Hyperboolean Algebras and Hyperboolean Modal Logic *†

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To the memory of our colleague and friend George Gargov

ABSTRACT. Hyperboolean algebras *are Boolean algebras with operators, constructed as algebras of complexes (or, power structures) of Boolean algebras. They provide an algebraic semantics for a modal logic (called here a* hyperboolean modal logic *with a Kripke semantics is accordingly based on frames in which the worlds are elements of Boolean algebras and the relations correspond to the Boolean operations. We introduce and give a complete axiomatization of the hyperboolean modal logic, and show that is lacks the finite model property. The technique of axiomatization hinges upon the fact that a "difference" operator is definable in hyperboolean algebras, and makes use of additional non-Hilbert-style rules. Finally, we discuss a number of open questions and directions for further research.*

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Introduction

In the abstract possible world semantics the worlds are objects without any internal structure and the accessibility relations between them satisfy some abstract conditions like reflexivity, transitivity and so on. For applications, however, we often need semantical structures in which possible worlds have some internal structure and the accessibility relations between possible worlds depend on that structure. For instance in many-dimensional modal logic [Venema 92] possible worlds have an internal structure of n-tuples; in some arrow logics [Vakarelov 92, Vakarelov 96] possible worlds have a structure of the arrows of a multigraph; in the interval tense logic [Venema 90] possible worlds have a structure of intervals of some ordered set.

The aim of this paper is to present a natural example of a modal logic having possible world semantics in which the worlds have the structure of sets, namely they are all or some subsets of some universe W and the accessibility relations between possible worlds are some natural relations between subsets. The first example of such a logic (known to the authors) is given in [Vakarelov 95] where the accessibility relations are the following basic Boolean relations between sets:

inclusion: $x \subseteq y$,

overlap, non-empty intersection: xOy iff $x \cap y \neq \emptyset$

underlap, non-full union: xUy iff $x \cup y \neq W$

Other examples can be found in [Vakarelov 96].

In this paper we consider a modal logic, called Hyperboolean Modal Logic, HBML for short, the standard semantical structures of which, called here *hyperframes*, consist of a Boolean algebra $W \subseteq \mathcal{P}(X)$ of subsets of a given set X and the accessibility relations are determined by the basic Boolean operations (x, y, z are subsets of X):

complement: $z = -x$

intersection: $z = x \cap y$,

union: $z = x \cup y$,

"implication": $z = x \to y = -x \cup y$,

zero, empty set: $0 = \emptyset$,

unit, full universe: $1 = X$

Let us note that the relations of complement, intersection and union,

and the unit constant are definable by implication and zero and this will be used later on for simplicity.

A motivation for this terminology comes from the theory of hypergraphs: any set of subsets of a given set X is called sometimes a hypergraph over X. Later on we will use a slight generalization of the above semantics: instead of a hyperframe we can take any Boolean algebra and define the corresponding accessibility relations by the Boolean operations as in the set-theoretical case. It is clear, on the base of the representation theory of Boolean algebras, that the standard and generalized semantics for HBML are equivalent.

Let $\underline{W}$ be a Boolean algebra and $\mathcal{P}(W)$ be the Boolean algebra of sets over W. Then the above defined relations determine the following normal and additive operators in $\mathcal{P}(W)$ in the sense of Jónsson and Tarski, which will be the set-theoretical counterparts of the modalities of HBML:

$$\langle - \rangle A =_{\text{def}} \{z \in W | (\exists x \in A)(z = -x)\},$$

$$A\langle \wedge \rangle B =_{\text{def}} \{z \in W | (\exists x \in A, y \in B)(z = x \cap y)\},$$

$$A\langle \vee \rangle B =_{\text{def}} \{z \in W | (\exists x \in A, y \in B)(z = x \cup y)\},$$

$$A\langle \rightarrow \rangle B =_{\text{def}} \{z \in W | (\exists x \in A, y \in B)(z = x \rightarrow y)\},$$

$$\langle 0 \rangle =_{\text{def}} \{0\},$$

$$\langle 1 \rangle =_{\text{def}} \{1\}.$$

The Boolean algebra $\mathcal{P}(W)$ with the above additional operators, named *internal operations*, will be called in this paper *"hyperboolean algebra"* (HBA) over W. In [Brink 84] HBA's are called second-order Boolean algebras. This construction of algebraic operations over the power set of an algebraic system can be carried out for arbitrary algebraic system and in [Brink 93] it is called *power algebra*. In the algebraic literature (for instance in group theory) the term *complex algebra* is also used. We choose the name "hyperboolean algebra" instead of "second-order Boolean algebra", because "second–order" in logic has another meaning.

HBML has modalities corresponding to the internal operations of a HBA. Its natural semantics is a Kripke semantics on hyperframes and Boolean algebras, and algebraic semantics over HBA's.

Such a logic can have various interpretations, one of them going back to an idea of Vakarelov and George Gargov to study a "logic of expert groups" (see [Gargov 87, Gargov and Radev 86, Vakarelov 89]) which can be simply presented as follows: groups of experts (a priori equally trustworthy) have their "opinions" (knowledge, beliefs, judgments, intuitions, etc.) on a certain matter, and these opinions are to be put together and coordinated in a way which

would enable some logical analysis on the "integrated opinions", in particular performing logical operations on them. These opinions can be simply presented as "yes-no", or "true-false", but they can have a more fine-grained Boolean structure. A reasonable formal approach seems to be to represent the groups' opinions as sets of Boolean values and adopt the internal Boolean operations for Boolean constructions on those opinions, e.g. form a conjunction of two group opinions by taking all conjunctions of opinions of individual agents from each group, etc.

The main difficulty in the axiomatization of HBML is that Boolean algebras are not modally definable in the sense of modal definability theory and that the standard canonical construction cannot be applied. But fortunately, in the language of HBML we can define the "difference" modality $\langle \neq \rangle$ and use a formal system with some "irreflexivity"-type rules for $\langle \neq \rangle$. A very general theorem for axiomatizing logics with difference modality has been given in [Venema 93], but in our case one of the conditions (versatility) of the Venema's theorem is not fulfilled. So, we have to give a direct proof of the completeness theorem. Our canonical construction, based on some irreflexivity rules, is different from those from Venema's proof and is similar to that of [Balbiani et al 97].

It should be noted that the construction of a "hyper" modal logic is not particular to Boolean algebras, but can be carried out for arbitrary structures with finitary operations and relations defined in it. Our technique for axiomatization is applicable to many of them, such as (ordered) groups, rings, pseudoboolean algebras etc. This is further discussed in the concluding section.

The structure of the paper is the following. In section 1 we introduce *hyperboolean algebras* and show that in every hyperboolean algebra a "difference" operator is definable, i.e. one that corresponds to the difference relation in the underlying Boolean algebra, and therefore a universal modality (easily defining a discriminator term) and an "only" operator which identifies singletons are definable, too. Using these, all axioms of Boolean algebras can be "uplifted" to identities in hyperboolean algebras. Section 2 introduces syntax and semantics of the hyperboolean modal logic HBML for which an axiomatic system is provided and some important syntactic results are proved. In section 3 the completeness of HBML is proved by applying an appropriately modified canonical model construction. We show that HBML does not have the finite model property. The paper ends with some specific and general remarks and open questions arising from the present study.

1 Hyperboolean algebras

For the purposes of this paper it will be convenient to adopt the following definition of a Boolean algebra based on the operation of implication $\rightarrow$ and

zero 0, coming from classical propositional logic. The system $\mathcal{W} = (W, 0, \rightarrow)$ is a Boolean algebra if $0 \in W$ and the implication $\rightarrow$ is a binary operation in W satisfying the following axioms for any $a, b, c \in W$, where $1 = 0 \rightarrow 0$:

B1 $a \rightarrow (b \rightarrow a) = 1$,

B2 $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1$,

B3 $((a \rightarrow 0) \rightarrow 0) \rightarrow a = 1$,

B4 If $1 \rightarrow a = 1$ then $a = 1$,

B5 If $a \rightarrow b = 1$ and $b \rightarrow a = 1$ then $a = b$.

Abbreviations: $-a = a \rightarrow 0$, $a \vee b = -a \rightarrow b$, $a \wedge b = -(a \rightarrow -b)$.

Boolean algebras with some definable operations in them will be used later on as standard semantical structures for HBML. In the case when the Boolean algebra is an algebra of subsets of a given universe X then it will be called a *hyperframe* (over X).

Let $\mathcal{W} = (W, 0, \rightarrow)$ be a Boolean algebra and let $\mathcal{P}(W) = (\mathcal{P}(W), \bot, \Rightarrow)$ be the Boolean algebra of sets over W, where $\bot = \emptyset$ is the zero and $\Rightarrow$ is the implication of $\mathcal{P}(W)$. By a *hyperboolean algebra* (HBA) over $\mathcal{W}$ we mean the structure $(\mathcal{P}(W), \bot, \Rightarrow, \langle 0 \rangle, \langle \rightarrow \rangle)$ where the additional operations are defined as follows, for every $(A, B \subseteq W)$:

- *Internal implication*: $A\langle \rightarrow \rangle B =_{\rm def} \{c \in W | (\exists a, b \in W)(a \in A, b \in B,$ and $c = a \rightarrow b)\}$ and
- *Internal zero*: $\langle 0 \rangle =_{\rm def} \{0\}$

Since HBAs will be used as algebraic semantics of a modal logic, we will use logical denotations of the standard Boolean operations in $\mathcal{P}(W)$:

- *complement*: $\neg A =_{\rm def} \mathcal{P}(W) - A$,
- *intersection*: $A \wedge B =_{\rm def} A \cap B$,
- *union*: $A \vee B =_{\rm def} A \cup B$,
- *bi-implication*: $A \Leftrightarrow B =_{\rm def} (A \Rightarrow B) \wedge (B \Rightarrow A)$
- *unit, the universe* W: $\top =_{\rm def} W$.

By means of $\langle \rightarrow \rangle$ and $\langle 0 \rangle$ we can also define the following natural internal operations:

- Internal unit: $\langle 1\rangle =_{\rm def} (\langle 0\rangle\langle\rightarrow\rangle\langle 0\rangle) = \{1\}$,
- Internal complement: $\langle -\rangle A =_{\rm def} A\langle\rightarrow\rangle\langle 0\rangle$,
- Internal join: $A\langle\vee\rangle B =_{\rm def} \langle -\rangle A\langle\rightarrow\rangle B$,
- Internal meet: $A\langle\wedge\rangle B =_{\rm def} \langle -\rangle(A\langle\rightarrow\rangle\langle -\rangle B)$.

The duals of the internal operations are defined as follows:

$[0] =_{\rm def} \neg\langle 0\rangle$, $[1] =_{\rm def} \neg\langle 1\rangle$,

$[-]A =_{\rm def} \neg\langle -\rangle\neg A = \{a \in W | (\forall b \in W)(\text{ if } a = -b \text{ then } b \in A)\}$,

$A[\rightarrow]B =_{\rm def} \neg(\neg A\langle\rightarrow\rangle\neg B)$

$= \{c \in W | (\forall a, b \in W)(\text{ if } c = a \rightarrow b \text{ then } a \in A \text{ or } b \in B)\}$.

The duals for the other binary internal operations $\langle\wedge\rangle$ and $\langle\vee\rangle$ can be defined in a similar way.

It is easy to see that any HBA is a Boolean algebra with operators (BAO) in the sense of [Jónsson and Tarski 51] with respect to its internal operations. So, the following holds.

Lemma 1.1 *Let $\langle\alpha\rangle$ denote any binary internal operation and $[\alpha]$ be the corresponding dual operation in a HBA $\mathcal{P}(W)$. Then for any $A, B, C \subseteq W$ we have:*

(i) $A\langle\alpha\rangle\bot = \bot\langle\alpha\rangle A = \bot$, $\langle -\rangle\bot = \bot$,

(i') $\top[\alpha]A = A[\alpha]\top = \top$, $[-]\top = \top$,

(ii) $A\langle\alpha\rangle(B \vee C) = (A\langle\alpha\rangle B) \vee (A\langle\alpha\rangle C)$,

$(A \vee B)\langle\alpha\rangle C = (A\langle\alpha\rangle C) \vee (B\langle\alpha\rangle C)$,

$\langle -\rangle(A \vee B) = \langle -\rangle A \vee \langle -\rangle B$,

(ii') $A[\alpha](B \wedge C) = (A[\alpha]B) \wedge (A[\alpha]C)$,

$(A \wedge B)[\alpha]C = (A[\alpha]C) \wedge (B[\alpha]C)$.

$[-](A \wedge B) = [-]A \wedge [-]B$,

(iii) $((A \Rightarrow B)[\alpha]C \Rightarrow (A[\alpha]C) \Rightarrow (B[\alpha]C)) = \top$,

$(A[\alpha](B \Rightarrow C) \Rightarrow ((A[\alpha]B) \Rightarrow (A[\alpha]C)) = \top$,

$([-](A \Rightarrow B) \Rightarrow ([-]A \Rightarrow [-]B)) = \top$.

The operations of the type $\langle\alpha\rangle$ and $\langle -\rangle$ will be called "diamond operations" and the operations of the type $[\alpha]$ and $[-]$ will be called "box operations". They correspond to diamond and box modalities in the modal logic HBML which we will study later on and we will sometimes call them also box and diamond modalities.

Let $(\mathcal{P}(W), \bot, \Rightarrow, \langle 0\rangle, \langle\rightarrow\rangle)$ be a HBA over some Boolean algebra W. The following operation is called *difference operator* in $\mathcal{P}(W)$: ($A \in \mathcal{P}(W)$)

$$\langle\neq\rangle A =_{\rm def} \{x \in W | \exists y \in A \text{ and } x \neq y\}$$

The next lemma shows that the difference operator is expressible by the basic operations of HBA .

Lemma 1.2 *Let $A \in \mathcal{P}(W)$. Then:*

(i) $$\langle\neq\rangle A = \begin{cases} \bot & \text{if} \quad |A| = 0 \\ \neg A & \text{if} \quad |A| = 1 \\ \top & \text{if} \quad |A| > 1 \end{cases},$$

(ii) $\langle\neq\rangle A = (\neg A \wedge ((A\langle\rightarrow\rangle\langle 1\rangle)\langle\rightarrow\rangle\top)) \vee ((((A\langle\rightarrow\rangle A) \wedge \neg\langle 1\rangle)\langle\rightarrow\rangle\langle 1\rangle)\langle\rightarrow\rangle\top)$,

(iii) The operations $\langle\neq\rangle A$ and its dual $[\neq]A = \neg\langle\neq\rangle\neg A$ satisfy the following algebraic identities:

(D1) $[\neq](A \Rightarrow B) \Rightarrow ([\neq]A \Rightarrow [\neq]B) = \top$,

(D2) $A \vee [\neq]\neg[\neq]A = \top$,

(D3) $\langle\neq\rangle\langle\neq\rangle A \Rightarrow (A \vee \langle\neq\rangle A) = \top$.

Proof. Statement (i) is another set-theoretical definition of $\langle\neq\rangle A$.

The proof of (ii) follows directly from (i) by inspecting the three cases of (i) (notice that the first disjunct of the formula defining $\langle\neq\rangle A$ is equivalent to $\neg A$ when $|A| \geq 1$ and to $\emptyset$ when $|A| = 0$; the second disjunct of that formula is equivalent to W when $|A| > 1$ and to $\bot$ when $|A| = 1$).

(iii) follows from the definition of difference and its dual. ■

By means of the difference operation $\langle\neq\rangle$ we define the following other operations:

the universal modalities: $[U]A =_{\rm def} A \wedge [\neq]A$ and $\langle U\rangle A =_{\rm def} \neg[U]\neg A = A \vee \langle\neq\rangle A$,

the "Only" operator: $\mathcal{O}A =_{\rm def} A \wedge [\neq]\neg A$.

The next lemma states some properties of $\langle U\rangle A$ and $\mathcal{O}A$.

Lemma 1.3 *Let $A, B \subseteq \mathcal{P}(W)$. Then:*

(i) The following is a set-theoretical definition of $\langle U\rangle A$:

$$\langle U\rangle A = \begin{cases} \bot & \text{if } A = \bot \\ \top & \text{if } A \neq \bot \end{cases},$$

(ii) The following is a set-theoretical definition of $\mathcal{O}A$:

$$\mathcal{O}A = \begin{cases} A & \text{if} \quad |A| = 1 \\ \bot & \text{if} \quad |A| \neq 1 \end{cases},$$

(iii) $\langle U\rangle$ and $\mathcal{O}$ satisfy the following algebraic identities:

(D4') $([U]A \Rightarrow (A[\rightarrow]B)) = \top$,

(D4") $([U]B \Rightarrow (A[\rightarrow]B)) = \top$,

(Bool1) $(\langle U\rangle\mathcal{O}A \wedge \langle U\rangle\mathcal{O}B \Rightarrow \langle U\rangle(\mathcal{O}A\langle\rightarrow\rangle\mathcal{O}B)) = \top$,

(Bool2) $((\mathcal{O}A\langle\rightarrow\rangle\mathcal{O}B) \Rightarrow \mathcal{O}(\mathcal{O}A\langle\rightarrow\rangle\mathcal{O}B)) = \top$,

(Bool3) $\langle U\rangle\langle 0\rangle = \top$,

(Bool4) $(\langle 0\rangle \Rightarrow \mathcal{O}\langle 0\rangle) = \top$,

(Bool5) $((\mathcal{O}A\langle\rightarrow\rangle(\mathcal{O}B\langle\rightarrow\rangle\mathcal{O}A)) \Rightarrow \langle 1\rangle) = \top$,

(Bool6) $(((\mathcal{O}A\langle\rightarrow\rangle(\mathcal{O}B\langle\rightarrow\rangle\mathcal{O}C))\langle\rightarrow\rangle((\mathcal{O}A\langle\rightarrow\rangle\mathcal{O}B)\langle\rightarrow\rangle(\mathcal{O}A\langle\rightarrow\rangle\mathcal{O}C)))$
$\Rightarrow \langle 1\rangle) = \top$,

(Bool7) $((((\mathcal{O}A\langle\rightarrow\rangle\langle 0\rangle)\langle\rightarrow\rangle\langle 0\rangle)\langle\rightarrow\rangle\mathcal{O}A) \Rightarrow \langle 1\rangle) = \top$,

(Bool8) $((\langle 1\rangle \wedge (\langle 1\rangle\langle\rightarrow\rangle\mathcal{O}A)) \Rightarrow \mathcal{O}A) = \top$,

(Bool9) $(\langle 1\rangle \wedge (\mathcal{O}A\langle\rightarrow\rangle\mathcal{O}B) \wedge (\mathcal{O}B\langle\rightarrow\rangle\mathcal{O}A) \Rightarrow \langle U\rangle(\mathcal{O}A \wedge \mathcal{O}B)) = \top$.

Proof. Statements (i) and (ii) follow from the definitions of $\langle U\rangle A$ and $\mathcal{O}A$ and (iii) follows from (i), (ii) and the axioms of Boolean algebra. ■

Remarks.

1. Using the universal modality one can define a *discriminator term* $t(x, y, z)$ for hyperboolean algebras as follows:

$$t(x, y, z) = ([U](x \Leftrightarrow y) \wedge z) \vee (\neg[U](x \Leftrightarrow y) \wedge x)$$

The existence of such a term has a significant impact on the algebraic properties of the hyperboolean algebras and the variety generated by them (see

e.g. [Burris and Sankappanavar 81]). In particular, it allows for reduction of all universal formulae of the language to identities.

2. Note that the identities (Bool5)-(Bool9) represent translations of the corresponding axioms B1-B5 for Boolean algebras into the language of hyperboolean algebras. They suggest a uniform translation of all universal formulae of the language L(BA) of Boolean algebras into identities of the language for hyperboolean algebras L(HBA), as follows.

First, note that every identity of L(BA) is equivalent to one of the form $t = 1$ where t is a term of that language. We define the following translation of the terms of L(BA) into terms of L(HBA):

- $\tau(x) = \mathcal{O}x$, for any variable x;
- $\tau(0) = \langle 0 \rangle$;
- $\tau(t \to s) = \tau(t)\langle \to \rangle \tau(s)$;

Note that the identity $t = 1$ holds in a Boolean algebra $\mathbf{B}$ iff the identity $(\tau(t) \Rightarrow \langle 1 \rangle) = \top$, or equivalently, $[U](\tau(t) \Rightarrow \langle 1 \rangle) = \top$ holds in the corresponding HBA P($\mathbf{B}$).

Further, every universal formula of L(BA) in a prenex form with a matrix (here we use $\supset$ for a logical implication in the language L(BA) to avoid confusion with the other implication symbols used in the text)

$$(t_1 = 1 \,\&\, \ldots \,\&\, t_n = 1) \supset (s_1 = 1 \vee \ldots \vee s_m = 1)$$

translates into an identity for L(HBA):

$$(\langle 1 \rangle \wedge \tau(t_1) \wedge \ldots \wedge \tau(t_n)) \Rightarrow ([U](\tau(s_1) \Rightarrow \langle 1 \rangle) \vee \ldots \vee [U](\tau(s_m) \Rightarrow \langle 1 \rangle)) = \top$$

Again, it is easy to see that the universal formula holds in a Boolean algebra $\mathbf{B}$ iff the corresponding identity holds in P($\mathbf{B}$).

Finally, note that every universal formula is equivalent to a conjunction of universal formulae of the type above.

2 Hyperboolean Modal Logic — HBML. Syntax and semantics

In this section we introduce Hyperboolean Modal Logic — HBML, its syntax, semantics and axiomatization.

Syntax of HBML

The language of HBML contains:

- $VAR = \{p_1, p_2, \ldots\}$ — a denumerable set of propositional variables,
- $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$, $\top$, $\bot$ — Boolean (classical) connectives,
- $\langle\rightarrow\rangle$ — a binary diamond modality "internal implication",
- $\langle 0\rangle$ — propositional constant "internal zero"
- $(,)$ — parentheses.

The notion of a formula is the usual one: besides the classical formulas, $\langle 0\rangle$ is a formula and if A and B are formulas $(A\langle\rightarrow\rangle B)$ is also a formula. We adopt the standard omission of parentheses. We also consider $\bot$ and $\Rightarrow$ as primitives and the rest of Boolean connectives as definable in a standard manner.

Abbreviations: we introduce the following additional internal operations and their duals:

$\langle 1\rangle =_{\rm def} \langle 0\rangle\langle\rightarrow\rangle\langle 0\rangle$,

$\langle -\rangle A =_{\rm def} A\langle\rightarrow\rangle\langle 0\rangle$,

$A\langle\vee\rangle B =_{\rm def} \langle -\rangle A\langle\rightarrow\rangle B$,

$A\langle\wedge\rangle B =_{\rm def} \langle -\rangle(A\langle\rightarrow\rangle\langle -\rangle B)$.

$[0] = \neg\langle 0\rangle$,

$[1] = \neg\langle 1\rangle$,

$A[\rightarrow]B = \neg(\neg A\langle\rightarrow\rangle\neg B)$,

$[-]A =_{\rm def} A\langle\rightarrow\rangle\langle 0\rangle$,

$A[\vee]B = \neg(\neg A\langle\vee\rangle\neg B)$,

$A[\wedge]B = \neg(\neg A\langle\wedge\rangle\neg B)$,

Important definable operations are:

difference operators:

$\langle\neq\rangle A =_{\rm def} (\neg A\wedge((A\langle\rightarrow\rangle\langle 1\rangle)\langle\rightarrow\rangle\top))\vee((((A\langle\rightarrow\rangle A)\wedge\neg\langle 1\rangle)\langle\rightarrow\rangle\langle 1\rangle)\langle\rightarrow\rangle\top)$,

$[\neq]A =_{\rm def} \neg\langle\neq\rangle\neg A$,

universal modalities:

$\langle U\rangle A =_{\rm def} A \vee \langle\neq\rangle A, \quad [U]A =_{\rm def} \neg\langle U\rangle\neg A,$

the operator "only":

$\mathcal{O}A =_{\rm def} A \wedge [\neq]\neg A.$

Semantics of HBML

The standard algebraic semantics of HBML is over hyperboolean algebras. Let $\mathcal{W} = (W, 0, \rightarrow)$ be a Boolean algebra and let $\mathcal{P}(W) = (\mathcal{P}(W), \perp, \Rightarrow)$ be the Boolean algebra of sets over W. By a valuation we mean any function v from the set VAR of propositional variables into $\mathcal{P}(W)$, i.e. for each $p \in VAR$ $v(p)$ is a subset of W. Each valuation v is then extended to arbitrary formulas by induction:

$v(\perp) = \perp = \emptyset,$

$v(A \Rightarrow B) = v(A) \Rightarrow v(B) = (W \backslash v(A)) \cup v(B),$

$v(A\langle\rightarrow\rangle B) = v(A)\langle\rightarrow\rangle v(B).$

$v(\langle 0\rangle) = \langle 0\rangle.$

We say that a formula A is *valid* in the HBA $\mathcal{P}(W)$ if for any valuation v we have $v(A) = \top = W$.

The above semantics can be reformulated as Kripke-style semantics on Boolean algebras and hyperframes based on the relation of satisfaction $x \vdash_v A$: "the formula A is true in x at the valuation v" being defined as follows: $x \vdash_v A$ iff $x \in v(A)$. This relation can be defined independently by induction on A as usual. Let W be a Boolean algebra and v be any valuation in W. Then:

$x \vdash_v A$ iff $x \in v(A)$, for $A \in VAR$,

$x \nvdash_v \perp$

$x \vdash_v A \wedge B$ iff $x \vdash_v A$ and $x \vdash_v B$,

$x \vdash_v A \vee B$ iff $x \vdash_v A$ or $x \vdash_v B$,

$x \vdash_v A\langle\rightarrow\rangle B$ iff $(\exists y, z \in W)(x = y \rightarrow z$ and $y \vdash_v A$ and $z \vdash_v B)$,

$x \vdash_v \langle 0\rangle$ iff $x = 0$.

The semantics of the definable internal operations can be obtained using their definitions and some obvious Boolean identities. For instance: $x \vdash_v \langle 1\rangle$ iff $x = 0 \rightarrow 0 = 1$,

$x \vdash_v A\langle\vee\rangle B$ iff $(\exists y, z \in W)(x = y \vee z$ and $y \vdash_v A$ and $z \vdash_v B)$.

Having in mind lemma 1.2 and lemma 1.3 we obtain the following standard clauses for the definable operations of "difference"," universal modalities" and "only":

$x \vdash_v \langle\neq\rangle A$ iff $(\exists y \in W)(x \neq y$ and $y \vdash_v A)$,

$x \vdash_v [\neq]A$ iff $(\forall y \in W)(x \neq y \to y \vdash_v A)$,

$x \vdash_v \langle U\rangle A$ iff $(\exists y \in W)(y \vdash_v A)$,

$x \vdash_v [U]A$ iff $(\forall y \in W)(y \vdash_v A)$,

$x \vdash_v \mathcal{O}A$ iff $x \vdash_v A$ and $(\forall y \neq x)(y \nvdash_v A)$.

A pair (W, v) of a Boolean algebra W and a valuation v in W is called a *model* (over W). A formula A is *valid in a model* (W, v) if for any $x \in W$, $x \vdash_v A$. A is *valid* in W if it is valid in all models over W.

Axiomatization of HBML

I. All propositional tautologies

II. The axioms of minimal modal logic for $[\to]$:

(K2') $(A \Rightarrow B)[\to]C \Rightarrow ((A[\to]C) \Rightarrow (B[\to]C))$,

(K2") $A[\to](B \Rightarrow C) \Rightarrow ((A[\to]B) \Rightarrow (A[\to]C))$,

III. The axioms for difference:

(D1) $[\neq](A \Rightarrow B) \Rightarrow ([\neq]A \Rightarrow [\neq]B)$,

(D2) $A \vee [\neq]\neg[\neq]A$,

(D3) $\langle\neq\rangle\langle\neq\rangle A \Rightarrow (A \vee \langle\neq\rangle A)$,

(D4') $[U]A \Rightarrow A[\to]B$,

(D4") $[U]B \Rightarrow A[\to]B$,

IV. Axioms for the internal Boolean structure:

(Bool1) $\langle U\rangle\mathcal{O}A \wedge \langle U\rangle\mathcal{O}B \Rightarrow \langle U\rangle(\mathcal{O}A\langle\to\rangle\mathcal{O}B)$,

(Bool2) $(\mathcal{O}A\langle\to\rangle\mathcal{O}B) \Rightarrow \mathcal{O}(\mathcal{O}A\langle\to\rangle\mathcal{O}B)$,

(Bool3) $\langle U\rangle\langle 0\rangle$,

(Bool4) $\langle 0\rangle \Rightarrow \mathcal{O}\langle 0\rangle$,

(Bool5) $(\mathcal{O}A\langle\rightarrow\rangle(\mathcal{O}B\langle\rightarrow\rangle\mathcal{O}A)) \Rightarrow \langle 1\rangle$,

(Bool6) $((\mathcal{O}A\langle\rightarrow\rangle(\mathcal{O}B\langle\rightarrow\rangle\mathcal{O}C))\langle\rightarrow\rangle((\mathcal{O}A\langle\rightarrow\rangle\mathcal{O}B)\langle\rightarrow\rangle(\mathcal{O}A\langle\rightarrow\rangle\mathcal{O}C))) \Rightarrow \langle 1\rangle$,

(Bool7) $(((\mathcal{O}A\langle\rightarrow\rangle\langle 0\rangle)\langle\rightarrow\rangle\langle 0\rangle)\langle\rightarrow\rangle\mathcal{O}A) \Rightarrow \langle 1\rangle$,

(Bool8) $(\langle 1\rangle \wedge (\langle 1\rangle\langle\rightarrow\rangle\mathcal{O}A)) \Rightarrow \mathcal{O}A$,

(Bool9) $\langle 1\rangle \wedge (\mathcal{O}A\langle\rightarrow\rangle\mathcal{O}B) \wedge (\mathcal{O}B\langle\rightarrow\rangle\mathcal{O}A) \Rightarrow \langle U\rangle(\mathcal{O}A \wedge \mathcal{O}B)$.

Rules of inference

Modus Ponens (MP):

$$\frac{A, A \Rightarrow B}{B}$$

Necessitation for $[U]$ (N[U]):

$$\frac{A}{[U]A}$$

Irreflexivity rules:

(IRR_0):

$$\frac{\mathcal{O}p \Rightarrow A \text{ for all } p \in \ VAR}{A}$$

($\mathrm{IRR}_{[\rightarrow]^r}$):

$$\frac{A \Rightarrow (B[\rightarrow](\mathcal{O}p \Rightarrow C)) \text{ for all } p \in VAR}{A \Rightarrow (B[\rightarrow]C)}$$

($\mathrm{IRR}_{[\rightarrow]^l}$):

$$\frac{A \Rightarrow ((\mathcal{O}p \Rightarrow B)[\rightarrow]C) \text{ for all } p \in VAR}{A \Rightarrow (B[\rightarrow]C)}$$

Let us note that the IRR-rules can be weakened assuming the premise not for all $p \in VAR$ but for some p not occurring in A, B, C.

Irreflexivity-like rules have been introduced for the first time in [Gabbay 81] and have been used by many authors: [Passy and Tinchev 91, Gargov and Goranko 91, Venema 92, Venema 93, de Rijke 92, de Rijke 93, Balbiani et al 97].

Lemma 2.1 *The following rules are derivable in HBML, where* $S \in \{U, \neq\}$*:*

$(N[\rightarrow]_l)$ $\frac{A}{A[\rightarrow]B}$,

$(N[\rightarrow]_r)$ $\quad \frac{B}{A[\rightarrow]B}$,

$(N[\neq])$ $\quad \frac{A}{[\neq]A}$.

$(Mono[\rightarrow]_l)$ $\quad \frac{A \Rightarrow B}{(A[\rightarrow]C) \Rightarrow (B[\rightarrow]C)}$,

$(Mono[\rightarrow]_r)$ $\quad \frac{A \Rightarrow B}{(C[\rightarrow]A) \Rightarrow (C[\rightarrow]B)}$,

$(Mono\langle\rightarrow\rangle_l)$ $\quad \frac{A \Rightarrow B}{(A\langle\rightarrow\rangle C) \Rightarrow (B\langle\rightarrow\rangle C)}$,

$(Mono\langle\rightarrow\rangle_r)$ $\quad \frac{A \Rightarrow B}{(C\langle\rightarrow\rangle A) \Rightarrow (C\langle\rightarrow\rangle B)}$,

$(Mono[S])$ $\quad \frac{A \Rightarrow B}{[S]A \Rightarrow [S]B}$,

$(Mono\langle S\rangle)$ $\quad \frac{A \Rightarrow B}{\langle S\rangle A \Rightarrow \langle S\rangle B}$.

Proof. Suppose A is a theorem. Then by (N[U]) $[U]A$ is also a theorem. Applying axiom D4' and (MP) we obtain that $A[\rightarrow]B$ is a theorem too. In the same way we prove the second rule. The rule $(N[\neq])$ follows directly from $(N[U])$. The rules of type "Mono" follow from the axioms for minimal modal logic. ■

Theorem 2.2 *(*Soundness theorem for HBML*) HBML is sound with respect to its algebraic and Kripke-style semantics.*

Proof. All axioms of HBML are valid by 1.1, 1.2 and 1.3. The proof that the rules of inference preserve validity is straightforward. ■

Lemma 2.3 *(i)* $[U]$ *and* $\langle U\rangle$ *are S5 box and diamond modalities,*

(ii) The following distributivity laws for $\langle\rightarrow\rangle$ *over disjunctions and* $[\rightarrow]$ *over conjunctions are theorems:*

$$(A_1\langle\rightarrow\rangle B) \vee (A_2\langle\rightarrow\rangle B) \Leftrightarrow (A_1 \vee A_2)\langle\rightarrow\rangle B,$$

$$(A\langle\rightarrow\rangle B_1) \vee (A\langle\rightarrow\rangle B_2) \Leftrightarrow A\langle\rightarrow\rangle(B_1 \vee B_2).$$

$$(A_1[\rightarrow]B) \wedge (A_2[\rightarrow]B) \Leftrightarrow (A_1 \wedge A_2)[\rightarrow]B,$$

$$(A[\rightarrow]B_1) \wedge (A[\rightarrow]B_2) \Leftrightarrow A[\rightarrow](B_1 \wedge B_2),$$

(iii) The following formulas are theorems of HBML, where p, q *are any formulas:*

(D5) $\quad \mathcal{O}p \wedge A \Rightarrow [U](\mathcal{O}p \Rightarrow A)$,

(D5') $\langle U\rangle(\mathcal{O}p \wedge A) \Rightarrow [U](\mathcal{O}p \Rightarrow A)$.

(D6) $((\mathcal{O}p \wedge A)\langle\rightarrow\rangle(\mathcal{O}q \wedge B)) \Leftrightarrow \langle U\rangle(\mathcal{O}p \wedge A) \wedge \langle U\rangle(\mathcal{O}q \wedge B) \wedge (\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q)$.

Proof. (i), (D5), (D5'). Let us note that the axioms (D1), (D2) and (D3) and the rule $(N[\neq])$ form a sublogic of HBML called the Modal Logic of Inequality. De Rijke in [de Rijke 92] proved that it is complete with respect to its standard semantics. Using this completeness result it is enough to verify (i) and (D5) and (D5') semantically.

(ii) These are standard facts about binary modalities.

(iii) The implication from the left to the right in D6 follows from the axioms (D4') and (D4") and the monotonicity of $\langle\rightarrow\rangle$.

The following is a derivation of the converse implication:

(1) $\langle U\rangle(\mathcal{O}p \wedge A)$ — assumption,

(2) $\langle U\rangle(\mathcal{O}q \wedge B)$ — assumption,

(3) $\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q$ — assumption,

(4) $\neg((\mathcal{O}p \wedge A)\langle\rightarrow\rangle(\mathcal{O}q \wedge B))$ — assumption,

(5) $(\mathcal{O}p \Rightarrow \neg A)[\rightarrow](\mathcal{O}q \Rightarrow \neg B)$ — by 4,

(6) $[U](\mathcal{O}p \Rightarrow A)$ — by 1 and D5',

(7) $[U](\mathcal{O}q \Rightarrow B)$ — by 2 and D5',

(8) $(\mathcal{O}p \Rightarrow A)[\rightarrow](\mathcal{O}q \Rightarrow \neg B)$ — by 6 and D4',

(9) $(\neg\mathcal{O}p)[\rightarrow](\mathcal{O}q \Rightarrow B)$ — by 7 and D4",

(10) $((\mathcal{O}p \Rightarrow \neg A) \wedge (\mathcal{O}p \Rightarrow A))[\rightarrow](\mathcal{O}q \Rightarrow \neg B)$ — by 5 and 8,

(11) $(\neg\mathcal{O}p)[\rightarrow](\mathcal{O}q \Rightarrow \neg B)$ — by 10,

(12) $(\neg\mathcal{O}p)[\rightarrow]((\mathcal{O}q \Rightarrow B) \wedge (\mathcal{O}q \Rightarrow \neg B))$ — by 9 and 11,

(13) $(\neg\mathcal{O}p)[\rightarrow](\neg\mathcal{O}q)$ — by 12,

(14) $\neg(\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q)$ — by 13,

(15) $\bot$ — by 3 and 14. ■

Later on, when applying (i) of 2.3 we will be saying "by S5", and similarly for (D5), (D5'), (D6).

Lemma 2.4 *(*Strong replacement lemma*) Let $\varphi(p)$ be a formula with unique occurrence of the variable p and $\varphi(A)$ be a replacement of p by A in $\varphi(p)$. Then for any A, B the following formula is a theorem of HBML:*

(SR) $\quad [U](A \Leftrightarrow B) \Rightarrow (\varphi(A) \Leftrightarrow \varphi(B))$

Proof. By induction on the complexity of $\varphi(p)$. The case $\varphi(p) = p$ is trivial. Suppose by induction hypothesis that the assertion is true for $\varphi(p)$. The Boolean cases of the induction do not present difficulties. Let $\psi(p) = \varphi(p)[\rightarrow]C$. We have to prove

(*) $\quad [U](A \Leftrightarrow B) \Rightarrow ((\varphi(A)[\rightarrow]C) \Leftrightarrow ((\varphi(B))[\rightarrow]C))$

First we will show that the following formula is a theorem of HBML:

(**) $[U](p \Leftrightarrow q) \Rightarrow ((p[\rightarrow]C) \Leftrightarrow ((q[\rightarrow]C))$.

The following is a derivation of (**):

1. $[U](p \Leftrightarrow q)$ — assumption,
2. $(p \Leftrightarrow q)[\rightarrow]C$ — by 1 and D4',
3. $(p \Rightarrow q)[\rightarrow]C$ — by 2 and 2.3 ,
4. $(q \Rightarrow p)[\rightarrow]C$ — likewise,
5. $(p[\rightarrow]C) \Rightarrow (q[\rightarrow]C)$ — by 3,
6. $(q[\rightarrow]C) \Rightarrow (p[\rightarrow]C)$ — by 4,
7. $(p[\rightarrow]C \Leftrightarrow (q[\rightarrow]C)$ — by 5 and 6.

The following is a derivation of (*):

1. $[U](A \Leftrightarrow B)$ — assumption,
2. $[U](A \Leftrightarrow B) \Rightarrow (\varphi(A) \Leftrightarrow \varphi(B))$ — by the induction hypothesis
3. $[U][U](A \Leftrightarrow B) \Rightarrow [U](\varphi(A) \Leftrightarrow \varphi(B))$ — by S5 and 2,
4. $[U](A \Leftrightarrow B) \Rightarrow [U](\varphi(A) \Leftrightarrow \varphi(B))$ — by S5 and 3,
5. $[U](\varphi(A) \Leftrightarrow \varphi(B))$ — by 1 and 4,
6. $[U](\varphi(A) \Leftrightarrow \varphi(B)) \Rightarrow ((\varphi(A)[\rightarrow]C) \Leftrightarrow (\varphi(B)[\rightarrow]C))$ — by (**),
7. $((\varphi(A)[\rightarrow]C) \Leftrightarrow (\varphi(B)[\rightarrow]C))$ — by 5 and 6.

The case $\psi = C[\rightarrow]\varphi(p)$ can be treated in a similar way. ■

Now we will introduce a stronger form of the irreflexivity rules, called DeepIRR and will show that they are derivable in HBML. We will use DeepIRR in the proof of the completeness theorem for HBML. First we introduce the so called *necessity forms*, NF (see [Goldblatt 82]). Let \$ be a symbol not in the language of HBML. Then by induction we define:

(1) \$ is an NF of depth 0,

(2) If A is a formula then $\$[\rightarrow]A$ and $A[\rightarrow]\$$ are NF of depth 0,

(3) If φ is an NF of depth n and A is a formula, then $(A \Rightarrow \varphi)$ is an NF of depth n,

(4) If φ is an NF of depth n and $S \in \{U, \neq\}$ then $[S]\varphi$ is an NF of depth $n+1$.

Note that any NF φ can be equivalently represented as $A_0 \Rightarrow [S_1](A_1 \Rightarrow \ldots [S_n](A_n \Rightarrow \psi)\ldots)$ where ψ is an NF of depth 0.

The result of replacement of \$ by a formula A in an NF φ is denoted by $\varphi(A)$. Let us note that $\varphi(A)$ acts like a normal modality over A: $\varphi(A \Rightarrow B) \Rightarrow (\varphi(A) \Rightarrow \varphi(B))$ is a theorem of HBML and if A is a theorem of HBML then $\varphi(A)$ is a theorem too.

Now for any NF φ we introduce the rule

DeepIRR(φ):

$$\frac{\varphi(\mathcal{O}p \Rightarrow A) \quad \text{for any } p \in VAR}{\varphi(A)}$$

Obviously the rules IRR_0, $IRR_{[\rightarrow]r}$ and $IRR_{[\rightarrow]l}$ are special cases of the DeepIRR-rule.

Theorem 2.5 *For any NF φ the rule $DeepIRR(\varphi)$, is derivable in HBML.*

Proof. (See [Gabbay and Hodkinson 90].) The theorem is true for any "temporalized modal language" having for each unary modality $[S]$ its "mirror image" $[S^{-1}]$ corresponding to the inverse of the relation semantically attached to $[S]$. In our case $[U^{-1}] = [U]$ and $[\neq^{-1}] = [\neq]$, because the relations corresponding to the modalities $[U]$ and $[\neq]$ are symmetric. ■

3 Completeness theorem for HBML

In this section we shall describe a canonical model for HBML similar to that from [Balbiani et al 97]. The logic HBML and the set of all theorems of HBML will be denoted by L.

A set of formulas Γ of HBML is called a *theory* if:

(i) $L \subseteq \Gamma$,

(ii) Γ is closed under MP, namely if $A, A \Rightarrow B \in \Gamma$ then $B \in \Gamma$,

(iii) Γ is closed under the DeepIRR rule, namely for every NF φ, if for every variable p, $\varphi(\mathcal{O}p \Rightarrow A) \in \Gamma$ then $\varphi(A) \in \Gamma$.

Obviously the set L of all theorems of HBML is a theory. If Γ is a theory and A is a formula then we denote by $\Gamma + A$ the set of formulas $\{B / A \Rightarrow B \in \Gamma\}$.

Lemma 3.1 *If Γ is a theory and A is a formula then $\Gamma + A$ is the smallest theory containing A and Γ.*

Proof. Straightforward. ■

Let Γ be a set of formulas, $R \in \{U, \neq\}$ and define $[R]\Gamma = \{A / [R]A \in \Gamma\}$.

Lemma 3.2 *If Γ is a theory then $[R]\Gamma$ is a theory too.*

Proof. Straightforward. ■

A theory Γ is *consistent* if $\bot \notin \Gamma$. Note that if Γ is a theory, then $\Gamma + A$ is inconsistent iff $\neg A \in \Gamma$.

A theory Γ is *maximal* if it is consistent and, for every formula A, either $A \in \Gamma$ or $\neg A \in \Gamma$.

Lemma 3.3 (Lindenbaum lemma) *Any consistent theory Γ can be extended to a maximal theory Δ.*

Proof. Let $A_0, A_1, \ldots$ be an enumeration of all formulas of HBML. We define inductively a sequence of consistent theories $\Gamma_0, \Gamma_1, \ldots$ in the following way. Define $\Gamma_0 = \Gamma$ and suppose that $\Gamma_0, \ldots, \Gamma_n$ are defined. For Γ_{n+1} we consider several cases.

Case 1: $\Gamma_n + A_n$ is consistent, then put $\Gamma_{n+1} = \Gamma_n + A_n$.

Case 2: $\Gamma_n + A_n$ is inconsistent. Then $\neg A_n \in \Gamma_n$. Note that A_n can be represented in the form $\varphi(B)$ for some NF φ (e.g. take $\varphi(\$) = \$$) and there are finitely many such representations: $\varphi_1(B_1), \ldots, \varphi_k(B_k)$. We define the finite sequence $\Gamma_n^0, \ldots, \Gamma_n^k$ inductively as follows.

Let $\Gamma_n^0 = \Gamma_n$ and suppose that $\Gamma_n^0, \ldots, \Gamma_n^i$, $i < k$ are defined. Then there exists a propositional variable p such that $\Gamma_n^i + \neg\varphi_i(\mathcal{O}p \Rightarrow B_i)$ is consistent. For, suppose the contrary. Then for any p $\Gamma_n^i + \neg\varphi_i(\mathcal{O}p \Rightarrow B_i)$ is inconsistent. Then $\varphi_i(\mathcal{O}p \Rightarrow B_i) \in \Gamma_n^i$ for any variable p and by DeepIRR(φ_i) we obtain that $\varphi_i(B_i) = A_n \in \Gamma_n^i$. But $\neg A_n$ also belongs to Γ_n^i, which implies that Γ_n^i is inconsistent — a contradiction. Let p_i be the first variable such that $\Gamma_n^i + \neg\varphi_i(\mathcal{O}p \Rightarrow B_i)$ is consistent. Then define $\Gamma_n^{i+1} = \Gamma_n^i + \neg\varphi_i(\mathcal{O}p \Rightarrow B_i)$. Define $\Gamma_{n+1} = \Gamma_n^k$. Put $\Delta = \bigcup_{n=0}^{\infty} \Gamma_n$. It is straightforward to show that Δ is a maximal theory containing Γ. ■

Corollary 3.4 *(i) If Γ is a theory and $A \notin \Gamma$ then there exists a maximal theory Δ such that $\Gamma \subseteq \Delta$ and $A \notin \Delta$.*

(ii) If a formula A is not a theorem of HBML then there exists a maximal theory Γ such that $A \notin \Gamma$

Proof. (i) Apply the Lindenbaum lemma to $\Gamma + \neg A$.

(ii) Notice that the set L of all theorems of HBML is a theory; then $A \notin L$ and apply (i). ■

Let W_L be the set of all maximal theories of HBML.

Define the following relations in W_L:

$\Gamma R_{\neq} \Delta$ iff $[\neq]\Gamma \subseteq \Delta$,

$\Gamma R_U \Delta$ iff $[U]\Gamma \subseteq \Delta$,

$\Gamma R_{\rightarrow} \Delta\Theta$ iff $(\exists p, q \in VAR)(\mathcal{O}p \in \Delta$ *and* $\mathcal{O}q \in \Theta$ *and* $(\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q) \in \Gamma$ *and* $\Gamma R_U \Delta$ *and* $\Gamma R_U \Theta$,)

$\mathrm{O} = \{\Gamma \in W_L / \langle 0\rangle \in \Gamma\}$.

Remark. It can be easily shown that the definition of $R_{\rightarrow}$ is equivalent to the "canonical" one, viz.: $\Gamma R_{\rightarrow} \Delta\Theta$ iff for every formula $A[\rightarrow]B \in \Gamma$, $A \in \Delta$ or $B \in \Theta$. One direction of that equivalence follows from lemma 3.7 (ii) proved further; the other — from axioms (D4') and (D4") and the Witness lemma below.

Lemma 3.5 (Witness Lemma) *For any $\Gamma \in W_L$ there exists a variable p such*

that $\mathcal{O}p \in \Gamma$. *If* $\mathcal{O}p, \mathcal{O}q \in \Gamma$ *then for any* $\Delta \in W_L$ *such that* $\Gamma R_U \Delta$ *we have* $[U](\mathcal{O}p \Leftrightarrow \mathcal{O}q) \in \Delta$.

Proof. Suppose for the sake of contradiction that $\Gamma \in W_L$ and for any variable p we have $\mathcal{O}p \notin \Gamma$. Then, by the maximality of Γ we have that for any variable p: $\mathcal{O}p \Rightarrow \bot \in \Gamma$ and hence by the IRR $\bot \in \Gamma$ — a contradiction.

Now let $\Gamma R_U \Delta$ and $\mathcal{O}p, \mathcal{O}q \in \Gamma$. Then by D5 and S5 (2.3) we obtain $[U][U](\mathcal{O}p \Leftrightarrow \mathcal{O}q) \in \Gamma$ and by $\Gamma R_U \Delta$ — that $[U](\mathcal{O}p \Leftrightarrow \mathcal{O}q) \in \Delta$. ■

Lemma 3.6 *Let* $S \in \{U, \neq\}$. *Then:*

(i) Let $\Gamma, \Delta \in W_L$. *Then:*

$\Gamma R_S \Delta$ iff $(\forall A)($ if $[S]A \in \Gamma$ then $A \in \Delta)$,

$\Gamma R_S \Delta$ iff $(\forall A)($ if $A \in \Delta$ then $\langle S \rangle A \in \Gamma)$,

(ii) For any formula A *and* $\Gamma \in W_L$*:*

(a) $[S]A \in \Gamma$ iff $(\forall \Delta \in W_L)($ if $\Gamma R_S \Delta$ then $A \in \Delta)$,

(b) $\langle S \rangle A \in \Gamma$ iff $(\exists \Delta \in W_L)(\Gamma R_S \Delta$ and $A \in \Delta)$.

(iii) R_U *is an equivalence relation, containing* $R_{\neq}$ *and* $R_{\rightarrow}$ *(in the sense if* $\Gamma R_{\rightarrow} \Delta\Theta$ *then* Γ, Δ *and* Θ *are* R_U*-equivalent).*

(iv) If $\Gamma R_U \Delta$ *and* $\mathcal{O}p \in \Gamma \cap \Delta$, *where* p *is any formula then* $\Gamma = \Delta$.

(v) For any Γ *there exists unique* Δ *such that* $\Gamma R_U \Delta$ *and* $\langle 0 \rangle \in \Delta$.

(vi) $\Gamma R_S \Delta$ *iff* $(\exists p \in VAR)(\mathcal{O}p \in \Delta \& \langle S \rangle \mathcal{O}p \in \Gamma)$.

(vii) If $\Gamma R_U \Delta$ *then* $\Gamma R_{\neq} \Delta$ *iff* $\Gamma \neq \Delta$,

Proof. (i) The proof follows from the definition of R_S.

(iia) $(\rightarrow)$: Suppose $\Gamma \in W_L$, $[S]A \in \Gamma$ and $\Gamma R_S \Delta$. Then by the definition of R_L we obtain $A \in \Delta$.

$(\leftarrow)$. Suppose $[S]A \notin \Gamma$. Then $A \notin [S]\Gamma$ and by 3.2 $[S]\Gamma$ is a theory. By 3.4 there exists $\Delta \in W_L$ such that $[S]\Gamma \subseteq \Delta$, hence $\Gamma R_S \Delta$, and $A \notin \Delta$. The second equivalence follows from the one just proved.

(iii) The fact that R_U is an equivalence relation in W_L follows from the fact that $[U]$ is an S5 modality.

To show that R_U contains $R_{\neq}$ suppose $\Gamma R_{\neq} \Delta$ and $[U]A \in \Gamma$. Then by the definition of $[\neq]$ we get $[\neq]A \in \Gamma$, which by $\Gamma R_{\neq} \Delta$ implies $A \in \Delta$.

That $R_{\rightarrow}$ is contained in R_U follows by the definition of $R_{\rightarrow}$ and the fact that R_U is an equivalence relation.

(iv) Suppose $\Gamma R_U \Delta$, $\mathcal{O}p \in \Gamma$ and $\mathcal{O}p \in \Delta$ and let $A \in \Gamma$. Then $\mathcal{O}p \wedge A \in \Gamma$ and by the theorem D5 of HBML (see 2.3) we obtain $[U](\mathcal{O}p \Rightarrow A) \in \Gamma$. Since $\Gamma R_U \Delta$ we get $\mathcal{O}p \Rightarrow A \in \Delta$ and by $\mathcal{O}p \in \Delta$ we obtain $A \in \Delta$. Therefore $\Gamma \subseteq \Delta$. Similarly we obtain the converse inclusion and hence $\Gamma = \Delta$.

(v) Let $\Gamma \in W_L$. Then by axiom (Bool3) $\langle U \rangle \langle 0 \rangle \in \Gamma$. By (ii)(b) there exists $\Delta \in W_L$ such that $\Gamma R_U \Delta$ and $\langle 0 \rangle \in \Delta$. To show that Δ is unique suppose $\Gamma R_U \Theta$ and $\langle 0 \rangle \in \Theta$. Then we obtain $\Delta R_U \Theta$. By axiom (Bool4) we get that $\mathcal{O}\langle 0 \rangle \in \Delta$ and $\mathcal{O}\langle 0 \rangle \in \Theta$. Then by (iv) we obtain $\Delta = \Theta$.

(vi) ($\rightarrow$) Suppose $\Gamma R_S \Delta$. By 3.5 there exists $p \in VAR$ such that $\mathcal{O}p \in \Delta$. Then by (i) $\langle S \rangle \mathcal{O}p \in \Gamma$.

($\leftarrow$) Suppose that for some $p \in VAR$, $\mathcal{O}p \in \Delta$ and $\langle S \rangle \mathcal{O}p \in \Gamma$. Then by (ii) there exists $\Theta \in W_L$ such that $\Gamma R_S \Theta$ and $\mathcal{O}p \in \Theta$. By (iv) $\Theta = \Delta$ and consequently $\Gamma R_S \Delta$. $\Delta R_U \Theta$ and by (iv) $\Delta = \Theta$. Hence $\Gamma R_S \Delta$.

(vii) Suppose $\Gamma R_U \Delta$.

($\rightarrow$) Let $\Gamma R_{\neq} \Delta$. By 3.5 $\mathcal{O}p \in \Gamma$, so $p \in \Gamma$ and $[\neq]\neg p \in \Gamma$ and hence $\neg p \in \Delta$. Consequently $\Gamma \neq \Delta$.

($\leftarrow$) Suppose $\Gamma \neq \Delta$. By 3.5 there exists a variable p such that $\mathcal{O}p \in \Delta$. To show $\Gamma R_{\neq} \Delta$ it is enough (on account of (vi)) to show that $\langle \neq \rangle \mathcal{O}p \in \Gamma$. By $\Gamma \neq \Delta$, $\mathcal{O}p \in \Delta$ and $\Gamma R_U \Delta$ we get $\mathcal{O}p \notin \Gamma$ and hence $\neg \mathcal{O}p \in \Gamma$. By $\Gamma R_U \Delta$ and $\mathcal{O}p \in \Delta$ we obtain from (i) that $\langle U \rangle \mathcal{O}p \in \Gamma$. So $\neg \mathcal{O}p \wedge \langle U \rangle \mathcal{O}p \in \Gamma$. This implies that $\langle \neq \rangle \mathcal{O}p \in \Gamma$, which has to be proved. ■

Lemma 3.7 *(i) Suppose $\Gamma \in W_L$. Then:*

(a) If $A\langle \rightarrow \rangle B \in \Gamma$ then for some propositional variables p, q we have $(\mathcal{O}p \wedge A)\langle \rightarrow \rangle(\mathcal{O}q \wedge B) \in \Gamma$.

(b) $A\langle \rightarrow \rangle B \in \Gamma$ iff there exist $\Delta, \Theta \in W_L$ such that $\Gamma R_{\rightarrow} \Delta\Theta$ and $A \in \Delta$ and $B \in \Theta$,

(c) $A[\rightarrow]B \in \Gamma$ iff $(\forall \Delta, \Theta \in W_L)(\Gamma R_{\rightarrow} \Delta\Theta \rightarrow A \in \Delta$ or $B \in \Theta)$.

(ii) If $\Delta R_U \Theta$ then there exists unique $\Gamma \in W_L$ such that $\Gamma R_{\rightarrow} \Delta\Theta$.

Proof. (i)(a) Suppose $A\langle \rightarrow \rangle B \in \Gamma$ and, for the sake of contradiction, that for any $p, q \in VAR$, $(\mathcal{O}p \wedge A)\langle \rightarrow \rangle(\mathcal{O}q \wedge B) \notin \Gamma$. Then for any $p, q \in VAR$ $(\mathcal{O}p \Rightarrow \neg A)[\rightarrow](\mathcal{O}q \rightarrow \neg B) \in \Gamma$ and by the DeepIRR we obtain that $(\neg A)[\rightarrow](\neg B) \in \Gamma$ and consequently $A\langle \rightarrow \rangle B \notin \Gamma$ — a contradiction.

(b)($\rightarrow$)Suppose $A\langle\rightarrow\rangle B \in \Gamma$. Then by (a) we have $(\mathcal{O}p \wedge A)\langle\rightarrow\rangle(\mathcal{O}q \wedge B) \in \Gamma$. Applying (D6) (2.3) we obtain:

$\langle U\rangle(\mathcal{O}p \wedge A) \in \Gamma$, $\langle U\rangle(\mathcal{O}q \wedge B) \in \Gamma$, $\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q \in \Gamma$.

From $\langle U\rangle(\mathcal{O}p \wedge A) \in \Gamma$, by 3.6 (ii) we get:
$(\exists\Delta \in W_L)(\Gamma R_U \Delta$ & $\mathcal{O}p \wedge A \in \Delta)$, and hence $\mathcal{O}p \in \Delta$ and $A \in \Delta$.

In the same way, from $\langle U\rangle(\mathcal{O}q \wedge B) \in \Gamma$ we obtain:
$(\exists\Theta \in W_L)(\Gamma R_U \Theta$ & $\mathcal{O}q \wedge B \in \Theta)$ and hence $\mathcal{O}q \in \Theta$ and $B \in \Theta$.

Now from $\mathcal{O}p \in \Delta$, $\mathcal{O}q \in \Theta$ and $\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q \in \Gamma$ we obtain $\Gamma R_{\rightarrow}\Delta\Theta$.

So we have just proved that $A\langle\rightarrow\rangle B \in \Gamma$ implies that for some $\Delta, \Theta \in W_L$ we have: $\Gamma R_{\rightarrow}\Delta\Theta$, $A \in \Delta$ and $B \in \Theta$.

($\leftarrow$) Suppose $\Gamma R_{\rightarrow}\Delta\Theta$, $A \in \Delta$ and $B \in \Theta$. Then for some $p, q \in VAR$ we have:

$\mathcal{O}p \in \Delta$, $\mathcal{O}q \in \Theta$ and $\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q \in \Gamma$, $\Gamma R_U \Delta$ and $\Gamma R_U \Theta$.

From here we obtain: $\mathcal{O}p \wedge A \in \Delta$ and by $\Gamma R_U \Delta$ — $\langle U\rangle(\mathcal{O}p \wedge A) \in \Gamma$. In the same way we obtain $\langle U\rangle(\mathcal{O}q \wedge B) \in \Gamma$. Consequently we have: $\langle U\rangle(\mathcal{O}p \wedge A) \wedge \langle U\rangle(\mathcal{O}q \wedge B) \wedge (\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q) \in \Gamma$. Applying D6 (2.3) we obtain $(\mathcal{O}p \wedge A)\langle\rightarrow\rangle(\mathcal{O}q \wedge B) \in \Gamma$ and by the monotonicity of $\langle\rightarrow\rangle$ we obtain $A\langle\rightarrow\rangle B \in \Gamma$.

(ii) (Uniqueness of Γ in $\Gamma R_{\rightarrow}\Delta\Theta$) Suppose $\Gamma R_{\rightarrow}\Delta\Theta$ and $\Gamma' R_{\rightarrow}\Delta\Theta$. Then for some $p, q, p', q' \in VAR$ we have: $\mathcal{O}p \in \Delta$, $\mathcal{O}q \in \Theta$, $\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q \in \Gamma$, $\Gamma R_U \Delta$, $\Gamma R_U \Theta$,

$\mathcal{O}p' \in \Delta$, $\mathcal{O}q' \in \Theta$, $\mathcal{O}p'\langle\rightarrow\rangle\mathcal{O}q' \in \Gamma$, $\Gamma' R_U \Delta$, $\Gamma' R_U \Theta$.

From these, by the *Witness lemma* we obtain that $[U](\mathcal{O}p \Leftrightarrow \mathcal{O}p') \wedge [U](\mathcal{O}q \Leftrightarrow \mathcal{O}q') \in \Gamma'$. From that and $\mathcal{O}p'\langle\rightarrow\rangle\mathcal{O}q' \in \Gamma'$ we obtain by the *Strong replacement lemma* (applied twice consecutively) that $\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q \in \Gamma'$ and by axiom (Bool2) we obtain that $\mathcal{O}(\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q) \in \Gamma'$ and (since $\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q \in \Gamma$) $\mathcal{O}(\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q) \in \Gamma$. But we have also $\Gamma R_U \Gamma'$ which by lemma 3.6 (iv) implies $\Gamma = \Gamma'$.

(Existence of Γ in $\Gamma R_{\rightarrow}\Delta\Theta$) Suppose $\Delta R_U \Theta$. Then for some Γ' we have $\Gamma' R_U \Delta$ and $\Gamma' R_U \Theta$. By the *Witness lemma* we have $\mathcal{O}p \in \Delta$, and $\mathcal{O}q \in \Theta$ for some variables p, q. From here we obtain $\langle U\rangle\mathcal{O}p \in \Gamma'$ and $\langle U\rangle\mathcal{O}q \in \Gamma'$. Then by axiom (Bool1) we obtain $\langle U\rangle(\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q) \in \Gamma'$. By 3.6 (ii) there exist $\Gamma \in W_L$ such that $\Gamma' R_U \Gamma$ and $(\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q) \in \Gamma$. We obtain also $\Gamma R_U \Delta$ and $\Gamma R_U \Theta$. Then $(\mathcal{O}p\langle\rightarrow\rangle\mathcal{O}q) \in \Gamma$, $\mathcal{O}p \in \Delta$, $\mathcal{O}q \in \Theta$, $\Gamma R_U \Delta$ and $\Gamma R_U \Theta$ imply $\Gamma R_{\rightarrow}\Delta\Theta$ and the lemma is proved. ■

Let $\Omega \in W_L$ and denote by W_Ω the R_U-equivalence class determined by Ω. By 3.6 (v) there exists unique $\Delta \in W_L(\Omega)$ such that $\langle 0\rangle \in \Delta$. We denote this Δ by 0_Ω. By 3.6 (vii) the restriction of the relation $R_{\neq}$ in W_Ω is the inequality relation $\neq$. By 3.7 (ii) for any $\Delta, \Theta \in W_\Omega$ there exists unique $\Gamma \in W_\Omega$ such that $\Gamma R_{\to} \Delta\Theta$. We denote this unique Γ by $\Delta \to_\Omega \Theta$. In this way W_Ω can be considered as an algebraic system $(W_\Omega, O_\Omega, \to_\Omega)$ with one zero-argument operation O_Ω and one two-argument operation $\to_\Omega$. We will call this system generated (by Ω) canonical structure (GCS) of HBML.

Lemma 3.8 (GCS-lemma) *Let $(W_\Omega, O_\Omega, \to_\Omega)$ be GCS for HBML. Then:*

(i) Let $\Gamma \in W_\Omega$ and A, B be formulas. Then the following equivalences are true:

$(\langle\neq\rangle)$ $\quad \langle\neq\rangle A \in \Gamma$ *iff* $(\exists \Delta \in W_\Omega(\Delta \neq \Gamma \ \& \ A \in \Delta)$,

$(\langle\to\rangle)$ $\quad A\langle\to\rangle B \in \Gamma$ *iff* $(\exists \Delta, \Theta \in W_\Omega)(\Gamma = \Delta \to_\Omega \Theta \ \& \ A \in \Delta \& B \in \Theta)$,

$(\langle 0\rangle)$ $\quad \langle 0\rangle \in \Gamma$ *iff* $\Gamma = 0_\Omega$.

(ii) There exists unique element $1_\Omega \in W_\Omega$ such that

$(\langle 1\rangle)$ $\quad \langle 1\rangle \in \Gamma$ *iff* $\Gamma = 1_\Omega$

(iii) W_Ω is a Boolean algebra.

Proof. (i) and (ii) follow from 3.6 and 3.7 and the definition of GCS. To proof (iii) we have to verify the axioms B1–B5 of Boolean algebra. As an example we will verify axiom B1 (in the derivations we will omit the subscript Ω in $\to_\Omega, 0_\Omega$ and 1_Ω):

B1: $\Gamma \to (\Delta \to \Gamma) = 1$.

Let $\Delta \to \Gamma = \Theta$ and $\Gamma \to \Theta = \Psi$, we have to show that $\Psi = 1$. By the witness lemma there exist variables p, q such that $\mathcal{O}p \in \Gamma$ and $\mathcal{O}q \in \Delta$. Then by (i) $(\mathcal{O}q\langle\to\rangle\mathcal{O}p) \in \Theta$ and $(\mathcal{O}p\langle\to\rangle(\mathcal{O}q\langle\to\rangle\mathcal{O}p)) \in \Psi$. By axiom (Bool5) we obtain $\langle 1\rangle \in \Psi$ and by (ii) $\Psi = 1$. Hence $\Gamma \to (\Delta \to \Gamma) = 1$. The axioms B2–B5 can be proved in a similar way. ■

By a generated canonical model (GCM) for HBML we mean any GCS W_Ω with the following canonical valuation

$v_\Omega(p) = \{\Gamma \in W_\Omega / p \in \Gamma\}, p \in \mathit{VAR}$.

Lemma 3.9 (Truth Lemma) *Let $M = (W_\Omega, v_\Omega)$ be a GCM for HBML, $v = v_\Omega$ and $\Gamma \in W_\Omega$. Then for any formula A we have:*

$\Gamma \vdash_v A$ *iff* $A \in \Gamma$.

Proof. By induction on the complexity of A and 3.8 (i) and (ii). ■

Lemma 3.10 (GCM-lemma) *The following two conditions are equivalent for any formula A of HBML:*

(i) A is a theorem of HBML,

(ii) A is true in any generated canonical model $M = (W_\Omega, v_\Omega)$.

Proof. $(i) \to (ii)$ follows from the soundness theorem.

$(ii) \to (i)$ Suppose that A is not a theorem of HBML. Then by the corollary of Lindenbaum lemma (3.4) there exists $\Omega \in W_L$ such that $A \notin \Omega$. Let $M = (W_\Omega, v_\Omega)$ be the GCM determined by Ω and $v = v_\Omega$. Then by the Truth Lemma $\Omega \not\Vdash_v A$ and hence A is not true in M. This by contraposition shows that (ii) implies (i). ■

Now we are ready for the main theorem of this paper.

Theorem 3.11 (Completeness Theorem for HBML) *The following are equivalent for any formula A in HBML:*

(i) A is a theorem of HBML,

(ii) A is valid in all Boolean algebras,

(iii) A is valid in all hyperframes.

Proof. $(i) \to (ii)$ — This is just the Soundness Theorem for HBML.

$(ii) \to (i)$ Suppose A is valid in all Boolean algebras. Then A is valid in all generated canonical models for HBML and by the GCM-lemma A is a theorem of HBML.

$(ii) \to (iii)$ — Obvious.

$(iii) \to (ii)$ — By the representation theorem for Boolean algebras. ■

Theorem 3.12 *HBML does not have the finite model property.*

Proof. Let us denote $\Diamond X = X\langle\vee\rangle\langle 1\rangle$. In every HBA $\Diamond X = \{x | \exists y \in X(x \geq y)\}$, i.e. $\Diamond X$ is a modal operator with a Kripke semantics corresponding to the partial ordering $\geq$ in the underlying BA. It is known (see [Bull and Segerberg 84]) that the modal logic of all partial orderings is S4, while the one of all *finite* partial orderings is a proper extension of S4 with Grzegorczyk's formula **Grz**:

$$\Box(\Box(p \to \Box p) \to p) \to p$$

where $\Box$ is the dual of $\Diamond$. Therefore that formula is valid on the class of all finite HBAs while it fails e.g. in $\mathcal{P}(\mathcal{P}(\mathbf{N}))$. ■

4 Concluding remarks

We shall conclude this paper with some specific and some general remarks and open questions.

The theory of hyperboolean algebras seem much more complex than the theory of Boolean algebras, both technically and computationally. Let us first note that the modal logic we have introduced encapsulates, via the standard translation a la van Benthem and the encoding of the lower level theory into the upper level by using the operator $\mathcal{O}$, the universal fragment of the monadic second-order theory of Boolean algebras. Our logic thus provides an axiomatization of that fragment, and therefore and explicit proof of its recursive enumerability which, as van Benthem has noted, is a priori predictable. Furthermore, the full first-order theory of hyperboolean algebras likewise encapsulates the full monadic second-order theory of Boolean algebras, which is know to be undecidable, whence the undecidability of the former, in contrast to the decidability of the first-order theory of Boolean algebras. It is however still an open problem for us whether HBML is decidable. In view of the lack of the finite model property, and the considerable complexity of the semantics, the negative answer is most likely.

Further, a natural problem arises to axiomatize the class of *finite* HBAs. Taking into account the fact that the logic S4 extended with Grzegorczyk's formula axiomatizes completely the class of all finite partial orderings, combined with the fact that every (finite) partial ordering is embeddable into an ordering of type $(P(X), \subseteq)$ gives some hope that adding Grzegorczyk's formula to our system will produce the desired complete axiomatization, but this is still an open problem. Some related ones:

- If recursively axiomatized, the logic of finite HBAs will be decidable. What is its complexity?
- Are the IRR rules in HBML replaceable by finitely many axioms?

A more ambitious program is to axiomatize and study the first-order theory of HBAs. It can be expressed in the first-order language of HBAs that

the underlying structure is an atomic Boolean algebra, which is "definably" complete, i.e. every definable set of elements has a supremum. Furthermore, by enforcing the "fullness" of the Boolean algebra on the upper level as much as possible, more sets of elements of the lower level are made available, which further enforces the completeness of the lower Boolean algebra, etc. How close does that describe the full fields of sets? Some model theoretic constructions, particularly techniques from [Doets 87] can possibly answer that question.

Let us note that from algebraic viewpoint, it is more natural to consider the class of all subalgebras of hyperboolean algebras and reserve the term "hyperboolean algebras" for them, while calling those isomorphic to a subalgebra of $\mathcal{P}(\mathcal{P}(X))$ e.g. *full* hyperboolean algebras. Thus the class of all full HBAs will have the same universal theory as the class of all $\mathcal{P}(\mathcal{P}(X))$, and will therefore be determined by the set of identities valid in the latter class. Likewise, the class of all HBAs will have the same universal theory as the class of all $\mathcal{P}(A)$ for any Boolean algebra A (we can now call these *hyperframes*). A number of open questions arise here:

- First of all, it is not known to us yet if the identities valid in the class of all HBAs are the same as those valid in the subclass of all full HBAs. Note that although every Boolean algebra A is embeddable into a field of sets $\mathcal{P}(X)$, the HBA $\mathcal{P}(A)$ is generally not embeddable into $\mathcal{P}(\mathcal{P}(X))$, neither reducible to it by means of any other obvious construction known to us that transfers validity of identities. That stratification, of course, only occurs in the classes of (possibly) infinite HBAs.
- The class of all HBAs, being a discriminator variety, is a priori known to have some nice properties, but its algebraic theory is still obscure. Are its identities finitely based? (This question is, of course, closely related to the one about eliminability of the IRR rules from HBML).
- Is the class of full HBAs a variety? Are its identities finitely based? Are the identities of all finite HBAs finitely based?
- An outstanding problem here, related to all above, is to find good representation results for (full) HBAs, or a natural class of representable ones.

We now turn to some general remarks and questions.

In this paper we have axiomatized the modal logic of "complexes" of specific structures, viz. Boolean algebras. It may seem at first sight that the nature of these structure has much to do with the logic, but this is not so. The interaction between the internal Boolean operations and the external ones is entirely superficial and technical one, and the situation rather resembles the

study of the first-order theory of Boolean algebras where they are first-order algebraic structures just like e.g. groups. It is therefore natural to consider the more general situation: given a class of (first-order) structures $\mathbf{K}$, regard them as frames, with accessibility relations corresponding to their basic functions and predicates. The class of resulting frames provides a semantics for a "modal logic of complexes of $\mathbf{K}$-structures" or, in our terminology, "hyper-$\mathbf{K}$ modal logic". Of course, the term "modal" here has a purely technical meaning, standing for the fragment of the monadic second-order theory of those structures, obtained by the standard translation from the propositional language of modal logic. From this viewpoint it is natural to look for generalizations of our axiomatization. As we have already noted, what makes our axiomatic system tick is the availability of the operator $\mathcal{O}$, by means of which the universal first-order theory of the underlying structures is readily encoded into the logic, and that takes care of the "algebraic" part of the axiomatization. It is therefore natural to expect that the modal logic of "hyper-groups", "hyper-rings", etc. structures where that operator is definable can be uniformly axiomatized in the same style. For instance, here is a definition (not the shortest one, but a straightforward adaptation of the one for Boolean algebras we use) of the "difference" operator for hyper-groups (where $e, ', \circ$ are resp. the group identity, inverse, and multiplication):[1]

$$\langle \neq \rangle A =$$
$$(\neg A \wedge ((A\langle \circ \rangle A^{(')})\langle \circ \rangle \top)) \vee ((((A\langle \circ \rangle A^{(')}) \wedge \neg\langle e \rangle)\langle \circ \rangle((A\langle \circ \rangle A^{(')}) \wedge \neg\langle e \rangle)^{(')})\langle \circ \rangle \top)$$

This technique is not directly applicable to other structures where the operator $\mathcal{O}$ does not seem to be defined, like semigroups and lattices. It is therefore worth exploring the boundaries of the applicability of our technique. It seems natural that they will have to do with presence or absence of sort of "residuals" of at least one of the basic operations.

Of course, the technique for axiomatization exploited in this article is not the only conceivable one. For simple enough structures all axioms may turn out to be modally definable and then a straightforward axiomatization available.

Finally, the model theory of "hyper-structure logics" naturally rests upon the theory of varieties of complex algebras (see [Goldblatt 89]), properly expanded to account for the complexes of structures with functions and relations. Such an expansion seems worth exploring.

[1] Yde Venema has meanwhile suggested a shorter and more elegant definition for "difference" in hyper-groups, which accordingly renders an alternative definition of that operator in hyperboolean algebras since every Boolean algebra can be regarded as a group with respect to the symmetric difference.

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