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Hierarchies of Modal and Temporal Logics with Reference Pointers

VALENTIN GORANKO
Department of Mathematics, Rand Afrikaans University, P.O. Box 524, Auckland Park 2006, Johannesburg, South Africa

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Abstract. We introduce and study hierarchies of extensions of the propositional modal and temporal languages with pairs of new syntactic devices: "point of reference – reference pointer" which enable semantic references to be made within a formula. We propose three different but equivalent semantics for the extended languages, discuss and compare their expressiveness. The languages with reference pointers are shown to have great expressive power (especially when their frugal syntax is taken into account), perspicuous semantics, and simple deductive systems. For instance, Kamp's and Stavi's temporal operators, as well as nominals (names, clock variables), are definable in them. Universal validity in these languages is proved undecidable. The basic modal and temporal logics with reference pointers are uniformly axiomatized and a strong completeness theorem is proved for them and extended to some classes of their extensions.

Key words: Modal and temporal logics, reference pointers, expressiveness, axiomatization, completeness

1. Introduction

The rapidly expanding scope of applications (actual or potential) of modal and temporal logics to theoretical computer science and artificial intelligence demands, inter alia, strengthening of the expressive power of their languages to make them really appropriate tools for adequate treatment of various phenomena, while keeping a relatively simple and efficient proof mechanism, convenient for applications and for implementation of automated deduction systems. This demand is particularly relevant for propositional languages. Their most valuable assets are the perspicuity of the syntax and deductive apparatus on the one hand, and on the other hand their *intensional* semantic nature which allows for representation of sophisticated first- or higher-order schemata on a propositional syntactic level. These two assets are in mutual controversy, reflecting the fundamental controversy in logic: "expressiveness vs. tractability". A number of languages and systems have been devised in seeking the best compromise in this controversy.

This article is intended as a further contribution to this trend. It proposes relatively simple but particularly strong extensions of modal and temporal logics (likewise applicable to dynamic and all other multimodal logics, too). As it usually happens, this enterprise was motivated from dissatisfaction with the expressiveness of the classical modal and temporal languages. One of their major drawbacks, at least from the point of view of natural language, is the lack of means to make references to points of the model (possible worlds, time instants, etc.), somehow specified in the formal context. For instance, one cannot say "now", "then", "when" in the classical temporal language. (For thorough discussion on "now" and "then" the reader is referred to e.g. Kamp (1971) and Vlach (1973).) As a consequence, very basic and natural features of the temporal frame cannot be expressed syntactically. To give just one example, the simple fact that "*now* will never occur again", or formally, that the flow of time is acyclic, is beyond the expressive abilities of the classical temporal language.

Various enrichments of modal and temporal languages have been proposed to improve its expressiveness. Let us mention three basic types of enrichments. The first one extends the language with new particular *operators* motivated from the specific semantics, e.g. Kamp's binary temporal modalities *Since* and *Until* and the more sophisticated Stavi's connectives $\mathbf{U}'(p,q)$ and $\mathbf{S}'(p,q)$ (see e.g. Burgess, 1982, 1984; and Gabbay, 1981b; see the latter as well for a general approach to this type of enrichments of the temporal language). The second one provides the language with new *sorts* of syntactic objects, (constants, restricted variables etc.), again naturally arising in the semantic framework. A characteristic example are Prior's "clock-variables", (Prior, 1967); (see also Bull, 1970; the *constants* in Passy and Tinchev, 1991; the *nominals* in Blackburn, 1989; and the *names* in Gargov and Goranko, 1993) which are bound to be true at *exactly one instant* of the flow of time. The third type of enrichment is implicit: the language remains the same but new rules of inference are added which are intended to depict semantic features not expressible by means of formulae by restricting the class of models to those for which the new rules are sound, and thus discarding unwanted non-standard models and therefore increasing the expressiveness of the language. A notable example here is Gabbay's "irreflexivity rule" (Gabbay, 1981a); see also Gabbay and Hodkinson (1990), Passy and Tinchev (1991), Gargov and Goranko (1993), Venema (1993).

What is proposed in this article can be attributed to each of these types of enrichments. We extend the language with a specific syntactic device intended to enable making references in the model but the result turns out to be a significant general improvement of the expressiveness of the language. The idea in a nutshell can be explained as follows. The process of evaluation of a formula consists of shuttling to and fro in the model and evaluating subformulae at points of that model. Thus, a snapshot of that process looks like this: the evaluating device is positioned at a point x and is evaluating certain subformula ϕ at that point (by looking at the main connective and taking an according action). Suppose, further in the process of evaluation we want to make references to that point x, saying for instance "A is true here if B was true at x". In order to do so we have prefixed the subformula ϕ by a *point of reference* $\downarrow$ thus telling the evaluating device "*Now*, remember this point" and it does so, after which proceeds to evaluate ϕ. Hereafter, whenever in that process we want to refer to the point x we put an atomic symbol $\uparrow$, called a *reference pointer* saying "being at the point indicated by $\downarrow$" (in a temporal setting, simply "*then*, or *at that instant*"), which the device will judge "true" if and only if it is positioned again at the point x. Schematically it looks like this: $\ldots \downarrow (\ldots \uparrow \ldots \uparrow \ldots \uparrow \ldots) \ldots$. For instance $\downarrow \mathbf{G} \neg \uparrow$ says "*Now* will never occur again". This construction can be iterated, e.g $\ldots \downarrow (\ldots \uparrow \ldots \downarrow (\ldots \uparrow \ldots) \ldots \uparrow \ldots) \ldots$ etc. The language can be further extended with more than one reference pairs "now$_i$-then$_i$" thus enabling cross-references. (Actually, the reference abilities of a language with more than two reference pairs already go beyond those of the utilized fragment of the natural language.) Adding λ reference pairs, for $\lambda = 1, 2, 3, \ldots, \omega$ to the modal or temporal language we obtain two hierarchies of increasingly expressive language. To indicate how strong they are it suffices to say that even one reference pair added to the temporal language makes it rich enough both to express Kamp's and Stavi's operators, and to simulate Prior's clock-variables.

Let us note that the idea of reference pointers, being a quite natural one, has been worked out by other authors, too, e.g in Gabbay's many-dimensional connectives (Gabbay, 1981b) and in the temporal logic IQ in Richards *et al.* (1989). Although employing different formalisms, the various approaches pursue similar ideas. Their relationships will be discussed elsewhere. Reference pointers bear also certain familiarity with nominals (names, constants) and can be partly simulated by quantifying over them (for temporal and dynamic logics with quantifiers over nominals see Bull (1970), Passy and Tinchev (1985, 1991).

In the article we introduce the two hierarchies: of modal languages $\mathcal{L}\updownarrow_m^\lambda$, and of temporal languages $\mathcal{L}\updownarrow_t^\lambda$, their semantics, discuss their expressiveness and prove that they have undecidable satisfiability problem. Then we give axiomatizations of their basic logics $\mathbf{K}\updownarrow_m^\lambda$ and $\mathbf{K}\updownarrow_t^\lambda$ for which we prove a strong completeness theorem and generalize this result to some classes of their extensions.

The reader is assumed to have some background in modal and temporal propositional logics (syntax, semantics, deductive systems and completeness theorem) within the bounds of e.g. Goldblatt (1987).

2. Syntax

The languages $\mathcal{L}_m$ and $\mathcal{L}_t$ of the propositional modal, resp. temporal, logic contain a countable set $P = \{p_1, p_2, \ldots\}$ of propositional variables, logical constants $\bot, \top$, connectives $\neg$ and $\wedge$, and a modality $\square$, resp. two temporal modalities $\mathbf{G}$ ("always in the future") and $\mathbf{H}$ ("always in the past"). The symbols $\vee, \rightarrow, \leftrightarrow, \diamond$, $\mathbf{F}$ ("sometime in the future") and $\mathbf{P}$ ("sometime in the past") are definable in a standard way. The temporal language $\mathcal{L}_t$ will be regarded as an extension of the modal language $\mathcal{L}_m$ by identifying $\square$ (resp. $\diamond$) with $\mathbf{G}$ (resp. $\mathbf{F}$).

We now introduce hierarchies of extensions of these languages:

$$\mathcal{L}_m \subset \mathcal{L}\updownarrow_m^1 \subset \mathcal{L}\updownarrow_m^2 \subset \ldots \subset \mathcal{L}\updownarrow_m^n \subset \ldots \subset \mathcal{L}\updownarrow_m^\omega,$$

and

$$\mathcal{L}_t \subset \mathcal{L}\updownarrow_t^1 \subset \mathcal{L}\updownarrow_t^2 \subset \ldots \subset \mathcal{L}\updownarrow_t^n \subset \ldots \subset \mathcal{L}\updownarrow_t^\omega,$$

where, for $\lambda = 1, 2, \ldots, \omega$, $\mathcal{L}\updownarrow_m^\lambda$ extends $\mathcal{L}_m$ with:

- a *universal modality* $\mathbf{A}$ ("always"), whose dual $\neg\mathbf{A}\neg$ is denoted by $\mathbf{E}$ ("sometime");
- and λ pairs of symbols $\{\downarrow_k$ *(point of reference)*, $\uparrow_k$ *(reference pointer)*$\}$, for $k < \lambda$.

$\mathcal{L}\updownarrow_t^\lambda$ extends $\mathcal{L}_t$ likewise.

Syntactically the reference pointer symbols $\uparrow_k$ behave like propositional variables, while the point of reference symbols $\downarrow_k$ are unary connectives which resemble quantifiers binding $\uparrow_k$.

The recursive definition of formulae in $\mathcal{L}_m$, (resp. in $\mathcal{L}_t$), is extended to $\mathcal{L}\updownarrow_m^\lambda$ (resp. to $\mathcal{L}\updownarrow_t^\lambda$) with the following clauses:

- $\uparrow_k$ is a formula, for $k < \lambda$;
- If φ is a formula then $\mathbf{A}\varphi$ is a formula;
- If φ is a formula then $\downarrow_k \varphi$ is a formula for any $k < \lambda$.

We need a few syntactic notions borrowed from first-order logic:

The first occurrence of $\downarrow_k$ in the formula $\downarrow_k \varphi$ has a *scope* φ.

An occurrence of $\uparrow_k$ in a formula φ is *bound* if it is in the scope of an occurrence of $\downarrow_k$; otherwise it is *free*.

If φ and ψ are formulae, $\varphi(\psi/\uparrow_k)$ denotes the result of simultaneous substitution of all free occurrences of $\uparrow_k$ in φ by ψ.

A formula φ is *closed* if there are no free occurrences of $\uparrow$'s in φ.

The complexity of a formula φ of $\mathcal{L}\updownarrow_t^\lambda$ is the number of logical connectives (including $\mathbf{A}$, $\uparrow$'s and $\downarrow$'s) in φ.

The reference depth of a formula φ is the largest number $r(\varphi)$ of nested occurrences of $\downarrow$'s in φ.

3. Semantics

We shall introduce relational (Kripke) semantics for the languages $\left\{\mathcal{L}\updownarrow_m^\lambda\right\}_{\lambda\leq\omega}$ and $\left\{\mathcal{L}\updownarrow_t^\lambda\right\}_{\lambda\leq\omega}$ in a uniform way, using as a paradigm $\mathcal{L}\updownarrow_t^\lambda$. The notions of *frame*, *valuation* and *model* are the standard ones. Given a model $\mathcal{M} = \langle T, R, V\rangle$ and a point (*world, instant*) $t \in T$ we have a recursive definition of truth $\mathcal{M} \models \varphi[t]$ for all formulae of $\mathcal{L}_t$, which we want to extend over the new symbols. However, we have no suitable way to define truth at a point for non-closed formulae of $\mathcal{L}\updownarrow_t^\lambda$, since that truth would depend on the "points of reference" which are not determined if the formula is not closed. We shall propose three ways to avoid this obstacle, which eventually yield the same semantics, but illuminated from different perspectives.

3.1. SEMANTICS I: STANDARD TRANSLATION ST

In order to define truth at a point of a model we extend the well-known standard translation ST, see van Benthem (1983), of modal and temporal languages as follows: Let L_1 be the first-order language containing a binary predicate R and a countable set of unary predicates $\{P_1, P_2, \ldots\}$. For convenience we split the set of individual variables of L_1 into two disjoint subsets: $W = \{w_k\}_{k<\lambda}$ and $Y = \{x, y_0, y_1, y_2, \ldots\}$ where each of x and w's will play special roles, viz.:

> x will represent the actual point in time (the current "now"), and w_k will represent the point of reference ("then$_k$)".

Now we define the standard translation ST inductively:

1. $ST(p_i) = P_i x$,
2. $ST(\uparrow_k) = (x = w_k)$,
3. $ST(\neg\varphi) = \neg ST(\varphi)$,
4. $ST(\varphi \wedge \psi) = ST(\varphi) \wedge ST(\psi)$,
5. $ST(\mathbf{G}\varphi) = \forall y(Rxy \rightarrow ST(\varphi)(y/x))$,
6. $ST(\mathbf{H}\varphi) = \forall y(Ryx \rightarrow ST(\varphi)(y/x))$,
7. $ST(\mathbf{A}\varphi) = \forall y(ST(\varphi)(y/x))$,
8. $ST(\downarrow_k \varphi) = ST(\varphi)(x/w_k)$.

In 5, 6, and 7 above y is the first variable from Y, different from x and not occurring in $ST(\varphi)$; u/v means uniform substitution of u for all free occurrences of v.

Note that if φ is in $\mathcal{L}\updownarrow^k_m$ or $\mathcal{L}\updownarrow^k_t$ then x and $w_1, \ldots, w_k$ can only have free occurrences in $ST(\varphi)$, where they are the only possibly free variables. Furthermore, φ is closed if and only if no w_i occurs in $ST(\varphi)$.

The model $\mathcal{M} = \langle \mathbf{T}, \mathbf{R}, \mathbf{V} \rangle$ can be regarded as an L_1-model where R is interpreted by $\mathbf{R}$ and P_i by $\mathbf{V}(p_i), i = 0, 1, 2, \ldots$. In order to distinguish validity in $\mathcal{M}$ as an L_1-model from validity in $\mathcal{M}$ as a Kripke model we shall use the symbol $\vdash$ for the former case and $\models$ for the latter. Now we define truth at a point for any *closed* formula φ:

$$\mathcal{M} \models_1 \varphi[s] \text{ if } \mathcal{M} \Vdash ST(\varphi)(s/x),$$

and then validity in a model:

$$\mathcal{M} \models_1 \varphi \text{ if } \mathcal{M} \models_1 \varphi[t] \text{ for every } t \in \mathbf{T}, \text{ i.e. if } \mathcal{M} \Vdash \forall x ST(\varphi).$$

Finally, φ is *valid in a frame* if it is valid in every model on the frame, and φ is (*universally*) *valid* if it is valid in every Kripke frame.

Here we only define validity for closed formulae since only they have a determined meaning, and we shall not be interested in non-closed formulae on their own.

3.2. Semantics II: Extensional of a Formula at a Point

Another way to introduce the semantics of languages with reference pointers is by introducing the *extensional of a formula* φ *at a point* s *of a model* $\mathcal{M} = \langle \mathbf{T}, \mathbf{R}, \mathbf{V} \rangle$, denoted by $\mathcal{E}(\mathcal{M}, s, \varphi)$, which is a subset of W^λ defined inductively as follows:

1. $\mathcal{E}(\mathcal{M}, s, p_i) = \begin{cases} W^\lambda & \text{if } s \in V(p_i), \\ \emptyset, & \text{otherwise} \end{cases}$
2. $\mathcal{E}(\mathcal{M}, s, \uparrow_k) = \{(v_1, \ldots, v_k, \ldots) : v_k = s\}$,
3. $\mathcal{E}(\mathcal{M}, s, \neg\varphi) = W^\lambda \setminus \mathcal{E}(\mathcal{M}, s, \varphi)$,
4. $\mathcal{E}(\mathcal{M}, s, \varphi \wedge \psi) = \mathcal{E}(\mathcal{M}, s, \varphi) \cap \mathcal{E}(\mathcal{M}, s, \psi)$,
5. $\mathcal{E}(\mathcal{M}, s, \mathbf{G}\varphi) = \bigcap_{sRt} \mathcal{E}(\mathcal{M}, t, \varphi)$,
6. $\mathcal{E}(\mathcal{M}, s, \mathbf{H}\varphi) = \bigcap_{tRs} \mathcal{E}(\mathcal{M}, t, \varphi)$,
7. $\mathcal{E}(\mathcal{M}, s, \mathbf{A}\varphi) = \bigcap_{t \in W} \mathcal{E}(\mathcal{M}, t, \varphi)$,
8. $\mathcal{E}(\mathcal{M}, s, \downarrow_k \varphi) = \{(v_1, \ldots, v_k, \ldots) : (v_1, \ldots, v_{k-1}, s, v_{k+1}, \ldots) \in \mathcal{E}(\mathcal{M}, s, \varphi)\}$.

Now, we define

$$\mathcal{M} \models_2 \varphi[s] \text{ if } \mathcal{E}(\mathcal{M}, s, \varphi) = W^\lambda$$

Validity in a model and in a frame are now defined as before.

Intuitively, $\mathcal{E}(\mathcal{M}, s, \varphi)$ consists of those strings $(v_1, \ldots, v_k, \ldots)$ which, if taken as strings of reference points for $\uparrow_1, \ldots, \uparrow_k, \ldots$ respectively, would render the formula φ true at s. Note that the extensional of a closed formula is either W^λ or $\emptyset$.

The advantage of this approach is that it defines truth of *all* formulae, not only the closed ones; the disadvantage is in its unfitness for syntactic manipulations.

3.3. Semantics III: Formulae with Parameters

Yet another approach, similar to the previous one is to define truth of *formulae with parameters*. This approach reveals familiarity with Gabbay's "many-dimensional connectives", see Gabbay (1981b).

Now we define inductively truth of a formula φ at a point s of a model $\mathcal{M} = \langle T, R, V \rangle$ *with respect to parameters* $v_1, v_2, \ldots \in T$, denoted $\mathcal{M} \models \varphi[s; v_1, v_2, \ldots]$ as follows:

1. $\mathcal{M} \models p_i[s; v_1, v_2, \ldots]$ if $s \in V(p_i)$,
2. $\mathcal{M} \models \uparrow_k [s; v_1, v_2, \ldots]$ if $v_k = s$,
3. $\mathcal{M} \models \neg\varphi[s; v_1, v_2, \ldots]$ if not $\mathcal{M} \models \varphi[s; v_1, v_2, \ldots]$,
4. $\mathcal{M} \models (\varphi \wedge \psi)[s; v_1, v_2, \ldots]$ if $\mathcal{M} \models \varphi[s; v_1, v_2, \ldots]$ and $\mathcal{M} \models \psi[s; v_1, v_2, \ldots]$.
5. $\mathcal{M} \models \mathbf{G}\varphi[s; v_1, v_2, \ldots]$ if $\forall t(sRt \Rightarrow \mathcal{M} \models \varphi[t; v_1, v_2, \ldots])$,
6. $\mathcal{M} \models \mathbf{H}\varphi[s; v_1, v_2, \ldots]$ if $\forall t(tRs \Rightarrow \mathcal{M} \models \varphi[t; v_1, v_2, \ldots])$,

7. $\mathcal{M} \models \mathbf{A}\varphi[s; v_1, v_2, \ldots]$ if $\forall t(\mathcal{M} \models \varphi[t; v_1, v_2, \ldots])$,
8. $\mathcal{M} \models\downarrow_k \varphi[s; v_1, v_2, \ldots]$ if $\mathcal{M} \models \varphi[s; v_1, \ldots, v_{k-1}, s, v_{k+1}, \ldots]$.

Clearly, the truth of a formula only depends on those parameters which correspond to reference pointers having free occurrences in the formula; in particular the truth of a closed formula does not depend on the parameters at all. We can therefore define for any closed formula φ:

$$\mathcal{M} \models_3 \varphi[s] \text{ if } \mathcal{M} \models \varphi[s; v_1, v_2, \ldots] \text{ for any string } v_1, v_2, \ldots \in \mathcal{M}.$$

Again, validity in models and frames is standard.

Remark 1. Although for uniformity we write infinite strings, when introducing semantics for $\mathcal{L}\updownarrow_m^\lambda$ or $\mathcal{L}\updownarrow_t^\lambda$ we use strings of length λ.

3.4. Equivalence of the Three Semantics

THEOREM 2.

1. *For every formula with reference pointers φ, model $\mathcal{M}$, point $s \in \mathcal{M}$, and string of points $v_1, v_2, \ldots \in \mathcal{M}$, the following are equivalent:*
   (a) $\mathcal{M} \Vdash ST(\varphi)(s, v_1, v_2, \ldots)$, *where $s, v_1, v_2, \ldots$ substitute respectively the free variables $x, w_1, w_2, \ldots$ in $ST(\varphi)$.*
   (b) $(v_1, v_2, \ldots) \in \mathcal{E}(\mathcal{M}, s, \varphi)$.
   (c) $\mathcal{M} \models_3 \varphi[s; v_1, v_2, \ldots]$.
2. *For every closed formula with reference pointers φ, model $\mathcal{M}$, and a point $s \in \mathcal{M}$ the following are equivalent:*
   (a) $\mathcal{M} \models_1 \varphi[s]$.
   (b) $\mathcal{M} \models_2 \varphi[s]$.
   (c) $\mathcal{M} \models_3 \varphi[s]$.

Proof.
1. A routine induction on φ.
2. Follows from 1.

Hereafter we shall only write $\models$ meaning e.g. $\models_1$. Besides, when a formula contains only one pair of reference pointers their indices will often be omitted when irrelevant.

4. Notes on Expressiveness and Definability in $\mathcal{L}\updownarrow_m^\lambda$ and $\mathcal{L}\updownarrow_t^\lambda$

THEOREM 3. *The languages $\mathcal{L}\updownarrow_m^\lambda$ and $\mathcal{L}\updownarrow_t^\lambda$ are ordered by expressiveness in models as follows:*

$$\mathcal{L}_m \subseteq \mathcal{L}_t \subseteq \mathcal{L}\updownarrow_m^1 \subseteq \mathcal{L}\updownarrow_t^1 \subseteq \mathcal{L}\updownarrow_m^2 \subseteq \mathcal{L}\updownarrow_t^2 \subseteq \ldots \subseteq \mathcal{L}\updownarrow_m^\omega = \mathcal{L}\updownarrow_t^\omega = L_1$$

Proof. For the inclusion $\mathcal{L}\updownarrow_t^k \subseteq \mathcal{L}\updownarrow_m^{k+1}$ it suffices to show how the past operator **H** can be expressed in a *modal* language by means of an *extra* reference pair $(\downarrow, \uparrow)$ which is not used for any other purposes:

$$\mathbf{H}\varphi = \downarrow \mathbf{A}(\mathbf{F}\uparrow \rightarrow \varphi).$$

As for $\mathcal{L}\updownarrow_t^\omega = L_1$, note that every L_1-formula in a prenex form

$$\alpha = Q_1 x_1 \dots Q_n x_n (Rx_i x_j, x_i x_j, P_i x_j)$$

is expressible in $\mathcal{L}\updownarrow_t^\lambda$ by

$$\phi_\alpha = \downarrow_{n+1} \hat{Q}_1 \downarrow_1 \dots \hat{Q}_n \downarrow_n (\mathbf{E}(\uparrow_i \wedge \Diamond \uparrow_j), \mathbf{E}(\uparrow_i \wedge \uparrow_j), \mathbf{E}(p_i \wedge \uparrow_j),$$

where $\hat{\forall} = \mathbf{A}$ and $\hat{\exists} = \mathbf{E}$. This means that $\forall x ST(\phi_\alpha)$ is equivalent to α, which is verified by a routine application of the translation ST.

As a corollary, the languages $\mathcal{L}\updownarrow_t^\lambda$ and $\mathcal{L}\updownarrow_m^\lambda$ cover the Π_1^1-fragment of the monadic second order language with one binary predicate with respect to definability in frames.

It is natural to expect that the above hierarchy is strict with respect to expressiveness. The strict inclusion $\mathcal{L}\updownarrow_t^k \subset \mathcal{L}\updownarrow_m^{k+1}$ can be easily shown as follows. Let F_n be the frame $\langle W_n, \emptyset \rangle$ consisting of n isolated irreflexive points and let M_n be the model over F_n such that all propositional variables are false everywhere. If $k \leq n$ we regard M_k as a submodel of M_n. Now, the models M_k and $M_{k+1}, k \geq 2$ are distinguished in $\mathcal{L}\updownarrow_m^k$ by the formula

$$\mathbf{E} \downarrow_1 \dots \mathbf{E} \downarrow_k \mathbf{A}(\uparrow_1 \vee \dots \vee \uparrow_k)$$

which is valid in the former but not in the latter model. These two models, however, cannot be distinguished by a (closed) formula of $\mathcal{L}\updownarrow_t^{k-1}$ because for every formula ϕ of that language, and for every point $s \in M_k$,

$$\mathcal{E}(M_k, s, \phi) = \mathcal{E}(M_{k+1}, s, \phi) \cap W_k^{k-1},$$

a fact which can be established by induction on ϕ.

As for the strictness of the inclusion $\mathcal{L}\updownarrow_m^k \subset \mathcal{L}\updownarrow_t^k$, it needs more elaborated distinguishing models and is left open. We also conjecture that the above hierarchy of languages is strict with respect to definability of frame properties.

Another open problem is to find proper notions of $\mathcal{L}\updownarrow_m^k$-bisimulation and $\mathcal{L}\updownarrow_t^k$-bisimulation, invariance under which would characterize expressiveness of first-order formulae in these languages.

Now we shall present a few eloquent testimonials to the strength of the languages with reference pointers.

1. Various postulates for Kripke frames which are beyond the scope of $\mathcal{L}_t$ are readily expressed in $\mathcal{L}\updownarrow_m^1$. Just two simple examples:

– irreflexivity: $F \Vdash \forall x \neg Rxx \;\Leftrightarrow\; F \models \downarrow \Box \neg \uparrow$,
– antisymmetry: $F \Vdash \forall x \forall y (x < y \rightarrow \neg y < x) \Leftrightarrow F \models \downarrow \Box\Box \neg \uparrow$.

2. A number of modalities not definable in $\mathcal{L}_t$ can be easily defined in $\mathcal{L}\updownarrow^1_m$. A simple but important example is the *difference modality* $[\neq]$ (see e.g. de Rijke, 1992):

$$[\neq]\varphi = \downarrow \mathbf{A}(\neg \uparrow \rightarrow \varphi).$$

3. Kamp's $\mathbf{S}(p,q)$ (*Since*) and $\mathbf{U}(p,q)$ (*Until*) and Stavi's $\mathbf{U}'(p,q)$ and $\mathbf{S}'(p,q)$ are explicitly definable in $\mathcal{L}\updownarrow^1_t$:

$$\mathbf{U}(p,q) = \downarrow \mathbf{F}(p \wedge \mathbf{H}(\mathbf{P}\uparrow \rightarrow q)) \text{ and } \mathbf{S}(p,q) = \downarrow \mathbf{P}(p \wedge \mathbf{G}(\mathbf{F}\uparrow \rightarrow q));$$

$$\mathbf{U}'(p,q) =$$

$$\downarrow \mathbf{FH}(\mathbf{P}\uparrow \rightarrow q) \;\wedge\; \neg \mathbf{U}(\neg q \vee \neg \mathbf{U}(\top, q), q)$$
$$\wedge\; \downarrow \mathbf{F}(\neg q \wedge p \wedge \mathbf{H}(\mathbf{P}\uparrow \wedge \mathbf{P}(\mathbf{P}\uparrow \wedge \neg q) \rightarrow p))$$

and likewise for $\mathbf{S}'(p,q)$.

4. The idea of *clock-variables* or *names for instants* can be adequately formalized in $\mathcal{L}\updownarrow^1_m$ without introducing a separate sort for them: The formula $\downarrow \mathbf{A}(p \leftrightarrow \uparrow)$ says "p is valid only now", and accordingly, $\mathbf{E}\downarrow \mathbf{A}(p \leftrightarrow \uparrow)$ means "p is valid at exactly one instant". Thus, in the consequent of the formula

$$\mathbf{E}\downarrow A(p \leftrightarrow \uparrow) \rightarrow \varphi(p, \ldots)$$

the variable p plays a role of a clock-variable. This fact will be essentially exploited in our axiomatic system.

We saw that $\mathcal{L}\updownarrow^1_m$ is at least as strong as a modal language with difference modality or with clock-variables (these two are equivalent with respect to definability, shown in Gargov and Goranko, 1993). Therefore some results about these two languages (see Gargov and Goranko, 1993; de Rijke, 1992) hold for $\mathcal{L}\updownarrow^\lambda_t$, too:

– every finite frame is described up to isomorphism in $\mathcal{L}\updownarrow^\lambda_t$.
– all universal sentences in the monadic second-order language for R and $=$ are definable in $\mathcal{L}\updownarrow^\lambda_t$.

In fact, $\mathcal{L}\updownarrow^1_m$ is even stronger than any of these languages. Indeed, the formula

$$\mathbf{A}(\Diamond \top \wedge \downarrow \Box \neg \uparrow) \rightarrow \mathbf{E}\downarrow \mathbf{E}(\Diamond\Diamond \uparrow \wedge \Box \neg \uparrow)$$

says that if a frame is irreflexive and every point has a successor then it is not transitive. It is valid in every *finite* model but not in the frame $\langle \mathcal{N}, < \rangle$. Therefore this condition is not definable in any of those languages, since their minimal logics enjoy the finite model property.

Moreover, as one could expect about such a powerful language, the set of valid formulae in $\mathcal{L}\updownarrow^1_m$ (and therefore in any other language from the hierarchies) is even not recursive.

THEOREM 4. *The satisfiability problem in $\mathcal{L}\updownarrow^1_m$ is Π^0_1-complete.*

Proof. We show that the *unbounded tiling problem* for $\mathcal{N} \times \mathcal{N}$, known to be Π^0_1-complete (see Harel, 1983), is reducible to the satisfiability problem in $\mathcal{L}\updownarrow^1_m$. The idea for doing this we borrow from de Rijke (1993).

First we define a formula GRID which is supposed to set the grid for tiling:

$$GRID = (p \wedge q) \wedge \varphi_1 \wedge \varphi_2 \wedge \varphi_3$$

where:

$$\begin{aligned}
\varphi_1 \;=\; & \mathbf{A}((p \wedge q \rightarrow \Diamond(p \wedge \neg q) \wedge \Diamond(\neg p \wedge q) \wedge \Box((p \wedge \neg q) \vee (\neg p \wedge q))) \wedge \\
& (p \wedge \neg q \rightarrow \Diamond(p \wedge q) \wedge \Diamond(\neg p \wedge \neg q) \wedge \Box((p \wedge q) \vee (\neg p \wedge \neg q))) \wedge \\
& (\neg p \wedge q \rightarrow \Diamond(\neg p \wedge \neg q) \wedge \Diamond(p \wedge q) \wedge \Box((\neg p \wedge \neg q) \vee (p \wedge q))) \wedge \\
& (\neg p \wedge \neg q \rightarrow \Diamond(\neg p \wedge q) \wedge \Diamond(p \wedge \neg q) \wedge \Box((\neg p \wedge q) \vee (p \wedge \neg q)))), \\
\varphi_2 \;=\; & \mathbf{A} \downarrow ((p \wedge q \rightarrow \mathbf{A}(\Diamond \uparrow \rightarrow \Box(p \wedge q \rightarrow \uparrow))) \;\wedge \\
& \quad (p \wedge \neg q \rightarrow \mathbf{A}(\Diamond \uparrow \rightarrow \Box(p \wedge \neg q \rightarrow \uparrow))) \;\wedge \\
& \quad (\neg p \wedge q \rightarrow \mathbf{A}(\Diamond \uparrow \rightarrow \Box(\neg p \wedge q \rightarrow \uparrow))) \;\wedge \\
& \quad (\neg p \wedge \neg q \rightarrow \mathbf{A}(\Diamond \uparrow \rightarrow \Box(\neg p \wedge \neg q \rightarrow \uparrow)))), \\
\varphi_3 \;=\; & \mathbf{A} \downarrow ((p \wedge q \rightarrow \mathbf{A}((\neg p \wedge \neg q \wedge \Diamond\Diamond \uparrow) \rightarrow \Box\Box(p \wedge q \rightarrow \uparrow))) \;\wedge \\
& \quad (p \wedge \neg q \rightarrow \mathbf{A}((\neg p \wedge q \wedge \Diamond\Diamond \uparrow) \rightarrow \Box\Box(p \wedge \neg q \rightarrow \uparrow))) \;\wedge \\
& \quad (\neg p \wedge q \rightarrow \mathbf{A}((p \wedge \neg q \wedge \Diamond\Diamond \uparrow) \rightarrow \Box\Box(\neg p \wedge q \rightarrow \uparrow))) \;\wedge \\
& \quad (\neg p \wedge \neg q \rightarrow \mathbf{A}((p \wedge q \wedge \Diamond\Diamond \uparrow) \rightarrow \Box\Box(\neg p \wedge \neg q \rightarrow \uparrow)))).
\end{aligned}$$

The formula $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ says that every point of the model has exactly two successors; at one of them the valuation of p changes and the valuation of q remains the same (that would be the move "to the right"), while at the other (the move "upwards") the opposite happens. Moreover, by φ_3, the routes "right;up" and "up;right" converge. That will be enough to embed a copy of $\mathcal{N} \times \mathcal{N}$ into any model of GRID.

Now, consider a tiling problem with a set of tiles $T = \{t_1, ..., t_m\}$ and colours $C = \{c_1, ..., c_k\}$. Every tile has four sides: "up", "down", "left" and "right", each coloured in one of the colours from C. To every colour c_i we assign four propositional variables u_i ("up"), d_i ("down"), l_i ("left"), and r_i ("right"). Each tile t with sides "up", "down", "left" and "right" coloured respectively in $c_{i_1}, c_{i_2}, c_{i_3}$, and c_{i_4}, we represent by the formula

$$\theta_t = (u_{i_1} \wedge \bigwedge_{j \neq i_1} \neg u_j) \wedge (d_{i_2} \wedge \bigwedge_{j \neq i_2} \neg d_j) \wedge (l_{i_3} \wedge \bigwedge_{j \neq i_3} \neg l_j) \wedge (r_{i_4} \wedge \bigwedge_{j \neq i_4} \neg r_j).$$

Now we define the formulae

$$COVER_T = \mathbf{A}(\bigvee_{i=1}^{m} \theta_i),$$

which says that the model is properly tiled, i.e. every point in the model is covered by exactly one tile (note that θ_i and θ_j are incompatible when $i \neq j$);

$$\begin{aligned} \mathit{MATCHUP} = \mathbf{A}(\bigwedge_{i=1}^{k}(u_i \to & (p \wedge q \to \Box(p \wedge \neg q \to d_i)) \wedge \\ & (p \wedge \neg q \to \Box(p \wedge q \to d_i)) \wedge \\ & (\neg p \wedge q \to \Box(\neg p \wedge \neg q \to d_i)) \wedge \\ & (\neg p \wedge \neg q \to \Box(\neg p \wedge q \to d_i))), \end{aligned}$$

which says that the colour "up" of each tile of the cover matches the colour "down" of the one above it;

$$\begin{aligned} \mathit{MATCHRIGHT} = \mathbf{A}(\bigwedge_{i=1}^{k}(r_i \to & (p \wedge q \to \Box(\neg p \wedge q \to l_i)) \wedge \\ & (p \wedge \neg q \to \Box(\neg p \wedge \neg q \to l_i)) \wedge \\ & (\neg p \wedge q \to \Box(p \wedge q \to l_i)) \wedge \\ & (\neg p \wedge \neg q \to \Box(p \wedge \neg q \to l_i))), \end{aligned}$$

which says that the colour "right" of each tile of the cover matches the colour "left" of the one to the right of it.

Finally, we put

$$\Phi_T = \mathit{GRID} \wedge \mathit{COVER}_T \wedge \mathit{MATCHUP} \wedge \mathit{MATHCHRIGHT}.$$

We claim that Φ_T is satisfiable if and only if $\mathcal{N} \times \mathcal{N}$ can be properly tiled by T.

Indeed, if $\mathcal{N} \times \mathcal{N}$ can be tiled by T we can define a model $\mathcal{M} = \langle \mathcal{N} \times \mathcal{N}, R, V \rangle$ where

$$\begin{aligned} \langle m_1, n_1 \rangle R \langle m_2, n_2 \rangle \quad & \text{iff} \quad m_2 = m_1 + 1, n_2 = n_1 \\ & \text{or} \quad m_2 = m_1, n_2 = n_1 + 1; \end{aligned}$$

$$V(p) = \{\langle 2m, n \rangle : m, n \in \mathcal{N}\}, \quad V(q) = \{\langle m, 2n \rangle : m, n \in \mathcal{N}\},$$

$$V(u_i) = \{\langle m, n \rangle : \text{the "up" colour of the tile at} \langle m, n \rangle \text{ is } c_i\},$$

and likewise for d_i, l_i, and r_i.

Then,

$$\mathcal{M} \models \Phi_T[\langle 0, 0 \rangle].$$

Conversely, if for some model $\mathcal{M} = \langle W, R, V \rangle$, $\mathcal{M} \models \Phi_T[x]$, we define a mapping $f : \mathcal{N} \times \mathcal{N} \to W$ as follows: $f(0,0) = x$; suppose that $f(m,n)$ is defined, then $f(m+1,n)$ is the unique "right"-successor of $f(m,n)$, and $f(m,n+1)$ is the unique "up"-successor of $f(m,n)$. The tiling of $\mathcal{N} \times \mathcal{N}$ is now determined by

COVER: for any $\langle m, n\rangle$ there is a unique θ_i which is true at $f(m,n)$; then we put the tile t_i at $\langle m, n\rangle$. Due to MATCHUP and MATCHRIGHT the tiling is a proper one, and this completes the proof.

5. Deductive Systems for $\mathcal{L}\updownarrow_m^\lambda$ and $\mathcal{L}\updownarrow_t^\lambda$

We first axiomatize the basic logics $\mathbf{K}\updownarrow_t^\lambda$ of the temporal languages $\mathcal{L}\updownarrow_t^\lambda$ as follows:

Axioms for $\mathbf{K}\updownarrow_t^\lambda$:

A1) The axioms of the basic temporal logic $\mathbf{K}_t$, written over propositional variables
A2) $\mathbf{S5}(\mathbf{A})$,
A3) $\mathbf{A}p \to (\mathbf{G}p \wedge \mathbf{H}p)$,
A4) $\downarrow_k \uparrow_k$,
A5) $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \to (q \to \mathbf{A}(p \to q))$
A6) $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \to (\downarrow_j \psi \leftrightarrow \psi(p/\uparrow_j))$, for any closed formula $\downarrow_j \psi$
where p, q are propositional variables and $j, k < \lambda$.

Rules:

1. **MP**:
$$\frac{\varphi, \varphi \to \psi}{\psi};$$
2. $\mathbf{NEC_A}$:
$$\frac{\varphi}{\mathbf{A}\varphi};$$
3. **CLSUB**:
$$\frac{\varphi}{clsub(\varphi)},$$
where *clsub*(φ) is a result of uniform substitution of *closed* formulae for propositional variables in φ.
4. **WITNESS**:
$$\frac{\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \to \varphi \quad \text{for every propositional variable } p}{\varphi},$$
where $k < \lambda$.

As for $\mathbf{K}\updownarrow_m^\lambda$, we replace A1 and A3 respectively by

A1') The axioms of the basic modal logic $\mathbf{K}$, written over propositional variables
A3') $\mathbf{A}p \to \Box p$
and add
A7) $p \to \Box \downarrow_k \mathbf{E}(\Diamond \uparrow_k \wedge p)$
The rules remain the same.

Hereafter we shall use $\mathcal{L}\updownarrow$ as a generic name for any of the languages $\mathcal{L}\updownarrow_m^\lambda$ and $\mathcal{L}\updownarrow_t^\lambda$ and respectively $\mathbf{K}\updownarrow$ for any of the logics $\mathbf{K}\updownarrow_m^\lambda$ and $\mathbf{K}\updownarrow_t^\lambda$, possibly with specifying the superscript λ.

Note the following:

- Only closed formulae are derivable in $\mathbf{K}\updownarrow$.
- In the presence of **CLSUB** the infinitary rule **WITNESS** can be replaced by a finitary version:

$$\mathbf{WITNESS}_f : \frac{\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \to \varphi \text{ for some var.} p \text{ not occurring in } \varphi}{\varphi}$$

- The rules

$$\mathbf{NEC_G} : \frac{\varphi}{\mathbf{G}\varphi} \text{ and } \mathbf{NEC_H} : \frac{\varphi}{\mathbf{H}\varphi}; \text{ resp. } \mathbf{NEC} : \frac{\varphi}{\Box\varphi}$$

  are derivable from $\mathbf{NEC_A}$, **MP** and A3.

Here are some important theorems of $\mathbf{K}\updownarrow^\lambda$:
(t1) $\downarrow_k \varphi \leftrightarrow \varphi$ for every closed formula φ;
(t2) $\downarrow_k \neg\varphi \leftrightarrow \neg \downarrow_k \varphi$;
(t3) $\downarrow_k (\varphi \wedge \psi) \leftrightarrow (\downarrow_k \varphi \wedge \downarrow_k \psi)$;
(t4) $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \to p$.
(t5) $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \leftrightarrow \downarrow_j \mathbf{A}(\uparrow_j \leftrightarrow p)$.
for any $j, k < \lambda$.

We exemplify derivations in $\mathbf{K}\updownarrow^\lambda$ by sketching a proof of (t2):

1. $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \to (\downarrow_k \neg\varphi \leftrightarrow \neg\varphi(p/\uparrow_k))$ by A6,
2. $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \to (\downarrow_k \varphi \leftrightarrow \varphi(p/\uparrow_k))$ by A6,
3. $\downarrow \mathbf{A}(\uparrow_k \leftrightarrow p) \to (\neg \downarrow_k \varphi \leftrightarrow \neg\varphi(p/\uparrow_k))$ by 2 and contrapositions,
4. $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \to (\neg \downarrow_k \varphi \leftrightarrow \downarrow_k \neg\varphi)$ by 1 and 3,
5. $\neg \downarrow_k \varphi \leftrightarrow \downarrow_k \neg\varphi$ by 4 and **WITNESS**.

The other derivations are similar.

Now we can show that axiom A7, translated to $\mathcal{L}\updownarrow_t^\lambda$, is derivable in $\mathbf{K}\updownarrow_t^\lambda$:

First, $\vdash p \to \mathbf{G}\downarrow_k \mathbf{E}(\mathbf{F}\uparrow_k \wedge p)$ iff $\vdash \mathbf{P}p \to \downarrow_k \mathbf{E}(\mathbf{F}\uparrow_k \wedge p)$;
Then, $\mathbf{K}_t \vdash q \to (\mathbf{P}p \to \mathbf{P}(\mathbf{F}q \wedge p))$,
hence $\mathbf{K}\updownarrow_t^\lambda \vdash q \to (\mathbf{P}p \to \mathbf{E}(\mathbf{F}q \wedge p))$.
Therefore, using (t4),
$\mathbf{K}\updownarrow_t^\lambda \vdash \downarrow_k \mathbf{A}(q \leftrightarrow \uparrow_k) \to (\mathbf{P}p \to \mathbf{E}(\mathbf{F}q \wedge p))$.
Now, apply A6, then (t2) and (t3) to obtain
$\mathbf{K}\updownarrow_t^\lambda \vdash \downarrow_k \mathbf{A}(q \leftrightarrow \uparrow_k) \to (\mathbf{P}p \to \downarrow_k \mathbf{E}(\mathbf{F}\uparrow_k \wedge p))$,
and finally, apply $\mathbf{WITNESS}_f$.

A few remarks are in order here.

1. The reason we design our axiomatic systems for closed formulae only is that we have clear meaning for these formulae only. Besides that, however, the formulae which are not closed behave rather irregularly. For instance, the rule for equivalent replacement

$$\frac{\varphi \leftrightarrow \psi}{\theta(\varphi/p) \leftrightarrow \theta(\psi/p)},$$

applied to such formulae, does not always preserve validity in a frame. The same happens with the substitution rule $\frac{\varphi}{sub(\varphi)}$, so that special provisions must be made, like "a formula free for substitution in a formula" etc. which would lead to the typical complications of the first-order machinery.
2. The intuition behind the rule **WITNESS** (which has a number of ancestors, e.g. some versions of rules for quantifiers in first-order logic, the ω-rule in arithmetic, the "irreflexivity rule" in (Gabbay, 1981a), COV in (Gargov and Goranko, 1993) etc.) is the following. If a formula is not valid, then it is false at some point t of some model $\mathcal{M}$. Then, a propositional variable p can be made a "t o'clock-variable" (in temporal setting) i.e. evaluated to be true exactly at that point t, and then p will be a "witness" of the falsity of φ. Therefore if all "clock-variables" testify that φ is true at the point (world, instant) in which they live, then φ must be valid.
3. Although **WITNESS** and **WITNESS**$_f$ infer the same theorems, they generate deductive systems different with respect to logical consequence, which is compact in the system with **WITNESS**$_f$, but not in the system with **WITNESS**.

Amongst the basic syntactic properties of **K** and $\mathbf{K}_t$ which are inherited, mutatis mutandis, in the extended logics we only mention the following:

LEMMA 5. *For any closed formulae α, β, φ and variable p,*
if

$$\mathbf{K}\updownarrow\vdash \alpha \leftrightarrow \beta$$

then

$$\mathbf{K}\updownarrow\vdash \varphi(\alpha/p) \leftrightarrow \varphi(\beta/p).$$

Proof. (Sketch) Induction on the reference depth $r(\varphi)$.

If $r(\varphi) = 0$, the proof repeats the standard one for **K**.

If $r(\varphi) = r > 0$ we assume that for all formulae with reference depth less than r the statement holds, and then do induction on the complexity of φ. The interesting case is $\varphi = \downarrow\theta$. Here we apply axiom $A6$ (with a variable q not occurring in φ, α, β) which replaces $\varphi(\alpha/p)$ by $\{\theta(\alpha/p)\}(q/\uparrow) = \{\theta(q/\uparrow)\}(\alpha/p)$ and likewise for $\varphi(\beta/p)$. The resulting formulae have reference depths lesser than r. Applying the inductive hypothesis for them, followed by application of **WITNESS**$_f$, completes the induction.

THEOREM 6 (Soundness theorem). *1. All axioms of* $\mathbf{K}\updownarrow$ *are valid.*

2. All rules of $\mathbf{K}\updownarrow$ *preserve validity in a frame, and therefore universal validity.*

Proof.

1. First, consider $\mathbf{K}\updownarrow_t^\lambda$. For A1 the result comes from $\mathbf{K}_t$; for A2 from $\mathbf{S5}$; for A3 and A4 it is quite simple. As for A5, it is enough to note that

$$\forall x ST(\downarrow \mathbf{A}(\uparrow\leftrightarrow p_i) \rightarrow (p_j \rightarrow \mathbf{A}(p_i \rightarrow p_j))) =$$

$$\forall x(\forall y(y = x \leftrightarrow P_i y) \rightarrow (P_j x \rightarrow \forall y(P_i y \rightarrow P_j y)))$$

which is universally valid.

Finally, take $A6$. $ST(\psi(p_i/\uparrow_j))$ is obtained from $ST(\psi)$ by replacing all occurrences of the kind $y = w_j$ (which are the only possible occurrences of w_j in $ST(\psi)$) by $P_i y$. Due to the antecedent $\forall y(y = x \leftrightarrow P_i y)$ this is equivalent to replace all occurrences of $y = w_j$ by $y = x$. The result of this substitution is exactly $ST(\psi(x/w_j)) = ST(\downarrow_j \psi))$.

The validity of axioms A1–A6 for $\mathbf{K}\updownarrow_m^\lambda$ is verified likewise. Now, A7:

$$\forall x ST(p_i \rightarrow \Box \downarrow_k \mathbf{E}(\Diamond \uparrow_k \wedge p_i) =$$

$$\forall x(P_i x \rightarrow \forall y(xRy \rightarrow \exists z(\exists t(zRt \wedge t = y) \wedge P_i z))),$$

equivalent to

$$\forall x(P_i x \rightarrow \forall y(xRy \rightarrow \exists z(zRy \wedge P_i z)))$$

which is universally valid.

2. The only interesting case is the rule **WITNESS**. Suppose that for some model $\langle T, V\rangle$ and point $t \in T$, $\langle T, V\rangle \not\models \varphi[t]$. Choose a variable p not occurring in φ and change the valuation V to V' as follows: $V'(p) = \{t\}$, and V' coincides with V elsewhere. Then $\langle T, V'\rangle \models \downarrow \mathbf{A}(\uparrow\leftrightarrow p)[t]$ and $\langle T, V'\rangle \not\models \varphi[t]$, hence $\langle T, V'\rangle \not\models \downarrow \mathbf{A}(\uparrow\leftrightarrow p) \rightarrow \varphi[t]$.

6. Completeness

Now we set ourselves to prove completeness of $\mathbf{K}\updownarrow$. We follow an elaborated version of the traditional in modal and temporal logic "canonical model technique", further developed in Passy and Tinchev (1991) and Gargov and Goranko (1993). The basic steps of the proof will be scrupulously outlined in a series of lemmata, but the standard details in their proofs will be usually omitted.

First we introduce yet another syntactic notion originating from the *admissible forms* in Goldblatt (1982); see also Gargov and Goranko (1993). Let $*$ be a symbol not belonging to $\mathcal{L}\updownarrow$. We define recursively *universal forms of* $*$ *in* $\mathcal{L}\updownarrow$ as follows:

1. $*$ is a universal form of $*$.
2. If $u(*)$ is a universal form of $*$, φ is a closed formula in $\mathcal{L}\updownarrow$, and $\mathbf{L}$ is a box-modality in $\mathcal{L}\updownarrow$ (i.e. **A, G, F** or $\Box$) then $\varphi \rightarrow u(*)$ and $\mathbf{L}u(*)$ are universal forms of $*$ in $\mathcal{L}\updownarrow$.

Every universal form of $*$ in $\mathcal{L}\updownarrow$ can be represented (up to tautological equivalence) in a uniform way:

$$u(*) = \varphi_0 \rightarrow \mathbf{L}_1(\varphi_1 \rightarrow \ldots \mathbf{L}_n(\varphi_n \rightarrow *)\ldots)$$

where $\mathbf{L}_1, \ldots, \mathbf{L}_n$ are box-modalities in $\mathcal{L}\updownarrow$ and some of $\varphi_1, \ldots, \varphi_n$ may be $\top$ if necessary. The number n is called a *depth* of the form u, denoted by $\partial(u)$.

For every universal form $u(*)$ and a formula θ we denote by $u(\theta)$ the result of substitution of θ for $*$ in $u(*)$. Obviously, if θ is a closed formula then $u(\theta)$ is a closed formula, too.

Now we introduce the rule

$$\mathbf{WITNESS}_U : \frac{u(\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \rightarrow \varphi) \quad \text{for every propositional variable } p}{u(\varphi)}$$

where u is an arbitrarily fixed universal form.

$\mathbf{WITNESS}_U$ seems much stronger than **WITNESS**, but:

THEOREM 7. *The rule* $\mathbf{WITNESS}_U$ *is derivable in* $\mathbf{K}\updownarrow$.

Proof. First, consider $\mathbf{K}\updownarrow_t^\lambda$. The key argument, see (Gabbay and Hodkinson, 1990), is the following. Given a universal form

$$u(*) = \varphi_0 \rightarrow \mathbf{L}_1(\varphi_1 \rightarrow \ldots \mathbf{L}_n(\varphi_n \rightarrow *)\ldots),$$

we define a form

$$u'(*) = \neg * \rightarrow (\varphi_n \rightarrow \mathbf{L}'_n(\varphi_{n-1} \rightarrow \ldots \mathbf{L}'_1 \neg\varphi_0)\ldots),$$

where $\mathbf{A}' = \mathbf{A}, \mathbf{G}' = \mathbf{H}$ and $\mathbf{H}' = \mathbf{G}$. Now, for every closed formula θ, $u(\theta)$ is deductively equivalent to $u'(\theta)$ in the sense that $\mathbf{K}\updownarrow_t^\lambda + u'(\theta) \vdash u(\theta)$ and $\mathbf{K}\updownarrow_t^\lambda + u(\theta) \vdash u'(\theta)$.

This trick does not work for $\mathbf{K}\updownarrow_m^\lambda$, but the axiom A7 comes to help there. We shall prove derivability of $\mathbf{WITNESS}_U$ in $\mathbf{K}\updownarrow_m^\lambda$ by induction on $\partial(u)$.

If $\partial(u) = 0$ then $u(*) = \varphi_0 \rightarrow *$ and $\mathbf{WITNESS}_U$ is obviously equivalent to **WITNESS** applied to $\varphi_0 \rightarrow \varphi$.

Now, let $u(*) = \varphi_0 \rightarrow \mathbf{L}_1(\varphi_1 \rightarrow \ldots \mathbf{L}_n(\varphi_n \rightarrow *)\ldots), n > 0$, and $\mathbf{WITNESS}_U$ is derivable for all universal forms of depth less than n. Denote

$$v(*) = \varphi_1 \rightarrow \mathbf{L}_2(\varphi_2 \rightarrow \ldots \mathbf{L}_n(\varphi_n \rightarrow *)\ldots),$$

i.e. $u(*) = \varphi_0 \rightarrow \mathbf{L}_1(v(*))$. Then $\partial(v) = n - 1$.

Let $v' = v(\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \rightarrow \varphi)$ and $v'' = v(\varphi)$.

We have to prove that $\vdash \varphi_0 \rightarrow \mathbf{L}_1(v')$ for all p implies $\vdash \varphi_0 \rightarrow \mathbf{L}_1(v'')$ for all p.

Case 1: $\mathbf{L}_1 = \mathbf{A}$. Note that $\vdash \varphi_0 \rightarrow \mathbf{A}(v')$ implies $\vdash \mathbf{E}\varphi_0 \rightarrow v'$ which is an instance of a universal form of depth $n-1$. Then, by the inductive hypothesis, $\vdash \mathbf{E}\varphi_0 \rightarrow v''$ which implies $\vdash \varphi_0 \rightarrow \mathbf{A}(v'')$.

Case 2: $\mathbf{L}_1 = \Box$. Let q be any variable. Here is a sketch of the derivation.

1. $\vdash \varphi_0 \rightarrow \Box(v')$, assumption;
2. $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow q) \rightarrow (\neg v' \rightarrow \mathbf{A}(q \rightarrow \neg v'))$, by A5;
3. $\vdash \mathbf{A}(q \rightarrow \neg v') \rightarrow \mathbf{A}(\Box(v') \rightarrow \Box\neg q)$, by easy derivation from A1–A3;
4. $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow q) \rightarrow (\neg v' \rightarrow \mathbf{A}(\Box(v') \rightarrow \Box\neg q))$, by 2, 3;
5. $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow q) \rightarrow (\neg v' \rightarrow \mathbf{A}(\varphi_0 \rightarrow \Box\neg q))$, by 1, 4;
6. $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow q) \wedge \mathbf{E}(\varphi_0 \wedge \Diamond q) \rightarrow v'$, by 5;
7. $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow q) \wedge \mathbf{E}(\varphi_0 \wedge \Diamond q) \rightarrow v''$, by 6, applying the inductive hypothesis to the form $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow q) \wedge \mathbf{E}(\varphi_0 \wedge \Diamond q) \rightarrow v(*)$, which has a depth $n-1$;
8. $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow q) \rightarrow (\mathbf{E}(\varphi_0 \wedge \Diamond q) \leftrightarrow \downarrow_k \mathbf{E}(\varphi_0 \wedge \Diamond \uparrow_k))$, by A6;
9. $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow q) \rightarrow (\downarrow_k \mathbf{E}(\varphi_0 \wedge \Diamond \uparrow_k) \rightarrow v'')$, by 7, 8;
10. $\downarrow_k \mathbf{E}(\varphi_0 \wedge \Diamond \uparrow_k) \rightarrow v''$, by 9, applying **WITNESS**;
11. $\Box \downarrow_k \mathbf{E}(\varphi_0 \wedge \Diamond \uparrow_k) \rightarrow \Box(v'')$, by 10;
12. $\vdash \varphi_0 \rightarrow \Box(v'')$, by 11 and A7.

This completes the induction and the proof.

Henceforth the completeness proofs for $\mathbf{K}\updownarrow_t^\lambda$ and $\mathbf{K}\updownarrow_m^\lambda$ are entirely analogous and we shall concentrate on $\mathbf{K}\updownarrow_t^\lambda$.

DEFINITION 8.

1. A *theory* in $\mathcal{L}\updownarrow_t^\lambda$ is a set of closed formulae of $\mathcal{L}\updownarrow_t^\lambda$, which contains all theorems of $\mathbf{K}\updownarrow_t^\lambda$ and is closed with respect to **MP**.
2. A *W-theory* (*witnessed theory*) is a theory in $\mathcal{L}\updownarrow_t^\lambda$ which is closed with respect to $\mathbf{WITNESS}_U$.

Note that for every set of closed formulae Γ there is a minimal W-theory WTh(Γ) /resp. a minimal theory Th(Γ)/ containing Γ. Indeed, the set of all closed formulae is a W-theory. Furthermore, the intersection of every family of W-theories is a W-theory. Then WTh(Γ) is the intersection of all W-theories containing Γ. Likewise for theories.

DEFINITION 9. A theory, resp. W-theory, is *consistent* if it does not contain $\bot$. A set of closed formulae Δ is *W-consistent* if WTh(Δ) is consistent.

The following property, well-known for theories, holds for W-theories, too.

LEMMA 10 (Deduction theorem for W-theories). *If Γ is a W-theory and φ, ψ are closed formulae then $\varphi \rightarrow \psi \in \Gamma$ iff $\psi \in WTh(\Gamma \cup \{\varphi\})$*

Proof. If $\varphi \to \psi \in \Gamma$ then, by **MP**, $\psi \in$WTh$(\Gamma \cup \{\varphi\})$. Vice versa, suppose that $\psi \in$ WTh$(\Gamma \cup \{\varphi\})$ and consider the set

$$\Delta = \{\theta : \theta \text{ is } a \text{ closed formula and } \varphi \to \theta \in \Gamma\}.$$

We shall prove that Δ is a W-theory containing $\Gamma \cup \{\varphi\}$. The proof goes as in the standard deduction theorem, with one additional step: closedness with respect to **WITNESS**$_U$, which follows form the fact that Γ is a W-theory, and $\varphi \to u(*)$ is a universal form whenever $u(*)$ is.

LEMMA 11. *If* Γ *is a set of closed formulae in which infinitely many propositional variables have no occurrences then* $WTh(\Gamma) = Th(\Gamma)$.

Proof. It is a standard fact that

$$\mathrm{Th}(\Gamma) = \{\theta : \gamma_1 \wedge \ldots \wedge \gamma_k \to \theta \in \mathbf{K}\updownarrow_t^\lambda \text{ for some } \gamma_1, \ldots, \gamma_k \in \Gamma\}.$$

Let us show that Th(Γ) is closed with respect to **WITNESS**$_U$. Suppose that for some universal form u,

$$u(\downarrow \mathbf{A}(\uparrow \leftrightarrow p) \to \varphi) \in \mathrm{Th}(\Gamma)$$

i.e.

$$\gamma_1^i \wedge \ldots \wedge \gamma_{k_i}^i \to u(\downarrow \mathbf{A}(\uparrow \leftrightarrow p_i) \to \varphi) \in \mathbf{K}\updownarrow_t^\lambda$$

for every propositional variable p_i. We can choose a propositional variable p_j which does not occur in either of u, φ, or Γ. Then, substituting any variable p for p_j in

$$\gamma_1^j \wedge \ldots \wedge \gamma_{k_j}^j \to u(\downarrow \mathbf{A}(\uparrow \leftrightarrow p_j) \to \varphi)$$

we obtain

$$\gamma_1^j \wedge \ldots \wedge \gamma_{k_j}^j \to u(\downarrow \mathbf{A}(\uparrow \leftrightarrow p) \to \varphi) \in \mathbf{K}\updownarrow_t^\lambda$$

for every p. Therefore $\gamma_1^j \wedge \ldots \wedge \gamma_{k_j}^j \to u(\varphi) \in \mathbf{K}\updownarrow_t^\lambda$ by **WITNESS**$_U$, hence $u(\varphi) \in \mathrm{Th}(\Gamma)$.

As a corollary, every set of formulae which satisfies the condition of Lemma 11 is W-consistent iff it is consistent.

DEFINITION 12. A W-theory Γ is *maximal* if for every closed formula φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ but not both.

Every maximal W-theory is consistent and cannot be extended to another consistent W-theory. Moreover, every maximal W-theory Γ contains a "witness" $\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow q)$ for some propositional variable q ("Γ o'clock variable") and every $k < \lambda$. Due to the $\mathbf{K}\updownarrow_t^\lambda$-theorem (t5), it is enough to find a q for *some* $k < \lambda$. Suppose otherwise. Then all $\neg \downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p)$ would be in Γ, and hence, by **WITNESS**$_U$, $\perp$ would belong to Γ.

LEMMA 13 (Lindenbaum lemma). *Every W-consistent set Γ_0 can be extended to a maximal W-theory.*

Proof. First, note that $\Gamma = \mathrm{WTh}(\Gamma_0)$ is a consistent W-theory. Let $\psi_1, \psi_2, \ldots$ be a list of all closed formulae of $\mathcal{L}\updownarrow_t^\lambda$ and $u_1, u_2, \ldots$ be a list of all universal forms in $\mathcal{L}\updownarrow_t^\lambda$. Then we can list all combinations $\{u_i(\psi_j)\}_{i,j=1}^\infty$ in a sequence $\theta_1, \theta_2, \ldots$. (obviously, with repetitions, but that does not matter). Now we shall define a sequence of consistent W-theories $T_0 \subseteq T_1 \subseteq \ldots$ as follows: $T_0 = \Gamma$; suppose that T_n is defined and consider $\mathrm{WTh}(T_n \cup \{\theta_n\})$. If it is consistent, this is T_{n+1}. Otherwise let $\theta_n = u_i(\psi_j)$. Then $\neg u_i(\psi_j) \in T_n$ by the deduction theorem. Therefore $u_i(\downarrow_k \mathbf{A}(\uparrow_k \leftrightarrow p) \to \psi_j)$ does not belong to T_n for some propositional variable p and some $k < \lambda$. Due to (t5), this is the case for *all* $k < \lambda$. Then we put

$$T_{n+1} = \mathrm{WTh}(T_n \cup \{\neg u_i(\downarrow_0 \mathbf{A}(\uparrow_0 \leftrightarrow p) \to \psi_j)\}).$$

T_{n+1} is consistent.

Finally, put $T = \bigcup_{n=0}^\infty T_n$.

By virtue of the construction, T is a maximal W-theory.

For any set of formulae Δ we define

$$\mathbf{G}\boldsymbol{\Delta} = \{\varphi : \mathbf{G}\varphi \in \boldsymbol{\Delta}\},\ \mathbf{H}\boldsymbol{\Delta} = \{\varphi : \mathbf{H}\varphi \in \boldsymbol{\Delta}\} \text{ and } \mathbf{A}\boldsymbol{\Delta} = \{\varphi : \mathbf{A}\varphi \in \boldsymbol{\Delta}\}.$$

LEMMA 14. *If Δ is a maximal W-theory then $\mathbf{G}\boldsymbol{\Delta}$, $\mathbf{H}\boldsymbol{\Delta}$ and $\mathbf{A}\boldsymbol{\Delta}$ are W-theories.*

Proof. We shall do the proof for $\mathbf{G}\boldsymbol{\Delta}$, the others are analogous.

That $\mathbf{G}\boldsymbol{\Delta}$ contains all theorems of $\mathbf{K}\updownarrow_t^\lambda$ and is closed with respect to **MP** is nothing new. $\mathbf{G}\boldsymbol{\Delta}$ is also closed with respect to $\mathbf{WITNESS}_U$ since Δ is closed and $\mathbf{G}u(*)$ is a universal form whenever $u(*)$ is.

LEMMA 15. *If Δ is a maximal W-theory and $\mathbf{F}\theta \in \boldsymbol{\Delta}$ (resp. $\mathbf{P}\theta \in \boldsymbol{\Delta}$, $\mathbf{E}\theta \in \boldsymbol{\Delta}$) then there is a maximal W-theory Δ' such that $\theta \in \Delta'$ and $\mathbf{G}\Delta \subseteq \Delta'$ (resp. $\mathbf{H}\boldsymbol{\Delta} \subseteq \boldsymbol{\Delta}'$, $\mathbf{A}\boldsymbol{\Delta} \subseteq \boldsymbol{\Delta}'$.*

Proof. By Lemma 14 $\mathbf{G}\boldsymbol{\Delta}$ is a W-theory. Moreover, $\mathbf{G}\neg\theta \notin \boldsymbol{\Delta}$ since Δ is consistent. Therefore $\neg\theta \notin \mathbf{G}\boldsymbol{\Delta}$, hence $\mathrm{WTh}(\mathbf{G}\boldsymbol{\Delta} \cup \{\theta\})$ is consistent. Then, by Lemma 13 it can be extended to a maximal W-theory Δ'. The other cases are analogous.

DEFINITION 16. A model $\langle T, R, V\rangle$ is called *clock-model* if for every $t \in T$ there is a "t o'clock-variable" p_t such that $V(p_t) = \{t\}$.

LEMMA 17 (Strong completeness theorem for W-consistent sets). *Every W-consistent set Γ_0 in $\mathbf{K}\updownarrow_t^\lambda$ is satisfiable in a clock-model.*

Proof. First, we extend Γ_0 to a maximal W-theory Γ. Then we define a *canonical* model $\mathcal{M} = \langle T, R, V\rangle$ as usual:

– $T = \{\Delta : \Delta \text{ is a maximal } W\text{-theory and } \mathbf{A}\Gamma \subseteq \boldsymbol{\Delta}\}$;

– for any $\Delta_1, \Delta_2 \in T$, $R\Delta_1\Delta_2$ if $\mathbf{G}\boldsymbol{\Delta}_1 \subseteq \Delta_2$;
– for any propositional variable p, $V(p) = \{\Delta \in T : p \in \Delta\}$.

It is a standard task to prove that for any Δ_1, Δ_2 from T, $R\Delta_1\Delta_2$ iff $\mathbf{H}\boldsymbol{\Delta}_2 \subseteq \Delta_1$ and that $\mathbf{A}\boldsymbol{\Delta}_1 \subseteq \Delta_2$.

Now we are going to prove that Γ_0 is satisfied at the point Γ of the model $\mathcal{M}$. This follows from the following sub-lemma:

LEMMA 18 (Truth-lemma). *For every closed formula θ and $\Delta \in T$,*

$\mathcal{M} \models \theta[\Delta]$ iff $\theta \in \Delta$.

Proof of Truth-lemma. Induction on $r(\theta)$. When $r(\theta) = 0$, i.e. θ contains no reference pointers, the proof goes by induction on the complexity of θ and repeats, mutatis mutandis, the proof the the truth-lemma for $\mathbf{K}_t$, as the universal modality $\mathbf{A}$ is dealt with in the same way as the temporal modalities. Now, let $r(\theta) > 0$ and assume that for all closed formulae with reference depth less than $r(\theta)$ the statement holds. Then we do again an induction on the complexity of θ. The only non-standard case is $\theta = \downarrow\psi$, for any fixed $\downarrow = \downarrow_k, k < \lambda$. Let $\downarrow\mathbf{A}(\uparrow\leftrightarrow p)$ be a "witness" in Δ. Then, by axiom $A6$, $\downarrow\psi \leftrightarrow \psi(p/\uparrow) \in \Delta$, i.e. $\downarrow\psi \in \Delta$ iff $\psi(p/\uparrow) \in \Delta$. Since $r(\downarrow\psi) > r(\psi(p/\uparrow))$, by the inductive hypothesis $\mathcal{M} \models \psi(p/\uparrow)[\Delta]$ iff $\psi(p/\uparrow) \in \Delta$. To complete the proof of the lemma it remains to show that $\mathcal{M} \models \psi(p/\uparrow)[\Delta]$ iff $\mathcal{M} \models\downarrow\psi[\Delta]$. Again by axiom $A6$ $\mathcal{M} \models\downarrow\mathbf{A}(\uparrow\leftrightarrow p) \to (\downarrow\psi \leftrightarrow \psi(p/\uparrow))$, hence it is enough to show that $\mathcal{M} \models\downarrow\mathbf{A}(\uparrow\leftrightarrow p)[\Delta]$. $ST(\downarrow\mathbf{A}(\uparrow\leftrightarrow p)) = \forall y(y = x \leftrightarrow Py)$ where P is the unary predicate symbol corresponding to p. Thus, $\mathcal{M} \models\downarrow\mathbf{A}(\uparrow\leftrightarrow p)[\Delta]$ iff $\mathbf{M} \Vdash \forall y(y = \Delta \leftrightarrow Py)$ which means that $V(p) = \{\Delta\}$, i.e. p is a "Δ o'clock variable". Let us see that this is the case indeed. First, $p \in \Delta$ by the $\mathbf{K}\updownarrow_t^\lambda$-theorem $(t2)$ and **MP**. Now, suppose that $p \in \Delta'$ for some $\Delta' \in$T. Take any $\chi \in \Delta$. According to axiom $A4$, $\mathbf{A}(p \to \chi) \in \Delta$, hence $p \to \chi \in \Delta'$, so $\chi \in \Delta'$. Thus $\Delta \subseteq \Delta'$, which implies $\Delta = \Delta'$. So, $p \in \Delta'$ iff $\Delta' = \Delta$. The truth-lemma is proved, which completes the proof of Lemma 17.

THEOREM 19 (Strong completeness theorem for $\mathbf{K}\updownarrow$). *Every $\mathbf{K}\updownarrow$-consistent set Γ of closed formulae in $\mathcal{L}\updownarrow$ is satisfiable.*

Proof. For $\mathbf{K}\updownarrow_t^\lambda$ we reduce the theorem to Lemma 17 with a simple trick. Let ρ be a "renaming" of the propositional variables in $\mathcal{L}\updownarrow_t^\lambda$ as follows: $\rho(p_i) = p_{2i+1}, i = 1, 2, \ldots$. If φ is a formula, denote by $\rho(\varphi)$ the result of uniform substitution $\rho(p_i)$ for each p_i in φ, and then put $\rho(\Gamma) = \{\rho(\varphi) : \varphi \in \Gamma\}$. Now, $\rho(\Gamma)$ is a consistent set, since consistency is not affected by the renaming ρ. Furthermore, since the variables with even indices do not occur in formulae of $\rho(\Gamma)$, it is W-consistent by Lemma 11, hence satisfiable at some instant t of a clock-model $\langle T, R, V\rangle$. Now we define a valuation V' in $< T, R >$ as follows: $V'(p) = V(\rho(p))$. The resulting model $\langle T, R, V'\rangle$ (which is not necessarily a clock-model) satisfies Γ at t.

The proof of strong completeness for $\mathbf{K}\updownarrow_m^\lambda$ is an accordingly simplified version of the one for $\mathbf{K}\updownarrow_t^\lambda$.

COROLLARY 20.
1. *For every* λ, $\mathbf{K}\updownarrow_t^\lambda$ *is a conservative extension of* $\mathbf{K}\updownarrow_m^\lambda$.
2. *For every* $\lambda > \kappa$, $\mathbf{K}\updownarrow_m^\lambda$ *(resp.* $\mathbf{K}\updownarrow_t^\lambda$*) is a conservative extension of* $\mathbf{K}\updownarrow_m^\kappa$ *(resp.* $\mathbf{K}\updownarrow_t^\kappa$*).*

7. Extensions

7.1. WITNESSED $\mathcal{L}\updownarrow$-LOGICS

DEFINITION 21. An $\mathcal{L}\updownarrow$*-logic* is a simple *closed* extension (extension by means of closed axioms only) of $\mathbf{K}\updownarrow$.

The strong completeness theorem is provable, mutatis mutandis, for all $\mathcal{L}\updownarrow$-logics:

THEOREM 22. *For each* $\mathcal{L}\updownarrow$*-logic* $\mathbf{L}$*, every consistent in* $\mathbf{L}$ *set of closed formulae is satisfied in some* $\mathbf{L}$*-model.*

Of course, a valuable completeness theorem would guarantee satisfiability in a model *based on an* $\mathbf{L}$*-frame*. Very few general results in that direction exist in modal and temporal logic, but for $\mathcal{L}\updownarrow$-logics there is an important one, stated in Theorem 24 below.

For any formula θ we denote $w(\theta) = \mathbf{E}\downarrow\mathbf{A}(\uparrow\leftrightarrow\theta)$. $w(\theta)$ says that θ is true at exactly one instant of the model.

DEFINITION 23.
1. A formula φ is *witnessed* if it is of the kind

$$\varphi = w(q_1) \wedge \ldots \wedge w(q_k) \rightarrow \psi(q_1, \ldots, q_k),$$

   where ψ is a closed formula which only contains propositional variables amongst $q_1, \ldots, q_k$. In particular, every formula without propositional variables is (equivalent to) a witnessed one.
2. An $\mathcal{L}\updownarrow$-logic is *witnessed* if it is axiomatized over $\mathbf{K}\updownarrow$ by means of witnessed formulae only.

THEOREM 24. *Let* $\mathbf{L}$ *be a witnessed* $\mathcal{L}\updownarrow$*-logic. Every* $\mathbf{L}$*-consistent set of closed formulae is satisfied in some model based on an* $\mathbf{L}$*-frame.*

Proof. Given a witnessed formula (which we can assume to be written over $p_1, \ldots, p_k$) $\varphi = w(p_1) \wedge \ldots \wedge w(p_k) \rightarrow \psi(p_1, \ldots, p_k)$ we define the first-order formula $\alpha(\varphi) = \forall x \forall z_1 \ldots \forall z_k \sigma(ST(\psi))$, where $z_1, \ldots, z_k$ are variables not occurring in $ST(\psi)$ and $\sigma(ST(\psi))$ is the result of uniform substitution in $ST(\psi)$ of all occurrences of atomic formulae of the kind $P_i y$ by $y = z_i$ respectively, for $i = 1, \ldots, k$. It is not difficult to verify the following: for every temporal frame F, $F \models \varphi$ iff $F \Vdash \alpha(\varphi)$. Now, let φ be an axiom of $\mathbf{L}$ and $\mathcal{M} = \langle T, R, V \rangle$ be a clock-model of $\mathbf{L}$. Then $\mathcal{M}$ satisfies all variants $w(p_{i_1}) \wedge \ldots \wedge w(p_{i_k}) \rightarrow \psi(p_{i_1}, \ldots, p_{i_k})$ of φ since they are theorems of $\mathbf{L}$. This implies that $\langle T, R \rangle \Vdash \alpha(\varphi)$ and hence $\langle T, R \rangle; \models \varphi$. Thus, every clock-model of a witnessed $\mathcal{L}\updownarrow$-logic $\mathbf{L}$ is based on a frame for $\mathbf{L}$. Therefore, by Lemma 17 and Theorem 19, every consistent in $\mathbf{L}$ set of closed formulae is satisfied in a model based on an $\mathbf{L}$-frame.

In fact, any witnessed formula

$$\varphi = w(q_1) \wedge \ldots \wedge w(q_k) \rightarrow \psi(q_1, \ldots, q_k),$$

is equivalent (with respect to validity in a frame) to a formula without propositional variables but with more reference pairs:

$$\varphi^\star = \mathbf{E} \downarrow_1 \ldots \mathbf{E} \downarrow_k \psi(\uparrow_1 / q_1, \ldots \uparrow_k / q_k)$$

where the pairs $(\downarrow_1, \uparrow_1), \ldots, (\downarrow_k, \uparrow_k)$ do not occur in ψ.

Therefore, in a language with sufficiently many reference pairs witnessed logics admit even simpler axiomatizations.

Here are a few simple examples of temporal logics whose complete axiomatizations in the classical language involve sophisticated completeness proofs, but are readily axiomatized as witnessed $\mathcal{L}\updownarrow_t$-logics.

COROLLARY 25. *The following extensions of* $\mathbf{K}\updownarrow_t$ *are strongly complete.*

1. *The* Logic of Linear (irreflexive) Time*:* $\mathbf{LT}\updownarrow_t = \mathbf{K}\updownarrow_t +$
   - *(irreflexivity)* $\downarrow \mathbf{G} \neg \uparrow$,
   - *(transitivity)* $\downarrow \mathbf{A}(\mathbf{FF} \uparrow \rightarrow \mathbf{F} \uparrow)$,
   - *(linearity)* $\downarrow \mathbf{A}(\mathbf{F} \uparrow \vee \uparrow \vee \mathbf{P} \uparrow)$.
2. *The* Logic of Forward Branching Time*:* $\mathbf{FBT}\updownarrow_t = \mathbf{K}\updownarrow_t +$
   - *(common histories)* $\downarrow \mathbf{APF} \uparrow$,
   - *(linear past)* $\downarrow \mathbf{GH}(\mathbf{F} \uparrow \vee \uparrow \vee \mathbf{P} \uparrow)$.
3. *The* Logic of Discrete Linear Time*:* $\mathbf{DLT}\updownarrow_t = \mathbf{LT}\updownarrow_t^+$
   - *(immediate predecessor)* $\downarrow \mathbf{PGG} \neg \uparrow$,
   - *(immediate successor)* $\downarrow \mathbf{FHH} \neg \uparrow$.
4. *The* Logic of Rational Time*:* $\mathbf{LRT}\updownarrow_t = \mathbf{LT}\updownarrow_t +$
   - *(no beginning)* $\mathbf{P}\top$,
   - *(no end)* $\mathbf{F}\top$,
   - *(density)* $\downarrow \mathbf{A}(\mathbf{F} \uparrow \rightarrow \mathbf{FF} \uparrow)$.

7.2. LOCALLY DEFINABLE OPERATORS

The languages with reference pointers have a built-in mechanism for self-extension with operators (modalities) which can be defined *locally* (at the point of evaluation) with an axiom but when added to a language with a limited number of reference pairs they increase its expressiveness.

Here is a general scheme for introduction of such unary operators. Let $\theta(p,q)$ be a formula containing the propositional variables p and q (and possibly others). Then we define a new operator $\#_\theta$ with the following semantic clause:

$$ST(\#_\theta\psi) = ST(\theta(p,\psi/q)(x = v/Pv),$$

where y ranges over *all* (free and bound) variables occurring in $ST(\theta(p,\psi/q)$.

Intuitively, θ is a scheme in which q is a placeholder for the argument and p a placeholder for a reference pointer to the point of evaluation.

Two examples:

1. $\theta(p,q) = \mathbf{E}(\Diamond p \wedge q)$. It is easy to see that $ST(\#_\theta\psi)$ is equivalent to $\exists y(Ryx \wedge ST(\psi)(y/x))$, i.e. $\#_\theta\psi)$ has the semantics of $\mathbf{P}\psi$.
2. $\theta(p,q) = \mathbf{E} \downarrow \mathbf{A}(p \to q)$. Then $ST(\#_\theta\psi)$ turns out equivalent to $\exists w ST(\psi)$, so the truth condition of $\#_\theta\psi$ at a point x says: there exists a reference point for the pointer $\downarrow$ which makes ψ true at x, i.e. this operator introduces the adverb "ever" in the language.

Note that in fact the locally definable operator $\#_\theta\psi$ can be explicitly defined as $\downarrow_\alpha \ \theta(\uparrow_\alpha \ /p, \psi/q)$ where $(\downarrow_\alpha, \uparrow_\alpha)$ is an *extra* reference pair. (otherwise an unwanted clash with unbound reference pointers in ψ may occur). Therefore the locally definable operators can be regarded as a "syntactic sugar" which avoids explicit over-referencing by introducing new linguistic structures and thus making the language more versatile.

The above observation enables us to uniformly axiomatize the operator $\#_\theta\psi$ by means of one additional scheme:

(A_θ):

$$\downarrow \mathbf{A}(p \leftrightarrow \uparrow) \to (\#_\theta\psi \leftrightarrow \theta).$$

The soundness of that schema is easily verified.

THEOREM 26. *The logic* $\mathbf{K}\updownarrow +(A_\theta)$ *is strongly complete with respect to the semantics of* $\#_\theta\psi$.

Proof. The only necessary addition to the proof of 19 is the clause for $\#_\theta\psi$ in the proof of the truth-lemma, which is treated similarly to $\downarrow \psi$, due to the additional axiom (A_θ).

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