A Road Map of Interval Temporal Logics and Duration Calculi

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ABSTRACT. *We survey main developments, results, and open problems on interval temporal logics and duration calculi. We present various formal systems studied in the literature and discuss their distinctive features, emphasizing on expressiveness, axiomatic systems, and (un)decidability results.*

KEYWORDS: *interval temporal logic, duration calculus, expressiveness, axiomatic system, decidability.*

1. Introduction

Interval-based temporal logics stem from four major scientific areas:

Philosophy. The philosophical roots of interval temporal logics can be traced back to Zeno and Aristotle. The nature of Time has always been a favourite subject in philosophy, and in particular, the discussion whether time instants or time periods should be regarded as the primary objects of temporal ontology has a distinct philosophical flavour. Some of the modern formal logical treatments of interval-based structures of time include: [HAM 72] providing a philosophical analysis of interval ontology and interval-based tense logics; [HUM 79] which elaborates on Hamblin's work, introducing a sequent calculus for an interval tense logic over precedence and sub-interval relations; [ROE 80], a follow-up on Humberstone's work, discussing and analyzing persistency (preservation of truth in sub-intervals) and homogeneity; [BUR 82] proposing axiomatic systems for interval-based tense logics of the rationals and reals, studied earlier in [ROE 80]. A comprehensive study and logical analysis of point-based and interval-based ontologies, languages, and logical systems can be found in [BEN 91].

Linguistics. Interval-based logical formalisms have featured in the study of natural languages since the seminal work of Reichenbach [REI 47]. They arise as suitable frameworks for modeling progressive tenses and expressing various language constructions involving time periods and event duration which cannot be adequately grasped by point-based temporal languages. Period-based temporal languages and logics have been proposed and studied in [DOW 79, KAM 79, RIC 88], to mention a few. The linguistic aspects of interval logics will not be treated here, apart from some discussion of the expressiveness about various interval-based temporal languages.

Artificial intelligence. Interval temporal languages and logics have sprung up from *expert systems*, *planning systems*, *theories of actions and change*, *natural language analysis and processing*, etc. as formal tools for temporal representation and reasoning in artificial intelligence. Some of the notable contributions in that area include: [ALL 83] proposing the thirteen Allen's relations between intervals in a linear ordering and a temporal logic for reasoning about them; [ALL 85] providing an axiomatization and a representation result for interval structures based on the *meets* relation between intervals, further studied and developed in [LAD 87], which also provides a completeness theorem and algorithms for satisfiability checking for Allen's calculus represented as a first-order theory; [GAL 90] critically analyzing Allen's framework and arguing the necessity of considering points and intervals on a par, and [ALL 94] developing interval-based theory of actions and events. A comprehensive survey on temporal representation and reasoning in artificial intelligence can be found in [CHI 00].

Computer science. One of the first applications of interval temporal logics to computer science, viz. for specification and design of hardware components, was proposed in [HAL 83, MOS 83] and further developed in [MOS 84, MOS 94, MOS 98, MOS 00a]. Later, other systems and applications of interval logics were proposed in [BOW 00, CHA 98, DIL 92a, DIL 92b, DIL 96a, DIL 96b, RAS 99]. Model checking tools and techniques for interval logics were developed and applied in [CAM 96, PEN 98]. Particularly suitable interval logics for specification and verification of real-time processes in computer science are the *duration calculi* (see [CHA 91, CHA 94, CHA 99, HAN 92, HAN 97, SØR 90]) introduced as extensions of interval logics, allowing representation and reasoning about time durations for which a system is in a given state. For an up-to-date survey on duration calculi see [CHA 04].

Intervals can be regarded as primitive entities or as definable in terms of their endpoints. Accordingly, interval-based temporal logics can be divided into two main classes: 'pure' interval logics, where the semantics is essentially interval-based, that is, formulas are directly evaluated with respect to intervals, and 'non-pure' interval logics, where the semantics is essentially point-based and intervals are only auxiliary entities. An important family of 'non-pure' interval logics is that of the logics in which the *locality* principle is imposed. Such a principle states that an atomic proposition is true at an interval if and only if it is true at the beginning point of that interval.

In this survey we outline (without claiming completeness) main developments, results, and open problems on interval temporal logics and duration calculi, focusing on 'pure' interval logics and on those non-pure ones which adopt locality. We present various formal systems studied in the literature and discuss their distinctive features, emphasizing on expressiveness, axiomatic systems, and (un)decidability results. Since duration calculi are discussed in more details in [CHA 04], we will present this topic in a rather succinct way, while going in more detail on interval logics, mainly on propositional level.

The paper is organized as follows. In Section 2 we introduce the basic syntactic and semantic ingredients of interval temporal logics and duration calculi, including interval temporal structures, operators, and languages with their syntax and semantics. In Section 3 we discuss propositional interval logics, in Section 4 we present a general tableau method for them, while in Section 5 we briefly survey first-order interval logics and duration calculi. Section 6 contains some concluding remarks and directions for future research.

2. Preliminaries

2.1. *Temporal ontologies, interval structures and relations between intervals*

Interval temporal logics are subject to the same ontological dilemmas as the instant-based temporal logic, viz.: should the time structure be considered *linear* or *branching? Discrete* or *dense? With* or *without beginning?* etc. In addition, however, new dilemmas arise regarding the nature of the intervals:

– *Should intervals include their end-points or not?*

– *Can they be unbounded?*

– *Are point-intervals (i.e. with coinciding endpoints) admissible or not?*

– *How are points and intervals related? Which is the primary concept? Should an interval be identified with the set of points in it, or there is more into it?*

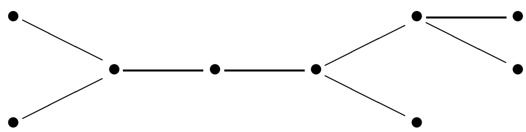
The last question is of particular importance for the semantics of interval logics.

Given a strict partial ordering $\mathbb{D} = \langle D, < \rangle$, an *interval* in $\mathbb{D}$ is a pair $[d_0, d_1]$ such that $d_0, d_1 \in D$ and $d_0 \leq d_1$. $[d_0, d_1]$ is a *strict* interval if $d_0 < d_1$. Often we will refer to all intervals on $\mathbb{D}$ as *non-strict intervals*, to distinguish from the latter. In particular, intervals $[d, d]$ will be called *point-intervals*. A point d *belongs to an interval* $[d_0, d_1]$ if $d_0 \leq d \leq d_1$ (i.e. the endpoints of an interval are included in it). The set of all non-strict intervals on $\mathbb{D}$ will be denoted by $\mathbf{I}(\mathbb{D})^+$, while the set of all strict intervals will be denoted by $\mathbf{I}(\mathbb{D})^-$. By $\mathbf{I}(\mathbb{D})$ we will denote either of these. For the purpose of this survey, we will call a pair $\langle \mathbb{D}, \mathbf{I}(\mathbb{D}) \rangle$ an *interval structure*.

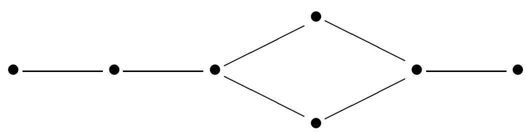
In all systems considered here the intervals will be assumed *linear*, although this restriction can often be relaxed without essential complications. Thus, we will concentrate on partial orderings with the *linear interval property:*

$$\forall x \forall y (x < y \rightarrow \forall z_1 \forall z_2 (x < z_1 < y \wedge x < z_2 < y \rightarrow z_1 < z_2 \vee z_1 = z_2 \vee z_2 < z_1)),$$

that is, orderings in which every interval is linear. Clearly every linear ordering falls here. An example of a non-linear ordering with this property is:



while a non-example is:



An interval structure is:

– **linear**, if every two points are comparable;

– **discrete**, if every point with a successor/predecessor has an immediate successor/predecessor along every path starting from/ending in it, that is,

$$\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z \leq y \wedge \forall w (x < w \wedge w \leq y \rightarrow z \leq w))),$$

and

$$\forall x \forall y (x < y \rightarrow \exists z (x \leq z \wedge z < y \wedge \forall w (x \leq w \wedge w < y \rightarrow w \leq z)));$$

– **dense**, if for every pair of different comparable points there exists another point in between:

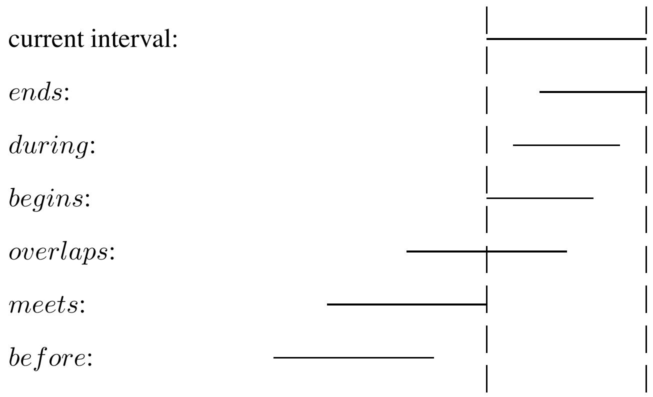
$$\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z < y));$$

– **unbounded above** (resp. **below**), if every point has a successor (resp. predecessor);

– **Dedekind complete**, if every non-empty and bounded above set of points has a least upper bound.

Besides interval logics over the classes of linear, (un)bounded, discrete, dense, and Dedekind complete interval structures, we will be discussing those interpreted on the single structures $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ with their usual orderings.

It is well known that there are 13 different binary relations between intervals on a linear ordering (and quite a few more on a partial ordering) [ALL 83]: *equals, ends, during, begins, overlaps, meets, before*, together with their inverses.



These relations lead to a rich interval algebra, the so-called Allen's Interval Algebra, which will not be discussed in detail here. A survey of Allen's Interval Algebra and of a number of its tractable fragments, including Vilain and Kautz's Point Algebra [VIL 86], van Beek's Continuous Endpoint Algebra [BEE 89], and Nebel and Bürckert's ORD-Horn Algebra [NEB 95], can be found in [CHI 00].

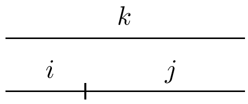
Another natural binary relation between intervals, definable in terms of Allen's relations, is the one of *sub-interval* which comes in three versions. Given a partial ordering $\mathbb{D}$ and intervals $[s_0, s_1]$ and $[d_0, d_1]$ in it:

– $[s_0, s_1]$ is a *sub-interval* of $[d_0, d_1]$ if $d_0 \leq s_0$ and $s_1 \leq d_1$. The relation of sub-interval will be denoted by $\sqsubseteq$;

– $[s_0, s_1]$ is a *proper sub-interval* of $[d_0, d_1]$, denoted $[s_0, s_1]\overline{\sqsubset}[d_0, d_1]$, if $[s_0, s_1] \sqsubseteq [d_0, d_1]$ and $[s_0, s_1] \neq [d_0, d_1]$;

– $[s_0, s_1]$ is a *strict sub-interval* of $[d_0, d_1]$, denoted $[s_0, s_1] \sqsubset [d_0, d_1]$, if $d_0 < s_0$ and $s_1 < d_1$.

Amongst the multitude of *ternary* relations between intervals there is one of particular importance for us, which corresponds to the binary operation of concatenation of meeting intervals. Such a ternary interval relation, which has been introduced by Venema in [VEN 91], can be graphically depicted as follows:



It is denoted by A and it is defined as follows:

– $Aijk$ if i *meets* j, i *begins* k, *and* j *ends* k,

that is, k is the concatenation of i and j.

2.2. *Propositional interval temporal languages and models*

The generic language of propositional interval logics includes the set of propositional letters $\mathcal{AP}$, the classical propositional connectives $\neg$ and $\wedge$ (all others, including the propositional constants $\top$ and $\bot$, are definable as usual), and a set of *interval temporal operators (modalities)* specific for each logical system.

There are two different natural semantics for interval logics, namely, a *strict* one, which excludes point-intervals, and a *non-strict* one, which includes them. A *non-strict interval model* is a pair $\mathbf{M}^+ = \langle \mathbb{D}, V \rangle$, where $\mathbb{D}$ is a partial ordering and $V : \mathbf{I}(\mathbb{D})^+ \rightarrow \mathbf{P}(\mathcal{AP})$ is a *valuation* assigning to each interval a set of atomic propositions considered true at it. Respectively, a *strict interval model* is a structure $\mathbf{M}^- = \langle \mathbb{D}, V \rangle$ defined likewise, where $V : \mathbf{I}(\mathbb{D})^- \rightarrow \mathbf{P}(\mathcal{AP})$. When we do not wish to specify the strictness, we will write simply $\mathbf{M}$, assuming either version.

Allen's relations give rise to respective unary modal operators, thus defining the modal logic of time intervals HS introduced by Halpern and Shoham in [HAL 91]. Some of these modal operators are definable in terms of others and it suffices to choose as basic the modalities corresponding to the relations *begins, ends*, and their inverses. Thus, the formulas of HS are generated by the following abstract syntax:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle B\rangle\phi \mid \langle E\rangle\phi \mid \langle\overline{B}\rangle\phi \mid \langle\overline{E}\rangle\phi.$$

The formal semantics of these modal operators (given in [HAL 91] in terms of non-strict models) is defined as follows:

($\langle B\rangle$) $\mathbf{M}^+, [d_0, d_1] \Vdash \langle B\rangle\phi$ if $\mathbf{M}^+, [d_0, d_2] \Vdash \phi$ for some d_2 such that $d_0 \leq d_2 < d_1$;

($\langle E\rangle$) $\mathbf{M}^+, [d_0, d_1] \Vdash \langle E\rangle\phi$ if $\mathbf{M}^+, [d_2, d_1] \Vdash \phi$ for some d_2 such that $d_0 < d_2 \leq d_1$;

($\langle\overline{B}\rangle$) $\mathbf{M}^+, [d_0, d_1] \Vdash \langle\overline{B}\rangle\phi$ if $\mathbf{M}^+, [d_0, d_2] \Vdash \phi$ for some d_2 such that $d_1 < d_2$;

($\langle\overline{E}\rangle$) $\mathbf{M}^+, [d_0, d_1] \Vdash \langle\overline{E}\rangle\phi$ if $\mathbf{M}^+, [d_2, d_1] \Vdash \phi$ for some d_2 such that $d_2 < d_0$.

A useful new symbol is the *modal constant* π *for point-intervals* interpreted in non-strict models as follows:

(π) $\mathbf{M}^+, [d_0, d_1] \Vdash \pi$ if $d_0 = d_1$.

Note that the constant π is definable as either $[B]\bot$ or $[E]\bot$, so it is only needed in weaker languages. The presence of π in the language allows one to interpret the strict semantics into the non-strict one by means of the translation:

– $\tau(p) = p$ for $p \in \mathcal{AP}$;
– $\tau(\neg\phi) = \neg\tau(\phi)$;
– $\tau(\phi \wedge \psi) = \tau(\phi) \wedge \tau(\psi)$;
– $\tau(\langle * \rangle \phi) = \langle * \rangle (\neg\pi \wedge \tau(\phi))$ for any (unary) interval diamond-modality $\langle * \rangle$.

The interpretation is effected by the following claim, proved by a straightforward induction on ϕ :

PROPOSITION 1. — *For every interval model* $\mathbf{M}$, *proper interval* $[d_0, d_1]$ *in* $\mathbf{M}$, *and formula* ϕ*:*

$$\mathbf{M}^-, [d_0, d_1] \Vdash \phi \textit{ iff } \mathbf{M}^+, [d_0, d_1] \Vdash \tau(\phi).$$

Usually, but not always, the non-strict semantics is taken by default.

Venema introduced in [VEN 91] three binary modalities C, D, and T, associated with the ternary relation A, with the following non-strict semantics:

(C) $\mathbf{M}^+, k \Vdash \phi C \psi$ if there exist two intervals i, j such that $Aijk$ and $\mathbf{M}^+, i \Vdash \phi$, and $\mathbf{M}^+, j \Vdash \psi$, that is,

$\mathbf{M}^+, [d_0, d_1] \Vdash \phi C \psi$ if $\mathbf{M}^+, [d_0, d_2] \Vdash \phi$, and $\mathbf{M}^+, [d_2, d_1] \Vdash \psi$ for some $d_2 \in \mathbb{D}$ such that $d_0 \leq d_2 \leq d_1$.

(D) $\mathbf{M}^+, j \Vdash \phi D \psi$ if there exist two intervals i, k such that $Aijk$ and $\mathbf{M}^+, i \Vdash \phi$, and $\mathbf{M}^+, k \Vdash \psi$, that is,

$\mathbf{M}^+, [d_0, d_1] \Vdash \phi D \psi$ if $\mathbf{M}^+, [d_2, d_0] \Vdash \phi$, and $\mathbf{M}^+, [d_2, d_1] \Vdash \psi$ for some $d_2 \in \mathbb{D}$ such that $d_2 \leq d_0$.

(T) $\mathbf{M}^+, i \Vdash \phi T \psi$ if there exist two intervals j, k such that $Aijk$ and $\mathbf{M}^+, j \Vdash \phi$, and $\mathbf{M}^+, k \Vdash \psi$, that is,

$\mathbf{M}^+, [d_0, d_1] \Vdash \phi T \psi$ if $\mathbf{M}^+, [d_1, d_2] \Vdash \phi$, and $\mathbf{M}^+, [d_0, d_2] \Vdash \psi$ for some $d_2 \in \mathbb{D}$ such that $d_1 \leq d_2$.

The interval logic CDT based on these modal operators will be discussed in the next section.

Similar modal operators for a relation of any arity were introduced in [NIS 80], and a logic with the 'Chop' operator T was studied in [ROS 86].

The semantics of interval temporal logics is sometimes subjected to restrictions justified by specific applications for which a logical system is designed, such as:

– *locality*, meaning that all atomic propositions are point-wise and truth at an interval is defined as truth at its initial point;

– *homogeneity*, requiring that truth of a formula at an interval implies truth of that formula at every sub-interval of it.

A different kind of restriction is imposed by the so-called *split-structures* (see Section 3.3). In the split-structures not all sub-intervals of an interval are 'available' but only those two which are determined by the 'split-point' in that interval.

We will not assume any semantic restrictions, unless otherwise specified.

2.3. *First-order languages and models for interval logics and duration calculi.*

The first-order languages for interval logics extend the propositional ones essentially the same way as in classical logic, but accounting for the fact that the first-order domain may change over time. Formally, these languages involve *terms* built as usual from variables, constants, and functional symbols. Constants and functional symbols are classified as *global* (or *rigid*), whose interpretation does not depend on the time, and *temporal* (or *flexible*), whose interpretation can vary over time. Predicate symbols, also classified as global or temporal, are denoted by $p^i, q^j, \ldots$, where $i, j \ldots$ represent the arities. The abstract syntax of formulas of a generic first-order interval language includes the clauses

$$\phi ::= p^n(\theta_1, \ldots, \theta_n) \mid \exists x\phi \mid \neg\phi \mid \phi \wedge \psi,$$

where $\theta_1, \ldots, \theta_n$ are terms, plus the clauses for the specific interval modal operators.

A formula is *global* (rigid) if it only contains global constant, functional, and predicate symbols.

Among the constants, there is a specific and important one, present in most of the first-order languages for interval logics and duration calculi, viz. the flexible constant l denoting the *length of the current interval*. Often it is combined with a structure of an additive group (typically, the additive group of reals) as part of the temporal domain, which allows for computing lengths of concatenated intervals, etc.

A specific additional feature of the syntax of duration calculi is the special category of terms called *state expressions* which are used to represent the duration for which a system stays in a particular state.

The semantics of first-order interval formulas is a combination of the standard semantics of a first-order (temporal) logic with the semantics of the specific underlying propositional interval logic.

3. Propositional Interval Logics

As already noted, every interval logic L has two versions, namely, the *non-strict* version L^+ and the *strict* one L^-, and when writing just L we will mean either one, as specified in the text.

3.1. *Monadic interval logics*

In this section we introduce and analyze the most well-known and/or interesting interval logics involving only unary modal operators, starting from the weakest. We will assume that the semantic structures are of the most general type we consider, viz. interval structures over partial orderings with the linear interval property, unless otherwise specified.

3.1.1. *The sub-interval logic* D

The logic D is the logic of the sub-interval relation. Since D allows one to look inside the current interval only, from the linear interval hypothesis, it follows that we can restrict ourselves to the class of linear structures.

The abstract syntax of the simplest version of D is:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle D \rangle \phi,$$

but one could also include in the language the modal constant π.

The sub-interval relation and the temporal logics associated with it were studied, from the perspective of philosophical temporal logics, in [HAM 72, ROE 80], [HUM 79] (together with precedence), and [BEN 91]. In the computer science literature, it was apparently first mentioned in [HAL 91] and its expressiveness (interpreted over linear non-strict models) discussed in [LOD 00].

Besides the strict and non-strict versions, the logic D allows essential semantic variations, depending on which sub-interval relation ($\sqsubseteq$, $\overline{\sqsubset}$, or $\sqsubset$) is assumed. Accordingly, the truth definition for D is based on the clause:

($\langle D \rangle$) $\mathbf{M}, [d_0, d_1] \Vdash \langle D \rangle \phi$ if there exists a sub-interval $[d_2, d_3]$ of $[d_0, d_1]$ such that $\mathbf{M}, [d_2, d_3] \Vdash \phi$.

At present, we are not aware of any specific published results about expressiveness, axiomatic systems, and decidability for any variants of the logic D, but we note that they all involve non-trivial valid formulas expressible in D, associated with ‘length vs depth’. To give some idea, here is an infinite scheme of valid formulas of the logic D, with a strict sub-interval relation, which says that if an interval contains sufficiently many distinct sub-intervals (and hence, sufficiently many distinct points), then it contains a chain of nested sub-intervals of pre-defined length:

$$\bigwedge_{i=1}^{d(n)} \langle D \rangle \left(p_i \wedge \bigwedge_{j \neq i} \neg p_j \right) \rightarrow \langle D \rangle^n \top,$$

for $d(n) \geq \binom{2n-1}{2} + 1$

3.1.2. *The logics* $\mathrm{B}\overline{\mathrm{B}}$ *and* $\mathrm{E}\overline{\mathrm{E}}$

Interval logics make it possible to express properties of *pairs* of time points, rather than *single* time points. In most cases, this feature prevents one from the possibility of reducing interval-based temporal logics to point-based ones without resorting to any kind of projection principle. However, there are a few exceptions where such a reduction can be defined thanks to a suitable choice of the interval modalities, thus allowing one to benefit from the good computational properties of point-based logics. This is the case of the logics $\mathrm{B}\overline{\mathrm{B}}$ and $\mathrm{E}\overline{\mathrm{E}}$ (and of their fragments).

The logic $\mathrm{B}\overline{\mathrm{B}}$ is generated by the following abstract syntax:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle B \rangle \phi \mid \langle \overline{B} \rangle \phi,$$

while $\mathrm{E}\overline{\mathrm{E}}$ is obtained from $\mathrm{B}\overline{\mathrm{B}}$ by substituting $\langle E \rangle$ for $\langle B \rangle$ and $\langle \overline{E} \rangle$ for $\langle \overline{B} \rangle$. In the following, we restrict our attention to $\mathrm{B}\overline{\mathrm{B}}$. However, all definitions and results can be easily adapted to $\mathrm{E}\overline{\mathrm{E}}$.

The decidability, as well as other logical properties, of $\mathrm{B}\overline{\mathrm{B}}$ can be obtained by translating it into the propositional temporal logic of linear time Lin-PTL with temporal modalities F (sometime in the future) and P (sometime in the past), which has the finite model property and is decidable (see e.g. [GAB 94]). The formulas of Lin-PTL are defined by

$$f ::= p \mid \neg f \mid f \wedge g \mid Pf \mid Ff,$$

and a model for Lin-PTL is a pair $\langle \mathbb{D}, \mathcal{V} \rangle$, where $\mathbb{D} = \langle D, < \rangle$ is a linearly ordered set and $\mathcal{V} : \mathbb{D} \mapsto \mathbf{P}(\mathcal{AP})$ is a valuation function. The semantics is standard:

– $M, d \Vdash p$ if $p \in \mathcal{V}(d)$;

– $M, d \Vdash \neg f$ if it is not the case that $M, d \Vdash f$;

– $M, d \Vdash f \wedge g$ if $M, d \Vdash f$ and $M, d \Vdash g$;

– $M, d \Vdash Pf$ if there exists d' such that $d' < d$ and $M, d' \Vdash f$;

– $M, d \Vdash Ff$ if there exists d' such that $d < d'$ and $M, d' \Vdash f$.

The formulas of $\mathrm{B}\overline{\mathrm{B}}$ are simply translated into formulas of Lin-PTL by a mapping τ which replaces $\langle B \rangle$ by P and $\langle \overline{B} \rangle$ by F.

Now, for every model $\mathbf{M} = \langle \mathbb{D}, V \rangle$ of $\mathrm{B\overline{B}}$, where $\mathbb{D} = \langle D, < \rangle$, and point $d \in D$, we construct a model for Lin-PTL $\mathbf{M}[d) = \langle [d), \mathcal{V} \rangle$, where $[d) = \{d' \in D \mid d \leq d'\}$ and the valuation $\mathcal{V}$ is defined as follows: for all $d' \in [d)$ and $p \in \mathcal{AP}$: $p \in \mathcal{V}(d')$ iff $p \in V([d, d'])$. Conversely, every model $\mathbf{M} = \langle D, \mathcal{V} \rangle$ for Lin-PTL based on a linear ordering with a least element can be obtained in such a way from some model of $\mathrm{B\overline{B}}$.

LEMMA 2. — *For every model* $\mathbf{M} = \langle \mathbb{D}, V \rangle$ *of* $\mathrm{B\overline{B}}$*, with* $\mathbb{D} = \langle D, < \rangle$*, point* $d \in D$*, and formula* $\phi \in \mathrm{B\overline{B}}$*:*

$$\mathbf{M}, [d, d'] \Vdash \phi \textit{ iff } \mathbf{M}[d), d' \Vdash \tau(\phi)$$

for any $d' \geq d$*.*

Proof: Structural induction on ϕ. For propositional variables the claim holds by definition. The cases of the propositional connectives are straightforward.

Let $\phi = \langle B \rangle \psi$. By definition, $\tau(\phi) = P\tau(\psi)$, and, by hypothesis, $\mathbf{M}, [d, d'] \Vdash \langle B \rangle \psi$, that is, there exists d'' such that $d \leq d'' < d'$ and $\mathbf{M}, [d, d''] \Vdash \psi$. By the inductive hypothesis, $\mathbf{M}[d), d'' \Vdash \tau(\psi)$, and thus $\mathbf{M}[d), d' \Vdash P\tau(\psi)$.

The case $\phi = \langle \overline{B} \rangle \psi$ is similar.

The claim of the lemma now follows immediately. ■

COROLLARY 3. — *A formula* $\phi \in \mathrm{B\overline{B}}$ *is satisfiable in a model* $\mathbf{M}$ *of* $\mathrm{B\overline{B}}$ *iff* $\tau(\phi)$ *is satisfiable in some model* $\mathbf{M}[d)$*.*

Given a linear ordering L we denote by ${}^+L$ the ordering obtained from L by adding a new least element. Accordingly, if C is a class of linear orderings, we define ${}^+C = \{{}^+L \mid L \in C\}$.

Consequently, we obtain the following theorem.

THEOREM 4. — *The satisfiability problem for the logic* $\mathrm{B\overline{B}}$*, interpreted in a given class of interval structures over a class of linear orderings* C*, is reducible to the satisfiability problem for the logic* Lin-PTL *interpreted over the class* ${}^+C$*.*

Thus, for instance, the decidability of $\mathrm{B\overline{B}}$ over the class of all linear orderings follows.

3.1.3. *The logic* BE

The logic BE features the two modalities $\langle B \rangle$ and $\langle E \rangle$, and its formulas are generated by the following abstract syntax:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle B \rangle \phi \mid \langle E \rangle \phi.$$

As we have already shown, the modal constant π is definable as $[B]\bot$. Accordingly, the point-intervals that respectively begin and end the current interval can be captured as follows:

– $[[BP]]\phi \triangleq (\phi \wedge \pi) \vee \langle B\rangle(\phi \wedge \pi)$, and
– $[[EP]]\phi \triangleq (\phi \wedge \pi) \vee \langle E\rangle(\phi \wedge \pi)$.

BE is strictly more expressive than (the non-strict version of) D. On the one hand, if we assume the sub-interval relation to be the strict one (the other two cases can be dealt with in a similar way), the modality $\langle D\rangle$ can be defined as follows:

– $\langle D\rangle\phi \triangleq \langle B\rangle\langle E\rangle\phi$.

On the other hand, the undefinability of $\langle B\rangle$ and $\langle E\rangle$ in D can be easily proved as follows. Let $\langle \mathbb{I}(\mathbb{D})^+, \sqsubset, V\rangle$ be a D-model, where $\mathbb{I}(\mathbb{D})^+$ is the set of all non-strict intervals over $\mathbb{D}$, $\sqsubset$ is the strict sub-interval relation over $\mathbb{I}(\mathbb{D})^+$, and V is the valuation function. The notions of p-morphism and bisimulation between D-models are defined in the usual way for modal logic (see e.g. [BLA 01]), and they satisfy the standard truth-preservation properties. Given two linearly ordered sets $\mathbb{D} = \{d_0, d_1\}$, with $d_0 < d_1$, and $\mathbb{D}' = \{d'_0\}$, we take into consideration two D-models $\mathbf{M}^+ = \langle \mathbb{I}(\mathbb{D})^+, \sqsubset, V\rangle$ and $\mathbf{M'}^+ = \langle \mathbb{I}'(\mathbb{D}')^+, \sqsubset, V'\rangle$ such that:

1) $\mathbb{I}(\mathbb{D})^+ = \{[d_0, d_0], [d_1, d_1], [d_0, d_1]]\}$ and $\mathbb{I}'(\mathbb{D}')^+ = \{[d'_0, d'_0]\}$;
2) the valuations of all intervals in both models are equal to $\{p\}$.

Let $R \subseteq \mathbb{D}\times\mathbb{D}'$ be the relation $\{(d_0, d'_0), (d_1, d'_0)\}$. It is immediate to show that such a relation induces a bisimulation $Z \subseteq \mathbb{I}(\mathbb{D})^+ \times \mathbb{I}'(\mathbb{D}')^+$ between $\mathbf{M}^+$ and $\mathbf{M'}^+$. First, all intervals of both models are evaluated to $\{p\}$, and thus any pair of Z-related intervals satisfies the same atomic propositions. Second, the strict sub-interval relation is empty in both models, and thus the back and the forth conditions are trivially satisfied.

Since $\mathbf{M}^+, [d_0, d_1]$ satisfies $\langle B\rangle p$ (resp., $\langle E\rangle p$), while $\mathbf{M'}^+, [d_0, d_1]$ does not, it immediately follows that $\langle B\rangle$ (resp., $\langle E\rangle$) cannot be defined in D.

BE is expressive enough to capture some relevant conditions on the underlying interval structure, as originally pointed out by Halpern and Shoham in the context of the logic HS [HAL 91]), from where the examples below are adapted. First, one can constrain an interval structure to be discrete by means of the formula:

– $\texttt{discrete} \triangleq \pi \vee \texttt{l1} \vee (\langle \texttt{B}\rangle\texttt{l1} \wedge \langle \texttt{E}\rangle\texttt{l1})$,

where $l1$ is true over an interval $[d_0, d_1]$ if and only if $d_0 < d_1$ and there are no points between d_0 and d_1. Such a condition can be expressed in BE as follows:

$l1 \triangleq \langle B\rangle\top \wedge [B][B]\bot$.

It is not difficult to show that an interval structure is discrete if and only if the formula *discrete* is valid in it. Furthermore, one can easily force an interval structure to be dense by constraining the formula

– $\texttt{dense} \triangleq \neg\texttt{l1}$.

to be valid. Finally, one can constrain an interval structure to be Dedekind complete by means of the formula

– $\texttt{Dedekind complete} \triangleq (\langle \texttt{B} \rangle \texttt{cell} \wedge [[\texttt{EP}]] \neg \texttt{q} \wedge [\texttt{E}]([[\texttt{BP}]]\texttt{q} \rightarrow \langle \texttt{B} \rangle \texttt{cell})) \rightarrow \langle \texttt{B} \rangle ([\texttt{E}](\neg \pi \rightarrow \langle \texttt{D} \rangle \texttt{cell}))$

where cell is true over an interval $[d_0, d_1]$ if and only if its endpoints satisfy a given proposition letter q (the cell delimiters), all sub-intervals satisfy a proposition letter p (the cell content), and there exists at least one sub-interval satisfying p, that is,

$$\texttt{cell} \triangleq [[\texttt{BP}]]\texttt{q} \wedge [[\texttt{EP}]]\texttt{q} \wedge [\texttt{D}]\texttt{p} \wedge \langle \texttt{D} \rangle \texttt{p}.$$

BE also allows one to define a modality $[All]$, referring to all sub-intervals of the given interval, which in that logic is essentially equivalent to the *universal modality* over the submodel generated by the current interval:

– $[All]\phi \triangleq \phi \wedge [B]\phi \wedge [E]\phi \wedge [B][E]\phi.$

As for (un)decidability results, Lodaya [LOD 00] proves the following theorem, which tailors the undecidability proof for HS provided by Halpern and Shoham (cf. Theorem 12) to BE.

THEOREM 5. — *The satisfiability problem for BE-formulas interpreted over non-strict dense linear structures is not decidable.*

Undecidability is proved by reducing the non-halting problem of a Turing Machine (TM) on a blank tape to the satisfiability problem for BE. According to Halpern and Shoham's approach, any computation of a TM is modeled as an infinite sequence of configurations of the machine, called instantaneous descriptions (IDs for short). Each ID is a finite sequence of tape cells that contain a unique tape symbol, and one of the cells has additional information representing the head position and the state of the machine. A suitable proposition is used to talk about consecutive IDs, e.g., to relate the n-th cell of a given ID to the same cell of the successive ID. By exploiting such a proposition, the transition function δ of a TM can be expressed by examining a group of three cells belonging to a given ID and determining the value of the same three cells in the successive ID. A suitable interval formula, parameterized by a TM, can then be built in such a way that such a formula is satisfiable if and only if the TM does not halt on a blank tape. As a matter of fact, most of Halpern and Shoham's proof is carried out in the BE fragment. The other modalities are only used to specify the sequence of IDs and to express the relationships between consecutive IDs. Lodaya shows how to treat the entire infinite computation as being inside a dense interval, which makes it possible to use the $\langle D \rangle$ modality to express the relationships between consecutive IDs as well as to talk about sequences of IDs.

Since density is expressible in BE by a constant formula, we have the following corollary of Theorem 5.

COROLLARY 6. — *The satisfiability problem for* BE *over the class of all non-strict linear structures is not decidable.*

The satisfiability of a formula ϕ in a dense model is indeed equivalent to the satisfiability of $[All]\neg l1 \wedge \phi$ in any non-strict model.

We conclude our description of BE by remarking that a number of meaningful problems, such as the decidability of the satisfiability problem for BE-formulas interpreted over special classes of linear orderings, or over strict models, and the definition of sound and complete axiomatic systems for BE, are, at the best of our knowledge, still open.

3.1.4. *Propositional neighbourhood logics*

The interval logics based on the *meets* relation and its inverse *met-by* are called *neighbourhood logics*. Notably, first-order neighbourhood logics were introduced and studied by Zhou and Hansen in [CHA 98], while their propositional variants, interpreted over linear structures (both strict and non-strict), were studied only quite recently by Goranko, Montanari, and Sciavicco [GOR 03b].

The language of propositional neighbourhood logics includes the modal operators $\Diamond_r$ and $\Diamond_l$ borrowed from [CHA 98]. Its formulas are generated by the following abstract syntax:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond_r\phi \mid \Diamond_l\phi.$$

The dual operators $\Box_r$ and $\Box_l$ are defined in the usual way. To make it easier to distinguish between the two semantics from the syntax, we will reserve this notation for the case of non-strict propositional neighbourhood logics, generically denoted by PNL$^+$, while for the strict ones, denoted by PNL$^-$, $\langle A\rangle$ and $\langle\overline{A}\rangle$ are used instead of $\Diamond_r$ and $\Diamond_l$, respectively. The class of non-strict propositional neighbourhood logics extended with the modal constant π will be denoted by PNL$^{\pi+}$.

The modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$ were originally introduced in the logic HS [HAL 91] as derived operators. The semantics of HS admits point-intervals and hence, according to our classification, it is non-strict. However, the modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$ only refer to strict intervals, and thus the semantics of the fragment A$\overline{\text{A}}$ can be considered essentially strict.

The formal semantics of the modal operators $\Diamond_r$ and $\Diamond_l$ is defined as follows:

($\Diamond_r$) $\mathbf{M}^+, [d_0, d_1] \Vdash \Diamond_r\phi$ if there exists d_2 such that $d_1 \leq d_2$ and $\mathbf{M}^+, [d_1, d_2] \Vdash \phi$;

($\Diamond_l$) $\mathbf{M}^+, [d_0, d_1] \Vdash \Diamond_l\phi$ if there exists d_2 such that $d_2 \leq d_0$ and $\mathbf{M}^+, [d_2, d_0] \Vdash \phi$,

while the semantic clauses for the operators $\langle A\rangle$ and $\langle\overline{A}\rangle$ are:

($\langle A\rangle$) $\mathbf{M}^-, [d_0, d_1] \Vdash \langle A\rangle\phi$ if there exists d_2 such that $d_1 < d_2$ and $\mathbf{M}^-, [d_1, d_2] \Vdash \phi$;

($\langle\overline{A}\rangle$) $\mathbf{M}^-, [d_0, d_1] \Vdash \langle\overline{A}\rangle\phi$ if there exists d_2 such that $d_2 < d_0$ and $\mathbf{M}^-, [d_2, d_0] \Vdash \phi$.

Propositional neighbourhood logics are quite expressive. For example, PNL$^-$ allows one to characterize various classes of linear structures:

(A-SPNL^u) $[A]p \rightarrow \langle A \rangle p$, in conjunction with its mirror image, defines the class of *unbounded* structures;

(A-SPNL^{de}) $(\langle A \rangle \langle A \rangle p \rightarrow \langle A \rangle \langle A \rangle \langle A \rangle p) \wedge (\langle A \rangle [A]p \rightarrow \langle A \rangle \langle A \rangle [A]p)$, in conjunction with its mirror image, defines the class of *dense* structures, extended with the 2-element linear ordering[1];

(A-SPNL^{di}) $([A]\bot \rightarrow [\overline{A}]([A][A]\bot \vee \langle A \rangle(\langle A \rangle \top \wedge [A][A]\bot))) \wedge ((\langle A \rangle \top \wedge [A](p \wedge [\overline{A}]\neg p \wedge [A]p)) \rightarrow [\overline{A}][\overline{A}]\langle A \rangle(\langle A \rangle \neg p \wedge [A][A]p))$, in conjunction with its mirror image, defines the class of *discrete* structures;

(A-SPNL^c) $\langle A \rangle \langle A \rangle [\overline{A}]p \wedge \langle A \rangle [A] \neg [\overline{A}]p \rightarrow \langle A \rangle (\langle A \rangle [\overline{A}]\ [\overline{A}]p \wedge [A]\ \langle A \rangle \neg\ [\overline{A}]\ p)$ defines the class of *Dedekind complete* structures.

Moreover, the language of PNL$^-$over unbounded structures is powerful enough to express the *difference* $[\neq]$ operator:

$$[\neq]q \triangleq [\overline{A}][\overline{A}][A]q \wedge [\overline{A}][A][A]q \wedge [A][A][\overline{A}]q \wedge [A][\overline{A}][\overline{A}]q,$$

saying that q is true at every interval different from the current one, and consequently to simulate *nominals* (the application of the operator n to q constrains q to hold over the current interval and nowhere else):

$$n(q) \triangleq q \wedge [\neq](\neg q).$$

It follows (see, e.g., [GAR 93]) that every universal property of strict unbounded linear structures can be expressed in PNL$^-$.

Sound and complete axiomatic systems for propositional neighbourhood logics have been obtained in [GOR 03b].

THEOREM 7. — *The following axiomatic system is sound and complete for the logic* PNL$^+$ *of non-strict linear structures:*

(A-NT) enough propositional tautologies;

(A-NK) the K axioms for $\Box_r$ and $\Box_l$;

(A-NNF0) $\Box_r p \rightarrow \Diamond_r p$, and its inverse;

(A-NNF1) $p \rightarrow \Box_r \Diamond_l p$, and its inverse;

(A-NNF2) $\Diamond_r \Diamond_l p \rightarrow \Box_r \Diamond_l p$, and its inverse;

(A-NNF3) $\Box_r \Diamond_l p \rightarrow \Diamond_l \Diamond_r \Diamond_r p \vee \Diamond_l \Diamond_l \Diamond_r p$, and its inverse;

(A-NNF4) $\Diamond_r \Diamond_r \Diamond_r p \rightarrow \Diamond_r \Diamond_r p$, and its inverse;

1. The 2-element linear ordering cannot be separated in the language of PNL$^-$.

(A-NNF$_\infty$) $\Box_r q \wedge \Diamond_r p_1 \wedge \ldots \wedge \Diamond_r p_n \rightarrow \Diamond_r(\Box_r q \wedge \Diamond_r p_1 \wedge \ldots \wedge \Diamond_r p_n)$, and its inverse, for each $n \geq 1$.

The rules of inference are Modus Ponens, Uniform Substitution, and $\Box_r$ and $\Box_l$ Generalization. Interestingly, some of these axioms, including the infinite scheme (A-NNF$_\infty$), were not included in the axiomatization of the first-order neighbourhood logic given in [BAR 00] as they could be derived using the first-order axioms.

THEOREM 8. — *[GOR 03b] A sound and complete axiomatic system for the logic* PNL$^{\pi+}$ *can be obtained from that for* PNL$^+$ *by adding the following axioms:*

(A-π1) $\Diamond_l \pi \wedge \Diamond_r \pi$*;*

(A-π2) $\Diamond_r(\pi \wedge p) \rightarrow \Box_r(\pi \rightarrow p)$*, and its inverse;*

(A-π3) $\Diamond_r p \wedge \Box_r q \rightarrow \Diamond_r(\pi \wedge \Diamond_r p \wedge \Box_r q)$*, and its inverse.*

Once $\Diamond_r, \Diamond_l$ are substituted by $\langle A \rangle, \langle \overline{A} \rangle$, and $\Box_r, \Box_l$ accordingly by $[A], [\overline{A}]$, the axioms for PNL$^-$ are very similar to those for PNL$^+$ (accordingly modified to reflect the fact that point-intervals are now excluded), except for the scheme (A-NNF$_\infty$) which is no longer valid.

THEOREM 9. — *[GOR 03b] The following axiomatic system is sound and complete for the logic* PNL$^-$ *of strict linear models:*

(A-ST) *enough propositional tautologies;*

(A-SK) *the K axioms for $[A]$ and $[\overline{A}]$;*

(A-SNF1) $p \rightarrow [A]\langle \overline{A} \rangle p$*, and its inverse;*

(A-SNF2) $\langle A \rangle \langle \overline{A} \rangle p \rightarrow [A]\langle \overline{A} \rangle p$*, and its inverse;*

(A-SNF3) $(\langle \overline{A} \rangle \langle \overline{A} \rangle \top \wedge \langle A \rangle \langle \overline{A} \rangle p) \rightarrow p \vee \langle \overline{A} \rangle \langle A \rangle \langle A \rangle p \vee \langle \overline{A} \rangle \langle \overline{A} \rangle \langle A \rangle p$*, and its inverse;*

(A-SNF4) $\langle A \rangle \langle A \rangle \langle A \rangle p \rightarrow \langle A \rangle \langle A \rangle p$*, and its inverse.*

Let us denote by PNL$^{\lambda-}$, with $\lambda \in \{u, de, di, c, ude, udi, uc\}$, PNL$^-$ interpreted respectively over unbounded, dense, discrete, Dedekind complete, dense and unbounded, discrete and unbounded, and Dedekind complete and unbounded linear structures, respectively. Likewise, PNL$^{\lambda+}$ denotes the respective class of non-strict models.

THEOREM 10. — *[GOR 03b] The following hold:*

1) For every $\lambda_1, \lambda_2 \in \{u, de, di, c, ude, udi, uc\}$, PNL$^{\lambda_1 -} \subsetneq$ PNL$^{\lambda_2 -}$ *iff the class of linear orders characterized by the condition λ_2 is strictly contained in the class of linear orders characterized by the condition λ_1.*

2) PNL$^{ude-}\subsetneq$PNL^{+}*, where the inclusion is in terms of the obvious translation between the two languages (which replaces the strict modalities with the non-strict ones, and vice versa).*

3) PNL $^{+}$= PNL^{u+} = PNL^{de+} = PNL^{ude+} = PNL^{di+} = PNL^{udi+}.

Note that the logic PNL^{udi-} does not yet characterize the interval structure of $\mathbb{Z}$, because the formula

$$\langle A\rangle p\wedge[A](p\rightarrow\langle A\rangle p)\wedge[A][A](p\rightarrow\langle A\rangle p)\rightarrow[A]\langle A\rangle\langle A\rangle p$$

is valid in $\mathbb{Z}$, but not in PNL^{udi-} since it fails in a PNL^{udi-}-model based on $\mathbb{Z}+\mathbb{Z}$.

THEOREM 11. — *[GOR 03b] The axiomatic system for* PNL^{-} *extended with (*A-SPNL^{u}*) (resp. (*A-SPNL^{de}*), (*A-SPNL^{di}*), (*A-SPNL^{ude}*), and (*A-SPNL^{udi}*)) is sound and complete for the class of unbounded (resp. dense, discrete, dense unbounded, and discrete unbounded) structures.*

Finally, we point out that most of the decidability problems related to propositional neighbourhood logics and their fragments are still open.

3.1.5. *The logic* HS

The most expressive propositional interval logic with unary modal operators studied in the literature is Halpern and Shoham's logic HS introduced in [HAL 91]. HS contains (as primitive or definable) all unary modalities introduced earlier. As mentioned in Section 2, HS features the modalities $\langle B\rangle, \langle E\rangle$ and their inverses $\langle\overline{B}\rangle, \langle\overline{E}\rangle$, which suffice to define all other modal operators, so that it can be regarded as the temporal logic of Allen's relations. Unlike most other previously studied interval logics, HS was originally interpreted in non-strict models not over linear orderings, but over all partial orderings with the linear interval property, and all results about HS stated below apply to that class of models, unless otherwise specified.

Formally, HS-formulas are generated by the following abstract syntax:

$$\phi ::= p \mid \neg\phi \mid \phi\wedge\psi \mid \langle B\rangle\phi \mid \langle E\rangle\phi \mid \langle\overline{B}\rangle\phi \mid \langle\overline{E}\rangle\phi.$$

Furthermore, as pointed out by Venema in [VEN 90], the neighbourhood modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$ are definable in the non-strict semantics as follows:

– $\langle A\rangle\phi \triangleq [[EP]]\langle\overline{B}\rangle\phi$, and
– $\langle\overline{A}\rangle\phi \triangleq [[BP]]\langle\overline{E}\rangle\phi$.

HS can express linearity of the interval structure by means of the following formula:

– `linear` $\triangleq$
$((\langle A\rangle p \rightarrow [A](p\vee\langle B\rangle p\vee\langle\overline{B}\rangle p))\wedge((\langle\overline{A}\rangle p \rightarrow [\overline{A}](p\vee\langle E\rangle p\vee\langle\overline{E}\rangle p)),$

as well as all conditions that can be expressed in its fragment BE.

As expected, HS is a highly undecidable logic. In [HAL 91] the authors have obtained important results about non-axiomatizability, undecidability and complexity of the satisfiability in HS for many natural classes of models. Their idea for proving undecidability is based on using an infinitely ascending sequence in the model to simulate the halting problem for Turing Machines. An *infinitely ascending sequence* is an infinite sequence of points $d_0, d_1, d_2, \ldots$ such that $d_i < d_{i+1}$ for all i. Any unbounded above ordering contains an infinite ascending sequence. A class of ordered structures contains an infinite ascending sequence if at least one of the structures in the class does.

THEOREM 12. — *The validity problem in* HS *interpreted over any class of ordered structures with an infinitely ascending sequence is r.e.-hard.*

From Theorem 12, it immediately follows that HS is undecidable for the class of all (non-strict) models, the class of all linear models, the class of all discrete linear models, the class of all dense linear models, the class of all dense and unbounded linear models, etc.

THEOREM 13. — *The validity problem in* HS *interpreted over any class of Dedekind complete ordered structures having an infinitely ascending sequence is* Π^1_1*-hard.*

For instance, the validity in HS in any of the orderings of the natural numbers, integers, or reals is not recursively axiomatizable.

Undecidability occurs even without existence of infinitely ascending sequences. We say that a class of ordered structures has *unboundedly ascending sequences* if for every n there is a structure in the class with an ascending sequence of length at least n.

THEOREM 14. — *The validity problem in* HS *interpreted over any class of Dedekind complete ordered structures having unboundedly ascending sequences is co-r.e. hard.*

Another proof of undecidability of HS, using a tiling problem, can be found in [MAR 99], see also [GAB 00].

In [VEN 90] (see also [MAR 97]) Venema has shown that HS interpreted over a linear ordering is at least as expressive as the universal monadic second-order logic (where second-order quantification is only allowed over monadic predicates) and there are cases where it is strictly more expressive. As a corollary, it can be proved that HS is strictly more expressive than every point-based temporal logic on linear orderings.

In the same paper Venema provided an interesting *geometrical* interpretation of HS, using which he obtained sound and complete axiomatic systems for HS with respect to relevant classes of structures. Here is the idea. An interval can be viewed as an ordered pair of coordinates over a $\langle D, < \rangle \times \langle D, < \rangle$ plane, where $\langle D, < \rangle$ is supposed to be linear. Since the ending point of an interval must be greater than or equal to the starting point, only the north-west half-plane is considered. Clearly, this geometrical interpretation has a good meaning only when HS-formulas are interpreted over linear frames. The geometrical operators are defined as follows:

– $\Diamond_{\downarrow}\phi \triangleq \langle B\rangle\phi$ (ϕ holds at a point right below the current one);

– $\Diamond_{\uparrow}\phi \triangleq \langle \overline{B}\rangle\phi$ (ϕ holds at a point right above the current one);

– $\Diamond_{\rightarrow}\phi \triangleq \langle E\rangle\phi$ (ϕ holds somewhere to the right of the current point);

– $\Diamond_{\leftarrow}\phi \triangleq \langle \overline{E}\rangle\phi$ (ϕ holds somewhere to the left of the current point);

– $\Diamond_{\updownarrow}\phi \triangleq \Diamond_{\downarrow}\phi \vee \phi \vee \Diamond_{\uparrow}\phi$ (ϕ holds at a point with the same longitude, i.e. on the same vertical line);

– $\Diamond_{\leftrightarrow}\phi \triangleq \Diamond_{\rightarrow}\phi \vee \phi \vee \Diamond_{\leftarrow}\phi$ (ϕ holds at a point with the same latitude, i.e. on the same horizontal line).

Notice that, in order to obtain the mirror image (inverse) of a formula written in the geometrical notation, one should simultaneously replace all $\Diamond_{\downarrow}$ by $\Diamond_{\leftarrow}$ and all $\Diamond_{\uparrow}$ by $\Diamond_{\rightarrow}$, and vice versa. Using this geometrical interpretation, Venema has provided sound and complete axiomatic systems for HS over the class of all structures, the class of all linear structures, the class of all discrete structures, and $\mathbb{Q}$. The basic axiomatic system **(A-HS)** for HS includes the following axioms and their mirror-images:

(A-HS1) enough propositional tautologies;

(A-HS2a) $\Box_{\downarrow}(p \to q) \to (\Box_{\downarrow}p \to \Box_{\downarrow}q)$;

(A-HS2b) $\Box_{\uparrow}(p \to q) \to (\Box_{\uparrow}p \to \Box_{\uparrow}q)$;

(A-HS3a) $\Diamond_{\downarrow}\Diamond_{\downarrow}p \to \Diamond_{\downarrow}p$;

(A-HS3b) $\Diamond_{\uparrow}\Diamond_{\uparrow}p \to \Diamond_{\uparrow}p$;

(A-HS4a) $\Diamond_{\downarrow}\Box_{\uparrow}p \to p$;

(A-HS4b) $\Diamond_{\uparrow}\Box_{\downarrow}p \to p$;

(A-HS5) $\Diamond_{\downarrow}\top \to \Diamond_{\downarrow}\Box_{\downarrow}\bot$;

(A-HS6) $\Box_{\downarrow}\bot \to \Box_{\rightarrow}\bot$;

(A-HS7a) $\Diamond_{\uparrow}\Diamond_{\leftarrow}p \to \Diamond_{\leftarrow}\Diamond_{\uparrow}p$;

(A-HS7b) $\Diamond_{\downarrow}\Diamond_{\rightarrow}p \leftrightarrow \Diamond_{\rightarrow}\Diamond_{\downarrow}p$;

(A-HS7c) $\Diamond_{\rightarrow}\Diamond_{\uparrow}p \to \Diamond_{\uparrow}\Diamond_{\rightarrow}p$;

(A-HS8) $(\Diamond_{\downarrow}p \wedge \Diamond_{\downarrow}q) \to [\Diamond_{\downarrow}(p \wedge \Diamond_{\downarrow}q) \vee \Diamond_{\downarrow}(p \wedge q) \vee \Diamond_{\downarrow}(\Diamond_{\downarrow}p \wedge q)]$,

and the following inference rules: Modus Ponens, Generalization for $\Box_{\downarrow}$, $\Box_{\uparrow}$, $\Box_{\rightarrow}$, and $\Box_{\leftarrow}$, and a pair of additional, un-orthodox rules which guarantee that all vertical and horizontal lines in the model are 'syntactically represented':

$$\frac{hor(p) \to \phi}{\phi} \quad \frac{ver(q) \to \psi}{\psi},$$

where p, q do not occur in ϕ, ψ respectively, and

– $hor(\phi) \triangleq \phi \wedge \Box_{\leftarrow}\phi \wedge \Box_{\rightarrow}\phi \wedge \Box_{\uparrow}(\neg\phi \wedge \Box_{\leftarrow}\neg\phi \wedge \Box_{\rightarrow}\neg\phi) \wedge \Box_{\downarrow}(\neg\phi \wedge \Box_{\leftarrow}\neg\phi \wedge \Box_{\rightarrow}\neg\phi)$;
– $ver(\phi) \triangleq \phi \wedge \Box_{\uparrow}\phi \wedge \Box_{\downarrow}\phi \wedge \Box_{\leftarrow}(\neg\phi \wedge \Box_{\uparrow}\neg\phi \wedge \Box_{\downarrow}\neg\phi) \wedge \Box_{\rightarrow}(\neg\phi \wedge \Box_{\uparrow}\neg\phi \wedge \Box_{\downarrow}\neg\phi)$.

The formula $hor(\phi)$ holds at an interval $[d_0, d_1]$ if and only if ϕ holds at any $[d_2, d_1]$ where $d_2 \leq d_1$ and nowhere else. Geometrically, it represents a horizontal line on which ϕ is true, and only there. Likewise $ver(\phi)$ says that ϕ is true exactly at the points of some vertical line.

THEOREM 15. — *The axiomatic system* ***(A-HS)*** *is sound and complete for the class of all non-strict interval structures.*

THEOREM 16. — *A sound and complete axiomatic system for the class of discrete structures can be obtained from* ***(A-HS)*** *by adding the following axiom:*

(A-HS^{z}) `discrete`.

A sound and complete axiomatic system for the class of linear structures can be obtained from ***(A-HS)*** *by replacing axiom (A-HS8) by the following axiom:*

(A-HS^{lin}) $(\Diamond_{\downarrow}\Diamond_{\uparrow}p) \rightarrow (\Diamond_{\downarrow}p \vee p \vee \Diamond_{\uparrow}p), (\Diamond_{\uparrow}\Diamond_{\downarrow}p) \rightarrow (\Diamond_{\downarrow}p \vee p \vee \Diamond_{\uparrow}p)$.

A sound and complete axiomatic system for $\mathbb{Q}$ can be obtained from the system for the class of linear structures by adding the following axiom:

(A-HS$^{\mathbb{Q}}$) $\Diamond_{\leftarrow}\top \wedge \Diamond_{\uparrow}\top \wedge$ `dense`.

In conclusion, we note that, besides D, B$\overline{\text{B}}$, E$\overline{\text{E}}$, BE, and A$\overline{\text{A}}$, there exist other interesting fragments of HS, such as, for instance, D$\overline{\text{D}}$, where $\overline{\text{D}}$ is the transpose of D (D$\overline{\text{D}}$ was already mentioned in [HAL 91]), and AD, which have not been investigated so far. Moreover, to the best of our knowledge, the strict logic HS^{-} has not been studied yet either, and thus no complete axiomatic systems and decidability/undecidablity results have been explicitly established for it.

3.2. *Interval logics with binary operators*

3.2.1. *The chop operator and (Local) Propositional Interval Logics.*

Arguably, the most natural binary interval modality is the *chop* operator C. As proved in [MAR 97], such an operator is not definable in HS. The logic that features the operator C and the modal constant π, interpreted according to the non-strict semantics, is the propositional fragment of the first-order Interval Temporal Logic (ITL) introduced by Moszkowski in [MOS 83] (cf. Section 5.1), usually denoted by PITL. PITL-formulas are defined as follows:

$$\phi ::= p \mid \pi \mid \neg\phi \mid \phi \wedge \psi \mid \phi C \psi.$$

The modalities $\langle B\rangle$ and $\langle E\rangle$ are definable in PITL as follows:

– $\langle B\rangle\phi \triangleq \phi C\neg\pi$, and
– $\langle E\rangle\phi \triangleq \neg\pi C\phi$.

As a matter of fact, the study of PITL was originally confined to the class of discrete linear orderings with finite time, with the *chop* operator paired with a *next* operator, denoted by $\bigcirc$, instead of π. Intervals in such structures will be identified with the (finite) sequences of points occurring in them. For any ϕ, $\bigcirc\phi$ holds at a given (discrete) interval $\sigma = s_1 s_2 \ldots s_n$, with $n \geq 1$, if ϕ holds at the interval $\sigma' = s_2 \ldots s_n$ (if any). It is immediate to see that, over discrete linear orderings, the modal constant π and the *next* operator are inter-changeable. On the one hand, $\pi \triangleq \bigcirc\bot$; on the other hand, for any ϕ, $\bigcirc\phi \triangleq l1C\phi$.

The logic PITL is quite expressive, as the following result from [MOS 83] testifies.

THEOREM 17. — *The satisfiability problem for* PITL *interpreted over the class of non-strict discrete structures is undecidable.*

The proof is actually an adaptation of a theorem by Chandra et al. [CHA 85] showing the undecidability of the satisfiability problem for a propositional process logic. Given two context-free grammars G_1 and G_2, one can build up a PITL-formula which is satisfiable if and only if the intersection of the languages generated by the two grammars is not empty. Since the latter problem is not decidable (see [HOP 79]), the claim immediately follows.

Since PITL is strictly more expressive than BE over the class of discrete linear structures, the above result does not transfer to the latter. On the contrary, the undecidability of the satisfiability problem for PITL over dense structures as well as over all linear structures immediately follows from the undecidability of BE over such structures.

COROLLARY 18. — *The satisfiability problem for* PITL*-formulas interpreted over the class of (non-strict) dense linear structures is undecidable.*

COROLLARY 19. — *The satisfiability problem for* PITL *interpreted over the class of (non-strict) linear structures is undecidable.*

The propositional counterpart of the fragment of ITL that only includes the *chop* operator, has not been investigated yet, as far as we know.

Decidable variants of PITL, interpreted over finite or infinite discrete structures, have been obtained by imposing the so-called *locality projection principle* [MOS 83]. Such a locality constraint states that each propositional variable is true over an interval if and only if it is true at its first state. This allows one to collapse all the intervals starting at the same state into the single interval consisting of the first state only.

Let Local PITL (LPITL for short) be the logic obtained by imposing the locality projection principle to PITL. The syntax of LPITL coincides with that of PITL, while its semantic clauses are obtained from PITL ones by modifying the truth definition of propositional variables as follows:

(loc-PS1) $\mathbf{M}^+, [d_0, d_1] \Vdash p$ iff $p \in V(d_0)$.

where the valuation function V has been adapted to evaluate propositional variables over points instead of intervals.

Various extensions of LPITL have been proposed in the literature. In [MOS 83], Moszkowski focused his attention on the extension of LPITL (over finite time) with quantification over propositional variables, and he proved the decidability of the resulting logic, denoted by QLPITL, by reducing its satisfiability problem to that of the point-based Quantified Propositional Temporal Logic QPTL, interpreted over discrete linear structures with an initial point. In fact, QLPITL is translated into QPTL over finite time, the decidability of which can be proved by a simple adaptation of the standard proof for QPTL over infinite time.

THEOREM 20. — QPTL *is at least as expressive as* QLPITL *interpreted over the class of (non-strict) discrete linear structures*.

Since the translation of QLPITL into QPTL is effective and QPTL is (non-elementarily) decidable, we have the following result.

COROLLARY 21. — *The satisfiability problem for the logic* QLPITL*, interpreted over the class of (non-strict) discrete linear structures is (non-elementarily) decidable*.

The (non-elementary) decidability of LPITL immediately follows from Corollary 21. A lower bound for the satisfiability problem for LPITL, and thus for any extension of it, has been given by Kozen (see [MOS 83]).

THEOREM 22. — *Satisfiability for LPITL is non-elementary.*

In several papers [MOS 83, MOS 94, MOS 98, MOS 00a, MOS 03], Moszkowski explored the extension of LPITL with the so-called *chop-star* modality, denoted by *. For any ϕ, ϕ^* holds over a given (discrete) interval if and only if the interval can be chopped into zero or more parts such that ϕ holds over each of them. The resulting logic, which we denote by LPITL*, is interpreted over either finite or infinite discrete linear structures. A sound and complete axiomatic system for LPITL* with finite time is given in [MOS 03].

THEOREM 23. — *The following axiomatic system is sound and complete for the class of (non-strict) discrete linear structures:*

(A-CLPITL1) *enough propositional tautologies;*

(A-CLPITL2) $(\phi C \psi) C \xi \leftrightarrow \phi C(\psi C \xi)$;

(A-CLPITL3) $(\phi \vee \psi) C \xi \rightarrow (\phi C \xi) \vee (\psi C \xi)$;

(A-CLPITL4) $\xi C(\phi \vee \psi) \rightarrow (\xi C \phi) \vee (\xi C \psi)$;

(A-CLPITL5) $\pi C \phi \leftrightarrow \phi$;

(A-CLPITL6) $\phi C\pi \leftrightarrow \phi$*;*

(A-CLPITL7) $p \rightarrow \neg(\neg pC\top)$, *with* $p \in \mathcal{AP}$*;*

(A-CLPITL8) $\neg(\neg(\phi \rightarrow \psi)C\top) \wedge \neg(\top C\neg(\xi \rightarrow \chi)) \rightarrow ((\phi C\xi) \rightarrow (\psi C\chi))$*;*

(A-CLPITL9) $\bigcirc\phi \rightarrow \neg\bigcirc\neg\phi$*;*

(A-CLPITL10) $\phi \wedge \neg(\top C\neg(\phi \rightarrow \neg\bigcirc\neg\phi)) \rightarrow \neg(\top C\neg\phi)$*;*

(A-CLPITL11) $\phi^* \leftrightarrow \pi \vee (\phi \wedge \bigcirc\top)C\phi^*$*,*

together with Modus Ponens and the following inference rules:

$$\frac{\phi}{\neg(\top C\neg\phi)}, \quad \frac{\phi}{\neg(\neg\phi C\top)}.$$

All axioms have a fairly natural interpretation. In particular, locality is basically dealt with by Axiom (A-CLPITL7).

The chop-star operator is a special case of a more general operator, called the *projection* operator. Such a binary operator, denoted by $proj$, yields general repetitive behaviour: for any given pair of formulas ϕ, ψ, $\phi\ proj\ \psi$ holds over an interval if such an interval can be partitioned into a series of sub-intervals each of which satisfies ϕ, while ψ (called the *projected formula*) holds over the new interval formed from the end points of these sub-intervals. Let us denote by LPITL$_{proj}$ the extension of LPITL with the projection operator `proj`. By taking advantage from such an operator, LPITL$_{proj}$ can express meaningful iteration constructs, such as `for` and `while` loops:

– `for` $n\ times$ `do` $p \triangleq p$ `proj` `len`(n);
– `while` p `do` $q \triangleq (p \wedge q)^* \wedge \neg(\top C(\texttt{len}(0) \wedge p))$,

where the formula p occurring in the `while` loop typically is a point formula, that is, a formula whose satisfaction is totally determined from the first state of the satisfying interval, and, for all $n \geq 0$, `len`(n) constrains the length of the current interval to be exactly n. `len`(n) is defined as follows:

– `len`$(n) \triangleq \bigcirc^n\top \wedge \bigcirc^{n+1}\bot$.

Furthermore, the chop-star operator can be easily defined in terms of projection operator as follows:

– $\phi^* \triangleq \phi$ `proj` $\top$.

LPITL$_{proj}$ was originally proposed by Moszkowski in [MOS 83] and later systematically investigated by Bowman and Thompson [BOW 98, BOW 03]. In particular, a tableau-based decision procedure and a sound and complete axiomatic system for LPITL$_{proj}$, interpreted over finite discrete structures, is given in [BOW 03].

The core of the tableau method is the definition of suitable normal forms for all operators of the logic. These normal forms provide inductive definitions of the operators. Then, in the style of [WOL 85], a tableau decision procedure to check satisfiability of LPITL$_{proj}$ formulas is established. Although the method has been developed at the propositional level, the authors advocate its validity also for first-order LPITL$_{proj}$.

The normal form for LPITL$_{proj}$ formulas has the following general format:

$$(\pi \wedge \phi_e) \vee \bigvee_i (\phi_i \wedge \bigcirc \phi'_i)$$

where ϕ_e and ϕ_i are point formulas and ϕ'_i is an arbitrary LPITL$_{proj}$ formula. The first disjunct states when a formula is satisfied over a point interval, while the second one states the possible ways in which a formula can be satisfied over a strict interval, namely, a point formula must hold at the initial point and then an arbitrary formula must hold over the remainder of the interval. This normal form embodies a recipe for evaluating LPITL$_{proj}$ formulas: the first disjunct is the base case, while the second disjunct is the inductive step. Bowman and Thomson showed that any LPITL$_{proj}$ formula can be equivalently transformed into this normal form.

In [BOW 03], Bowman and Thomson also provided a sound and complete axiomatic system for LPITL$_{proj}$, interpreted over discrete linear structures. Let ϕ, ψ, ξ be arbitrary formulas and $p \in \mathcal{AP}$. The proposed system includes the following axioms:

(A-LPITL1) enough propositional tautologies;

(A-LPITL2) $\neg\pi \leftrightarrow \bigcirc\top$;

(A-LPITL3) $\bigcirc\phi \rightarrow \neg \bigcirc \neg\phi$;

(A-LPITL4) $\bigcirc(\phi \rightarrow \psi) \rightarrow \bigcirc\phi \rightarrow \bigcirc\psi$;

(A-LPITL5) $(\bigcirc\phi)C\psi \leftrightarrow \bigcirc(\phi C\psi)$;

(A-LPITL6) $(\phi \vee \psi)C\xi \leftrightarrow \phi C\xi \vee \psi C\xi$;

(A-LPITL7) $\phi C(\psi \vee \xi) \leftrightarrow \phi C\psi \vee \phi C\xi$;

(A-LPITL8) $\phi C(\psi C\xi) \leftrightarrow (\phi C\psi)C\xi$;

(A-LPITL9) $(p \wedge \phi)C\psi \leftrightarrow p \wedge (\phi C\psi)$, with $p \in \mathcal{AP}$;

(A-LPITL10) $\pi C\phi \leftrightarrow \phi C\pi \leftrightarrow \phi$;

(A-LPITL11) $\phi\ proj\ \pi \leftrightarrow \pi$;

(A-LPITL12) $\phi\ proj\ (\psi \vee \xi) \leftrightarrow (\phi\ proj\ \psi) \vee (\phi\ proj\ \xi)$;

(A-LPITL13) $\phi\ proj\ (p \wedge \psi) \leftrightarrow p \wedge (\phi\ proj\ \psi)$;

(A-LPITL14) $\phi\, proj\ \bigcirc\psi \leftrightarrow (\phi \wedge \neg\pi)C(\phi\, proj\ \psi)$.

The inference rules, besides Modus Ponens and $\bigcirc$-generalization, include the following rule:

$$\frac{\phi \rightarrow \bigcirc^k\phi}{\neg\phi}.$$

THEOREM 24. — *The above axiomatic system is sound and complete for the class of (non-strict) discrete structures.*

Finally, Kono [KON 95] presents a tableau-based decision procedure for QLPITL with *projection*, which has been successfully implemented. The method generates a deterministic state diagram as a verification result. Although it has been argued that the associated axiomatic system is unsound (see [MOS 03]), Kono's work actually inspired Bowman and Thompson's one.

3.2.2. *The logics* CDT *and* BCDT^+

The most expressive propositional interval logic over (non-strict) linear orderings proposed in the literature is Venema's CDT [VEN 91]. A generalization of CDT to (non-strict) partial orderings with the linear interval property, called BCDT^+ has been recently investigated by Goranko, Montanari, and Sciavicco [GOR 03a]. The language of CDT and BCDT^+ contains the three binary operators C, D, and T, together with the modal constant π. Formulas of CDT are generated by the following abstract grammar:

$$\phi ::= \pi \mid p \mid \neg\phi \mid \phi \wedge \psi \mid \phi C\psi \mid \phi D\psi \mid \phi T\psi.$$

The semantics of both CDT and BCDT^+ is non-strict.

The following result links the expressiveness of CDT in terms of definable binary operators to that of the fragment $\mathrm{FO}_3[<](x_i, x_j)$ of first-order logic over linear orderings with at most three variables, at most two of which, viz x_i and x_j are free [VEN 91].

THEOREM 25. — *Every binary modal operator definable in $FO_3[<](x_i, x_j)$ has an equivalent in* CDT*, and vice versa.*

As for the relationships with the other propositional interval logics, interpreted over linear orderings, CDT is strictly more expressive than PITL, since the latter is not able to access any interval which is not a sub-interval of the current interval. Moreover, it is immediate to show that CDT subsumes HS:

– $\langle\rightarrow\rangle\phi = (\neg\pi)C\phi$;
– $\langle\leftarrow\rangle\phi = (\neg\pi)D\phi$;
– $\langle\uparrow\rangle\phi = (\neg\pi)T\phi$;
– $\langle\downarrow\rangle\phi = \phi C(\neg\pi)$.

A sound and complete axiomatic system for CDT over (non-strict) linear structures has been defined by Venema in [VEN 91]. Let us define $hor(\phi)$ as in the case of HS. The axiomatic system for CDT includes the following axioms, and their inverses (obtained by exchanging the arguments of all C occurrences, and replacing each occurrence of T by D and vice versa):

(A-CDT1) enough propositional tautologies;

(A-CDT2a) $(\phi \vee \psi)C\xi \leftrightarrow \phi C\xi \vee \psi C\xi$;

(A-CDT2b) $(\phi \vee \psi)T\xi \leftrightarrow \phi T\xi \vee \psi T\xi$;

(A-CDT2c) $\phi T(\psi \vee \xi) \leftrightarrow \phi T\psi \vee \phi T\xi$;

(A-CDT3a) $\neg(\phi T\psi)C\phi \rightarrow \neg\psi$;

(A-CDT3b) $\neg(\phi T\psi)D\psi \rightarrow \neg\phi$;

(A-CDT3c) $\phi T\neg(\psi C\phi) \rightarrow \neg\psi$;

(A-CDT4) $\neg\pi C\top \leftrightarrow \neg\pi$;

(A-CDT5a) $\pi C\phi \leftrightarrow \phi$;

(A-CDT5b) $\pi T\phi \leftrightarrow \phi$;

(A-CDT5c) $\phi T\pi \rightarrow \phi$;

(A-CDT6) $[(\pi \wedge \phi)C\top \wedge ((\pi \wedge \psi)C\top)C\top] \rightarrow (\pi \wedge \psi)C\top$;

(A-CDT6a) $(\phi C\psi)C\xi \leftrightarrow \phi C(\psi C\xi)$;

(A-CDT6b) $\phi T(\psi C\xi) \leftrightarrow (\psi C(\phi T\xi) \vee (\xi T\phi)T\psi)$;

(A-CDT6c) $\psi C(\phi T\xi) \rightarrow \phi T(\psi C\xi)$;

(A-CDT7d) $(\phi T\psi)C\xi \rightarrow ((\xi D\phi)T\psi \vee \psi C(\phi D\xi))$;

and the following derivation rules: Modus Ponens, Generalization:

$$\frac{\phi}{\neg(\neg\phi C\psi)}, \quad \frac{\phi}{\neg(\neg\phi T\psi)}, \quad \frac{\phi}{\neg(\psi T\neg\phi)}, \text{ and their inverses,}$$

and the Consistency rule: if $p \in \mathcal{AP}$ and p does not occur in ϕ, then

$$\frac{\text{hor}(p) \rightarrow \phi}{\phi}.$$

THEOREM 26. — *The above axiomatic system is sound and complete for the class of (non-strict) linear orderings.*

THEOREM 27. — *A sound and complete axiomatic system for the class of (non-strict) dense linear orderings can be obtained from the system for the class of (non-strict) linear orderings by adding the following axiom:*

(A-CDT^d) $\neg\pi \rightarrow (\neg\pi C \neg\pi)$.

A sound and complete axiomatic system for the class of (non-strict) discrete linear orderings can be obtained from the system for the class of (non-strict) linear orderings by adding the following axiom:

(A-CDT^z) $\pi \vee ((l1C\top) \wedge (\top Cl1))$;

A sound and complete axiomatic system for $\mathbb{Q}$ can be obtained from the system for the class of (non-strict) linear orderings by adding the following axiom:

(A-CDT$^{\mathbb{Q}}$) $(\neg\pi \rightarrow (\neg\pi C \neg\pi)) \wedge (\neg\pi T \top) \wedge (\neg\pi D \top)$.

In [VEN 91], Venema has also developed a sound and complete natural deduction system for CDT, similar to the natural deduction system for relation algebras earlier developed by Maddux [MAD 92].

Finally, as a consequence from previous results for HS and PITL, the satisfiability (resp. validity) for CDT is not decidable over almost all interesting classes of linear orderings, including all, dense, discrete, etc. Again, the strict versions of CDT and BCDT$^+$ have not been explicitly studied yet, but it is natural to expect that similar results apply there, too.

3.3. *Restricted interval logics: split logics*

Split Logics (SLs for short) can be viewed as an attempt of identifying expressive, yet decidable, propositional interval logics without resorting to any locality principle. We have already seen that, in the interval logic setting, decidability can be gained by reducing the set of modal operators (this is the case of B$\overline{\text{B}}$ and E$\overline{\text{E}}$) or by imposing locality conditions (this is the case of LPITL). In the case of SLs, decidability is achieved by imposing suitable constraints on the interval structures over which formulas are interpreted. In the following, we briefly describe the basic features of SLs, and we provide a short summary of the relevant results about them.

SLs have been proposed by Montanari, Sciavicco, and Vitacolonna in [MON 02] as the interval logic counterparts of the monadic first-order (MFO) theories of time granularity studied in [MON 96, FRA 02] (as a matter of fact, there exist also interesting connections between SLs and the propositional dense time logic proposed by Ahmed and Venkatesh in [AHM 93]). SLs are propositional interval logics equipped with operators borrowed from HS and CDT, but interpreted over specific structures, called *split structures*. Models based on split structures are called *split models*. The distinctive feature of split structures is that every interval can be 'chopped' in at most one way (obviously, there is no way to constrain the length of the two resulting subintervals). In [MON 02], the authors show that such a restriction does not prevent

SLs from the possibility of expressing a number of meaningful temporal properties. Furthermore, they prove the decidability of various SLs by embedding them into decidable MFO theories of time granularity as well as their completeness with respect to the guarded fragment of these theories.

Formulas of SLs are generated by the following abstract syntax:

$$\phi ::= p \mid \phi \wedge \phi \mid \neg\phi \mid \langle D\rangle\phi \mid \langle \overline{D}\rangle\phi \mid \langle F\rangle\phi \mid \langle \overline{F}\rangle\phi \mid \phi C\phi \mid \phi D\phi \mid \phi T\phi.$$

A split structure is a pair $\langle \mathbb{D}, \mathbf{H}(\mathbb{D})\rangle$, where $\mathbf{H}(\mathbb{D})$ is proper subset of $\mathbf{I}(\mathbb{D})$ (a precise characterization of $\mathbf{H}(\mathbb{D})$ can be found [MON 02]). A *split model* is a pair $\mathbf{M}= \langle \mathbb{D}, V\rangle$, where $V : \mathbf{H}(\mathbb{D}) \rightarrow \mathbf{P}(\mathcal{AP})$. The semantic clauses for the modalities $\langle D\rangle$, $\langle \overline{D}\rangle$, $\langle F\rangle$, and $\langle \overline{F}\rangle$ are the following ones (the semantic clauses for C, D, and T have already been given):

($\langle D\rangle$) $\mathbf{M}, [d_0, d_1] \Vdash \langle D\rangle\phi$ if there exist d_2, d_3 such that $[d_2, d_3]\overline{\sqsubset}[d_0, d_1]$, and $\mathbf{M}$, $[d_2, d_3] \Vdash \phi$;

($\langle \overline{D}\rangle$) $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{D}\rangle\phi$ if there exist d_2, d_3 such that $[d_0, d_1]\overline{\sqsubset}[d_2, d_3]$, and $\mathbf{M}$, $[d_2, d_3] \Vdash \phi$;

($\langle F\rangle$) $\mathbf{M}, [d_0, d_1] \Vdash \langle F\rangle\phi$ if there exist d_2, d_3 such that $d_1 < d_2$, $d_2 < d_3$, and $\mathbf{M}$, $[d_2, d_3] \Vdash \phi$;

($\langle \overline{F}\rangle$) $\mathbf{M}, [d_0, d_1] \Vdash \langle \overline{F}\rangle\phi$ if there exist d_2, d_3 such that $d_3 < d_2$, $d_2 < d_0$, and $\mathbf{M}$, $[d_3, d_2] \Vdash \phi$.

The modal constant π can also be introduced as a useful shorthand.

In the following we sketch the correspondence between split logics and MFO theories of time granularity. In particular, we enlighten the close relationship that exists between split structures and the temporal structures for time granularity, called layered (or granular) structures [MON 96]. Layered structures replace the single 'flat' temporal domain of linear, point-based temporal logics by a (possibly infinite) set of temporal layers. Each layer is a discrete, linear, point-based domain bounded in the past and infinite in the future. The relationships between time points belonging to the same layer are governed by the usual order relation, while those between points belonging to different layers are expressed by means of suitable projection relations. A formal definition of layered structures can be found in [MON 96, FRA 02]. Here we give an intuitive account of them. The domain of layered structures is a set $\bigcup_{i\in I} T^i$, where $I \subseteq \mathbb{Z}$, which consists of many copies of $\mathbb{N}$ (possibly infinitely many), denoted T^i, each one being a *layer* of the structure. If there is a finite number n of layers, the structure is called *n-layered* (n-LS), otherwise, the structure is called *ω-layered*. Among *ω-layered* structures, we consider the *upward unbounded* layered structure (UULS), which consists of a finest layer and an infinite sequence of coarser and coarser layers, and the *downward unbounded* one (DULS), which consists of a coarsest layer and an infinite sequence of finer and finer ones. In all cases, layers are totally ordered according to their degree of 'coarseness/fineness', and each point of a given layer is associated with k points of the immediately finer layer, if any (*k-refinability*). This accounts for a view of layered structures as (possibly infinite) sequences of (possibly infinite) complete k-ary trees. In the case of the UULS, there is only one infinite tree built up from leaves, which form the finest layer of the structure. In the case of the DULS (resp. n-LS), the infinite sequence of infinite trees (resp. finite) is ordered according to the ordering of the roots, which form the coarsest layer of the structure. In [MON 96, FRA 02], monadic second-order (MSO) theories of layered structures have been systematically studied and the decidability of a number of them has been proved.

SLs can be viewed as the interval logic counterparts of the first-order fragments of the MSO theories of 2-refinable layered structures. More precisely, we focus our attention on the theories MFO$[\bigcup_i T^i, <_1, <_2, \downarrow_0, \downarrow_1]$, interpreted over the 2-refinable n-LS, MFO$[\bigcup_i T^i, <_1, <_2, \downarrow_0, \downarrow_1]$, interpreted over the 2-refinable DULS, and MFO$[\bigcup_i T^i,$ $<_2, \downarrow_0, \downarrow_1]$ interpreted over the 2-refinable UULS. The symbols in the square brackets are (pre)interpreted as follows: $\downarrow_0 (x, y)$ (resp. $\downarrow_1 (x, y)$) is a binary projection relation such that y is the first (resp. second) point in the refinement of x; $<_1$ is a strict partial order such that $x <_1 y$ if x belongs to a tree that precedes the tree y belongs to; $x <_2 y$ holds if y is a descendant of x. As for split structures, we consider (i) the class of bounded below, unbounded above, dense, and with maximal intervals split structures, (ii) the class of bounded below, unbounded above, discrete, and with maximal intervals split structures, and (iii) the class of bounded below, unbounded above, discrete split structures. A split structure with maximal intervals is a split structure $\langle \mathbb{D}, \mathbf{H}(\mathbb{D}) \rangle$, such that, for every $[d_0, d_1] \in \mathbf{H}(\mathbb{D})$ there exists $[d_2, d_3] \in \mathbf{H}(\mathbb{D})$ such that $[d_0, d_1] \sqsubseteq [d_2, d_3]$ and there is no $[d_4, d_5] \in \mathbf{H}(\mathbb{D})$ such that $[d_2, d_3] \overline{\sqsubset} [d_4, d_5]$ (the interval $[d_2, d_3]$ is called a *maximal interval*).

THEOREM 28. — *The following results hold:*

1) SL *interpreted over the class of bounded below, unbounded above, dense, and with maximal intervals split structures can be embedded into* MFO$[\bigcup_i T^i, <_1, <_2, \downarrow_0, \downarrow_1]$ *interpreted over the* 2*-refinable* DULS*;*

2) SL *interpreted over the class of bounded below, unbounded above, discrete, and with maximal intervals split structures can be embedded into* MFO$[\bigcup_i T^i, <_1, <_2, \downarrow_0, \downarrow_1]$ *interpreted over the* 2*-refinable* n-LS*;*

3) SL *interpreted over the class of bounded below, unbounded above, discrete split structures can be embedded into* MFO$[\bigcup_i T^i, <_2, \downarrow_0, \downarrow_1]$ *interpreted over the* 2*-refinable* UULS.

Since such MFO theories of time granularity are decidable, we have the following corollary.

COROLLARY 29. — *The satisfiability problem for* SL *formulas, interpreted over the above classes of split structures, is decidable.*

4. A general tableau method for propositional interval logics

In this section we describe a sound and complete tableau method for BCDT$^+$, developed by Goranko, Montanari and Sciavicco in [GOR 03a], which combines features of tableau methods for modal logics with constraint label management and the classical tableau method for first-order logic. The proposed method can be adapted to variations and subsystems of BCDT$^+$, thus providing a general tableau method for propositional interval logics.

First, some basic terminology. A *finite tree* is a finite directed connected graph in which every node, apart from one (the *root*), has exactly one incoming arc. A *successor* of a node $\mathbf{n}$ is a node $\mathbf{n'}$ such that there is an edge from $\mathbf{n}$ to $\mathbf{n'}$. A *leaf* is a node with no successors; a *path* is a sequence of nodes $\mathbf{n_0}, \ldots, \mathbf{n_k}$ such that, for all $i = 0, \ldots, k-1$, $\mathbf{n_{i+1}}$ is a successor of $\mathbf{n_i}$; a *branch* is a path from the root to a leaf. The *height* of a node $\mathbf{n}$ is the maximum length (number of edges) of a path from $\mathbf{n}$ to a leaf. If $\mathbf{n}, \mathbf{n'}$ belong to the same branch and the height of $\mathbf{n}$ is less than or equal to the height of $\mathbf{n'}$, we write $\mathbf{n} \prec \mathbf{n'}$.

Let $\mathbb{C} = \langle C, < \rangle$ be a finite partial order. A *labelled formula*, with label in $\mathbb{C}$, is a pair $(\phi, [c_i, c_j])$, where $\phi \in$ BCDT$^+$ and $[c_i, c_j] \in \mathbb{I}(\mathbb{C})^+$.

For a node $\mathbf{n}$ in a tree, the *decoration* $\nu(\mathbf{n})$ is a triple $((\phi, [c_i, c_j]), \mathbb{C}, u_{\mathbf{n}})$, where $\mathbb{C}$ is a finite partial order, $(\phi, [c_i, c_j])$ is a labelled formula, with label in $\mathbb{C}$, and $u_{\mathbf{n}}$ is a *local flag function* which associates the values 0 or 1 with every branch B containing $\mathbf{n}$. Intuitively, the value 0 for a node $\mathbf{n}$ with respect to a branch B means that $\mathbf{n}$ can be expanded on B (in fact, $\mathbf{n}$ must be expanded on B, sooner or later, in order to saturate the current decorated tree). For the sake of simplicity, we will often assume the interval $[c_i, c_j]$ to consist of the elements $c_i < c_{i+1} < \cdots < c_j$, and sometimes, with a little abuse of notation, we will write $\mathbb{C} = \{c_i < c_k, c_m < c_j, \ldots\}$. A *decorated tree* is a tree in which every node has a decoration $\nu(\mathbf{n})$. For every decorated tree, we define a *global flag function* u acting on pairs *(node, branch through that node)* as $u(\mathbf{n}, B) = u_{\mathbf{n}}(B)$. Sometimes, for convenience, we will include in the decoration of the nodes the global flag function instead of the local ones. For any branch B in a decorated tree, we denote by $\mathbb{C}_B$ the ordered set in the decoration of the leaf of B, and for any node $\mathbf{n}$ in a decorated tree, we denote by $\Phi(\mathbf{n})$ the formula in its decoration. If B is a branch, then $B \cdot \mathbf{n}$ denotes the result of the expansion of B with the node $\mathbf{n}$ (addition of an edge connecting the leaf of B to $\mathbf{n}$). Similarly, $B \cdot \mathbf{n_1} \mid \ldots \mid \mathbf{n_k}$ denotes the result of the expansion of B with k immediate successor nodes $\mathbf{n_1}, \ldots, \mathbf{n_k}$ (which produces k branches extending B). A tableau for BCDT$^+$ will be defined as a special decorated tree. We note again that $\mathbb{C}$ remains finite throughout the construction of the tableau.

DEFINITION 30. — *Given a decorated tree $\mathcal{T}$, a branch B in $\mathcal{T}$, and a node $\mathbf{n} \in B$ such that $\nu(\mathbf{n}) = ((\phi, [c_i, c_j]), \mathbb{C}, u)$, with $u(\mathbf{n}, B) = 0$, the* branch-expansion rule *for B and $\mathbf{n}$ is defined as follows (in all the considered cases, $u(\mathbf{n'}, B') = 0$ for all new pairs $(\mathbf{n'}, B')$ of nodes and branches).*

– If $\phi = \neg\neg\psi$, then expand the branch to $B \cdot \mathbf{n_0}$, with $\nu(\mathbf{n_0}) = ((\psi, [c_i, c_j]), \mathbb{C}_B, u)$.

– If $\phi = \psi_0 \wedge \psi_1$, then expand the branch to $B \cdot \mathbf{n_0} \cdot \mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\psi_0, [c_i, c_j]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\psi_1, [c_i, c_j]), \mathbb{C}_B, u)$.

– If $\phi = \neg(\psi_0 \wedge \psi_1)$, then expand the branch to $B \cdot \mathbf{n_0}|\mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_i, c_j]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c_i, c_j]), \mathbb{C}_B, u)$.

– If $\phi = \neg(\psi_0 C \psi_1)$ and c is the least element of $\mathbb{C}_B$, with $c_i \leq c \leq c_j$, which has not been used yet to expand the node $\mathbf{n}$ on B, then expand the branch to $B \cdot \mathbf{n_0}|\mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_i, c]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c, c_j]), \mathbb{C}_B, u)$.

– If $\phi = \neg(\psi_0 D \psi_1)$, c is a minimal element of $\mathbb{C}_B$ such that $c \leq c_i$, and there exists $c' \in [c, c_i]$ which has not been used yet to expand the node $\mathbf{n}$ on B, then take the least such $c' \in [c, c_i]$ and expand the branch to $B \cdot \mathbf{n_0}|\mathbf{n_1}$, with $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c', c_i]), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c', c_j]), \mathbb{C}_B, u)$.

– If $\phi = \neg(\psi_0 T \psi_1)$, c is a maximal element of $\mathbb{C}_B$ such that $c_j \leq c$, and there exists $c' \in [c_j, c]$ which has not been used yet to expand the node $\mathbf{n}$ on B, then take the greatest such $c' \in [c_j, c]$ and expand the branch to $B \cdot \mathbf{n_0}|\mathbf{n_1}$, so that $\nu(\mathbf{n_0}) = ((\neg\psi_0, [c_j, c']), \mathbb{C}_B, u)$ and $\nu(\mathbf{n_1}) = ((\neg\psi_1, [c_i, c']), \mathbb{C}_B, u)$.

– If $\phi = (\psi_0 C \psi_1)$, then expand the branch to $B \cdot (\mathbf{n_i} \cdot \mathbf{m_i})| \ldots |(\mathbf{n_j} \cdot \mathbf{m_j})|(\mathbf{n'_i} \cdot \mathbf{m'_i})| \ldots |(\mathbf{n'_{j-1}} \cdot \mathbf{m'_{j-1}})$, where:

1) for all $c_k \in [c_i, c_j]$, $\nu(\mathbf{n_k}) = ((\psi_0, [c_i, c_k]), \mathbb{C}_B, u)$ and $\nu(\mathbf{m_k}) = ((\psi_1, [c_k, c_j]), \mathbb{C}_B, u)$;

2) for all $i \leq k \leq j-1$, let $\mathbb{C}_k$ be the interval structure obtained by inserting a new element c between c_k and c_{k+1} in $[c_i, c_j]$, $\nu(\mathbf{n'_k}) = ((\psi_0, [c_i, c]), \mathbb{C}_k, u)$, and $\nu(\mathbf{m'_k}) = ((\psi_1, [c, c_j]), \mathbb{C}_k, u)$.

– If $\phi = (\psi_0 D \psi_1)$, then repeatedly expand the current branch, once for each minimal element c (where $[c, c_i] = \{c = c_0 < c_1 < \cdots c_i\}$), by adding the decorated sub-tree $(\mathbf{n_0} \cdot \mathbf{m_0})| \ldots |(\mathbf{n_i} \cdot \mathbf{m_i})|(\mathbf{n'_1} \cdot \mathbf{m'_1})| \ldots |(\mathbf{n'_i} \cdot \mathbf{m'_i})|(\mathbf{n''_0} \cdot \mathbf{m''_0})| \ldots |(\mathbf{n''_i} \cdot \mathbf{m''_i})$ to its leaf, where:

1) for all c_k such that $c_k \in [c, c_i]$, $\nu(\mathbf{n_k}) = ((\psi_0, [c_k, c_i]), \mathbb{C}_B, u)$ and $\nu(\mathbf{m_k}) = ((\psi_1, [c_k, c_j]), \mathbb{C}_B, u)$;

2) for all $0 < k \leq i$, let $\mathbb{C}_k$ be the interval structure obtained by inserting a new element c' immediately before c_k in $[c, c_i]$, and $\nu(\mathbf{n'_k}) = ((\psi_0, [c', c_i]), \mathbb{C}_k, u)$ and $\nu(\mathbf{m'_k}) = ((\psi_1, [c', c_j]), \mathbb{C}_k, u)$;

3) for all $0 \leq k \leq i$, let $\mathbb{C}_k$ be the interval structure obtained by inserting a new element c' in $\mathbb{C}_B$, with $c' < c_k$, which is incomparable with all existing predecessors of c_k, $\nu(\mathbf{n''_k}) = ((\psi_0, [c', c_i]), \mathbb{C}_k, u)$, and $\nu(\mathbf{m''_k}) = ((\psi_1, [c', c_j]), \mathbb{C}_k, u)$.

– If $\phi = (\psi_0 T \psi_1)$, then repeatedly expand the current branch, once for each maximal element c (where $[c_j, c] = \{c_j < c_{j+1} < \cdots c_n = c\}$), by adding the decorated sub-tree $(\mathbf{n_j} \cdot \mathbf{m_j})| \ldots |(\mathbf{n_n} \cdot \mathbf{m_n})|(\mathbf{n'_j} \cdot \mathbf{m'_j})| \ldots |(\mathbf{n'_{n-1}} \cdot \mathbf{m'_{n-1}})|(\mathbf{n''_j} \cdot \mathbf{m''_j})| \ldots |(\mathbf{n''_n} \cdot \mathbf{m''_n})$ to its leaf, where:

1) for all c_k *such that* $c_k \in [c_j, c]$, $\nu(\mathbf{n_k}) = ((\psi_0, [c_j, c_k]), \mathbb{C}_B, u)$ *and* $\nu(\mathbf{m_k}) = ((\psi_1, [c_i, c_k]), \mathbb{C}_B, u)$*;*

2) for all $j \leq k < n$*, let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c' *immediately after* c_k *in* $[c_j, c]$*, and* $\nu(\mathbf{n'_k}) = ((\psi_0, [c_j, c']), \mathbb{C}_k, u)$ *and* $\nu(\mathbf{m'_k}) = ((\psi_1, [c_i, c']), \mathbb{C}_k, u)$*;*

3) for all $j \leq k \leq n$*, let* $\mathbb{C}_k$ *be the interval structure obtained by inserting a new element* c' *in* $\mathbb{C}_B$*, with* $c_k < c'$*, which is incomparable with all existing successors of* c_k*,* $\nu(\mathbf{n''_k}) = ((\psi_0, [c_j, c']), \mathbb{C}_k, u)$*, and* $\nu(\mathbf{m''_k}) = ((\psi_1, [c_i, c']), \mathbb{C}_k, u)$*.*

Finally, for any node $\mathbf{m}$ $(\neq \mathbf{n})$ *in* B *and any branch* B' *extending* B*, let* $u(\mathbf{m}, B')$ *be equal to* $u(\mathbf{m}, B)$*, and for any branch* B' *extending* B*,* $u(\mathbf{n}, B') = 1$*, unless* $\phi = \neg(\psi_0 C \psi_1)$*,* $\phi = \neg(\psi_0 D \psi_1)$*, or* $\phi = \neg(\psi_0 T \psi_1)$ *(in such cases* $u(\mathbf{n}, B') = 0$*).*

Let us briefly explain the expansion rules for $\psi_0 C \psi_1$ and $\neg(\psi_0 C \psi_1)$ (similar considerations hold for the other temporal operators). The rule for the (existential) formula $\psi_0 C \psi_1$ deals with the two possible cases: either there exists c_k in $\mathbb{C}_B$ such that $c_i \leq c_k \leq c_j$ and ψ_0 holds over $[c_i, c_k]$ and ψ_1 holds over $[c_k, c_j]$ or such an element c_k must be added. The (universal) formula $\neg(\psi_0 C \psi_1)$ states that, for all $c_i \leq c \leq c_j$, ψ_0 does not hold over $[c_j, c]$ or ψ_1 does not hold over $[c, c_j]$. As a matter of fact, the expansion rule imposes such a condition for a single element c in $\mathbb{C}_B$ (the least element which has not been used yet), and it does not change the flag (which remains equal to 0). In this way, all elements will be eventually taken into consideration, including those elements in between c_i and c_j that will be added to $\mathbb{C}_B$ in some subsequent steps of the tableau construction.

Let us define now the notions of open and closed branch. We say that a node $\mathbf{n}$ in a decorated tree $\mathcal{T}$ is *available on a branch* B to which it belongs if and only if $u(\mathbf{n}, B) = 0$. The branch-expansion rule is *applicable* to a node $\mathbf{n}$ on a branch B if the node is available on B and the application of the rule generates at least one successor node with a new labelled formula. This second condition is needed to avoid looping of the application of the rule on formulas $\neg(\psi_0 C \psi_1)$, $\neg(\psi_0 D \psi_1)$, and $\neg(\psi_0 T \psi_1)$.

DEFINITION 31. — *A branch* B *is* closed if some of the following conditions holds:

(i) there are two nodes $\mathbf{n}, \mathbf{n'} \in B$ such that $\nu(\mathbf{n}) = ((\psi, [c_i, c_j]), \mathbb{C}, u)$ and $\nu(\mathbf{n'}) = ((\neg\psi, [c_i, c_j]), \mathbb{C}', u)$ for some formula ψ and $c_i, c_j \in \mathbb{C} \cap \mathbb{C}'$;

(ii) there is a node $\mathbf{n}$ such that $\nu(\mathbf{n}) = ((\pi, [c_i, c_j]), \mathbb{C}, u)$ and $c_i \neq c_j$; or

(iii) there is a node $\mathbf{n}$ such that $\nu(\mathbf{n}) = ((\neg\pi, [c_i, c_j]), \mathbb{C}, u)$ and $c_i = c_j$.

If none of the above conditions hold, the branch is open.

DEFINITION 32. — *The* branch-expansion strategy *for a branch* B *in a decorated tree* $\mathcal{T}$ *is defined as follows:*

1) Apply the branch-expansion rule to a branch B *only if it is open;*

2) If B *is open, apply the branch-expansion rule to the closest to the root available node in* B *for which the branch-expansion rule is applicable.*

DEFINITION 33. — *A* tableau *for a given formula* $\phi \in$BCDT$^+$ *is any finite decorated tree* $\mathcal{T}$ *obtained by expanding the three-node decorated tree built up from an empty-decoration root and two leaves with decorations* $((\phi, [c_b, c_e]), \{c_b < c_e\}, u)$ *and* $((\phi, [c_b, c_b]), \{c_b\}, u)$*, where the value of* u *is* 0*, through successive applications of the branch-expansion strategy to the existing branches.*

It is easy to show that if $\phi \in$ BCDT$^+$, $\mathcal{T}$ is a tableau for ϕ, $\mathbf{n} \in \mathcal{T}$, and $\mathbb{C}$ is the ordered set in the decoration of $\mathbf{n}$, then $\langle \mathbb{C}, < \rangle$ is an interval structure.

THEOREM 34 (SOUNDNESS AND COMPLETENESS). — *If* $\phi \in$ BCDT$^+$ *and a tableau* $\mathcal{T}$ *for* ϕ *is closed, then* ϕ *is not satisfiable. Moreover, if* $\phi \in$ BCDT$^+$ *is a valid formula, then there is a closed tableau for* $\neg\phi$.

5. First-Order Interval Logics and Duration Calculi

Research on interval temporal logics in computer science was originally motivated by problems in the field of specification and verification of hardware protocols, rather than by abstract philosophical or logical issues. Not surprisingly, it focused on first-order, rather than propositional, interval logics. In this section, we summarize some of the most-important developments in first-order interval logics and duration calculi, referring the interested reader to respectively [MOS 03] and [CHA 04] for more details.

5.1. *The logic* ITL

First-order ITL, interpreted over discrete linear orderings with finite time intervals, was originally developed by Halpern, Manna, and Moszkowski in [MOS 83, HAL 83]. The language of ITL includes terms, predicates, Boolean connectives, first-order quantifiers, and the temporal modalities C and $\bigcirc$. Terms are built on variables, constants, and function symbols in the usual way. Constants and function symbols are classified as *global/rigid* and *temporal/ flexible*. Terms are usually denoted by $\theta_1, \ldots, \theta_n$. Predicate symbols are also partitioned into global and temporal ones. They are denoted by $p^i, q^j, \ldots$, where p^i is a predicate of arity i, q^j is a predicate of arity j, and so on. The abstract syntax of ITL formulas is:

$$\phi ::= \theta \mid p^n(\theta_1, \ldots, \theta_n) \mid \exists x\phi \mid \; \neg\phi \mid \phi \wedge \psi \mid \; \bigcirc \phi \mid \phi C \psi.$$

The semantics of ITL-formulas is a combination of the standard semantics of a first-order temporal logic with the semantics of PITL. An account of possible uses and applications is e.g. [MOS 86].

In [DUT 95a] Dutertre studies the fragment of ITL which we will denote here by ITL$_D$, involving only the *chop* operator. First, ITL$_D$ is considered over abstract, Kripke-style models $\mathbf{M}^+ = \langle W, R, I \rangle$, where W is a set of worlds (abstract intervals), R is a ternary relation corresponding to Venema's ternary relation A (cf. Section

2.1, and I is a first-order interpretation. Further, Dutertre considers a more concrete semantics, over interval structures with associated 'length' measure represented by a special temporal variable l which takes values in a commutative group $\langle \mathbb{D}, +, -, 0 \rangle$. The language is assumed to have the flexible constant l, and the rigid symbols 0 and $+$, respectively interpreted as the neutral element and the addition in $\langle \mathbb{D}, +, 0 \rangle$. The semantics of ITL$_D$-formulas is a combination of the semantics of ITL (without *next*), and the interpretation of l in a model $\mathbf{M}^+$ for an interval $[d_0, d_1]$ is $d_1 - d_0$.

As for the expressive power of ITL$_D$, note that one can easily define the modal constant π (cf. Section 2.2) by means of l:

– $\pi \triangleq (l = 0)$.

Hence, the HS modalities corresponding to *begins* and *ends* are also definable in the language, and thus, from the results of Section 3.1.3, we can conclude that ITL$_D$ is at least as expressive as PITL. The undecidability of the logic easily follows.

Dutertre developed a sound and complete axiomatic system for ITL$_D$ (the details of the soundness and completeness proof can be found in [DUT 95a]). In addition to the standard axioms of first-order classical logic, including the axioms of identity and the axioms describing the properties for the temporal domain $\mathbb{D}$, Dutertre's systems involves the following specific axioms for ITL$_D$:

(A-ITL1) $(\phi C \psi) \wedge \neg(\phi C \xi) \rightarrow \phi C(\psi \wedge \neg\xi)$;

(A-ITL2) $(\phi C \psi) \wedge \neg(\xi C \psi) \rightarrow (\phi \wedge \neg\xi) C \psi$;

(A-ITL3) $((\phi C \psi) C \xi) \leftrightarrow (\phi C(\psi C \xi))$;

(A-ITL4) $(\phi C \psi) \rightarrow \phi$ if ϕ is a rigid formula;

(A-ITL5) $(\phi C \psi) \rightarrow \psi$ if ψ is a rigid formula;

(A-ITL6) $((\exists x)\phi C \psi) \rightarrow (\exists x)(\phi C \psi)$ if x is not free in ψ;

(A-ITL7) $(\phi C(\exists x)\psi) \rightarrow (\exists x)(\phi C \psi)$ if x is not free in ϕ;

(A-ITL8) $((l = x) C \phi) \rightarrow \neg((l = x) C \neg\phi)$;

(A-ITL9) $(\phi C(l = x)) \rightarrow \neg(\neg\phi C(l = x))$;

(A-ITL10) $(l = x + y) \leftrightarrow ((l = x) C(l = y))$;

(A-ITL11) $\phi \rightarrow (\phi C(l = 0))$;

(A-ITL12) $\phi \rightarrow ((l = 0) C \phi)$.

The inference rules are Modus Ponens, Generalization, Necessitation, and the following Monotonicity rule:

$$\frac{\phi \rightarrow \psi}{\phi C \xi \rightarrow \psi C \xi},$$

together with the symmetric one. It should be noted that certain restrictions apply to the instantiation with flexible terms in quantified formulas.

As in the propositional case, variants of ITL obtained by imposing the locality constraint have been explored in the literature. Sound and complete axiomatic systems for local variants of ITL for finite and infinite time have been established in [DUT 95a, DUT 95b, MOS 00b], while automata-theoretic techniques for proving completeness of ITL have been applied in [MOS 00a, MOS 03].

For more details about completeness and decidability results on ITL see [MOS 03]. See also [MOS 86] and [DUA 96], for applications of ITL to temporal logic programming, and [MOS 96b, MOS 98], where the ITL-based programming language Tempura is described in detail.

5.1.1. *Some extensions and variations of* ITL

An extension of ITL with projection has been studied in [GUE 00b] where a complete axiomatic system for it has been established. A probabilistic extension of ITL has been studied in [GUE 00d].

An interesting variation of ITL is the Signed Interval Logic (SIL) introduced by Rasmussen [RAS 99, RAS 02]. The semantics of SIL is based on *signed intervals*, i.e., intervals provided with a *direction* (forward or backward). A sound and complete axiomatic system for SIL was established in [RAS 99], a natural deduction system in [RAS 01b], and a sequent calculus in [RAS 01a].

Dillon, Kutty, Moser, Melliar-Smith, and Ramakrishna introduce and study in a series of publications [RAM 92, DIL 92a, DIL 92b, DIL 93, DIL 94c, DIL 94b, DIL 95, MOS 96a, DIL 96a, DIL 96b, DIL 94a] the so-called Future Interval Logics. These employ the locality principle and feature 'interval modalities' encoded by pairs of formulas and refer to intervals whose endpoints satisfy these formulas. Notably, these logics are more tractable and have lower complexity than e.g. ITL. Complexity results for Future Interval Logic have been obtained by Aaby and Narayana [AAB 85], while applications of these logics have been explored in Ramakrishna's PhD thesis [RAM 93].

5.2. ***The logic NL***

The logic ITL has an intrinsic limitation: its modalities do not allow one to 'look' outside the current interval (modalities with this characteristic are called *contracting* modalities). To overcome such a limitation, Zhou and Hansen [CHA 91] proposed the first-order logic of *left* and *right* neighbourhood modalities, called *neighbourhood logic* (NL for short), whose propositional fragment has been analyzed in Section 3.1.4.

First-order syntactic features are as in the ITL case. Right and left neighbourhood modalities are denoted by $\Diamond_r$ and $\Diamond_l$, respectively. The abstract syntax of NL formulas is:

$$\phi ::= \theta \mid p^n(\theta_1, \ldots, \theta_n) \mid \neg\phi \mid \phi \wedge \psi \mid \Diamond_l \phi \mid \Diamond_r \phi \mid \exists x \phi,$$

where terms $\theta_1, \ldots, \theta_n$ are defined as in ITL.

The semantic clauses for the neighbourhood modalities $\Diamond_l$ and $\Diamond_r$ are defined as in the propositional case. The rest of the semantics of NL is defined exactly as in the ITL case. While practically meant to be the ordered additive group of the real numbers, the temporal domain is abstractly specified by means of a set of first-order axioms defining the so-called $\mathcal{A}$-*models* [CHA 98].

The first-order neighbourhood logic NL is quite expressive. In particular, it allows one to express the *chop* modality as follows:

– $\phi C \psi \triangleq \exists x, y(l = x + y) \wedge \Diamond_l \Diamond_r((l = x) \wedge \phi \wedge \Diamond_r((l = y) \wedge \psi)),$

as well as any of the modalities corresponding to Allen's relations. Consequently, NL can virtually express all interesting properties of the underlying linear ordering, such as discreteness, density, etc.

Here we give an axiomatic system for NL, due to Barua, Roy, and Zhou [BAR 00], where the soundness and completeness proofs can be found. In what follows, the symbol $\Diamond$ stands for either $\Diamond_l$ or $\Diamond_r$, while $\overline{\Diamond}$ stands for $\Diamond_r$ (resp., $\Diamond_l$) when $\Diamond$ stands for $\Diamond_l$ (resp., $\Diamond_r$). The axiomatic system consists of the following axioms:

(A-NL1) $\Diamond\phi \rightarrow \phi$, where ϕ is a global formula;

(A-NL2) $l \geq 0$;

(A-NL3) $x \geq 0 \rightarrow \Diamond(l = x)$;

(A-NL4) $\Diamond(\phi \vee \psi) \rightarrow \Diamond\phi \vee \Diamond\psi$;

(A-NL5) $\Diamond\exists x\phi \rightarrow \exists x\Diamond\phi$;

(A-NL6) $\Diamond((l = x) \wedge \phi) \rightarrow \Box((l = x) \rightarrow \phi)$;

(A-NL7) $\Diamond\overline{\Diamond}\phi \rightarrow \Box\overline{\Diamond}\phi$;

(A-NL8) $(l = x) \rightarrow (\phi \leftrightarrow \overline{\Diamond}\Diamond((l = x) \wedge \phi)$;

(A-NL9) $((x \geq 0) \wedge (y \geq 0)) \rightarrow (\Diamond((l = x) \wedge \Diamond((l = y) \wedge \Diamond\phi)) \leftrightarrow \Diamond((l = x + y) \wedge \Diamond\phi))$,

plus the axioms for the domain $\mathbb{D}$ (axioms for $=, +, \leq$, and $-$), and the usual axioms for first-order logic. The same restrictions that have been made for the ITL concerning the instantiation of quantified formulas still apply here. The inference rules are, as usual, Modus Ponens, Necessitation, Generalization, and the following rule for Monotonicity:

$$\frac{\phi \rightarrow \psi}{\Diamond\phi \rightarrow \Diamond\psi}.$$

In [BAR 97], NL has been extended to a 'two-dimensional' version, called NL^2, where two modalities $\Diamond_u$ and $\Diamond_d$ have been added and interpreted as 'up' and 'down' neighbourhoods. NL^2 can be used to specify super-dense computations, taking vertical time as virtual time, and horizontal time as real time.

The relationship between the Neighbourhood Logic and tractable fragments of Allen's Interval Algebra has been studied in [PUJ 97].

5.3. *Duration calculi*

Duration Calculus (DC for short) is an interval temporal logic endowed with the additional notion of *state*. Each state is denoted by means of a state expression, and it is characterized by a *duration*. The duration of a state is (the length of) the time period during which the system remains in the state. DC has been successfully applied to the specification and verification of real-time systems. For instance, it has been used to express the behaviour of communicating processes sharing a processor and to specify their scheduler, as well as to specify the requirements of a gas burner [SØR 90].

DC has originally been developed as an extension of Moszkowski's ITL, and thus denoted by DC/ITL. Since the seminal work by Zhou, Hoare, and Ravn [CHA 91], various meaningful fragments of DC/ITL have been isolated and analyzed. Recently, an alternative Duration Calculus, based on the logic NL, and thus denoted by DC/NL, has been proposed by Roy in [ROY 97]. As a matter of fact, most results for DC/ITL and its fragments transfer to DC/NL and its fragments. In the following we introduce the basic notions and we summarize the main results about DC/ITL. Further details can be found in [CHA 04].

5.3.1. *The calculus DC/ITL*

Zhou, Hoare, and Ravn's DC/ITL is based on Moszkowski's ITL interpreted over the class of non-strict interval structures based on $\mathbb{R}$. Its only interval modality is *chop*. Its distinctive feature is the notion of state. States are represented by means of a new syntactic category, called *state expression*, which is defined as follows: the constants 0 and 1 are state expressions, a state variable X is a state expression, and, for any pair of state expressions S and T, $\neg S$ and $S \vee T$ are state expressions (the other Boolean connectives are defined in the usual way). Furthermore, given a state expression S, the duration of S is denoted by $\int S$. DC/ITL terms are defined as in ITL, provided that temporal variables are replaced by state expressions. DC/ITL formulas are generated by the following abstract syntax:

$$\phi ::= p^n(r_1, \ldots, r_n) \mid \top \mid \neg\phi \mid \phi \vee \psi \mid \phi C \psi \mid \exists x \phi$$

where $r_1, \ldots, r_n$ are terms, p^n is a n-ary (global) predicate, C is the *chop* modality, and x is a (global) variable.

Any state (expression) S is associated with a total function $S : \mathbb{R} \mapsto \{0, 1\}$, which has a finite number of discontinuity points only. For any time point t, the state expression interpretation $\mathcal{I}$ is defined as follows:

– $\mathcal{I}[0](t) = 0$;
– $\mathcal{I}[1](t) = 1$;
– $\mathcal{I}[S](t) = S(t)$;
– $\mathcal{I}[\neg S](t) = 1 - \mathcal{I}[S](t)$;
– $\mathcal{I}[S \vee T](t) = 1$ if $\mathcal{I}[S](t) = 1$ or $\mathcal{I}[T](t) = 1, 0$ otherwise.

The semantics of a duration $\int S$ in a given (non-strict) model, with respect to an interval $[d_0, d_1]$, can be defined using the Riemann definite integral $\int_{d_0}^{d_1} \mathcal{I}[S](t)dt$. The semantics of the other syntactic constructs is given as in the case of ITL.

A number of useful abbreviations can be defined in DC/ITL. In particular, $\lceil S \rceil$ stands for: "S holds almost everywhere over a strict interval", and it is defined as follows:

– $\lceil S \rceil \triangleq (\int S = \int 1) \wedge \neg(\int 1 = \int 0)$.

$\int 1$ is usually abbreviated by l, and it can be viewed as the length of the current interval; finally, $\lceil\,\rceil$, which holds over point-intervals, can be defined as $l = 0$.

The satisfiability problem for both first-order DC/ITL (full DC/ITL) and its fragment devoid of first-order quantification (Propositional DC/ITL) has been shown to be undecidable. First-order DC/ITL, provided with, at least, the functional symbol $+$ and the predicate symbol $=$, with the usual interpretation, has been completely axiomatized in [HAN 92]. The axiomatic system includes the following specific axioms:

(A-DC1) $\int 0 = 0$;

(A-DC2) $\int S \geq 0$;

(A-DC3) $\int S + \int T = \int(S \vee T) + \int(S \wedge T)$;

(A-DC4) $((\int S = x)C(\int S = y)) \leftrightarrow (\int S = x + y)$;

(A-DC5) $\int S = \int T$ provided that $S \leftrightarrow T$ holds in propositional logic

and the following inference rule (where $S_1 \ldots S_n$ are state expressions and $\bigvee_{i=1}^{n} S_i \leftrightarrow 1$):

$$\frac{H(\lceil\,\rceil),\; H(\phi) \;\rightarrow\; H(\phi \vee \bigvee_{i=1}^{n}(\phi C \lceil S_i \rceil))}{H(\top)},$$

in conjunction with its inverse (obtained by exchanging the ordering of the formulas in every *chop*), where $H(\phi)$ represents the formula obtained from $H(X)$ by replacing every occurrence of X in H by ϕ.

Duration calculus on abstract domains has been studied and axiomatized in [GUE 98].

Various interesting fragments of DC have been investigated by Zhou, Hansen, and Sestoft in [CHA 93a]. First, they consider the possibility of interpreting DC formulas over different classes of structures. In particular, the fragment of DC *interpreted over* $\mathbb{N}$ is the set of DC formulas interpreted over $\mathbb{R}$ evaluated with respect of $\mathbb{N}$-intervals, that is, intervals whose endpoints are in $\mathbb{N}$. The fragment of DC *interpreted over* $\mathbb{Q}$ is similarly defined. Then, the authors take into consideration some syntactic sub-fragments of the above calculi and they atudy the decidability/undecidability of their satisfiability problem. It turns out that the fragments of propositional DC whose formulas are built up from primitive formulas of the type $\lceil S \rceil$ only have a decidable satisfiability problem when interpreted over $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$. A validity checking procedure for some of these fragments was developed in [SKA 94]. By adding to the set of primitive formulas those of the form $l = k$, the problem remains decidable over $\mathbb{N}$, but it becomes undecidable over the other classes of structures. The same fragment at the first-order level is undecidable in all the considered cases. Finally, the fragment of propositional DC whose formulas are built up from primitive formulas of the type $\int S = \int T$ only is also undecidable.

As for the complexity of the satisfiability problem, in [RAB 98] Rabinovich reports a result by Sestoft (personal communication) stating that the satisfiability problem for the fragment of DC whose formulas are built up from primitive formulas of the type $\lceil S \rceil$ only, interpreted over $\mathbb{N}$, has a non-elementary complexity. Rabinovich shows that the satisfiability problem for the same fragment, interpreted over $\mathbb{R}$, also is non-elementarily decidable, by providing a linear time reduction from the equivalence problem for star-free expressions to the validity problem for the considered fragment of DC.

In [CHE 00], Chetcuti-Sperandio and Fariñas del Cerro isolate another fragment of propositional DC by imposing suitable syntactic restrictions. Formulas of such a fragment are generated by the following abstract syntax:

$$\phi ::= \top \mid \bot \mid lPk \mid I = 0 \mid I = l \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi C \psi,$$

where k is a constant, $P \in \{<, \leq, =, \geq, >\}$, and I is $\int S$, for a given state S. The resulting logic is shown to be expressive enough to capture Allen's Interval Algebra. The authors propose a sound, complete, and terminating tableau system for the logic, thus showing that its satisfiability problem is decidable. The tableau system is a mixed procedure, combining standard tableau techniques with temporal constraint network resolution algorithms.

5.3.2. *Some extensions and variations of Duration Calculus*

In [CHA 98] (see also [ROY 97]) Duration Calculus and the first-order neighbourhood logic (NL) have been combined into the (clearly, undecidable) DC/NL which has been completely axiomatized by merging the axiomatic systems for DC and NL. The fragment of DC/NL obtained by restricting the formulas to be built up only from primitive formulas of the type $\lceil S \rceil$ has been proved to be decidable, while the extension of the latter with primitive formulas of the type $l = k$ is undecidable, as already mentioned.

Duration Calculus with infinite intervals has been studied in [CHA 95]. Other extensions of Duration Calculus include: Extended Duration Calculus for real-time systems [CHA 93b], Mean Value Calculus of Durations [CHA 94], Duration Calculus with Iteration [HUN 99c, GUE 00c], Duration Calculus with Projection [GUE 02, GUE 03], higher-order Duration Calculus [GUE 00a, NAI 00], probabilistic Duration Calculus for continuous time [HUN 99b].

Another variation of DC is Pandya's Interval Duration Logic [PAN 96] the models of which are timed state sequences in dense time structures.

Applications of Duration Calculus to real-time and hybrid systems have been developed in [HUN 99a, HUN 02, HUO 02, SIE 01, THA 01].

Automatic verification and model-checking tools for interval logics and duration calculi have been developed and analyzed in [KON 92, SKA 94, HAN 94, CAM 96, YON 02] and program synthesis from DC specifications has been studied in [SIE 01].

Finally, in [FRÄ 96, FRÄ 02, FRÄ 98] Fränzle describes model checking methods for DC and he argues that, despite its undecidability, if the class of models is restricted to the possible behaviours of embedded real-time systems, model-checking procedures are feasible for rich subsets of Duration Calculus and related logics.

For further details, recent results, and applications of DC see [CHA 04].

6. Summary and concluding remarks

In this survey paper, we have attempted to give a general picture of the extensive and rather diverse research done in the areas of interval temporal logics and duration calculi. Among all important issues in the field, we have mainly focused on expressiveness, proof systems, and decidability/undecidability results.

To summarize, sound and complete axiomatic systems on propositional level are known for CDT, with respect to certain classes of linear orderings, for HS, with respect to the class of partial orderings with the linear interval property, for the family of logics in $\mathcal{PNL}$, with respect to various classes of linear orderings, both in the strict and non-strict semantics, and for ITL and NL with respect to general semantics, while the problem of finding an axiomatic system for specific linear orderings is still largely unexplored.

Furthermore, sound and complete tableau systems have been developed for $BCDT^+$ and for some local variants of ITL. Given the generality of $BCDT^+$, the tableau method for such a logic is in fact a tableau method for a large variety of propositional interval logics.

The satisfiability/validity problem has been shown to be undecidable for HS, CDT, ITL, and NL, with respect to most classes of structures. As a matter of fact, rather weak subsystems of HS turn out to be (highly) undecidable for some classes of structures. Decidable fragments have been obtained by imposing severe restrictions on the expressive power or the semantics of the logics (as an example, by imposing the locality projection principle).

Finally, we point out once more that, to the best of our knowledge, the problems of constructing axiomatic systems, tableau systems, and (un)decidability proofs have not been explicitly addressed yet for the strict semantics variants of most of the existing interval logics (with the exceptions of PNL^- and its subsystems).

In conclusion, the single major challenge in the area of interval temporal logics is to identify expressive enough, yet decidable, fragments and/or logics which are genuinely interval-based, that is, not explicitly translated into point-based logics and not invoking locality or other semantic restrictions reducing the interval-based semantics to the point-based one.

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