Valentin Goranko

Axiomatizations with Context Rules of Inference in Modal Logic

To the memory of my teacher, collaborator, and friend Georgi Gargov

Abstract. A certain type of inference rules in (multi-) modal logics, generalizing Gabbay's Irreflexivity rule, is introduced and some general completeness results about modal logics axiomatized with such rules are proved.

Keywords: modal logic, inference rules, axiomatizations, completeness.

1. Introduction

This article is a technical contribution to the theory and methodology of axiomatizations and completeness proofs in (multi-) modal logics. It is a belated and corrected full paper on the results announced in the abstract [9].

The paper deals with axiomatizations of modal logics by means of a particular type of rules (called here *context rules*) added to traditional Hilbert style axiomatic systems. Context rules generalize the idea of Gabbay's Irreflexivity rule used in [4] to axiomatize temporal logics on irreflexive time flows, the class of which is well known to be non-definable by means of temporal formulae. Various modifications of the rule have since been successfully applied to produce complete axiomatic systems in [16, 5, 25, 21, 22, 23, 17, 6, 20, 11], etc. A scheme of context rules, called "non-ξ rules" has been studied in detail in [24] where a quite general completeness result about logics axiomatized with such rules and Sahlqvist axioms in modal languages of tense similarity type with additional "difference" modality has been proved.

The present work gives semantic sufficient conditions, expressed in terms of *r-persistent formulae*, for the applicability of the method in arbitrary (multi-)modal languages (possibly extended with a "universal" modality). It therefore suggests a shift of emphasis in model theory of modal logic from descriptive frames and d-persistent (i.e. canonical) logics to refined frames and r-persistent logics and formulae. That shift has also been advocated in [18] from another viewpoint: refined frames are sufficient to describe logical consequence while descriptive ones are not. Again, from model-theoretic perspective the class of refined frames seems more natural since it is closed

Presented by **Robert Goldblatt**; *Received* December 12, 1996

Studia Logica **61**: 179–197, 1998.

under ultraproducts and, as a consequence, all r-persistent formulae are elementary, while there are d-persistent non-elementary ones. However, not much is known about r-persistent formulae and logics and they are still awaiting a thorough study.

The paper is organized as follows. In the preliminary section 2 we introduce the basic definitions and notions used in the main text, and we illustrate with some facts and examples the notions which play a central role in the paper, viz. refined frames and r-persistent formulae and logics. Section 3 introduces context rules. Section 4 is the main one, where we prove some general completeness results about logics axiomatized with context rules. Section 5 mentions some easy generalizations. Section 6 discusses the scope and the limitations of the method. The last two short sections include some open questions and concluding remarks.

2. Preliminaries

The reader is assumed to have a general background in modal logic regarding syntax and Kripke semantics, incl. frames, general frames, and models; truth, satisfiability, and validity in them; maximal consistent theories, canonical frames and canonical general frames of a modal logic. Good sources are e.g. [1, 7, 13].

An arbitrary multi-modal language $\mathcal{L}$ with a set of (unary) box-modalities $\{\mathbf{B}_i\}_{i\in I}$ is fixed hereafter. We use metavariables as follows: α, β for first-order formulae, the other lowercase Greek letters for modal formulae; Γ, Δ, Σ for sets of modal formulae.

Given a frame $F = \langle W, \{R_i\}_{i\in I}\rangle$, we refer to the elements of W as "states" (rather than the more traditional but somewhat restrictive "possible worlds"). Sometimes we write $w \in F$ instead of $w \in W$. For any frame F, valuation V in F, $w \in F$, and a modal formula ϕ, $\langle F, V\rangle \vDash \phi[w]$ means "ϕ is true at w in the model $\langle F, V\rangle$"; $F \vDash \phi[w]$ means "ϕ is true at w in every model $\langle F, V\rangle$"; and likewise for validity (truth at all states) in a model and a frame.

Given a general frame $\langle W, \{R_i\}_{i\in I}, \mathcal{P}\rangle$, for any $w \in W$ and $i \in I$ we denote $R_i(w) = \{v \in W \mid wR_iv\}$, $\mathcal{P}_w = \{X \in \mathcal{P} \mid w \in X\}$, and for any $X \in \mathcal{P}$, $\mathbf{B}_iX = \{w \in W \mid R_i(w) \subseteq X\}$.

By a *frame property* we mean a property (formalized or not) which applies to frames; by a *local frame property* we mean a property which applies to states in a frame.

DEFINITION. (a) A modal formula ϕ *defines the frame property* P if for every frame F, $F \vDash \phi$ iff F satisfies P.

(b) A modal formula ϕ *locally defines the local frame property* $P(x)$ if for every frame F and $w \in F$, $F \models \phi[w]$ iff $P(w)$ holds in F.

There is an extensive literature on modally definable properties of frames, see e.g. [2, 1], as well as [14, 15] on the more intimate relationships between definability and completeness.

Clearly, if ϕ locally defines $P(x)$ then ϕ defines $\forall x P(x)$. The question whether there are modally definable Π^0_n frame properties which are not locally modally definable seems to be open (see [1], ch. 3).

DEFINITION. ([19, 3, 7]) A general frame $\mathcal{F} = \langle W, \{R_i\}_{i \in I}, \mathcal{P} \rangle$ is:

(a) *refined* (*natural*, in [3]) if the following two conditions hold for all $x, y \in W$:

   (i) $\mathcal{P}_x = \mathcal{P}_y$ implies $x = y$,

   (ii) For any $i \in I$, if for all $X \in \mathcal{P}$, $x \in \mathbf{B}_i X$ implies $y \in X$, then xR_iy.

(b) *descriptive*, if it is refined and satisfies the additional condition:

   (iii) every ultrafilter on $\mathcal{P}$ is $\mathcal{P}_w$ for some $w \in W$. (Note that every $\mathcal{P}_w$ *is* an ultrafilter on $\mathcal{P}$).

SOME FACTS AND EXAMPLES (most of which can be found, unless indicated otherwise, in [7]):

- Every *full* general frame (i.e., one which contains all subsets of the universe) is refined and every finite refined frame is full.
- Moreover, every *discrete* general frame (which contains all singletons) is refined.
- Every finite full frame, but no infinite one, is descriptive.
- Every canonical general frame is descriptive.
- The class of refined frames is closed under ultraproducts.

The class of refined frames has also been studied in [18] from a topological perspective and characterized in terms of topological spaces.

DEFINITION. ([7]) (a) A modal formula ϕ is *r-persistent* (resp. *d-persistent*) if for every refined (resp. descriptive) general frame $\mathcal{F} = \langle F, \mathcal{P} \rangle$, if $\mathcal{F} \models \phi$ then $F \models \phi$.

(b) Likewise, a modal logic L is *r-persistent* (resp. *d-persistent*) if for every refined (resp. descriptive) general frame $\mathcal{F} = \langle F, \mathcal{P} \rangle$, if $\mathcal{F} \models L$ then $F \models L$.

In [3] r-persistent formulae and logics are called *natural.*

DEFINITION. A modal formula ϕ is *locally r-persistent* if for every refined general frame $\mathcal{F} = \langle F, \mathcal{P}\rangle$ and $w \in F$, if $\mathcal{F} \vDash \phi[w]$ then $F \vDash \phi[w]$.

SOME FACTS AND EXAMPLES:

- Clearly, every pure formula (with no variables) is locally r-persistent.
- Many well-known formulae axiomatizing natural frame conditions like reflexivity, symmetry, tranzitivity, linearity, etc. are locally r-persistent.
- Every locally r-persistent formula is r-persistent.

  The converse does not hold, witness the example in Corollary. 6.3.
- Fine has proved in [3] that every natural logic is Δ-elementary; respectively, every finitely axiomatizable one is elementary.
- Independently, Goldblatt has proved (see [7], sect. 18) that every r-persistent formula is d-persistent and elementary and has given an example showing that the converse is not true.
- Moreover, Fine has shown in [3] that there are elementary and complete modal logics which are not r-persistent, witness the logic $\mathbf{S4.1} = \mathbf{S4} + \Box\Diamond p \to \Diamond\Box p$ introduced by McKinsey.
- An even simpler and stronger example is given in [24]: the formula $\Diamond\Box p \to \Box\Diamond p$ defining Church-Rosser's property is of Sahlqvist type but it fails in the underlying frame of a certain discrete general frame.
- An example of a locally r-persistent formula which (most probably) is not equivalent to a Sahlqvist one will be given in section 6.

DEFINITION. We define *universal form of* $*$ in $\mathcal{L}$ (see [8]) recursively as follows:

1. $*$ is a universal form of $*$.
2. If $\mathbf{u}(*)$ is a universal form of $*$, ϕ is a formula in $\mathcal{L}$ and $\mathbf{B}$ is a box-modality in $\mathcal{L}$, then $\phi \to \mathbf{u}(*)$ and $\mathbf{Bu}(*)$ are universal forms of $*$ in $\mathcal{L}$.

Every universal form of $*$ can be represented (up to tautological equivalence) in a uniform shape:

$$\mathbf{u}(*) = \phi_0 \to \mathbf{B}^1(\phi_1 \to \ldots \mathbf{B}^n(\phi_n \to *)\ldots)$$

where $\mathbf{B}^1, \ldots, \mathbf{B}^n$ are box-modalities in $\mathcal{L}$ and some of $\phi_1, \ldots, \phi_n$ may be $\top$ if necessary.

For every universal form $\mathbf{u}(*)$ and a formula θ we denote by $\mathbf{u}(\theta)$ the result of substitution of θ for $*$ in $\mathbf{u}(*)$. Note that the result of substitution applied to a universal form is again a universal form.

DEFINITION. Given a universal form $\mathbf{u}(*)$ and a model $\mathcal{M} = \langle F, \{R_i\}_{i\in I}, V\rangle$ we define *reachability of a state v from a state w by a $\mathbf{u}$-path in $\mathcal{M}$* recursively as follows:

1. If $\mathbf{u}(*) = *$ then v is reachable from w by a $\mathbf{u}$-path if $v = w$.
2. If $\mathbf{u}(*) = \phi \to \mathbf{u}'(*)$ then v is reachable from w by a $\mathbf{u}$-path if $\mathcal{M} \models \phi[w]$ and v is reachable from w by a $\mathbf{u}'$-path.
3. If $\mathbf{u}(*) = \mathbf{B}_i\mathbf{u}'(*)$ then v is reachable from w by a $\mathbf{u}$-path if v is reachable from w' by a $\mathbf{u}'$-path for some w' such that R_iww'.

3. Context rules

In this paper we only consider finitary inference rules, which can therefore be assumed to have only one premise (viz. the conjunction of all premises).

An inference rule

$$\frac{\phi}{\psi}$$

in propositional logics is traditionally interpreted as a schema with respect to uniform substitutions: for every substitution σ, if $\sigma(\phi)$ is a theorem then $\sigma(\psi)$ is a theorem. Such rules are sometimes called *structural.*

Here we introduce a more general type of rule schemata with certain restrictions on the allowed substitutions.

DEFINITION. 1. Let $p_1, \ldots, p_k$ be propositional variables and σ a substitution. σ is said to be *indifferent to the variables* $p_1, \ldots, p_k$ if it neither affects these variables, nor introduces any new occurrences of any of them.

2. Let ζ, η be formulae and $p_1, \ldots, p_k$ be propositional variables possibly occurring in ζ or η. The rule

$$\frac{\zeta}{\eta}$$

is said to *operate in the context of* $p_1, \ldots, p_k$ if it is applied as follows: for every substitution σ indifferent to $p_1, \ldots, p_k$, if $\sigma(\zeta)$ is a theorem, then $\sigma(\eta)$ is a theorem.

Such a rule will be called a *context(-dependent) rule.* (The analogy with context-dependent rules in generating grammars is clear.)

An example: the so called in [24] "non-ξ rule":

$$\frac{\neg\xi \rightarrow \phi}{\phi}$$

where ξ shares no variables with ϕ, is equivalent to the rule

$$\frac{\neg\xi \rightarrow p}{p}$$

where p is a variable not occurring in ξ, operating in the context of all variables in ξ.

Henceforth we shall presume that all logics considered contain the rule for uniform substitution.

A uniform substitution will be called a *renaming* if it substitutes different variables for variables.

4. Axiomatizations with context rules

We have already noted Gabbay's result on axiomatization of irreflexive classes of frames and the other subsequent completeness results using context rules.

In this section we present some axiomatization results for conditional frame properties which have inspired the introduction of context rules. Here we shall focus on modal logics axiomatized with one additional context rule. Some easy generalizations will be mentioned in the next section.

THEOREM 4.1. *Let L be an r-persistent modal logic and*

1. *ϕ be a modal formula which locally defines $\alpha(x)$,*
2. *ψ be a locally r-persistent modal formula which locally defines $\beta(x)$;*

Then L extended with the rule schema

$$\mathbf{R}_{\frac{\phi}{\psi}}: \qquad \frac{\mathbf{u}(\phi)}{\mathbf{u}(\psi)}$$

operating in the context of all variables of ϕ, where $\mathbf{u}$ is any universal form sharing no variables with ϕ, completely axiomatizes the class of frames

$$\mathcal{C} = \mathrm{FR}(L) \cap \mathrm{FR}(\forall x(\alpha(x) \rightarrow \beta(x))).$$

PROOF. SOUNDNESS: We only need to check that the context rule $\mathbf{R}_{\phi/\psi}$ preserves validity in the class $\mathcal{C}$. In fact, we can show more: it preserves validity in *each* frame from $\mathcal{C}$. Indeed, suppose $F \in \mathcal{C}$ and $F \not\models \sigma(\mathbf{u}(\psi))$

for some universal form $\mathbf{u}$ sharing no variables with ϕ and a substitution σ indifferent to the variables in ϕ. Then $\langle F, V\rangle \not\models \sigma(\mathbf{u}(\psi))[v]$ for some valuation V and a state v, hence $\langle F, V\rangle \not\models \sigma(\psi)[w]$ for some state w reachable from v by a $\sigma(\mathbf{u})$-path. Therefore $F \not\models \psi[w]$, hence $F \not\models \beta(w)$, so $F \not\models \alpha(w)$. Thus, $\langle F, V'\rangle \not\models \phi[w]$ for some valuation V'. We can assume that V' agrees with V on the variables in $\sigma(\mathbf{u})$. Therefore, going backwards from w to v along the $\sigma(\mathbf{u})$-path, it follows that $\langle F, V'\rangle \not\models \sigma(\mathbf{u})(\phi)[v]$, i.e. $\langle F, V'\rangle \not\models \sigma(\mathbf{u}(\phi))[v]$, hence $F \not\models \sigma(\mathbf{u}(\phi))$. Thus, every theorem is valid in $\mathcal{C}$.

COMPLETENESS: We first introduce the following infinitary version of $\mathbf{R}_{\phi/\psi}$:

$$\mathbf{R}^{\infty}_{\phi/\psi}: \quad \frac{\mathbf{u}(\rho(\phi)) \quad \text{for all substitutions } \rho}{\mathbf{u}(\tau(\psi)) \quad \text{for any substitution } \tau}$$

where $\mathbf{u}$ is *any* universal form.

We can show that $\mathbf{R}_{\phi/\psi}$ and $\mathbf{R}^{\infty}_{\phi/\psi}$ are deductively equivalent, being derivable from each other. Indeed,

- $\mathbf{R}_{\phi/\psi}$ is derivable from $\mathbf{R}^{\infty}_{\phi/\psi}$: assume $\vdash \sigma(\mathbf{u})(\phi)$ for some substitution σ indifferent to the variables in ϕ. Then, by the substitution rule, $\vdash \sigma(\mathbf{u})(\rho(\phi))$ for any substitution ρ (which can be assumed not to affect $\vdash \sigma(\mathbf{u})$). Therefore, by $\mathbf{R}^{\infty}_{\phi/\psi}$, $\vdash \sigma(\mathbf{u})(\psi)$.
- $\mathbf{R}^{\infty}_{\phi/\psi}$ is derivable from $\mathbf{R}_{\phi/\psi}$: assume $\vdash \mathbf{u}(\rho(\phi))$ for all substitutions ρ, in particular for a ρ being a renaming of the variables of ϕ such that $\rho(\phi)$ shares no variables with ψ or with $\mathbf{u}$. Now let σ be a renaming which on $\rho(\phi)$ agrees with ρ^{-1} and such that $\sigma(\mathbf{u})$ shares no variables with ϕ or ψ. Then $\vdash \sigma(\mathbf{u}(\rho(\phi)))$, i.e. $\vdash \sigma(\mathbf{u})(\phi)$ therefore, by $\mathbf{R}_{\phi/\psi}$, $\vdash \sigma(\mathbf{u})(\psi)$, hence $\vdash \sigma(\mathbf{u})(\tau(\psi))$ for any substitution τ. ■

DEFINITION. A set of formulae T is a *theory* in $L + \mathbf{R}_{\phi/\psi}$ if it is closed under Modus Ponens; T is an *R-theory* if it is a theory in $L + \mathbf{R}_{\phi/\psi}$, closed under $\mathbf{R}^{\infty}_{\phi/\psi}$.

Note that for every set of formulae Γ there is a minimal R-theory $\mathrm{RTh}(\Gamma)$ containing Γ, viz. the intersection of all R-theories containing Γ.

DEFINITION. A theory, resp. R-theory, is *consistent* if it does not contain $\bot$. A set of formulae Δ is R-*consistent* if $\mathrm{RTh}(\Delta)$ is consistent.

In particular, an R-theory is R-consistent iff it is consistent.

LEMMA 4.2. (Deduction theorem for R-theories) *If Γ is an R-theory and ζ, η are formulae then $\zeta \to \eta \in \Gamma$ iff $\eta \in \mathrm{RTh}(\Gamma \cup \{\zeta\})$.*

PROOF. For the non-trivial part, suppose that $\eta \in \mathrm{RTh}(\Gamma \cup \{\zeta\})$ and consider the set of formulae

$$\Delta - \{\delta : \zeta \to \delta \in \Gamma\}.$$

Δ is an R-theory containing $\Gamma \cup \{\zeta\}$. The proof of this goes as usual, with one additional step: closedness under $\mathbf{R}^{\infty}_{\zeta/\eta}$ which follows form the fact that Γ is an R-theory, and $\zeta \to \mathbf{u}(*)$ is a universal form whenever $\mathbf{u}(*)$ is. ■

The following lemma essentially repeats Lemma 11 in [11].

LEMMA 4.3. *If* Γ *is a set of formulae in which infinitely many propositional variables have no occurrences then* $\mathrm{RTh}(\Gamma) = \mathrm{Th}(\Gamma)$.

As a corollary, every set of formulae which satisfies the condition of the lemma is R-consistent iff it is consistent.

DEFINITION. An R-theory Γ is *maximal* if for every formula ϕ, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$ but not both.

LEMMA 4.4. (Lindenbaum Lemma) *Every R-consistent set* Γ *can be extended to a maximal R-theory.*

PROOF. First, note that $\mathrm{RTh}(\Gamma)$ is a consistent R-theory. Let $\xi_1, \xi_2, \ldots$ be a list of all formulae, $\tau_1, \tau_2, \ldots$ a list of all substitutions of the variables of ψ, and $u_1, u_2, \ldots$ a list of all universal forms. Then we can list (with repetitions) all combinations $\{u_i(\tau_j(\psi))\}^{\infty}_{i,j=1}$ alternated with all formulae $\xi_1, \xi_2, \ldots$ in a sequence $\theta_1, \theta_2, \ldots$. Now we define a sequence of consistent R-theories $T_0 \subseteq T_1 \subseteq \ldots$ as follows: $T_0 = \mathrm{RTh}(\Gamma)$; suppose that T_n is defined and consider $\mathrm{RTh}(T_n \cup \{\theta_n\})$. If it is consistent, then this is T_{n+1}, else $T_{n+1} = T_n$ in case θ_n is not one of $\{u_i(\tau_j(\psi))\}^{\infty}_{i,j=1}$; otherwise suppose $\theta_n = u_i(\tau_j(\psi))$. Then $\neg u_i(\tau_j(\psi)) \in T_n$ by the deduction theorem. Therefore $u_i(\rho(\phi))$ does not belong to T_n for some substitution ρ. Then we put

$$T_{n+1} = \mathrm{RTh}(T_n \cup \{\neg u_i(\rho(\phi))\}.$$

T_{n+1} is R-consistent.

Finally, put $T = \bigcup_{n=0}^{\infty} T_n$. By virtue of the construction, T is a maximal consistent R-theory. ■

For any set of formulae Δ we define

$$\mathbf{B}\Delta = \{\xi : \mathbf{B}\xi \in \Delta\}$$

for each modal box-operator $\mathbf{B}$ in the language.

LEMMA 4.5. *If* Δ *is a maximal R-theory then* $\mathbf{B}\Delta$ *is an R-theory.*

PROOF. It is well known that $\mathbf{B}\Delta$ is a theory. It is closed under $\mathbf{R}^{\infty}_{\phi/\psi}$ since Δ is closed and $\mathbf{Bu}(*)$ is a universal form whenever $\mathbf{u}(*)$ is. ■

LEMMA 4.6. *If Δ is a maximal R-theory and $\mathbf{B}\neg\theta \notin \Delta$ then there is a maximal R-theory Δ' such that $\theta \in \Delta'$ and $\mathbf{B}\Delta \subseteq \Delta'$.*

PROOF. By Lemma 4.5 $\mathbf{B}\Delta$ is an R-theory and $\neg\theta \notin \mathbf{B}\Delta$, hence $\mathrm{RTh}(\mathbf{B}\Delta \cup \{\theta\})$ is R-consistent by Lemma 4.2. Then, by Lemma 4.4 it can be extended to a maximal R-theory Δ'. ■

Now we can embark on the completeness proof. Let a formula χ be consistent (and therefore, R-consistent, by Lemma 4.3) in $L + \mathbf{R}_{\phi/\psi}$. Following the standard canonical construction we shall define a model based on a standard frame for $L + \mathbf{R}_{\phi/\psi}$ (a *standard model*) which satisfies χ. Moreover, we shall prove the following lemma.

LEMMA 4.7. (Strong completeness theorem for R-consistent sets) *Every R-consistent set Γ_0 in $L + \mathbf{R}_{\phi/\psi}$ is satisfiable in a standard model of that logic.*

PROOF. For technical convenience we do the proof for a monomodal language; the generalization to a multimodal language is straightforward.

First, we extend Γ_0 to a maximal R-theory Γ. Then we define a *thinned out canonical* model $\mathcal{M} = \langle F, V\rangle$ accordingly:

• In the set of all maximal R-theories we define the *canonical relation*

$$R\Delta_1\Delta_2 \text{ iff } \mathbf{B}\Delta_1 \subseteq \Delta_2 .$$

• Then we define W to be the set of all maximal R-theories R-accessible from Γ and $F = \langle W, R\rangle$.

• The valuation V is defined canonically: for any propositional variable p, $V(p) = \{\Delta \in W : p \in \Delta\}$. ■

The following *truth lemma* is proved as usual.

LEMMA 4.8. *For every formula θ and $\Delta \in W$,*

$$\mathcal{M} \models \theta[\Delta] \textit{ iff } \theta \in \Delta .$$

In particular, Γ_0 is satisfied at the state Γ of $\mathcal{M}$.

Finally, we shall prove that $F \in \mathcal{C}$. Let us consider the general frame

$$\mathcal{F} = \langle W, R, \{V(\theta) \mid \theta \in \mathrm{FOR}\}\rangle .$$

Some facts:

- $\mathcal{F}$ is refined (but not descriptive!). The proof is the same as for any canonical general frame (cf. [7]).
- $\mathcal{F} \models L$.

- Therefore $F \models L$ since L is r-persistent.
- $F \models \forall x(\alpha(x) \to \beta(x))$. Indeed, let $\Delta \in W$ and $F \models \alpha(\Delta)$. Then $F \models \phi[\Delta]$, therefore $F \models \sigma(\phi)[\Delta]$, hence $\sigma(\phi) \in \Delta$, for any substitution σ. Since Δ is an R-theory it follows that $\tau(\psi) \in \Delta$, hence $\mathcal{M} \models \tau(\psi)[\Delta]$ for any substitution τ. Therefore $\mathcal{F} \models \psi[\Delta]$, hence $F \models \psi[\Delta]$ since ψ is locally r-persistent. So, $F \models \beta(\Delta)$.

Thus, $F \in \mathcal{C}$. This completes the proofs of Lemma 4.7 and hence the completeness theorem.

In particular, $K + \mathbf{R}_{\phi/\bot}$ axiomatizes the class $\mathrm{FR}(\forall x \neg\alpha(x))$ — a generalization of Gabbay's irreflexivity lemma. While the rule $\mathbf{R}_{\phi/\bot}$ is equivalent to Venema's "non-ϕ rule", the result is incomparable with Venema's result (see section 6).

REMARK. The completeness proof above amounts to proving a certain omitting types theorem in respect of the types

$$\Sigma_{\mathbf{u};\tau} = \{ST(\mathbf{u}(\rho(\phi))) \mid \rho \text{ is a substitution}\} \cup \{\mathbf{u}(\tau(\psi))\}$$

and the first-order theory of $\mathrm{FR}(L)$ extended with the translations of the formulae in the R-consistent theory for which a standard model is sought. However, a direct attempt to apply the well-known omitting types theorem for first-order logic fails: these types may be locally realized by a formula which is not a translation of a modal formula. ■

Hereafter we assume that the language contains (definable) the *universal modality* which will be denoted by $\sharp$.

THEOREM 4.9. *Let L be an r-persistent modal logic and:*

1. *ϕ be a modal formula which defines α,*
2. *ψ be an r-persistent modal formula which locally defines $\beta(x)$.*

Then L extended with the rule schema

$$\mathbf{R}_{\frac{\sharp\phi}{\psi}}: \qquad \frac{\mathbf{u}(\sharp\phi)}{\mathbf{u}(\psi)}$$

operating in the context of all variables of ϕ, where $\mathbf{u}$ is any universal form sharing no variables with ϕ, completely axiomatizes the class of frames

$$\mathcal{C} = \mathrm{FR}(L) \cap \mathrm{FR}(\alpha \to \forall x \beta(x)).$$

PROOF. SOUNDNESS: Again, we show that the rule $\mathbf{R}_{\sharp\phi/\psi}$ preserves validity in each frame from the class $\mathcal{C}$. Suppose $F \in \mathcal{C}$ and $F \not\models \sigma(\mathbf{u}(\psi))$ for some universal form $\mathbf{u}$ sharing no variables with ϕ and a substitution σ indifferent to the variables in ϕ. Then $\langle F, V\rangle \not\models \sigma(\mathbf{u}(\psi))[v]$ for some valuation V and a state v, hence $\langle F, V\rangle \not\models \sigma(\psi)[w]$ for some state w, $\sigma(\mathbf{u})$-accessible from v. Therefore $F \not\models \beta(w)$, hence $F \not\models \forall x\beta(x)$. So, $F \not\models \alpha$, hence $\langle F, V'\rangle \not\models \phi$ for some valuation V'. We can assume that V' agrees with V on the variables in $\sigma(\mathbf{u})$. Therefore, $\langle F, V'\rangle \not\models \sharp\phi$, so $\langle F, V'\rangle \not\models \sharp\phi[w]$, hence $\langle F, V'\rangle \not\models \sigma(\mathbf{u})(\sharp\phi)[v]$. Thus, $F \not\models \sigma(\mathbf{u}(\sharp\phi))$.

COMPLETENESS: The proof essentially repeats the previous one. The final step is to show for the thinned out canonical frame F that $F \models \alpha \to \forall x\beta(x)$. Indeed, let $F \models \alpha$. Then $F \models \phi$, hence for any substitution σ, $F \models \sigma(\phi)$, so $\langle F, V\rangle \models \sigma(\phi)$ for the canonical valuation V, hence $\sigma(\phi) \in \Delta$ for each $\Delta \in F$. Therefore $\tau(\psi) \in \Delta$ for each substitution τ and $\Delta \in F$, so $\langle F, V\rangle \models \tau(\psi)[\Delta]$ for each τ and Δ, hence $\mathcal{F} \models \psi[\Delta]$, so $F \models \psi[\Delta]$ for each $\Delta \in F$ since ψ is locally r-persistent. Therefore $F \models \forall x\beta(x)$. ■

In particular, if ϕ defines α then the rule

$$\frac{\mathbf{u}(\sharp\phi)}{\mathbf{u}(\bot)}$$

axiomatizes the class of frames in which α is not valid.

Likewise, the following theorem holds.

THEOREM 4.10. *Let L be an r-persistent modal logic and:*

1. *ϕ be a modal formula which defines α,*
2. *ψ be an r-persistent modal formula which defines β.*

Then L extended with the rule schema

$$\mathbf{R}_{\frac{\sharp\phi}{\sharp\psi}} \qquad \frac{\mathbf{u}(\sharp\phi)}{\mathbf{u}(\sharp\psi)}$$

operating in the context of all variables of ϕ, where $\mathbf{u}$ is any universal form sharing no variables with ϕ, completely axiomatizes the class of frames

$$\mathcal{C} = \mathrm{FR}(L) \cap \mathrm{FR}(\alpha \to \beta).$$

5. Some generalizations

1. Context rules in temporal and other extended modal languages. While the results of the previous section remain valid in various multi-modal logics, the context rules used here are essentially simplified in temporal languages, as noted in [5], and, more generally, in languages of *versatile* types, see [24], since universal forms are no more necessary there. In particular in a language of *tense* similarity type, where every modality has a counterpart semantically corresponding to the inverse relation, the rule

$$\frac{\mathbf{u}(\zeta)}{\mathbf{u}(\eta)}$$

collapses to

$$\frac{\theta \to \zeta}{\theta \to \eta}$$

from which the former one is derivable.

The other rules used in the previous section are likewise simplified.

A similar simplification can be obtained in case of expressive enough (e.g. with definable difference modality) non-temporal types, as shown in [10].

2. Many rules together. With no essential complications, the completeness proofs from the previous section apply to logics with more than one (even countably many) additional context rules.

6. The scope and limitations of the method

We have proved some general completeness results united by the same proof method. Variations of the idea can produce similar results for other schemata of context rules, thus extending the scope of applicability of the method. We shall not pursue that further here, but shall rather discuss the scope of the obtained results by taking a closer look at the basic notions involved in their formulations.

6.1. Stronger languages

The completeness results presented here hold for arbitrary (multi-) modal languages (if applicable, extended with a universal modality). This generality is at the expense of imposing demanding semantic conditions on the underlying logic L and the formulae occurring in the additional context rules, as well as the form of these rules. As already noted, in richer languages the rules can be essentially simplified. Moreover, as shown in [24], in a language with (definable) difference modality an appropriate additional rule makes the canonical general frames discrete, and therefore in such languages the present results can be strengthened in the spirit of [24] to apply to all logics axiomatized with formulae persistent with respect to all discrete frames. These (as pointed out by the referee) obviously include all r-persistent logics, as well as all tense Sahlqvist logics, as proved by Venema. In arbitrary languages, however, canonical general frames need not be discrete, hence there is no obvious relaxation of the requirement for r-persistence, and therefore the results in [24] are generally incomparable with the present results. (We leave aside the fact that the rule schemata considered here are more general than Venema's non-ξ rules, since the latter results can be accordingly extended to cover our schemata.)

6.2. R-persistent vs. Sahlqvist logics

The semantic conditions imposed in the present results are sufficient but, in general, unlikely to be necessary (and certainly not in strong enough languages, as noted above). Alternatively, they can be possibly replaced by appropriate syntactic conditions of the type of Sahlqvist formulae. In that respect it is interesting to note that, at least in classical modal language, the sets of r-persistent and Sahlqvist formulae are not comparable. To see that, on one hand we have the formula $\Diamond\Box p \to \Box\Diamond p$ (defining Church-Rosser's property) which is of Sahlqvist type but not r-persistent. On the other hand, there are r-persistent (even locally r-persistent) formulae which are not (likely to be equivalent to ones) of Sahlqvist type. (Of course, since Sahlqvist type formulae are defined purely syntactically and have no exact semantic characterization, it is not quite clear how a negative result like that can be proved.) As an example, let us consider the logic $L_{2,1}$ determined by the class of frames satisfying the following two properties: every state in the frame

(i) has at most two successors;

(ii) has at least one successor which has at most one successor.

That logic can be axiomatized by the (Sahlqvist) formula **Alt**$_2$: $(\Diamond p_1 \land \Diamond p_2 \land \Diamond p_3) \to (\Diamond(p_1 \land p_2) \lor \Diamond(p_1 \land p_3) \lor \Diamond(p_2 \land p_3))$ added to McKinsey's formula $\Box\Diamond p \to \Diamond\Box p$ (alternatively, the formula $\Box(\Diamond p_1 \land \Diamond p_2) \to \Diamond\Diamond(p_1 \land p_2)$ can be used; note that both alternatives are not of Sahlqvist type). It is a standard exercise to show that these indeed characterize $L_{2,1}$. Moreover, it is easy to check that the conditions (i) and (ii) hold at every state in every refined frame which satisfies both formulae at that state. Thus, $L_{2,1}$ is locally r-persistent without being of Sahlqvist type.

6.3. More on r-persistent formulae and logics

A better way to outline the scope of the obtained results is to look for more precise description of the notions of refined frames, (locally) r-persistent formulae, (locally) definable properties. Some results were mentioned in the preliminary section, but no systematic study has been done so far. In this paper we just initiate such a study by underlining its relevance to the completeness theory in modal logic and by mentioning below some easy observations and (counter)examples following from results available in the existing literature.

PROPOSITION 6.1. *Let ϕ and ψ be (locally) r-persistent modal formulae. Then:*

1. *$\phi \wedge \psi$ is (locally) r-persistent;*
2. *if ϕ and ψ share no variables then $\#\phi \vee \#\psi$ is r-persistent (resp. $\phi \vee \psi$ is locally r-persistent).*
3. *$\Box\phi$ is (locally) r-persistent.*

PROOF. The only less trivial case is to prove that $\Box$ preserves r-persistence. (For local r-persistence that is quite easy.) Let $\mathcal{F} = \langle F, \mathcal{P}\rangle$ be a refined frame, $\mathcal{F} \models \Box\phi$, $u \in F$, and Ruw. We have to show that $F \models \phi[w]$. Let $\mathcal{F}_w$ be the general subframe of $\mathcal{F}$ generated by w. Then $\mathcal{F}_w$ is refined and every state in $\mathcal{F}_w$ is an R-successor, hence $\mathcal{F}_w \models \phi$. Therefore $F_w \models \phi$ where F_w is the underlying Kripke frame for $\mathcal{F}_w$. In particular, $F_w \models \phi[w]$, hence $F \models \phi[w]$. ∎

NOTE. It is easy to see that the negation and the implication do not preserve (local) r-persistence. Neither does $\Diamond$. Example: the formula $\phi = \Box\neg p \vee \Box p$ is easily seen to be (locally) r-persistent, and $\Diamond\phi$ is equivalent to McKinsey's formula $\Box\Diamond p \to \Diamond\Box p$ which is not elementary, hence not (locally) r-persistent. ∎

Of course, the proposition above is by no means an attempt to provide a comprehensive syntactic description of (local) r-persistence. These notions are still awaiting their "Sahlqvist theorem".

PROPOSITION 6.2. *Every locally r-persistent modal formula is locally first-order definable.*

PROOF. Due to Theotem 8.7 from [1], it is sufficient to prove that ϕ is locally preserved under ultrapowers, i.e. if $F \models \phi[w_i]$ for all $i \in I$ and D is an ultrafilter on I then $\prod_D F \models \phi[w/D]$, where $w(i) = w_i$, for each $i \in I$. Indeed, the ultrapower $\prod_D\langle F, \mathbf{2}^W\rangle$ of the full general frame $\langle F, \mathbf{2}^W\rangle$ is a discrete, hence refined general frame and $\prod_D\langle F, \mathbf{2}^W\rangle \models \phi[w/D]$ (see Theorem 4.12 in [1]). ∎

Note that Church-Rosser formula $\Diamond\Box p \to \Box\Diamond p$ is locally first-order definable by $\forall y\forall z(Rxy \wedge Rxz \to \exists t(Ryt \wedge Rzt))$ but not locally r-persistent, so the converse of the previous statement does not hold.

COROLLARY 6.3. *Not every r-persistent modal formula is locally r-persistent.*

PROOF. The formula $\gamma = \Box\Diamond\Box\Box p \to \Diamond\Diamond\Box\Diamond p$ has been proved in [1, Theorem 7.1] to be first-order definable by the formula $\alpha = \forall x \exists y Rxy$, but not locally first-order definable, and hence, by Propoposition 6.2 not locally r-persistent. However, γ is r-persistent: if $\langle W, R, V, \mathcal{P}\rangle \models \gamma$ where $V(p) = W$ then it easily follows that $\langle W, R\rangle \models \alpha$. ■

Finally, a semi-formal sufficient condition for r-persistence is described below.

DEFINITION. A frame $F = \langle W_F, R_F\rangle$ is *isomorphically embeddable* into a frame $G = \langle W_G, R_G\rangle$ if there is an injective mapping $h: W_F \longrightarrow W_G$ satisfying the condition: for every $x, y \in W_F$,

$$R_F xy \text{ iff } R_G h(x)h(y).$$

Many important modally definable frame conditions, such as reflexivity, symmetry, transitivity, linearity, Alt_n, etc., are expressible in terms of *non-embeddability* of certain finite frames into any frame satisfying the condition. For instance, a frame G is linear if the frame $\langle\{x, y\}, \emptyset\rangle$ is not isomorphically embeddable into G; G is transitive if the frame $\langle\{x, y, z\}, \{(x, y), (y, z)\}\rangle$ is not embeddable into G, etc. For the related concept of *sketch-omission* and logics determined in terms of it see [14].

Let us observe that if a modal formula θ determines the "F non-embeddability" property for some finite frame F then it is r-persistent. Indeed, if some valuation V on a frame G falsifies the formula θ, i.e. enforces the existence of a subframe F' (not necessarily generated) of G, isomorphic to F, then any other valuation of the variables of θ which coincides with V on F' would do the same. Then, in every refined frame $\mathcal{G}$ based on G an *admissible* valuation with that property can be found , i.e. θ would be refuted in $\mathcal{G}$.

A similar criterion for local r-persistence can be formulated.

6.4. A note on frame conditions defined and axiomatized by context rules

It is clear that context rules can axiomatize frame conditions which are beyond the expressiveness of the modal language. Furthermore, it is easy to give meaningful examples of frame conditions not expressible by Venema's non-ξ rules but expressible by means of a context rule. For instance, the property *All reflexive states are end-points*, expressible by the first-order formula $\forall x(Rxx \to \forall y(Rxy \to x = y))$ which falls within the scope of

Theorem 8, is axiomatizable above any r-persistent logic by means of the rule

$$\frac{\mathbf{u}(\Box p \to p)}{\mathbf{u}(p \to \Box p)}$$

operating in the context p where $\mathbf{u}$ is any universal form not containing p.

Furthermore, Theorem 19 gives a general method to axiomatize over any r-persistent logic the complement of any class of frames definable by means of a modal formula.

In the light of the completeness results discussed here, the following problem of characterizing *modal definability by means of context rules* naturally arises. We say that the context rule $\frac{\zeta}{\eta}$ operating in the context of variables $p_1, \ldots, p_k$ is *valid in a frame* F if for every state $s \in F$ and every substitution σ indifferent to $p_1, \ldots, p_k$, if $\sigma(\zeta)$ is valid at the state s in F then $\sigma(\eta)$ is valid at s in F. The notion of a class of frames *definable by a context rule*, or a set of context rules, is accordingly introduced. In the case of non-ξ rules, this is the notion of *negative definability* introduced in [24]. A study of negative definability has been initiated in [12] where some model-theoretic characterizations in the spirit of Goldblatt and Thomason's classical results on modal definability of classes of frames have been obtained. The definability by means of context rules extends both the usual modal definability and the negative definability. Note that the classes of frames axiomatized in section 4 are precisely those defined by the respective rules relatively to the class of frames of the underlying logic. Therefore, characterizations of definability with context rules could shed additional light on the scope of applicability of the method of axiomatization with such rules.

7. Some open problems and avenues for further research

- The thinned out canonical frames constructed in the completeness proofs above are special types of refined frames. Therefore, a further study of (perhaps an appropriate refinement) of the notion of refined frame is justified from the view point of completeness theory.
- The verification of (local) r-persistence is not always an easy task. Therefore, more precise descriptions and good sufficient conditions (syntactic, as well as semantic) for (local) r-persistence are desirable. In particular, which Sahlqvist formulae are (locally) r-persistent? And, is there and upper bound for the arithmetic hierarchy complexity of (locally) r-persistent formulae? (A positive answer has been suggested by M. Kracht).
- Likewise, better descriptions of (locally) modally definable frame properties are desirable.

- Finally, the idea of using semantically motivated additional rules of inference for axiomatizations in non-classical logics has so far been only marginally exploited. It is still awaiting systematic exploration.

8. Concluding remarks

Based on the results in the paper we argue that context rules are useful for the construction of complete axiomatizations in quite general situation. What is largely unknown, however, is to what extent those rules are *really necessary*. In some cases they are well-known to be redundant, i.e. Gabbay's irreflexivity rule added to **K** or **K4** produces no new theorems. In other cases they are not; examples are given e.g. in [6, 24]. However, these are narrow margins surrounding the large grey area of logics axiomatized with the help of context rules for the sake of proving completeness, while it is not known if those rules can be omitted or replaced by finitely many additional axioms. The only general results in that respect seem to be that a context rule can be replaced by a recursive set of axioms (due to its finitary nature).

Finally, contrary to a popular (though unjustified) belief, context rules *are* tractable as derivation rules (see some examples of derivations in [11]). However, (according to my knowledge) no successful attempts to develop really convenient and efficient proof systems like semantic tableaux for systems involving context rules have been made so far, and this seems to be a challenge worth exploring.

Acknowledgements

This work was partly supported by FRD research grant GUN 2034353.

Thanks are due to Yde Venema for discussions and comments on the issues in this paper, to the referee for the careful reading, some corrections and suggestions, and in particular for correcting (with the help of Rob Goldblatt) the author's comparison of the present results with those of [24].

References

[1] VAN BENTHEM, J., *Modal Logic and Classical Logic*, Bibliopolis, Napoli, 1985.

[2] VAN BENTHEM, J., 'Correspondence theory', in: D. Gabbay, F. Guenthner (eds.), *Handbook on Philosophical Logic, vol. II*, Reidel, Dordrecht, 1984, 167–247.

[3] FINE, K., 'Some connections between modal and elementary logic', in: S. Kanger (ed.), *Proc. Third Scandinavian Logic Symposium*, North-Holland, Amsterdam, 1975, 15–31.

[4] GABBAY, D., 'An Irreflexivity lemma with applications to axiomatizations of conditions on tense frames', in: U. Monnich (ed.), *Aspects of Philosophical Logic*, Reidel, Dordrecht, 1981, 67–89.

[5] GABBAY, D., & I. HODKINSON, 'An axiomatization of the temporal logic with Since and Until over the real numbers', *J. of Logic and Computation* 1, 1990, 229–259.

[6] GARGOV, G., & V. GORANKO, 'Modal logic with names', *J. of Philosophical Logic* 22(6), 1993, 607–636.

[7] GOLDBLATT, R., 'Metamathematics of modal logic', *Reports on Mathematical Logic* 6, 41–78 (Part I); 7, 21–52 (Part II).

[8] GOLDBLATT, R. I., *Axiomatizing the Logic of Computer Programming*, Springer LNCS 130, 1982.

[9] GORANKO, V., 'Applications of quasi-structural rules to axiomatizations in modal logic', in: *Abstracts of the 9th Intern. Congress of Logic, Methodology and Philosophy of Science*, Uppsala, 1991, Vol. 1: Logic, p. 119.

[10] GORANKO, V., 'A note on derivation rules in modal logic', *Bull. of the Sect. of Logic*, Univ. of Lódź, 24(2), 1995, 98–104.

[11] GORANKO, V., 'Hierarchies of modal and temporal logics with reference pointers', *J. of Logic, Language and Information*, 5(1), 1996, 1–24.

[12] HOLLENBERG, M., 'Negative definability in modal logic', manuscript, Department of Philosophy, Utrecht University, 1994, to appear in *Studia Logica*.

[13] HUGHES, G., & M. CRESSWELL, *A New Introduction to Modal Logic*, Routledge, London and New York, 1996.

[14] KRACHT, M., *Internal Definability and Completeness in Modal Logic*, Ph. D. Thesis, Free University, Berlin, 1990.

[15] KRACHT, M., 'How completeness and correspondence theory got married', in: M. de Rijke (ed.), *Colloquium on Modal Logic*, Dutch Network for Language, Logic and Information, 1991, 161–186.

[16] PASSY, S., & T. TINCHEV, 'PDL with data constants', *Information Processing Letters* 20, 1985, 35–41.

[17] DE RIJKE, M., 'The modal logic of inequality', *J. of Symbolic Logic* 57(2), 1992, 566–584.

[18] SAMBIN, J., & V. VACCARO, 'Topology and duality in modal logic', *Annals of Pure and Applied Logic* 44, 1988, 173–242.

[19] THOMASON, S. K., 'Semantic analysis of tense logic', *J. of Symbolic Logic* 37, 1972, 150–158.

[20] VAKARELOV, D., 'Modal rules for intersection', *Bull. of Symbolic Logic* 1(2), 1995, 264–265.

[21] VENEMA, Y., 'Expressiveness and completeness of an interval tense logic', *Notre Dame J. of Formal Logic* 31, 1990, 529–547.

[22] VENEMA, Y., 'A modal logic for chopping intervals', *J. of Logic and Computation* 1, 1991, 453–476.

[23] VENEMA, Y., *Many-dimensional Modal Logic*, Ph. D. Dissertation, Univ. of Amsterdam, 1991.

[24] VENEMA, Y., 'Derivation rules as anti-axioms in modal logic', *J. of Symbolic Logic* 58(3), 1993, 1003–1034.

[25] ZANARDO, A., 'A complete deductive system for Since-Until branching-time logic', *J. of Philosophical Logic* 20, 1991, 131–148.

VALENTIN GORANKO
Department of Mathematics
Rand Afrikaans University
PO Box 524
Auckland Park 2006, South Africa