Symmetry, Invariance, and Imprecise Probability

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It is tempting to think that a process of choosing a point at random from the surface of a sphere can be probabilistically symmetric, in the sense that any two regions of the sphere which differ by a rotation are equally likely to include the chosen point. Isaacs, Hájek and Hawthorne (2022) argue from such symmetry principles and the mathematical paradoxes of measure to the existence of imprecise chances and the rationality of imprecise credences. Williamson (2007) has argued from a related symmetry principle to the failure of probabilistic regularity. We contend that these arguments fail, because they rely on auxiliary assumptions about probability which are inconsistent with symmetry to begin with. We argue, moreover, that symmetry should be rejected in light of this inconsistency, and because it has implausible decision-theoretic implications. The weaker principle of probabilistic invariance says that the probabilistic comparison of any two regions is unchanged by rotations of the sphere. This principle supports a more compelling argument for imprecise probability. We show, however, that invariance is incompatible with mundane judgements about what is probable. Ultimately, we find reason to be suspicious of the application of principles like symmetry and invariance to non-measurable regions.

1. Introduction

At the centre of a perfectly symmetrical hollow sphere lies a perfectly symmetrical particle emitter, which will soon emit a lone particle in a random direction. The particle will travel outwards and hit exactly one point on the surface of the sphere. By the symmetry of the case, it is tempting to think that the particle emitter is probabilistically perfectly fair in the following sense:

> SYMMETRY If two sets of points on the surface of the sphere differ by a rotation, then they are exactly equally likely landing places for the particle.

Principles like Symmetry have played an important role in a number of arguments for controversial conclusions in the philosophy of probability.

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For example, Williamson (2007) argues from an analogous symmetry principle about coin flips to the view that no infinite sequence of heads and tails is more likely than a contradiction.

Most recently, Isaacs, Hájek and Hawthorne (2022)—henceforth, 'IHH'—have argued from Symmetry to the existence of imprecise chances and the rationality of imprecise credences. Their arguments appeal to the mathematical phenomenon of non-measurable sets. Their most powerful arguments, in our opinion, appeal to paradoxical decompositions of objects like our sphere, based on Hausdorff (1914) and Banach and Tarski (1924). Here is one result of this form (see Tomkowicz and Wagon 2016 for exposition):

> *Theorem* 1 (Robinson 1947). The surface of a sphere in three-dimensional Euclidean space can be partitioned into four sets A, B, C, D, such that B is a rotation of $B \cup C \cup D$ and C is a rotation of $A \cup B \cup C$.

The 'paradox' is that A and a rotation of B together cover the entire surface of the sphere, and the same is true for D and a rotation of C. So the sphere can be decomposed into sets that can then be rearranged so as to cover the entire surface of the sphere twice over. This shows that there is no finitely additive probability function defined on all sets of points on the surface of the sphere that, in accordance with Symmetry, assigns the same probability to sets which differ by a rotation.[1] IHH conclude that some sets of points do not have real-valued probabilities.

In orthodox probabilistic theorizing, paradoxes such as these are avoided by restricting attention to the *Lebesgue-measurable* sets of points on the sphere. As a first gloss, a Lebesgue-measurable set is one that can be approximated by repeatedly taking complements, countable unions, and countable intersections of rectangular patches of the surface of the sphere (see Heil 2019, ch. 2). A hardline but popular orthodox stance denies that there are any probability comparisons to be had beyond the Lebesgue-measurable sets. For example, van Fraassen (1989, p. 329) claims that 'the requirement to have probability defined everywhere would be unacceptable'. On this view, the sets of points in Theorem 1 are not more or less likely than each other (or any other set of points) to be the landing place of the particle, because they are not

[1] For a finitely additive probability function, the probabilities of A through D as in Theorem 1 must sum to one. But by symmetry, so must the probabilities of A and B and of C and D—a contradiction.

Lebesgue-measurable; they are simply beyond the realm of probabilistic comparison.

This hardline orthodoxy is difficult to maintain. It contradicts the overwhelmingly plausible idea that every set, measurable or not, is at least as likely a landing place as any of its subsets. Hardliners also face the awkward question of how 'measurable' is to be understood in real-world applications of probabilistic theorizing. Meteorologists talk about the probability that it will rain tomorrow, seemingly assuming that rain tomorrow is at least as likely as heavy rain tomorrow, without worrying whether the propositions in question are 'measurable' in any standard sense. So we view orthodoxy as avoiding non-measurable sets, not because there is a fundamental probabilistic difference between measurable and non-measurable sets, but because paradoxes of probability such as those posed by Theorem 1 are best ignored for many purposes (cf. Hoek 2021).

We therefore agree with IHH that there are probabilistic comparisons to be made between arbitrary sets of points on the sphere. However, we deny that these comparisons should respect Symmetry.[2]

More specifically, we do three things. First, we show that there is no good argument from Symmetry and finite additivity to imprecise probability, since these premisses are inconsistent, given uncontroversial auxiliary assumptions. Second, we argue that Symmetry should be rejected in light of this inconsistency, and because it has implausible decision-theoretic implications. Third, we provide a better argument for imprecise probability, which does not appeal to Symmetry, and explain why we nonetheless do not find this argument compelling.

2. Symmetry versus Additivity

Let Ω denote the set of all points on the surface of the sphere, which we assume to be isometric to the unit sphere in $\mathbb{R}^3$. Let $\succ$ denote the relation that holds between sets of points X and Y when X is a more likely landing place for the particle than Y. (Our arguments are not intended to be sensitive to whether 'likely' is interpreted in a subjective or objective way.)

[2] Principles like Symmetry are also rejected by Hoek (2021) and Maudlin (2021); our arguments are quite different from theirs. Dorr (2024) rejects a principle like Symmetry for rational credences while granting, at least for the sake of argument, that it applies to objective chance. The existence of imprecise objective chances, however, is arguably inconsistent with the thesis of Dorr, Nebel, and Zuehl (2023), which is the linchpin of their (2021) argument against imprecise credence.

We define the relation $\sim$ (equally likely) as follows:

$X \sim Y$ if and only if for any $Z \subseteq \Omega$, $Z \succ X$ if and only if $Z \succ Y$, and $X \succ Z$ if and only if $Y \succ Z$.

Note that $\sim$, so defined, is an equivalence relation.

Symmetry can then be formulated as follows:

SYMMETRY $X \sim \pi X$, where π is any rotation of the sphere.[3]

Our argument appeals to two principles which are even more central to our notion of probability than Symmetry:

NON-TRIVIALITY $\Omega \nsim \varnothing$.

ADDITIVITY If $X \cap Z = Y \cap Z = \varnothing$, then
(a)$X \succ Y$ if and only if $X \cup Z \succ Y \cup Z$, and
(b)$X \sim Y$ if and only if $X \cup Z \sim Y \cup Z$.

We take Non-triviality to be obvious, and Additivity to be highly desirable.

The principles above imply that Symmetry is false:

Theorem 2. Non-triviality, Additivity, and Symmetry are inconsistent.[4]

Proof. Take A, B, C, D as in Theorem 1. By Symmetry, $B \sim B \cup C \cup D$, so by Additivity we have $A \cup B \sim \Omega$ and hence $C \cup D \sim \varnothing$. By a parallel argument, $C \cup D \sim \Omega$. Thus $\Omega \sim \varnothing$, contradicting Non-triviality. □

We take Theorem 2 to cast considerable doubt on Symmetry, given the plausibility of Non-triviality and Additivity. IHH, however, appear to favour a formalization of imprecise probability which violates Additivity while satisfying the other principles. They suggest taking the

[3] IHH appeal to the idea that 'any rotation of a given set of points must have the same probability as that set of points' (p. 893). We assume that having the same probability is equivalent to being equally probable—otherwise, there must be more to probability than whatever probabilities IHH assign. Following Keynes (1921), de Finetti (1937), Koopman (1940), Savage (1954), and others, we prefer to theorize in terms of the relations $\succ$ and $\sim$ in order to more easily formulate principles that are neutral between different numerical representations of probability.

[4] A similar result is proved by Thong (2024, Theorem 2), though his involves additional axioms and is specific to IHH's view. His result can be regarded as a corollary of ours. Thong draws the lesson that IHH's view violates either Additivity or Transitivity. However, Transitivity plays no role in our result, so IHH cannot preserve Additivity by rejecting Transitivity.

probability of an event to be the interval bounded by the event's Lebesgue inner and outer measures (see, for example, Heil 2019, ch. 2 for definitions). Supposing that events are equally likely if and only if they have the same probability, Symmetry and Non-triviality will hold on this view, so Additivity must fail. (Where A, B, C, D are as in Theorem 1, the inner probability of each such set will be 0 and the outer probability 1, so they are all equally likely on IHH's view.)

This puts them, however, in an awkward dialectical situation. If Additivity is rejected, Theorem 1 is no barrier to assigning real-number probabilities to every set of points. For example, the ordering that holds X to be more probable than Y when X has a greater Lebesgue outer measure satisfies Symmetry and Non-triviality, as would the ordering similarly induced by the Lebesgue inner measure, or by the midpoint (or any other convex combination) of the Lebesgue inner and outer measures. IHH provide no argument against a precise probability assignment that satisfies Symmetry but not Additivity, so the argument for imprecision is incomplete.

IHH would accept a restriction of Additivity to the Lebesgue measurable sets. Such a restriction is also satisfied by any of the precise views mentioned above. But we find all of these views inferior to one which satisfies Additivity in full generality. In our view, it is Symmetry that should be restricted in light of the paradoxes of measure.

Theorem 2 also complicates the argument of Williamson (2007) mentioned in §1. He argues from a Symmetry-like principle about sequences of coin flips,[5] along with Additivity, to the falsity of

Regularity   If X is not empty, then $X \nsim \varnothing$.

Williamson's argument, applied to the sphere, can be put as follows.

Theorem 3. Symmetry, Additivity, and Regularity are inconsistent.

Proof. In the proof of Theorem 2 we derive $C \cup D \sim \varnothing$ from Symmetry and Additivity, in contradiction with Regularity. □

[5] A Hausdorff-like result for an infinite sequence of coin flips requires a suitably rich class of putatively probability-preserving symmetries on the set of possible results of the sequence. A sufficient condition is if every permutation of the flips induces a symmetry on the set of possible results, which is to say that for any permutation π of flips, the function that maps the sequence of results $\langle r_i \rangle_{i=1}^{\infty}$ to $\langle r_{\pi(i)} \rangle_{i=1}^{\infty}$ is counted as a symmetry. Williamson's argument relies on one such permutation being probability-preserving, and his justification for this premiss extends to an arbitrary permutation if the case is suitably modified.

Theorem 3 might seem to provide a powerful argument against Regularity. But Theorem 2 casts some doubt on this impression, because—given the obvious auxiliary assumption of Non-triviality—Additivity and Symmetry are not even consistent.

Though our proof of Theorem 3 appeals to the non-measurable sets C and D, it is important to emphasize that this is not necessary for Williamson's result (his own argument appeals only to sets of measure zero). It is therefore possible to run Williamson's argument with Symmetry and Additivity restricted to measurable sets. Such an argument would not be directly impugned by Theorem 2. However, once the principles are restricted in this way, and they are no longer the sweeping, powerful constraints they initially appeared to be, we think there is considerably less pressure on the proponent of Regularity to accept them (the restriction of Symmetry in particular).

3. In defence of Additivity

The conflict between Symmetry and Additivity poses an important choice point for probabilistic theorizing.[6] We are strongly inclined to retain Additivity and reject Symmetry. As we will see in this section, this choice can be supported on decision-theoretic grounds. Moreover, we will find in §4 that there is a natural weakening of Symmetry—Invariance—which is compatible with Additivity and provides a more compelling argument for imprecise probability, so we recommend this choice to IHH as well.

Let a *prospect* be a function which assigns an *outcome* to each point on the sphere—intuitively, the prize you would receive if that prospect were chosen and the emitted particle lands at that point. We assume

NON-TRIVIAL OUTCOMES Some outcome is better than another.

[6] There are other arguments against Additivity of a quite different nature. Consider the case famously described by Ellsberg (1988): a ball is chosen at random from an urn containing 30 red balls and 60 blue or green balls of unknown proportion. Many people prefer a gamble which pays out some prize if the ball is red to one which pays out if it's blue, while also preferring a gamble which pays out that same prize if the ball is blue or green to one which pays out if it's red or green. Now suppose that someone has these preferences, while always and only preferring gambles with a higher probability of a pay-off (other things being equal). It follows that red $\succ$ blue and blue-or-green $\succ$ red-or-green, in violation of Additivity (Fishburn 1986).

Even if the described pattern of preferences were rational, it seems bizarre to try to explain it in purely probabilistic terms. In particular, we see no reason to accept that an agent with these preferences always prefers a higher probability of receiving the prize. Ambiguity aversion seems better explained by the agent's *not knowing*, or even judging, whether red is more likely than blue, but preferring to bet on events with known rather than unknown probability.

For any prospect f and rotation π of the sphere, $f \circ \pi$ is the composition of f with π—that is, $(f \circ \pi)(x) = f(\pi(x))$ for every point $x \in \Omega$. A proponent of Symmetry should accept

Symmetric Value f is not better than $f \circ \pi$, for any rotation π.

It would be bizarre to insist that devices like our particle emitter can be *perfectly fair* in the sense of satisfying Symmetry while thinking it wouldn't be fair to get a prize on some set of points rather than on some rotation of that set. Symmetric Value also follows, given Symmetry, from the more general principle of 'stochastic equivalence', which asserts that equally distributed prospects are equally good (see Joyce 1999; Bader 2018; Russell and Isaacs 2021; Goodsell 2024; Russell forthcoming).

Say that f *statewise dominates* g if $f(x)$ is better than $g(x)$ for all x in some set $E \not\sim \varnothing$, and $f(x) = g(x)$ for all $x \in E^{\complement}$ (E's complement in Ω).

Statewise Dominance If f statewise dominates g, then f is better than g.

That is, if f guarantees an outcome at least as good as the outcome of g, and has a non-zero chance (that is, $E \not\sim \varnothing$) of a strictly better outcome, then f is a better option than g.

We appeal also to a significant weakening of Additivity (given Nontrivality) that is entailed by IHH's inner/outer view (assuming, again, that having equal probability is the same as being equally probable):

Weak Complementation If $X \sim \varnothing$, then $X^{\complement} \not\sim \varnothing$.

Theorem 4. Non-trivial Outcomes, Symmetric Value, Statewise Dominance, and Weak Complementation are inconsistent.

Proof. Take A, B, C, D as in Theorem 1, and for some better outcome and some worse one, let $\mathbf{1}_X$ yield the better outcome on $X \subseteq \Omega$ and the worse one elsewhere.

By Symmetric Value, $\mathbf{1}_{B \cup C \cup D}$ is not better than $\mathbf{1}_B$. It follows that $\mathbf{1}_{B \cup C \cup D}$ does not statewise dominate $\mathbf{1}_B$, hence that $C \cup D \sim \varnothing$. By an analogous argument, $A \cup B \sim \varnothing$, contradicting Weak Complementation. □

If IHH insist on retaining Symmetry, we surmise that, among the other principles, they would be best advised to reject Statewise Dominance.

They might take issue with it on the grounds that f can statewise dominate g, on our definition, even if the set E of points on which f is better is not *more* likely than $\varnothing$. It's open to them to say that $E \not\succ \varnothing$ even while $E \nsim \varnothing$ (that is, a non-zero chance may not be a *greater than zero* chance). But Symmetric Value is also inconsistent with a principle which weakens Statewise Dominance, supposing the space of values of outcomes to be sufficiently rich. This is established by an argument of Pruss (2023).

Say that f *almost strictly* statewise dominates g if $f(x)$ is a strictly better outcome than $g(x)$ for all x in Ω besides at most two points x_1 and x_2, where $f(x_1) = g(x_1)$ and $f(x_2) = g(x_2)$. A prospect that almost strictly statewise dominates another is all-but-guaranteed to yield a strictly better outcome, so we feel safe in assuming

> Almost Strict Statewise Dominance If f almost strictly statewise dominates g, then f is better than g.

Pruss's argument also relies, plausibly, on a bidirectionally infinite sequence of ever-better outcomes.

> Infinite Outcomes There is a non-empty set of outcomes that, for each of its elements, contains at least one that is better and one that is worse.

Theorem 5 (Pruss 2023). Symmetric Value is inconsistent with Almost Strict Statewise Dominance and Infinite Outcomes.

Proof. Let ϕ be a rotation of the sphere by an irrational angle (in degrees), so that for every $x \in \Omega$ not on the axis of rotation, $\phi^i(x) \neq \phi^j(x)$ for any distinct (possibly negative) integers i, j (where ϕ^i is the i-fold self-composition of ϕ if i is positive, of ϕ^{-1} if i is negative, and the identity function if i is zero).

Using Infinite Outcomes (and the Axiom of Choice), we construct a prospect as follows. First, assign any outcomes to the two points on the axis of ϕ. Then, when x is not on the axis of the rotation of ϕ and has not yet been assigned an outcome, assign outcomes to $\{\phi^i(x) : i \in \mathbb{Z}\}$ that are strictly increasing in value, so that $\phi^i(x)$ is assigned a better outcome than $\phi^j(x)$ if and only if $i > j$.

Repeating this process transfinitely, we are left with a prospect f such that $f(\phi(x))$ is better than $f(x)$ unless x is on the axis of ϕ, hence $f \circ \phi$ almost strictly statewise dominates f, so is better, by Almost Strict Statewise Dominance. But $f \circ \phi$ cannot be better than f by Symmetric Value. □

Pruss himself does not conclude that Symmetric Value should be rejected, in part because he seems to like Williamson's argument against Regularity. But, when considered alongside the violation of Additivity in Theorem 2, it seems clear to us that Symmetry and Symmetric Value have to go. We recommend this choice to IHH as well, because they do not need Symmetry to argue for imprecise probability.

4. Invariance versus Totality

A better argument for imprecise probability appeals to the following weakening of Symmetry:

Invariance For any rotation π, $X \succ Y$ if and only if $\pi X \succ \pi Y$.

A consequence of this principle (given our definition of $\sim$) is that $X \sim Y$ if and only if $\pi X \sim \pi Y$, for any rotation π.[7]

Unlike Symmetry, Invariance is compatible with Additivity because any rotation of the sphere preserves strict subsethood, in the sense that $X \subset Y$ if and only if $\pi X \subset \pi Y$. And it is enough for IHH's central purpose because, given other plausible principles, it is inconsistent with

Totality Either $X \succ Y$, $X \sim Y$, or $Y \succ X$.

Totality is a principle that characteristically divides those who believe probability to be 'precise' from those who do not. We also assume

Minimality $\varnothing \not\succ X$.
Transitivity If $X \succ Y$ and $Y \succ Z$ then $X \succ Z$.

Theorem 6. Invariance and Totality are inconsistent, given Non-triviality, Additivity, Minimality, and Transitivity.

The proof appeals repeatedly to the following simple lemma (where $\succsim$ is the disjunction of $\succ$ with $\sim$):

Lemma 1. Totality, Minimality, and Additivity entail that $X \succsim Y$ whenever $Y \subseteq X$.

[7] *Proof.* Suppose first that $\pi X \sim \pi Y$. Given Invariance, this implies that for any $Z \subseteq \Omega$, $Z \succ X$ if and only if $\pi Z \succ \pi X$ if and only if $\pi Z \succ \pi Y$ if and only if $Z \succ Y$. By exactly similar reasoning $X \succ Z$ if and only if $Y \succ Z$. Thus $X \sim Y$.

Suppose next that $X \sim Y$. Given Invariance, this implies that for any $Z \subseteq \Omega$, $Z \succ \pi X$ if and only if $\pi^{-1} Z \succ X$ if and only if $\pi^{-1} Z \succ Y$ if and only if $Z \succ \pi Y$. By exactly similar reasoning, $\pi X \succ Z$ if and only if $\pi Y \succ Z$. Thus $\pi X \sim \pi Y$. □

Proof. Suppose $Y \subseteq X$ but $X \not\gtrsim Y$. By Totality, $Y \succ X$, so $\varnothing \succ X \setminus Y$ by Additivity, in violation of Minimality. □

Proof of Theorem 6. Take A, B, C, D as in Theorem 1. By Totality, we have $B \gtrsim C$ or $C \gtrsim B$. Suppose $B \gtrsim C$ without loss of generality, and let π be a rotation with $\pi C = A \cup B \cup C$. By Invariance,

$$\pi B \gtrsim A \cup B \cup C. \tag{1}$$

Since B is disjoint from C and disjointness is preserved under rotations, πB is disjoint from $A \cup B \cup C$, and is therefore a subset of D. Thus $D \gtrsim \pi B$ by Lemma 1. So from (1) and Transitivity we have

$$D \gtrsim A \cup B \cup C. \tag{2}$$

Now let σ be a rotation with $\sigma B = B \cup C \cup D$, hence $\sigma D \subseteq A$. By similar reasoning,

$$A \gtrsim B \cup C \cup D. \tag{3}$$

Putting (2) and (3) together with Lemma 1, we have

$$D \gtrsim A \cup B \cup C \gtrsim A \gtrsim B \cup C \cup D \gtrsim D. \tag{4}$$

If any of these inequalities were strict, then by Transitivity (and the definition of $\sim$) we'd have $D \succ D$, which by Additivity would imply $\varnothing \succ \varnothing$, contradicting Minimality. Hence $B \cup C \cup D \sim D$, so $B \cup C \sim \varnothing$ by Additivity, which implies $B \sim C \sim \varnothing$ by Lemma 1 and Minimality.

Since $\varnothing$ is invariant under any rotation, Invariance then implies both

$$B \cup C \cup D \sim \varnothing, \text{ and} \tag{5}$$

$$A \cup B \cup C \sim \varnothing \tag{6}$$

Hence $\Omega \sim \varnothing$ by Additivity, contradicting Non-triviality. □

Theorem 6, in our view, provides a more compelling argument for imprecise (that is, not totally ordered) probability than IHH's argument from Symmetry. Still, we are inclined to hold onto Totality. Mere Invariance does not seem to capture the intuition that motivates IHH and others. (Here we are in agreement with Pruss 2023.) The intuition repeatedly expressed by IHH (pp. 893, 895, 897, 901, 903, 911) is that devices like our particle emitter can be fair in the sense that congruent sets of points have equal probability. The package that retains Additivity by replacing Symmetry with Invariance does not capture this judgement; it straightforwardly violates it. Notably, Invariance does not support a Williamson-style argument against Regularity.

We also worry that Invariance makes a mess of qualitative probabilistic reasoning. The following principles seem overwhelmingly plausible, for some n and some uniform reading of 'likely' (the argument would also work with 'very likely', 'extremely likely', and so on):

n-Likeliness There is some set of points X such that
(a) X is likely,
(b) X can be partitioned into n equally probable subsets that are equally probable as $X^{\complement}$ (intuitively, X has probability $n/(n+1)$), and
(c) X is equally likely as any of its rotations.

Likeliness Delineation If X is likely and Y is not, then X is more likely than Y.

Likeliness Exclusion No set and its complement are both likely.

To illustrate these principles, it seems possible to divide the sphere into a thousand equal-sized segments, all but one of which are painted red, in such a way that the particle is likely to land on a red point (999-Likeliness part (a)); each of the thousand segments is equally likely to be hit (part (b)); and the particle is equally likely to land in any rotation of the red region (part (c)). If the particle is no less likely to land in some set X than it is to land on a red point, then it must be likely to land in X too (Likeliness Delineation; see Dorr, Nebel, and Zuehl 2023 for extensive discussion and defence of this sort of principle). So it can't be likely to land outside X (Likeliness Exclusion). Of course, there may be very weak readings of 'likely' on which intuitively low-probability events can count as likely (making their high-probability complements likely too, by Likeliness Delineation). But we only need there to be some reading of 'likely' (or 'extremely likely', …) on which all of the above principles are true. We find this hard to deny.

These principles, however, are untenable, given the failures of Totality required by Invariance. The argument appeals to some further premisses which seem very plausible on the assumption of Additivity and which, we expect, would be acceptable even to opponents of Totality.[8]

SUPERSET If $X \succ Y \supseteq Z$ or $X \supseteq Y \succ Z$, then $X \succ Z$.
COMPLEMENTATION $X \succ Y$ if and only if $Y^{\complement} \succ X^{\complement}$.[9]

Theorem 7. *n*-Likeliness, Likeliness Delineation, and Likeliness Exclusion are inconsistent with Invariance, given Superset, Complementation, Additivity, Minimality, and Transitivity.

The proof relies on the following lemmata.

Lemma 2. Suppose Additivity and Transitivity hold and $X_i \succ Y_i$ and $X_i \cap X_j = Y_i \cap Y_j = \varnothing$ for all i, j from 1 to n. Then $X_1 \cup \cdots \cup X_n \succ Y_1 \cup \cdots \cup Y_n$.

Proof. For $n = 2$, following Krantz et al. (1971, pp. 211–12, Lemma 2), suppose $X_1 \succ Y_1$, $X_2 \succ Y_2$, and $X_1 \cap X_2 = Y_1 \cap Y_2 = \varnothing$. Then by Additivity,

$$(X_1 \setminus Y_2) \cup X_2 \succ (X_1 \setminus Y_2) \cup Y_2 = X_1 \cup (Y_2 \setminus X_1) \succ Y_1 \cup (Y_2 \setminus X_1). \tag{7}$$

Then by Transitivity,

$$(X_1 \setminus Y_2) \cup X_2 \succ Y_1 \cup (Y_2 \setminus X_1). \tag{8}$$

Notice that $X_1 \cap Y_2$ is disjoint from both sides of (8), so by another application of Additivity we have the required

$$\begin{aligned} X_1 \cup X_2 = (X_1 \cap Y_2) \cup (X_1 \setminus Y_2) \cup X_2 \succ Y_1 \cup (Y_2 \setminus X_1) \cup (X_1 \cap Y_2) \\ = Y_1 \cup Y_2. \end{aligned} \tag{9}$$

For the case where $n > 2$, the result follows from repeated application of Transitivity and the result for the $n = 2$ case. □

Lemma 3. Where A, B, C, D are as in Theorem 1, there are infinitely many disjoint rotations of B.

[8] These axioms all follow from the theory of Insua (1992, p. 89), which is developed to accommodate imprecise probability. If we could help ourselves to Totality, then Superset and Complementation could both be derived from Minimality, Transitivity, and Additivity—see Krantz et al. (1971, pp. 211–12).

[9] The same follows for $\sim$. *Proof.* $X \sim Y$ implies, by Complementation, that for any Z, $Z \succ Y^{\complement}$ if and only if $Y \succ Z^{\complement}$ if and only if $X \succ Z^{\complement}$ if and only if $Z \succ X^{\complement}$; and, similarly, $Y^{\complement} \succ Z$ if and only if $X^{\complement} \succ Z$. Thus $X \sim Y$ implies $X^{\complement} \sim Y^{\complement}$, which in turn implies $X \sim Y$. □

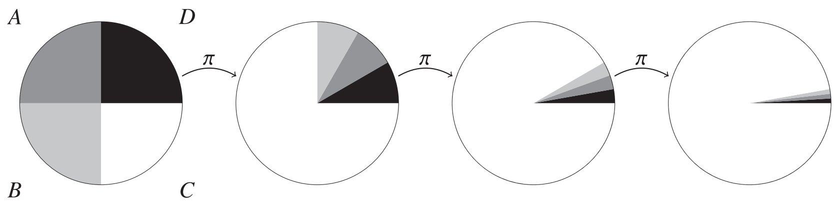

Fig. 1. Illustration of Lemma 3, with segments of the circle representing A, B, C and D, and their images under successive applications of π. Notice how the light grey segment is always disjoint from its position in all previous circles.

Proof. Let π be a rotation such that $\pi C = A \cup B \cup C$, so that $\pi(A \cup B \cup D) = D$. $\pi^m B$ and $\pi^n B$ are disjoint for $m \neq n \in \mathbb{N}$, because, if $m > n$, $\pi^m B$ is a subset of $\pi^n D$ which is disjoint from $\pi^n B$. See Figure 1. □

Proof of Theorem 7. Suppose for *reductio* that B is likely. Then $A \cup C \cup D$ is not likely by Likeliness Exclusion, so by Likeliness Delineation we have

$$B \succ A \cup C \cup D. \tag{10}$$

By Invariance it follows, where π is a rotation with $\pi C = A \cup B \cup C$, that

$$\pi B \succ A \cup B \cup C \cup \pi A \cup \pi D. \tag{11}$$

So by Superset, $\pi B \succ B$. Recall from the proof of Theorem 6 that $\pi B \subseteq D$, so $\pi B \subseteq A \cup C \cup D$. Thus, by Superset again, we also have

$$A \cup C \cup D \succ B. \tag{12}$$

By (10), (12), and Transitivity we have $B \succ B$. By Additivity we then have $\varnothing \succ \varnothing$, which is impossible given Minimality. So B is not likely.

Let S witness n-Likeliness (where $n > 1$). By Likeliness Delineation $S \succ B$, so by Invariance and part (c) of n-Likeliness, $S \succ B \cup C \cup D$. We then have $S \succ C$ by Superset, and so $S \succ A \cup B \cup C$ by Invariance and part (c) of n-Likeliness. We then have $D \succ S^{\complement}$ by Complementation, so $B \cup C \cup D \succ S^{\complement}$ by Superset, so $B \succ S^{\complement}$ by Invariance, part (c) of n-Likeliness, and Complementation for $\sim$ (see note 9 above).

Now, by Lemma 3, there are $n + 1$ disjoint rotations of B, which by a familiar argument are each more likely than $S^{\complement}$. But by the choice of S, Ω can be partitioned into $n + 1$ sets which are each equally as likely as $S^{\complement}$.

Hence the union of those rotations of B is more likely than Ω given Additivity and Transitivity, by Lemma 2—a contradiction given Minimality and Complementation. □

Those who wish to reject Totality on the basis of Invariance therefore face two tasks. The first is to motivate Invariance in a way that does not extend to Symmetry. The second is to explain why we should reject one or more of the above principles for reasoning about what's likely. We do not insist that these tasks are insurmountable, but we are inclined to place our bets against Invariance and in favour of Totality.

5. Conclusion

We have argued that Symmetry is not nearly as obvious as it has seemed. We therefore do not find IHH's argument for imprecise probability compelling. A better argument for imprecise probability appeals to Invariance. But we have seen reason to question Invariance, on the basis of principles which support Totality.

We acknowledge that failures of Symmetry and especially of Invariance may seem highly counterintuitive. IHH point out that real-valued probability functions defined on all sets of points are committed to arbitrarily severe violations of these principles (p. 911). We agree that this situation is extremely surprising and strange. But the measure-theoretic paradoxes which give rise to these results are themselves extremely surprising and strange. And our results suggest that any view about the comparative likelihood of propositions related to non-measurable regions will have highly counterintuitive implications.[10]

References

Bader, Ralf M. 2018: 'Stochastic Dominance and Opaque Sweetening'. *Australasian Journal of Philosophy*, 96(3), pp. 498–507.

Banach, Stefan, and Alfred Tarski 1924: 'Sur la décomposition des ensembles de points en parties respectivement congruentes'. *Fundamenta Mathematicae*, 6, pp. 244–77.

de Finetti, Bruno 1937: 'La Prévision: Ses Lois Logiques, Ses Sources Subjectives'. *Annales de l'Institut Henri Poincaré*, 17, pp. 1–68.

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Dorr, Cian 2024: 'Does Non-Measurability Favour Imprecision?' *Mind*, 133(530), pp. 472–503.

—— Jacob M. Nebel, and Jake Zuehl 2021: 'Consequences of Comparability'. *Philosophical Perspectives*, 35(1), pp. 70–98.

—— 2023: 'The Case for Comparability'. *Noûs*, 57(2), pp. 414–53.

Ellsberg, Daniel 1988: 'Risk, Ambiguity, and the Savage Axioms'. In Peter Gärdenfors and Nils-Eric Sahlin (eds.), *Decision, Probability, and Utility*, pp. 245–69. Cambridge: Cambridge University Press.

Fishburn, Peter C. 1986: 'The Axioms of Subjective Probability'. *Statistical Science*, 1(3), pp. 335–45.

Goodsell, Zachary 2024: 'Decision Theory Unbound'. *Noûs*, 58(3), 669–95.

Hausdorff, F. 1914: 'Bemerkung über den Inhalt von Punktmengen'. *Mathematische Annalen*, 75(3), pp. 428–33.

Heil, Christopher 2019: *Introduction to Real Analysis*. Cham: Springer.

Hoek, Daniel 2021: 'Chance and the Continuum Hypothesis'. *Philosophy and Phenomenological Research*, 103(3), pp. 639–60.

Insua, David Rios 1992: 'On the Foundations of Decision Making under Partial Information'. *Theory and Decision*, 33(1), pp. 83–100.

Isaacs, Yoaav, Alan Hájek, and John Hawthorne 2022: 'Non-Measurability, Imprecise Credences, and Imprecise Chances'. *Mind*, 131(523), pp. 892–916.

Joyce, James M. 1999: *The Foundations of Causal Decision Theory*. Cambridge: Cambridge University Press.

Keynes, John Maynard 1921: *A Treatise on Probability*. New York: Dover Publications.

Koopman, B. O. 1940: 'The Bases of Probability'. *Bulletin of the American Mathematical Society*, 46(10), pp. 763–74.

Krantz, David et al. 1971: *Foundations of Measurement, Vol. I: Additive and Polynomial Representations*. New York: Academic Press.

Maudlin, Tim 2021: 'Credence—and Chance—Without Numbers (and with the Euclidean Property)'. Unpublished manuscript, New York University. https://philsci-archive.pitt.edu/19237/.

Pruss, Alexander R. 2023: 'Strict Dominance and Symmetry'. *Philosophical Studies*, 180(3), pp. 1017–29.

Robinson, Raphael 1947: 'On the Decomposition of Spheres'. *Fundamenta Mathematicae*, 34(1), pp. 246–60.

Russell, Jeffrey Sanford forthcoming: 'Fixing Stochastic Dominance'. To appear in *British Journal for the Philosophy of Science*.

Russell, Jeffrey Sanford, and Yoaav Isaacs 2021: 'Infinite Prospects'. *Philosophy and Phenomenological Research*, 103(1), pp. 178–98.

Savage, Leonard 1954: *The Foundations of Statistics*. New York: Wiley.
Thong, Joshua 2024: 'Consequences of Assigning Non-Measurable Sets Imprecise Probabilities'. *Mind*, 133(531), pp. 793–804.
Tomkowicz, Grzegorz, and Stan Wagon 2016: *The Banach-Tarski Paradox*, 2nd edn. Cambridge: Cambridge University Press.
van Fraassen, Bas C. 1989: *Laws and Symmetry*. Oxford: Oxford University Press.
Williamson, Timothy 2007: 'How Probable Is an Infinite Sequence of Heads?' *Analysis*, 67(3), pp. 173–80.

