gamma \cup \Delta$ are all atoms.) Roughly, the thought here is that the atoms are conceptually independent, and so an assertion/denial concerning of one atom cannot have any bearing (as far as coherence goes) on an assertion/denial of another. Let me nevertheless point out that, as hinted above, nothing hangs on this thought, as there is a trade-off between the atomic sequents and anti-sequents. Thus, to enrich our calculus with "material inferences" associated with incoherent poisitions of the form $p \vdash q$, all we have to do is add this sequent axiom and remove the corresponding anti-sequent axiom $p \nvdash q$. Such trade-offs affect neither the results discussed below, nor their philosophical significance.

Now, for a comprehensive account of coherence and incoherence, we should, indeed must, have not only sequent rules governing incoherence conditions, but also anti-sequent rules governing coherence conditions. For example, we need the following anti-sequent rules for negation

$$\frac{\Gamma \nvdash A, \Delta}{\Gamma, \neg A \nvdash \Delta} -\neg L \qquad \frac{\Gamma, A \nvdash \Delta}{\Gamma \nvdash \neg A, \Delta} -\neg R$$

whose rationale is that if denying (asserting) A is coherent in some context, then so is asserting (denying) $\neg A$ in that context.

Before reviewing the other anti-sequent rules, I would like to stress the last point, because there is more to it than meets the eye. On the face of it, there is a plain and simple way of adding anti-sequents: declare $\Gamma \nvdash \Delta$ as derivable whenever $\Gamma \vdash \Delta$ is not derivable, and only in this case, for every Γ and Δ.[14] Notice that this suggestion merely extends our picking up of the anti-sequent axioms. Notice too that the suggestion makes the candidates—in effect, all the candidates—for anti-sequent rules *admissible*, i.e., ones whose addition to the sequent rules doesn't change the set of derivable sequents and anti-sequents. For example, suppose that $\Gamma \nvdash A, \Delta$ is derivable. It follows that $\Gamma \vdash A, \Delta$ is not derivable, in which case $\Gamma, \neg A \vdash \Delta$ is not derivable either.[15] Hence, $\Gamma, \neg A \nvdash \Delta$ is derivable. Consequently, $-\neg L$ is admissible. Dwelling on this result, one might be tempted to argue, on the basis of the law of parsimony, that the above suggestion is better than adding specific anti-sequent rules such as $-\neg L$ and $-\neg R$.

Yet reading through the literature, one cannot help but notice that this suggestion is not even considered a viable option. A reasonable explanation for this goes as follows. It seems that the explanatory power of a hybrid calculus—such as the one we are trying to develop here—rests on *making explicit* the derivability relations that hold between various sequents and anti-sequents. Moreover, it seems that making such relations explicit in a given calculus amounts to making them *derivable*, rather than merely admissible. I shall call this explanatory line of thought the derivability assumption:

> *The derivability assumption*: A given calculus makes explicit the derivability relations that hold between different sequents and anti-sequents iff those relations are derivable in that calculus.

[14] This suggestion is feasible because propositional logic is decidable, and so there is an effective procedure that determines (in a finite amount of time) whether $\Gamma \vdash \Delta$ is derivable or not.

[15] This is so because, as mentioned above, the operational sequent rules that are at stake here are all invertible.

Given the derivability assumption, it is clear that the above suggestion doesn't hold water, as it undermines the explanatory power of the calculus.

Of independent interest is the question on what grounds the derivability assumption may be justified. I shall not address this question here, simply because there does not appear to be any objection to the derivability assumption in the literature.[16] Rather, I shall make the assumption and go on without further ado to discuss various anti-sequent rules.

Regarding such rules, it seems like we do not have much of a choice here. For, maintaining the extensional assumption behooves us to pick up anti-sequent rules that are the complementary of the sequent rules. Thus, in addition to $-\neg L$ and $-\neg R$, we must pick up the following two left rules for disjunction

$$\frac{\Gamma, A \nvdash \Delta}{\Gamma, A \vee B \nvdash \Delta}\ {-\vee L1} \qquad \frac{\Gamma, B \nvdash \Delta}{\Gamma, A \vee B \nvdash \Delta}\ {-\vee L2}$$

whose bilateralist rationale is that if it is coherent to assert A (B) in some context, then it is coherent to assert $A \vee B$ in that context. Likewise, we must pick up the following right rule for disjunction

$$\frac{\Gamma \nvdash A, B, \Delta}{\Gamma \nvdash A \vee B, \Delta}\ {-\vee R}$$

whose bilateralist rationale is that if it is coherent to deny A and B in some context, then it is coherent to deny $A \vee B$ in that context.

As for structural rules, the following anti-weakening rule suggests itself:

$$\frac{\Gamma, \Sigma \nvdash \Pi, \Delta}{\Gamma \nvdash \Delta}\ {Anti{-}Weak}$$

The bilateralist rationale here is that if a given position is coherent, then making fewer assertions and denials—that is, asserting and denying fewer sentences—is coherent too.

One may also expect a corresponding cut rule for anti-sequents. The situation here is more tricky, though. Tiomkin (1988) has suggested two cut rules that result from a contrapositive reading of the cut rule for sequents:

$$\frac{\Gamma \nvdash \Delta \quad \Gamma \vdash A, \Delta}{\Gamma, A \nvdash \Delta}\ {Tiomkin{-}cut} \qquad \frac{\Gamma \nvdash \Delta \quad \Gamma, A \vdash \Delta}{\Gamma \nvdash A, \Delta}\ {Tiomkin{-}cut}$$

[16] An anonymous referee to this journal suggests that this is not just a matter of taste but of the structural difference between admissibility and derivability. A rule might be admissible in a calculus, but no longer be admissible when the calculus is extended by further axioms or rules. Consider cut-free classical calculi, for which cut is admissible, but is no longer admissible when extended with a naïve truth predicate. Leaving the coherence of operational rules as hostage to admissibility may thus not be enough to reassure us that these rules remain in place when the system is extended with new concepts and their corresponding new rules.

Notice, however, that this argument does not go through when it comes to the above suggestion. For, if we simply dictate non-compositionally what anti-sequents are derivable, i.e., by declaring that $\Gamma \nvdash \Delta$ as derivable whenever $\Gamma \vdash \Delta$ is not derivable regardless of the structure of Γ, Δ, then the operational anti-sequent rule will remain admissible even if the language is extended. For example, the above proof that $-\neg L$ is admissible goes through independently of whether we extend our language with new axioms or rules.

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

TABLE 2 The Anti-sequent rules.

Axioms (where $\Gamma \cup \Delta$ consists of atoms, and $\Gamma \cap \Delta = \emptyset$):

$$\Gamma \nvdash \Delta$$

Structural Rules:

$$\frac{\Gamma, \Sigma \nvdash \Pi, \Delta}{\Gamma \nvdash \Delta}\ \textit{Anti-Weak}$$

Operational Rules:

$$\frac{\Gamma \nvdash A, \Delta}{\Gamma, \neg A \nvdash \Delta}\ -\neg L \qquad \frac{\Gamma, A \nvdash \Delta}{\Gamma \nvdash \neg A, \Delta}\ -\neg R$$

$$\frac{\Gamma, B \nvdash \Delta}{\Gamma, A \vee B \nvdash \Delta}\ -\vee L \qquad \frac{\Gamma, A \nvdash \Delta}{\Gamma, A \vee B \nvdash \Delta}\ -\vee L \qquad \frac{\Gamma \nvdash A, B, \Delta}{\Gamma \nvdash A \vee B, \Delta}\ -\vee R$$

Yet, as Carnielli and Pulcini mention (Carnielli & Pulcini, 2017), both these rules are proof-theoretically unsatisfactory, as they do not display any cut formula.[17] Alternatively, Carnielli and Pulcini suggest that the following two instances of anti-weakening

$$\frac{\Gamma \nvdash A, \Delta}{\Gamma \nvdash \Delta} \qquad \frac{\Gamma, A \nvdash \Delta}{\Gamma \nvdash \Delta}$$

be regarded as unary cut rules, since A plays in each of them the role of a cut formula. Following this line of thought, anti-weakening will be our one and only structural anti-sequent rule. The Tiomkin-cut rules will show up at some point in Section 5, but other than that nothing in what follows depends on my choice here.

Taken together, the above discussed rules form the anti-sequent calculus for the invalidities of classical logic that appears in Table 2. Combining the axioms and rules of Table 1 and Table 2, we get the hybrid calculus $\mathcal{H}$. Rosenblatt (2019, 2021) proves that $\mathcal{H}$ is both *externally complete* and *externally consistent*: for every Γ, Δ, either $\Gamma \vdash \Delta$ is derivable or $\Gamma \nvdash \Delta$ is derivable, but not both. Thus, a bilateralist account of coherence and incoherence based on the calculus $\mathcal{H}$ maintains the extensional assumption.

Interesting as all this may be, I have serious reservations about $\mathcal{H}$ and, *ipso facto*, about any bilateralist account that is based on it. Most importantly, notice that the rules of $\mathcal{H}$ are all (what might be called) *segregated*, i.e., they do not mix sequents and anti-sequents. As a result, there is no way in this system to generate mixed derivations combining both sequents and anti-sequents. This is a matter of concern because, on the face of it, a satisfactory account of coherence and incoherence should shed light on, among other things, the interrelations between coherent and incoherent positions. My concern here is not merely abstract. Consider for instance the following suggested mixed structural rules:

$$\frac{A \vdash}{\nvdash A}\ \textit{Mix1} \qquad \frac{\vdash A}{A \nvdash}\ \textit{Mix2}$$

[17] An anonymous referee pointed out that this is only a reason not to think of the Tiomkin cut rules as genuine anti-sequent versions of cut. Yet, there is no problem with having these rules on board. In that case, however, it is at least questionable whether these rules should be among the basic rules of the system. Indeed, in my system (presented in the next chapter) the Tiomkin cut rules can be derived from more basic rules.

1933159 2, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$Mix1$ says that if asserting A is in and of itself—that is, where no other assertions and denials are involved—incoherent, then denying A must in and of itself be coherent. $Mix2$ says that if denying A is in and of itself incoherent, then asserting A must in and of itself be coherent. These rules are intuitively justified: the rationale behind them is that if a sentence is unassertible then it must be deniable, and if a sentence is undeniable then it must be assertible. Yet, $Mix1$ and $Mix2$ cannot be derived in the above system, precisely because its rules are all segregated.

To appreciate how strong this point is, notice that that $Mix1$ and $Mix2$ are admissible in the above calculus. To show that $Mix1$ is admissible (the proof that $Mix2$ is admissible is analogous), suppose that $A \vdash$ is derivable. Therefore, $\vdash A$ is not derivable: otherwise, the empty sequent $\vdash$ would be derivable by one application of cut, and so every sequent would be derivable by one application of weakening, thus violating external consistency. By external completeness, $\nvdash A$ is derivable, as required. Given the derivability assumption, it follows that $\mathcal{H}$ does not make explicit the derivability relations that hold between different sequents and anti-sequents.[18] Hence, $\mathcal{H}$ falls short of underlying a comprehensive account of coherence and incoherence.

The segregated nature of $\mathcal{H}$'s rules has an additional side effect that I find problematic: it results in a proliferation of operational rules. $\mathcal{H}$ has no less than four rules for each connective: one pair of sequent rules and one pair of anti-sequent rules. This is problematic for at least two reasons. First, it is difficult to justify such a proliferation of rules not only in the face of parsimony, but also from a bilateralist point of view that is committed to the extensional assumption.[19] After all, under this assumption these rules should be interdependent, and thus interderivable in any calculus that genuinely reflects the nature of the notions of coherence and incoherence.

Second, the proliferation of operational rules violates a well-entrenched tenet in proof theory, which I call the *structural tenet*. According to this tenet, in developing a given proof system, we should do our best not to modify or augment the operational rules, but rather let structural features—i.e., both the structural rules and some restrictions on the structure of derivations—do as much of the heavy lifting as possible.[20] Adding anti-sequents is not different in this regard, and so it should hopefully be obtained by playing around with the system's structural features, rather than by augmenting the operational rules.

Summing up, we have good reasons to look for a better bilateralist account of coherence and incoherence, based on a calculus that employs rules—preferably, structural rules—that allow us to generate mixed derivations combining both sequents and anti-sequents. In the next section, I develop such an account and explore some of its consequences.

---

[18] $Mix1$ and $Mix2$ are not the only problem here. In fact, there are infinitely many admissible yet underivable rules in $\mathcal{H}$. For one thing, it is clear that the following rule:

$$\frac{A \vee B \vdash}{\nvdash A}\ Mix\vee$$

is also admissible yet underivable, and we may thus generate infinitely many other such rules by weakening the disjunction in the premise-sequent even further: $A \vee B \vee C, A \vee B \vee C \vee D$, etc.

[19] To the best of my knowledge, the first criticism of bilateralism on the basis of parsimony is due to Dummett (1973, p. 317), who argues that having denial as an independent speech act must result in a multiplicity of inference rules. A version of this criticism applies to the kind of bilateralism associated with Smiley (1996) and Rumfitt (2000) (see Restall, 2020, p. 12; for responses, see, e.g., Price, 1990; Restall, 2005; Ripley, 2011). Notice that as mentioned above, the extensional assumption makes the criticism stronger in the case of hybrid proof systems.

[20] There are numerous examples illustrating this tenet, the first one (to the best of my knowledge) being Gentzen's sequent calculi LJ and LK for intuitionistic and classical logics, respectively, which differ only structurally, on the basis of whether we allow for multiple-conclusions to appear on the right-hand side of the turnstile (Gentzen, 1964).

4 | THE ALTERNATIVE ACCOUNT

In developing the calculus below, I do my best to meet the structural tenet. In particular, instead of adding specific anti-sequent rules, I introduce rules governing the very notion that is at stake here, namely, the coherence turnstile $\nvdash$.

To be sure, the coherence turnstile $\nvdash$ is not to be conflated with some piece of vocabulary governed by some operational rules. For one thing, $\nvdash$ cannot be embedded in complex expressions; for another, $\nvdash$ is not introduced on either side of the incoherence turnstile $\vdash$, but rather has the status of a complementary, equally legitimate, turnstile. Nonetheless, the $\nvdash$ rules introduced below are modeled after well-known operational rules, namely, Gentzen's (natural deduction) introduction and elimination rules for negation (Gentzen, 1964). This is so because the $\nvdash$ rules aim at capturing one of the extensional assumption's implications, i.e., that the coherence turnstile $\nvdash$ may roughly be thought of as negating the incoherence turnstile $\vdash$.

Let me begin with the elimination rule, which is the simpler. Gentzen's elimination rule, we may recall, is $\dfrac{A \quad \neg A}{\bot}$, whose rationale is that A and $\neg A$ are mutually exclusive. Analogously, we get the following elimination rule for anti-sequents, which I label $\nvdash -Out$:

$$\dfrac{\Gamma \vdash \Delta \quad \Gamma \nvdash \Delta}{\vdash}\ {\scriptstyle \nvdash - Out}$$

As long as the empty sequent $\vdash$ remains underivable—and the extensional assumption dictates that we insist on this—the $\nvdash -Out$ rule guarantees that no position is both coherent and incoherent.

As for the introduction rule, things are a bit more complex. Gentzen's rule goes as follows: $\dfrac{\begin{matrix}{}^{1:}[A] \\ \vdots \\ \bot\end{matrix}}{A}\,[1]$. That is, to introduce $\neg A$ we have to show that assuming A leads to absurdity. Analogously, we get the following rule, which I label $\nvdash -In$:

$$\dfrac{\begin{matrix}{}^{1:}[\Gamma \vdash \Delta] \\ \vdots \\ \vdash\end{matrix}}{\Gamma \nvdash \Delta}\ {\scriptstyle \nvdash - In}\ [1]$$

A few clarificatory comments on this rule are now in order. Most notably, the rule makes use of assuming and discharging sequents, thereby presenting us with a system that integrates natural deduction-like elements into a sequent calculus. This is technique is not new: it was first used in a paper by Schroeder-Heister about forty years ago (Schroeder-Heister, 1984), and has received much attention in recent years (see, e.g., Golan & Hlobil, 2022; Golan, 2023; Hlobil, 2019). The assumption $\Gamma \vdash \Delta$ is presented inside square brackets and the superscript 1 is not part of the syntax, but rather is meant to keep track of the assumptions that need to be discharged. Assumption 1 is discharged in the last step of the derivation, immediately below the empty sequent, and the derivation marked by the four dots—namely, the derivation of the empty sequent $\vdash$ from $\Gamma \vdash \Delta$—may involve more than one leaf (and hence more than one assumption). Though the rule is technically intriguing, the bilateralist rationale underlying it is obvious: the statement that the position $< \Gamma, \Delta >$ is coherent is established by way of *reductio ad absurdum*; that is, the assumption that $< \Gamma, \Delta >$ is incoherent leads to absurdity.

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

TABLE 3 The Calculus $\mathbb{H}$.

Sequent Axioms (where $\Gamma \cup \Delta$ consists of atoms):

$$\Gamma, p \vdash p, \Delta$$

Anti-sequent Axioms (where $\Gamma \cup \Delta$ consists of atoms, and $\Gamma \cap \Delta = \emptyset$):

$$\Gamma \nvdash \Delta$$

Structural Rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Sigma \vdash \Delta, \Pi}\ Weak$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma \vdash A, \Delta}{\Gamma \vdash \Delta}\ Cut$$

$$\frac{\begin{array}{c} {}^{1:}[\Gamma \vdash \Delta] \\ \vdots \\ \vdash \end{array}}{\Gamma \nvdash \Delta}\ \nvdash\!-\!In\ [1]$$

$$\frac{\Gamma \vdash \Delta \quad \Gamma \nvdash \Delta}{\vdash}\ \nvdash\!-\!Out$$

Operational Rules:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ {+\neg L} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ {+\neg R}$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ {+\vee L} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ {+\vee R}$$

We can now define the the hybrid calculus $\mathbb{H}$ whose rules and axioms are given in Table 3. The calculus consists of (i) the sequent axioms, (ii) the anti-sequent axioms, (iii) the sequent structural rules, with the addition of $\nvdash -In$ and $\nvdash -Out$, and (iv) the sequent operational rules. It would be helpful at this point to demonstrate some derivations in $\mathbb{H}$, particularly derivations that make use of $\nvdash -In$ and $\nvdash -Out$. To begin with, here are the derivations of the above-discussed rules $Mix1$ and $Mix2$:

$$\dfrac{\dfrac{{}^{1:}[A \vdash] \qquad \vdash A}{\vdash}\ Cut}{A \nvdash}\ \nvdash\!-\!In\ [1] \qquad\qquad \dfrac{\dfrac{{}^{1:}[\vdash A] \qquad A \vdash}{\vdash}\ Cut}{\nvdash A}\ \nvdash\!-\!In\ [1]$$

Recall that it is impossible to derive $Mix1$ and $Mix2$ in $\mathcal{H}$, due to the segregated nature of $\mathcal{H}$'s rules. By contrast, the derivations of $Mix1$ and $Mix2$ in $\mathbb{H}$ are straightforward, since $\nvdash -In$ and $\nvdash -Out$ allow us to incorporate both sequents and anti-sequents in a single proof tree.

What is more, we can derive in $\mathbb{H}$ all the anti-sequent rules of $\mathcal{H}$—operational and structural rules alike—as established by the following proposition:

Proposition 1. *All the anti-sequent rules of $\mathcal{H}$ are derivable in $\mathbb{H}$.*

Proof. The derivations of these rules in $\mathbb{H}$ all have a similar pattern: a reductio proof that rests on $\nvdash -In$, in which we also draw on the derivable invertibility of the sequent rules. Hence, I shall give only three examples; the rest are left to the reader as an exercise.

First, we derive $-\neg L$ as follows:

$$\dfrac{\dfrac{\Gamma \nvdash A,\Delta \qquad \dfrac{\dfrac{{}^{1:}[\Gamma,\neg A\vdash\Delta]}{\Gamma,\neg A\vdash A,\Delta}\,\textit{Weak} \qquad \dfrac{\Gamma, A\vdash A,\Delta}{\Gamma\vdash A,\neg A,\Delta}\,{+\neg R}}{\Gamma\vdash A,\Delta}\,\textit{Cut}}{\vdash}\,{\nvdash\!-\!\textit{Out}}}{\Gamma\neg A\nvdash\Delta}\,{\nvdash\!-\!\textit{In}}\ [1]$$

Second, we derive $- \vee R$ as follows:

$$\dfrac{\dfrac{\Gamma \nvdash A,B,\Delta \qquad \dfrac{\dfrac{{}^{1:}[\Gamma\vdash A\vee B,\Delta]}{\Gamma\vdash A\vee B,A,B,\Delta}\,\textit{Weak} \qquad \dfrac{\Gamma, A\vdash A,B,\Delta \qquad \Gamma, B\vdash A,B,\Delta}{\Gamma, A\vee B\vdash A,B,\Delta}\,{+\vee R}}{\Gamma\vdash A,B,\Delta}\,\textit{Cut}}{\vdash}\,{\nvdash\!-\!\textit{Out}}}{\Gamma\nvdash A\vee B,\Delta}\,{\nvdash\!-\!\textit{In}}\ [1]$$

Finally, we derive anti-weakening as follows:

$$\dfrac{\dfrac{\Sigma,\Gamma \nvdash \Delta,\Pi \qquad \dfrac{{}^{1:}[\Gamma\vdash\Delta]}{\Sigma,\Gamma\vdash\Delta,\Pi}\,\textit{Weak}}{\vdash}\,{\nvdash\!-\!\textit{Out}}}{\Gamma\nvdash\Delta}\,{\nvdash\!-\!\textit{In}}\ [1]$$

□

Corollary. *$\mathbb{H}$ is externally complete: for every Γ, Δ, either $\Gamma \vdash \Delta$ is derivable or $\Gamma \nvdash \Delta$ is derivable.*

Proof. By **Proposition 1**, $\mathbb{H}$ is at least as strong as $\mathcal{H}$, and the latter calculus is externally complete. □

The second half of the extensional assumption—namely, external consistency—is established by the following proposition:

Proposition 2. *$\mathbb{H}$ is externally consistent: for every Γ, Δ, it is not the case that both $\Gamma \vdash \Delta$ and $\Gamma \nvdash \Delta$ are derivable in $\mathbb{H}$.*

Proof. It is enough to show that $\mathcal{H}$ is as strong as $\mathbb{H}$, because $\mathcal{H}$ is externally consistent. To do so, it is enough to show that $\nvdash -In$ and $\nvdash -Out$ are admissible in $\mathcal{H}$, i.e., that adding these rules to $\mathcal{H}$ has no effect on the set of derivable sequents and anti-sequents.

For $\nvdash -Out$, this is straightforward: $\mathcal{H}$ is externally consistent, and so it cannot be that both $\Gamma \vdash \Delta$ and $\Gamma \nvdash \Delta$ are derivable in $\mathcal{H}$. Therefore, $\nvdash -Out$ is vacuously admissible in $\mathcal{H}$. As for $\nvdash -In$, suppose that there is a derivation of $\vdash$ from $\Gamma \vdash \Delta$ that makes use only of $\mathcal{H}$'s rules. But $\vdash$ is not derivable in $\mathcal{H}$, and so $\Gamma \vdash \Delta$ is not derivable either. The external completeness of $\mathcal{H}$ then guarantees that $\Gamma \nvdash \Delta$ is derivable in $\mathcal{H}$, as required. □

1933592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Together, **Proposition 1** and **Proposition 2** show that $\mathbb{H}$ is exactly as strong as $\mathcal{H}$ at the level of sequents and anti-sequents: one may derive in $\mathbb{H}$ precisely those sequents and anti-sequents that are derivable in $\mathcal{H}$. However, $\mathbb{H}$ is clearly stronger at the level of meta-sequents—or "metainferences," as they are sometimes referred to—as one may derive in $\mathbb{H}$ rules such as $Mix1$ and $Mix2$ that are underivable in $\mathcal{H}$.[21] Moreover, $\mathbb{H}$ results from adding only two structural rules to the sequent rules, namely, $\nvdash -In$ and $\nvdash -Out$. Thus, I believe that $\mathbb{H}$ is clearly preferable to $\mathcal{H}$ for three different reasons: (i) it more closely captures the nature of coherence and incoherence, as well as the interrelations between these two notions, (ii) it does so by enriching the basic sequent system only with structural rules, thereby satisfying the structural tenet, and (iii) it does so with only two additional rules—unlike the proliferation of rules in $\mathcal{H}$—thereby satisfying the law of parsimony. Granted that $\mathbb{H}$ is preferable to $\mathcal{H}$, I finally turn to my ultimate goal in this paper, i.e., arguing that the notion of incoherence takes conceptual priority over the notion of coherence.

5 | WHY INCOHERENCE TAKES PRIORITY

What I established in the previous section may briefly be put as follows: starting off with the sequent rules, we may derive all the anti-sequent rules—structural and operational rules alike—with the sole help of $\nvdash -In$ and $\nvdash -Out$. To put it bilateralistically, starting off with the notion of incoherence, we may derive not only the information which positions are coherent, but also all the principles governing the notion of coherence.

Such a derivation is highly important. For, to show mastery of a given concept, one is sometimes required not only to specify the concept's extension, but also to grasp its *intension*, i.e., the rules that fix the extension. Suppose, for example, that a little girl is shown a basket with green and red apples, and asked to identify the latter. Suppose that she succeeds in doing so. Does her success establish the claim that she has mastery of the concept "the red apples in the basket"? Not at all. It may well be that, if we add more apples to the basket—this time, yellow ones—the girl will identify them as red as well. In such a case, we would say that, even though the girl did at first successfully specify the extension of the concept "the red apples in the basket," her failure to do so the second time shows that she has yet to grasp how that extension is fixed, and thus fails to master the concept under consideration.

Mutatis mutandis, to show mastery of the notions of coherence and incoherence, it is not enough to specify the extensions of these notions—namely, which positions are coherent and which are incoherent—but also their intensions. In proof theory, such intensions are associated with the rules that help us derive sequents and anti-sequents. Schroeder-Heister has recently made this point quite nicely:

> "[I]n general proof theory we are not solely interested in whether B follows from A, but in the way by means of which we arrive at B starting from A. In this sense general

[21] For extensive discussions of metainferences, see (Dicher & Paoli, 2019; Golan, 2021). In effect, $\mathbb{H}$ is not only stronger than $\mathcal{H}$ at the level of metainferences, but as strong as possible: there are no admissible yet underivable rules in $\mathbb{H}$. To show this, we have to prove that at the (first) level of metainferences, $\mathbb{H}$ is *complete* with respect to the locally-valid mixed metainferences of classical logic (the proof is similar in nature to some completeness proofs given in Dicher & Paoli, 2019; Golan, 2021.) Yet dealing with such a proof clearly exceeds the scope of the present paper, and so I leave it for another occasion. For my purposes here, it is enough that $\mathbb{H}$ is stronger than $\mathcal{H}$ at the level of metainferences, and thus provides us with a better account of the notions of coherence and incoherence.

proof theory is intensional and epistemological in character" (Schroeder-Heister, 2023, p. 2).[22]

That is why the results from the previous section are so important: they establish the claim that mastery of the notion of incoherence *ipso facto* implies mastery of the notion of coherence.

Now, for the conceptual assumption to hold, one has to show that the notion of coherence is no less primitive. Namely, one has to show that mastery of the notion of coherence implies mastery of the notion of incoherence; otherwise, one could not resist an argument to the effect that the notion of incoherence takes conceptual priority. Unfortunately, this opposite move cannot be carried out in full. That is, there is no way to start off with the anti-sequent rules, and derive from them all the sequent rules with the sole help of structural rules that, analogously to $\nvdash -In$ and $\nvdash -Out$, introduce and eliminate the incoherence turnstile $\vdash$. Having established the last claim in the rest of this section, my conclusion will follow straightforwardly: the notion of incoherence takes priority over the notion of coherence, and so we should deny the conceptual assumption.[23]

Let me explain why the opposite move fails. Suppose that we have all the anti-sequent rules and axioms at our disposal. Having also added the sequent axioms, we first need to find structural rules analogous to $\nvdash -In$ and $\nvdash -Out$ for introducing and eliminating sequents. As for the introduction rule, the following candidate—which I label $\vdash -In$—naturally presents itself:

$$\dfrac{\begin{array}{c} {}^{1:}[\Gamma \nvdash \Delta] \\ \vdots \\ \vdash \end{array}}{\Gamma \vdash \Delta}\ \vdash\!-In\ [1]$$

I take it that $\vdash -In$ speaks for itself, and needs no further justification. Regarding the elimination rule, we may simply rebrand $\nvdash -Out$ as $\vdash -Out$, and use it to eliminate sequents:

$$\dfrac{\Gamma \vdash \Delta \quad \Gamma \nvdash \Delta}{\vdash}\ \vdash\!-Out$$

My rationale here is that sequents and anti-sequents have an equal status in this rule, and so it should be considered an elimination rule for both.

One may object at this point that the conclusion-sequent of $\vdash -Out$ is a *sequent*—namely, the empty sequent—and so this rule cannot be genuinely used to eliminate sequents in our derivations. As an alternative, we may suggest the following rule

$$\dfrac{\Gamma \vdash \Delta \quad \Gamma \nvdash \Delta}{p \nvdash p}\ \vdash\!-Out\ Alt$$

where p is an arbitrary atomic sentence; after all, the axioms were picked up so that $p \nvdash p$ is underivable. If $\vdash -Out\ Alt$ is adopted instead of $\vdash -Out$, then the introduction rule of $\vdash$ has to

[22] See also (Schroeder-Heister, 2022) for an extensive discussion of this point.

[23] An anonymous referee pointed out that such a formal result (about a specific system) may not suffice to establish the *general* claim that incoherence is more primary (*tout court*) than coherence. While I admit that more must be done to make that general claim, I do believe that, at the very least, my argument shifts the burden of proof to the opponent.

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

be adjusted to it, by way of taking the following form:

$$\frac{\begin{array}{c}{}^{1:}[\Gamma \nvdash \Delta]\\ \vdots \\ p \nvdash p\end{array}}{\Gamma \vdash \Delta}\ \vdash\!-\textit{In Alt}\ [1]$$

As it turns out, the two pairs of rules $<\vdash -In, \vdash -Out>$ and $<\vdash -In\ Alt, \vdash -Out\ Alt>$ are equivalent, and the choice between them is somewhat arbitrary. To forestall the above objection, I shall adopt the latter pair of rules; the reader is invited to make sure that this choice of mine does not matter when it comes to the status of the derivations below.

At any rate, with these rules in hand, we expect to be able to conduct derivations of the sequent rules analogous to the derivations of the anti-sequent rules given in the proof of **Proposition 1**. Indeed, the rule of weakening may be derived as follows:

$$\cfrac{\Gamma \vdash \Delta \qquad \cfrac{{}^{1:}[\Gamma, \Sigma \nvdash \Delta, \Pi]}{\Gamma \nvdash \Delta}\ \textit{Anti}-\textit{Weak}}{\cfrac{p \nvdash p}{\Gamma, \Sigma \vdash \Delta, \Pi}\ \vdash\!-\textit{In Alt}\ [1]}\ \vdash\!-\textit{Out Alt}$$

But that is where the similarity ends. To begin with, it is not at all clear that the rule of cut can be derived along the same lines. It seems that to do so, we have to add to the anti-sequent rules at least one of Tiomkin's cut rules discussed (and rejected) above, and then reason as follows:

$$\cfrac{\Gamma, A \vdash \Delta \qquad \cfrac{\Gamma \vdash A, \Delta \qquad {}^{1:}[\Gamma \nvdash \Delta]}{\Gamma, A \nvdash \Delta}\ \textit{Tiomkin}-\textit{cut}}{\cfrac{p \nvdash p}{\Gamma \vdash \Delta}\ \vdash\!-\textit{In Alt}\ [1]}\ \vdash\!-\textit{Out Alt}$$

Worse yet, the situation is impossible to remedy when it comes to the operational rules. This is so because, as opposed to the operational sequent rules, the operational anti-sequent rules are not derivable-invertible. Since the derivations of the operational anti-sequent rules in the proof of **Proposition 1** draw on this derivable-invertibility, there are no analogous derivations of the sequent rules. That is to say, it is impossible to derive the operational sequent rules from the corresponding operational anti-sequent rules.

Let me demonstrate this point with the left rule for negation. On the face of it, a derivation of $+\neg L$ from $-\neg L$ should go along the following lines:

$$\cfrac{\Gamma \vdash A, \Delta \qquad \cfrac{{}^{1:}[\Gamma, \neg A \nvdash \Delta]}{\Gamma \nvdash A, \Delta}\ ?}{\cfrac{p \nvdash p}{\Gamma \neg A \vdash \Delta}\ \vdash\!-\textit{In}\ [1]}\ \vdash\!-\textit{Out}$$

Yet, there is no way to justify the derivation of $\Gamma \nvdash A, \Delta$ from the assumption $\Gamma, \neg A \nvdash \Delta$, as $-\neg L$ is not derivable-invertible.[24] To be even more specific, the proof that $+\neg L$ is derivable-invertible—in

[24] As a matter of fact (see Rosenblatt, 2021, pp. 1047–1053 for a proof), $-\neg L$ is admissible-invertible: if $\Gamma, \neg A \nvdash \Delta$ is derivable then so is $\Gamma \nvdash A, \Delta$. But admissible-invertibility is of no help here, because $\Gamma, \neg A \nvdash \Delta$ is only *assumed* in the above derivation, and we have no guarantee that $\Gamma \nvdash A, \Delta$ is derivable under the mere assumption $\Gamma, \neg A \nvdash \Delta$.

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

effect, the proof that operational sequent rules are all derivable-invertible—crucially relies on an application of cut in the last step:

$$\dfrac{\dfrac{\Gamma, \neg A \vdash \Delta}{\Gamma, \neg A \vdash A, \Delta}\ Weak \qquad \dfrac{\Gamma, A \vdash A, \Delta}{\Gamma \vdash A, \neg A, \Delta}\ +\neg R}{\Gamma \vdash A, \Delta}\ Cut$$

but no anti-cut rule—Tiomkin's or otherwise—has a similar effect on $-\neg L$ or any other operational anti-sequent rule.

Now, as far as the negation rules are concerned, there is an *ad hoc* solution to this problem, by forcing the rules to be derivable-invertible. Indeed, in several places, e.g., (Golan, 2021; Hlobil, 2019), the operational rules are formulated with double lines, indicating that one may derive not only the conclusion-sequent from the premise-sequents, but also each premise-sequent from the conclusion-sequent. In particular, if we alternatively formulate $-\neg L$ as follows:

$$\frac{\Gamma \nvdash A, \Delta}{\Gamma, \neg A \nvdash \Delta}\ -\neg L\ Alt \quad \text{(double line)}$$

then the above derivation of $+\neg L$ may be straightforwardly completed.

Yet this suggestion is not only *ad hoc*; the fact of the matter is that it does not apply to all the rules. For one thing, the left rules for disjunction

$$\frac{\Gamma, B \nvdash \Delta}{\Gamma, A \vee B \nvdash \Delta}\ -\vee L \qquad \frac{\Gamma, A \nvdash \Delta}{\Gamma, A \vee B \nvdash \Delta}\ -\vee L$$

are not even admissible-invertible: it may well be that $\Gamma, A \vee B \nvdash \Delta$ is derivable, but $\Gamma, B \nvdash \Delta$ (or $\Gamma, A \nvdash \Delta$) is not. To put things bilateralistically, it may well be that $A \vee B$ is coherently assertible but one of the disjuncts, say A, is not coherently assertible (in which case B must be coherently assertible.) Which is to say, $- \vee L$ cannot be formulated with double lines, and so $+ \vee L$ cannot be derived from it.[25]

Wrapping up, the sequent rules cannot be derived from the anti-sequent rules in the same way that the latter were derived from the former. As I said, this result has the philosophical effect that the notion of incoherence takes conceptual priority over the notion of coherence.

6 | CONCLUSION

I have laid out in this paper a bilateralist account of the notions of coherence and incoherence based on the calculus $\mathbb{H}$. I take it that my account is preferable to any account that is based on the calculus $\mathcal{H}$ (such as the one given by Rosenblatt, 2019) for at least three reasons: (i) it more closely captures the nature of coherence and incoherence, as well as the interrelations between these two notions, (ii) it does so by enriching the basic sequent system only with structural rules, thereby satisfying the structural tenet, and (iii) it does so with only two additional rules—unlike

[25] As Rosenblatt points out (Rosenblatt (2021), pp. 1047–1053), $- \vee L$ is *semi-invertible*: if $\Gamma, A \vee B \nvdash \Delta$ is derivable, then so is at least one of $\Gamma, A \nvdash \Delta$ and $\Gamma, B \nvdash \Delta$ (he actually points this out for the right conjunction rule, but the proof is analogous.) But semi-invertibility will not help us either, for the same reason that admissibile-invertibility does not help in the case of $-\neg L$: in a purported derivation of $+ \vee L$, $\Gamma, A \vee B \nvdash \Delta$ may only be *assumed*, and there is no guarantee that one of $\Gamma, A \nvdash \Delta$ and $\Gamma, B \nvdash \Delta$ is derivable under this assumption.

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

the proliferation of rules in $\mathcal{H}$—thereby satisfying the law of parsimony. Moreover, my account provides us with an argument to the effect that the notion of incoherence takes conceptual priority over the notion of coherence.

I believe that these results are interesting in their own right. Yet to conclude, I would like to point out one potential implication of my account that is worth looking into. Recall that $\nvdash -In$ and $\nvdash -Out$ are modeled after Gentzen's natural deduction rules for negation. Taken by themselves, these rules generate intuitionistic negation rather than classical negation. Consequently, it is not at all obvious that $\nvdash -In$ and $\nvdash -Out$ guarantee external completeness, as opposed to external consistency that is guaranteed by $\nvdash -Out$. Thus, even though $\mathbb{H}$ is externally complete, it may cease to be so once we enrich our language with expressive resources rich enough to generate self-referential paradoxes of the kind discussed in Rosenblatt (2019, 2021). Hence, I am inclined to believe that, upon enriching our language with such resources, gap-theorists—i.e., those who argue that certain positions are neither coherent nor incoherent—will have the upper hand over glut-theorists who maintain that such positions are both coherent and incoherent. Yet this conjecture of mine is clearly in need of further investigation, and so I leave it for another occasion.

ACKNOWLEDGEMENTS

Many thanks to Robert Brandom, Ulf Hlobil, Ryan Simonelli, and Shuhei Shimamura for fruitful discussions. I would also like to thank an anonymous referee for this journal, for helpful comments and suggestions.

ORCID

Rea Golan https://orcid.org/0000-0003-4923-9998

REFERENCES

Brandom, R. (2000). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Carnielli, W., & Pulcini, G. (2017). Cut-elimination and deductive polarization in complementary classical logic. *Logic Journal of the IGPL*, *25*(3), 273–282.

Descartes, R. (2008). *Meditations on first philosophy*. Oxford University Press.

Dicher, B. & Paoli, F. (2019). ST, LP and tolerant metainferences. In C. Başkent & T. Ferguson (Eds.), *Graham priest on dialetheism and paraconsistency* (pp. 383–407). Springer.

Dummett, M. (1973). *Frege: Philosophy of language*. Duckworth.

Frege, G. (1960). Negation. *Translations from the philosophical writings of Gottlob Frege* (pp. 117–36). Oxford University Press.

Geach, P. (1965). Assertion. *The Philosophical Review*, *74*(4), 449–465.

Gentzen, G. (1964). Investigations into logical deduction. *American Philosophical Quarterly*, *1*(4), 288–306.

Golan, R. (2021). Metainferences from a proof-theoretic perspective, and a hierarchy of validity predicates. *Journal of Philosophical Logic*, *51*(6), 1295–1325.

Golan, R. & Hlobil, U. (2022). Minimally nonstandard K3 and FDE. *Australasian Journal of Logic*, *19*(5), 182–213.

Golan, R. (2023). A simple sequent system for minimally inconsistent LP. *Review of Symbolic Logic*, *16*(4), 1296–1311.

Hlobil, U. (2019). Faithfulness for naive validity. *Synthese*, *196*, 4759–4774.

Negri, S., & von Plato, J. (2001). *Structural proof theory*. Cambridge University Press.

Price, H. (1990). Why 'Not'? *Mind*, *99*(394), 221–238.

Restall, G. (2005). Multiple conclusions. In P. Hájek, L. Valdés-Villanueva, & D. Westerståhl (Eds.), *Logic, methodology, and philosophy of science: Proceedings of the twelfth international congress* (pp. 189–205). College Publications.

Restall, G. (2020). Speech acts & the question for a natural account of classical proof.

Ripley, D. (2011). Negation, denial, and rejection. *Philosophy Compass*, *6*(9), 622–629.

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Ripley, D. (2013). Paradoxes and failures of cut. *Australasian Journal of Philosophy*, *91*(1), 139–164.
Ripley, D. (2015). Embedding denial. In C. Caret & O. Hjortland (Eds.), *Foundations of logical consequence* (pp. 289–309). Oxford University Press.
Rosenblatt, R. (2019). Bilateralism and invalidities. *Inquiry*, *64*(4), 481–510.
Rosenblatt, R. (2021). Towards a non-classical meta-theory for substructural approaches to paradox. *Journal of Philosophical Logic*, *50*, 1007–1055.
Rumfitt, I. (2000). 'Yes' and 'no'. *Mind*, *104*(436), 781–823.
Schroeder-Heister, P. (1984). A natural extension of natural deduction. *Journal of Symbolic Logic*, *49*(4), 1284–1300.
Schroeder-Heister, P. (2022). Axiomatic thinking, identity of proofs and the quest for an intensional proof-theoretic semantics. In F. Ferreira, R. Kahle, & G. Sommaruga (Eds.), *Axiomatic thinking I*. Springer.
Schroeder-Heister, P. (2023). Proof-theoretic semantics. In E. N. Zalta & U. Nodelman (Eds.), *The Stanford Encyclopedia of Philosophy* (Fall 2023 ed.). https://plato.stanford.edu/archives/fall2023/entries/proof-theoretic-semantics/.
Smiley, T. (1996). Rejection. *Analysis*, *56*(1), 1–9.
Tiomkin, M. L. (1988). Proving unprovability. *Proceedings of the 3rd Annual Symposium on Logic in Computer Science* (LICS '88) (pp. 22–26).

How to cite this article: Golan, R. (2024). Bilateralism, coherence, and incoherence. *Philosophy and Phenomenological Research*, 1–20. https://doi.org/10.1111/phpr.13115

19331592, 0, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/phpr.13115 by Cochrane Israel, Wiley Online Library on [15/10/2024]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License