

Modal Logic and Universal Algebra

I. Modal axiomatizations of structures. *

Valentin Goranko¹ and Dimiter Vakarelov²

¹Department of Mathematics, Rand Afrikaans University
PO Box 524, Auckland Park 2006,
Johannesburg, South Africa
E-mail: vfg@na.rau.ac.za,
URL: http://www.rau.ac.za/staff/goranko_vf

²Department of Mathematical Logic
Faculty of Mathematics and Computer Science,
Sofia University
blvd James Bouchier 5, 1126 Sofia, Bulgaria
E-mail: dvak@fmi.uni-sofia.bg

ABSTRACT. We study the general problem of axiomatizing structures in the framework of modal logic and present a uniform method for complete axiomatization of the modal logics determined by a large family of classes of structures of any signature.

Keywords: *complex algebras of structures, modal logics, axiomatizations, completeness, definability.*

Introduction

By a structure we mean a non-empty universe with a collection of functions and predicates of finite arity. Every such a structure can be regarded as a generalized Kripke frame with a collection of accessibility relations corresponding to the principal predicates and the graphs of the principal

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functions. Given a class of structures \mathbf{C} of a signature σ (σ -structures), we consider the induced multi-modal language M_σ with modalities corresponding to these accessibility relations, and the multi-modal logic $ML_{\mathbf{C}}^\sigma$ in that language which captures the modal formulae valid in \mathbf{C} . On the other hand, with every class of σ -structures we associate the class of their *complex algebras* (*power structures* in Brink 1993) which provides the algebraic semantics for that logic.

In this paper we give uniform schemes for complete axiomatization of the modal logics for a broad family of elementary classes (incl. all Π_2^0 -classes) of σ -structures, assuming that the *difference* operator is present in the corresponding modal language. That operator is definable in many important classes of structures, such as all expansions of implicative lattices (e.g. Heyting and Boolean algebras) and all expansions of groups (e.g. rings and modules). In other cases, such as arbitrary lattices and semigroups, it has to be added to the language. The difference operator has recently been widely studied and used as a useful tool both for boosting the expressiveness of modal languages and for axiomatizing extended modal logics, combined with using specific non-Hilbert style *context dependent* rules, often referred to as “Gabbay-style” rules. For some related results see Goranko 1990, Gargov and Goranko 1993, de Rijke 1992, de Rijke 1993, Venema 1993, Goranko 1998.

This paper, in particular, contributes further in that direction, by presenting a general completeness proof which does not assume versatility of the languages, as in Venema 1993. It is based on a modified canonical construction, very close to the Henkin completeness method for the first-order logic. The idea of the construction is taken from Passy and Tinchev 1991 where similar construction has been applied to PDL with so called “data constants”.

Besides the various specific applications for particular classes of structures, the obtained results can be appreciated from a few general viewpoints.

First, the modal logic $ML_{\mathbf{C}}^\sigma$ corresponding to a class of structures \mathbf{C} encapsulates the universal fragment of the monadic second-order theory of \mathbf{C} , and thus provides an axiomatization for that fragment.

Second, that modal logic can be regarded as an axiomatization of the class of corresponding *complex algebras* (see Goldblatt 1989) of the structures in \mathbf{C} .

Third, from viewpoint of modal logic, the structures in \mathbf{C} can be regarded as Kripke frames in which the universe of possible worlds has a specific algebraic structure, besides the traditional relational component. In cases when suitable representation theorems exist, (e.g. for groups, lattices, Boolean algebras etc.) that algebraic structure can accordingly

reflect some internal structure of the possible worlds themselves. A numbers of recent papers contain particular studies of such “structured” Kripke frames. For instance in many-dimensional modal logic Venema 1992 possible worlds have an internal structure of n -tuples; in some arrow logics Vakarelov 1992, Vakarelov 1996 possible worlds have a structure of the arrows of a multigraph; in the interval tense logic Venema 1990 they have a structure of intervals of some ordered set. We should also mention Orłowska’s work Orłowska 1996 where relational framework for modalization of semigroups and various extensions is developed.

The main precursor of the present paper is Goranko and Vakarelov 1998 where the method presented here is applied to axiomatize the modal logic of Boolean algebras, i.e. the class of frames (called there hyperboolean algebras) the worlds of which are elements of a Boolean algebra, and the frame is endowed with the usual Boolean operations.

The structure of the paper is as follows. In section 1 we introduce σ -structures and complex σ -algebras, as well as the syntax and semantics of the corresponding multimodal languages. We also discuss the role of the difference operator and define a translation of the universal first-order formulae on σ -structures to identities in complex σ -algebras. Section 2 introduces an axiomatic system for the minimal multimodal logic of the class of all σ -structures and presents a detailed proof of its completeness. Section 3 considers modal logics for classes of σ -structures. It gives a uniform axiomatization, by means of additional axioms, of the modal logic of any universal first-order class of σ -structures, which is then extended to a uniform axiomatization, by means of additional rules, of the modal logic of any universal-existential class. We also briefly discuss second-order definability on structures and second-order quantification over substructures or congruences, definable in the modal languages and their expressiveness as universal fragments of monadic second-order languages. The paper ends with some concluding remarks and questions arising from the present study.

The reader is assumed to have a technical knowledge on modal logics and some background in abstract algebra.

1 Preliminaries

1.1 σ -structures and complex σ -algebras.

Let σ be an arbitrary signature. For notational convenience we shall assume that it consists of finitely many functional, constant and predicate symbols resp. $f_1, \dots, f_k; c_1, \dots, c_l; r_1, \dots, r_m$, and let ρ be the arity function.

A σ -structure \underline{W} is any non-empty set with designated functions, constants and relations corresponding to the symbols in σ :

$$\underline{W} = \langle W; F_1, \dots, F_k; C_1, \dots, C_l; R_1, \dots, R_m \rangle$$

With every signature σ we associate an algebraic (purely functional) signature σ^* which extends the signature of Boolean algebras, containing the usual Boolean symbols \perp, \neg, \wedge (and $\top, \vee, \Rightarrow, \Leftrightarrow$ accordingly definable), with a set of operators $\{\langle f_1 \rangle, \dots, \langle f_k \rangle, \langle c_1 \rangle, \dots, \langle c_l \rangle, \langle r_1 \rangle, \dots, \langle r_m \rangle\}$ of arities respectively $\rho(f_i), 0$, and $\rho(r_i) - 1$ corresponding to the functional, constant, and predicate symbols.

Further, with every σ -structure \underline{W} we associate a σ^* -structure called here its *complex σ -algebra*:

$$\mathcal{P}(\underline{W}) = \langle \mathcal{P}(W); \emptyset, -, \cap; \langle F_1 \rangle, \dots, \langle F_k \rangle; \langle C_1 \rangle, \dots, \langle C_l \rangle; \langle R_1 \rangle, \dots, \langle R_m \rangle \rangle$$

which is the Boolean algebra of sets over W endowed with the corresponding *power operations*, respectively defined as follows:

$$\langle F_i \rangle(X_1, \dots, X_n) = \{F_i(x_1, \dots, x_n) \mid x_1 \in X_1, \dots, x_n \in X_n\},$$

$$\langle C_i \rangle = \{C_i\},$$

$$\text{and } \langle R_i \rangle(X_1, \dots, X_n) =$$

$$\{x \in W \mid (\exists x_1 \in X_1) \dots (\exists x_n \in X_n)(R_i(x, x_1, \dots, x_n))\}.$$

In particular, for unary relational symbols, $\langle R_i \rangle = \{x \in W \mid R_i(x)\}$.

For every non-constant symbol s in σ we introduce a “box” operator, dual to the “diamond” operator:

$$[s](A_1, \dots, A_n) = \neg \langle s \rangle(\neg A_1, \dots, \neg A_n).$$

Given a complex algebra $\mathcal{P}(\underline{W})$ we further extend it to a *differentiated* complex algebra by adding to its signature a *difference operator* $\langle \neq \rangle$ defined as follows:

$$\langle \neq \rangle X =_{\text{def}} \{x \in W \mid \exists y \in X \text{ and } x \neq y\}.$$

We denote its dual by $[\neq]$.

Note that differentiated complex algebras are Boolean algebras with (normal and additive) operators in the sense of Jónsson and Tarski (Jónsson and Tarski 1951).

It turns out that for a large family of important structures the difference operator is definable by means of the other operators in the complex algebras. Two important general cases are given below.

- **Implicative lattices and expansions.**

As shown in Goranko and Vakarelov 1998, the difference operator can be defined in complex Boolean algebras (called there “hyperboolean algebras”) as follows: $\langle \neq \rangle A =$

$$(\neg A \wedge ((A \langle \rightarrow \rangle \langle 1 \rangle) \langle \rightarrow \rangle \top)) \vee \\ (((A \langle \rightarrow \rangle A) \wedge \neg \langle 1 \rangle) \langle \rightarrow \rangle \langle 1 \rangle) \langle \rightarrow \rangle \top)$$

This definition obviously works in all implicative lattices, hence all pseudo-Boolean (Heyting) algebras, as well as all structures expanding pseudo-Boolean or Boolean algebras with additional operators, such as Post algebras, modal, relation, cylindric algebras, etc.

- **Groups and expansions.**

The difference operator can be defined for complexes of groups, too:¹

$$\langle \neq \rangle A = (\neg A \langle \circ \rangle (A \langle \circ \rangle A^{(\prime)})) \vee (((A \langle \circ \rangle A^{(\prime)}) \wedge \neg \langle e \rangle) \langle \circ \rangle \top).$$

where $\langle \circ \rangle$, $\langle \prime \rangle$, and $\langle e \rangle$ are the power operators of the group operations and identity.

Thus, $\langle \neq \rangle$ is definable in all expansions of groups, such as ordered groups, rings, modules, vector spaces, (ordered) fields etc.

On the other hand, the difference operator seems not definable in other important classes of structures such as lattices (even distributive lattices with top and bottom) and semigroups (even monoids).

Henceforth, we shall assume that the difference operator is part of the signature of any complex algebra, unless otherwise indicated.

1.2 Modal σ -languages and σ -frames. Syntax and semantics of multimodal σ -logics.

Every signature σ determines a propositional multi-modal language M_σ containing a denumerable set $VAR = \{p_1, p_2, \dots\}$ of propositional variables, the usual Boolean symbols \perp, \neg, \wedge (and $\top, \vee, \Rightarrow, \Leftrightarrow$ accordingly definable) and a set of modalities $\{\langle \neq \rangle, \langle f_1 \rangle, \dots, \langle f_k \rangle, \langle c_1 \rangle, \dots, \langle c_l \rangle, \langle r_1 \rangle, \dots, \langle r_m \rangle\}$ corresponding to the algebraic operators of σ^* . For technical convenience we identify the language M_σ with the algebraic language for complex σ -algebras and use the same symbols in both languages. The notion of a formula of M_σ is defined as usual: besides the classical formulae, $\langle c \rangle$ is a formula for every constant symbol c , and if A_1, \dots, A_n are formulae and $\langle \alpha \rangle$ is an n -ary modality, then $\langle \alpha \rangle(A_1, \dots, A_n)$ is a formula, too. $[\alpha](A_1, \dots, A_n)$ is an abbreviation of $\neg \langle \alpha \rangle(\neg A_1, \dots, \neg A_n)$ as usual. We adopt the standard omission of parentheses. The set of formulae of M_σ will be denoted by FOR_σ .

¹A similar formula was independently proposed by Yde Venema

According to the notational convention above, every modal formula of M_σ can also be regarded as an algebraic term for complex σ -algebras.

Complex σ -algebras provide an algebraic semantics for the language M_σ in the usual way: by a *valuation* in a complex σ -algebra $\mathcal{P}(\underline{W})$ we mean any function v from VAR into $\mathcal{P}(\underline{W})$. Each valuation v is then extended to arbitrary formulae by a straightforward induction:

$$\begin{aligned}
v(\perp) &= \emptyset, \\
v(A \wedge B) &= v(A) \cap v(B) \\
v(\neg A) &= W - v(A) \\
v(\langle \neq \rangle A) &= \langle \neq \rangle(v(A)), \\
v(\langle s_i \rangle) &= \langle S_i \rangle \text{ for any constant or unary relational symbol } s_i, \\
v(\langle f_i \rangle(A_1, \dots, A_n)) &= \langle F_i \rangle(v(A_1), \dots, v(A_n)) \text{ for any } n\text{-ary functional} \\
\text{symbol } f_i, \\
v(\langle r_i \rangle(A_1, \dots, A_n)) &= \langle R_i \rangle(v(A_1), \dots, v(A_n)) \text{ for any } n+1\text{-ary predicate} \\
\text{symbol } r_i.
\end{aligned}$$

A formula A is *valid* in the complex algebra $\mathcal{P}(\underline{W})$ if $v(A) = W$ for any valuation v .

Given a class \mathbf{C} of σ -structures, by \mathbf{C}^* we denote the class of complex algebras of the structures from \mathbf{C} .

A formula of M_σ is valid in \mathbf{C} if it is valid in every complex algebra from \mathbf{C}^* . A formula of M_σ is *valid* if it is valid in every complex σ -algebra.

The semantics given above can be regarded as a Kripke-style semantics on σ -structures by considering every complex σ -algebra as a Kripke frame, called here a σ -*frame*:

$$\mathbf{F}_{\underline{W}} = \langle W; \neq, G_{F_1}, \dots, G_{F_k}, \{C_1\}, \dots, \{C_l\}, R_1, \dots, R_m \rangle$$

where G_F is the graph of the function F , though taken with the last argument first: $G_F = \{(F(x_1, \dots, x_n), x_1, \dots, x_n) | x_1, \dots, x_n \in W\}$ (just in order to comply with the customary notation in modal logic).

Then the truth definitions above become standard truth definitions of the Kripke semantics respectively generalized for n -ary modalities. They can be reformulated in terms of the satisfaction relation $x \models_v A$: “the formula A is true at x under the valuation v ”, defined as follows: $x \models_v A$ iff $x \in v(A)$. This relation can be defined independently by induction on A as usual.

A pair $(\mathbf{F}_{\underline{W}}, v)$ for a σ -structure \underline{W} and a valuation v in W is called a *model* (over \underline{W}). A formula A is *valid in a model* $(\mathbf{F}_{\underline{W}}, v)$ if for any $x \in W$, $x \models_v A$. A is *valid in $\mathbf{F}_{\underline{W}}$* if it is valid in all models over \underline{W} . Clearly, any formula is valid in a σ -frame $\mathbf{F}_{\underline{W}}$ if and only if it is valid in the complex algebra of \underline{W} .

Finally, a formula is *valid in a σ -structure \underline{W}* if it is valid in its complex algebra.

Hereafter, an arbitrary finitary signature σ is fixed, unless otherwise indicated.

1.3 Using the difference operator

The presence of the difference operator in the complex algebras of a class of structures renders the modal language very expressive (see e.g. Gargov and Goranko 1993, de Rijke 1992, de Rijke 1993). Here we shall only mention some algebraic consequences which are of importance for the axiomatization of the corresponding modal logic.

The following are definable by means of the difference operator $\langle \neq \rangle$:

the universal modalities:

$$\langle U \rangle X =_{\text{def}} X \vee \langle \neq \rangle X \text{ and } [U]X =_{\text{def}} \neg \langle U \rangle \neg X = X \wedge [\neq]X,$$

and *the “Only” operator:* $\mathcal{O}X =_{\text{def}} X \wedge [\neq] \neg X$.

Remark 1.1 Taking the universal modality and the Only operator as primitives we can define the difference operator: $\langle \neq \rangle X =_{\text{def}} \langle U \rangle X \wedge \neg \mathcal{O}X$. This makes possible to base our investigations not on difference but on the universal modality and the Only operator. In some sense this will be more natural, because in the applications of difference we mainly use the universal modality and the Only operator. We prefer however to preserve the tradition, because difference is better known and more popular in the area of modal logic.

Remark 1.2 Using the universal modality we can define a *discriminator term* $t(x, y, z)$ in the complex algebras:

$$t(x, y, z) = ([U](x \Leftrightarrow y) \wedge z) \vee (\neg [U](x \Leftrightarrow y) \wedge x).$$

The existence of such a term has a significant impact on the algebraic properties of the complex algebras and the variety generated by them (see e.g. Burris and Sankappanavar 1981). In particular, it allows for reduction of all universal formulae of the language of complex σ -algebras to identities.

Moreover, using the universal modality $[U]$ and the operator \mathcal{O} we can construct a uniform translation τ of all universal formulae of the first-order language L_σ of σ -structures into identities in the first-order language L_σ^* of their complex σ -algebras.

For simplicity and further applications we assume that the atomic formulae of L_σ are the following: $x = x_j$, $x = c_i$, $x = f_i(x_1, \dots, x_n)$, $r_i(x, x_1, \dots, x_n)$, where x, x_1, \dots, x_n are individual variables of L_σ . We may do that without loss of generality since we consider languages with equality, hence every term can be “unnested” into a conjunction of such atomic formulae. (That assumption, however, is not essential for the existence of the translation.) Also, for notational convenience we use the same variables for the languages L_σ and L_σ^* .

Definition 1.3 The translation τ is defined as follows (x, x_1, \dots, x_n are variables):

For atomic formulae:

- $\tau(x = x_j) = \langle U \rangle (\mathcal{O}x \wedge \mathcal{O}x_j)$,
- $\tau(x = c_i) = \langle U \rangle (\mathcal{O}x \wedge \langle c_i \rangle)$ for any constant symbol c_i ,
- $\tau(x = f_i(x_1, \dots, x_n)) = \langle U \rangle (\mathcal{O}x \wedge \langle f_i \rangle (\mathcal{O}x_1, \dots, \mathcal{O}x_n))$ for any n -ary functional symbol f_i ,
- $\tau(r_i(x, x_1, \dots, x_n)) = \langle U \rangle (\mathcal{O}x \wedge \langle r_i \rangle (\mathcal{O}x, \mathcal{O}x_1, \dots, \mathcal{O}x_n))$ for any relational symbol r_i .

For open formulae:

- $\tau(\neg\varphi) = \neg\tau(\varphi)$,
- $\tau(\varphi \wedge \psi) = \tau(\varphi) \wedge \tau(\psi)$.

For universal formulae:

Let $\psi = (\forall x_1) \dots (\forall x_n) \varphi(x_1, \dots, x_n)$ where $\varphi(x_1, \dots, x_n)$ is an open formula. Then

- $\tau(\psi) = \langle U \rangle (\mathcal{O}x_1) \wedge \dots \wedge \langle U \rangle (\mathcal{O}x_n) \Rightarrow \tau(\varphi(x_1, \dots, x_n))$.

Lemma 1.4 (Modal definability lemma for universal classes) *A closed universal formula γ is valid in a σ -structure \underline{W} iff the corresponding identity $\tau(\gamma) = \top$ holds in $P(\underline{W})$.*

Proof. Straightforward. ♣

2 The minimal modal logic for the class of all σ -structures. σ -logics.

In this section we axiomatize the minimal modal logic ML^σ consisting of the valid formulae of the class of all complex algebras of σ -structures for any signature σ , and prove completeness of that axiomatic system.

2.1 Axiomatic system for ML^σ .

2.1.1 Axioms:

I. Enough propositional tautologies.

II. Axioms of the minimal multimodal logic:

$$(K[\alpha]) \quad [\alpha](A_1, \dots, A_{i-1}, B \Rightarrow C, A_{i+1}, \dots, A_n) \Rightarrow \\ ([\alpha](A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n) \Rightarrow [\alpha](A_1, \dots, A_{i-1}, C, A_{i+1}, \dots, A_n)),$$

and

$$(Dual) \quad \langle \alpha \rangle(A_1, \dots, A_n) \Leftrightarrow \neg[\alpha](\neg A_1, \dots, \neg A_n)$$

for every n -ary operator $[\alpha]$ and $i \in \{1, \dots, n\}$.

III. Axioms for $[\neq]$:

$$(D1) \quad A \vee [\neq]\neg[\neq]A, \\ (D2) \quad \langle \neq \rangle \langle \neq \rangle A \Rightarrow (A \vee \langle \neq \rangle A),$$

IV. Axioms for $[U]$:

$$(U) \quad [U]A \Rightarrow [\alpha](A_1, \dots, A_{i-1}, A, A_{i+1}, \dots, A_n),$$

V. Axioms for the constant and functional modalities:

(c) For every constant symbol c from σ ,

$$(c1) \quad \langle U \rangle \mathcal{O} \langle c \rangle, \\ (c2) \quad \langle c \rangle \Rightarrow \mathcal{O} \langle c \rangle,$$

(f) For every n -ary functional symbol f from σ ,

$$(f1) \quad (\langle U \rangle \mathcal{O} A_1 \wedge \dots \wedge \langle U \rangle \mathcal{O} A_n) \Rightarrow \langle U \rangle \langle f \rangle (\mathcal{O} A_1, \dots, \mathcal{O} A_n)$$

and

$$(f2) \quad \langle f \rangle (\mathcal{O} A_1, \dots, \mathcal{O} A_n) \Rightarrow \mathcal{O} \langle f \rangle (\mathcal{O} A_1, \dots, \mathcal{O} A_n).$$

2.1.2 Rules of inference:

Uniform substitution (SUB):

$$\frac{A}{sub(A)}$$

where $sub(A)$ is the result of application of any uniform substitution of formulae for variables in A .

Modus Ponens (MP):

$$\frac{A, A \Rightarrow B}{B}$$

Necessitation for $[U]$ (N[U]):

$$\frac{A}{[U]A}$$

Witness rule schema:

(WIT):

$$\frac{A \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, \mathcal{O}p \Rightarrow B, A_{i+1}, \dots, A_n)), \text{ for all } p \in \text{VAR}}{A \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, B, A_{i+1}, \dots, A_n))}$$

for any n -ary modality $[\alpha]$ and $i \in \{1, \dots, n\}$.

Let us note that, as usual, the Witness rules can be replaced by finitary rules assuming the proviso not for all $p \in \text{VAR}$ but for some p *not occurring* in A, B, A_1, \dots, A_n .

The Witness rule used here shares the idea of Gabbay's Irreflexivity rule in Gabbay 1981 and originates from Passy & Tinchev's COV rule (see Passy and Tinchev 1991, Gargov and Goranko 1993). Such rules and have been used in a number of recent publications, e.g. Venema 1992, Venema 1993, Gargov and Goranko 1993, de Rijke 1992, de Rijke 1993, Balbiani et al. 1997.

Remark 2.1 As we have mentioned earlier, the difference operator $\langle \neq \rangle$ and its dual can be defined by the universal modality $[U]$ and \mathcal{O} : $\langle \neq \rangle A = \langle U \rangle A \wedge \neg \mathcal{O}A$. So, the logic $ML_{\mathcal{C}}^{\neq}$ can be alternatively axiomatized using only the operators $[U]$ and \mathcal{O} . Then, instead of the axioms for $\langle \neq \rangle$ (D1 and D2), we can use the following list of axiom schemes:

- (K[U]) $[U](A \Rightarrow B) \Rightarrow ([U]A \Rightarrow [U]B)$
- (S5[U]) $[U]A \Rightarrow A, A \vee [U]\neg[U]A, [U]A \Rightarrow [U][U]A,$
- (O1) $[U](A \Leftrightarrow B) \wedge \mathcal{O}A \Rightarrow \mathcal{O}B,$
- (O2) $\mathcal{O}A \Rightarrow A,$
- (O3) $\mathcal{O}A \Rightarrow \mathcal{O}\mathcal{O}A,$
- (O4) $A \wedge \langle U \rangle \mathcal{O}A \Rightarrow \mathcal{O}A,$
- (O5) $\langle U \rangle A \wedge \mathcal{O}(A \vee B) \Rightarrow \mathcal{O}A.$

The rules are as before, only for the rule (WIT) with $\alpha = [\neq]$ we use the definition of $[\neq]$.

For the proof of completeness of the axiomatic system for ML^{σ} we need some preparatory work, which will be done in the following subsections.

2.2 σ -logics. The sublogic ML^{\neq} .

By a σ -logic we mean any set L of formulae closed under substitution of propositional variables, containing the axioms and closed under the rules of the logic ML^{σ} . A σ -logic L is *consistent* if \perp is not a theorem of L .

An important sublogic of ML^{σ} is based on the axioms from the groups I-IV, Modus Ponens and Necessitation for $[U]$ (witness rules are dropped).

It is the minimal normal polymodal logic with difference modality and will be denoted by ML^\neq . All the axioms of this logic are in a Sahlqvist form, so it is complete in the semantics, definable by its axioms. The semantics for the difference modality now is not standard. If we denote the accessibility relation for $\langle \neq \rangle$ by R_\neq , we see that it satisfies the following conditions:

- $xR_\neq y \rightarrow yR_\neq x$,
- If $xR_\neq y$ and $yR_\neq z$ then $x = z$ or $xR_\neq z$.
- If $R_\alpha(x, x_1, \dots, x_n)$ then $xR_U x_i$ for each $i = 1, \dots, n$.
- If we denote the accessibility relation of the modality $[U]$ by R_U then we have: $xR_U y$ iff $x = y$ or $xR_\neq y$. It is easy to see that R_U is an equivalence relation containing R_\neq and hence $[U]$ is an S5 modality. We will use this fact later on saying “by S5”.
- The semantics of \mathcal{O} now is also non-standard:
 $x \models \mathcal{O}A$ iff $x \models A$ and $(\forall y)(xR_\neq y \Rightarrow y \not\models A)$.

Later on we make use of the completeness theorem of ML^\neq with respect to the above nonstandard semantics instead of making formal derivations from the axioms. We will refer to that as: “by the nonstandard semantics of ML^\neq ”.

Lemma 2.2 *The following rules are derivable in ML^\neq :*

(i) *Necessitation for arbitrary modality $[\alpha]$ ($N[\alpha]$):*

$$\frac{A}{[\alpha]A},$$

(ii) *Monotonicity for $[\alpha]$ and $\langle \alpha \rangle$:*

$$\frac{A \Rightarrow B}{[\alpha](P_1, \dots, P_{i-1}, A, \dots, P_n) \Rightarrow [\alpha](P_1, \dots, P_{i-1}, B, \dots, P_n)},$$

and

$$\frac{A \Rightarrow B}{\langle \alpha \rangle (P_1, \dots, P_{i-1}, A, \dots, P_n) \Rightarrow \langle \alpha \rangle (P_1, \dots, P_{i-1}, B, \dots, P_n)},$$

(iii a)

$$\frac{A \Rightarrow [U]B}{\langle U \rangle A \Rightarrow B}$$

(iii b)

$$\frac{\langle U \rangle A \Rightarrow B}{A \Rightarrow [U]B}$$

Proof. (i) follows from the Necessitation for $[U]$ ($N[U]$) and axiom IV.
(ii) follows from (i) and axiom II.
(iii) follows by S5. ♣

Lemma 2.3 *The following variations of the Witness rule are derivable in the logic ML^σ .*

(i) Basic Witness rule (WIT_0):

$$\frac{\mathcal{O}p \Rightarrow A, \text{ for all } p \in \text{VAR}}{A}$$

(ii) Extended Witness rule (WIT_1):

$$\frac{A \Rightarrow [U](B \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, \mathcal{O}p \Rightarrow C, \dots, A_n))), \text{ for all } p \in \text{VAR}}{A \Rightarrow [U](B \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, C, \dots, A_n)))}$$

for any n -ary modality $[\alpha]$ and $i \in \{1, \dots, n\}$,

Proof. (i) Easily obtained from (WIT) by taking $\alpha = U, i = 1, B = A$, and $A = \top$, using the Necessitation for $[U]$ and the ML^σ -theorem $[U]A \Rightarrow A$.
(ii) Suppose

$$ML^\sigma \vdash A \Rightarrow [U](B \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, \mathcal{O}p \Rightarrow C, A_{i+1}, \dots, A_n)))$$

for all $p \in \text{VAR}$. Then, by the rule 2.2(iia),

$$ML^\sigma \vdash \langle U \rangle A \Rightarrow (B \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, \mathcal{O}p \Rightarrow C, A_{i+1}, \dots, A_n)))$$

for all $p \in \text{VAR}$, hence

$$ML^\sigma \vdash (\langle U \rangle A \wedge B) \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, \mathcal{O}p \Rightarrow C, A_{i+1}, \dots, A_n)).$$

Then, by (WIT),

$$ML^\sigma \vdash (\langle U \rangle A \wedge B) \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, C, A_{i+1}, \dots, A_n))$$

so

$$ML^\sigma \vdash \langle U \rangle A \Rightarrow (B \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, C, A_{i+1}, \dots, A_n)))$$

hence, by the rule 2.2(iiib),

$$ML^\sigma \vdash A \Rightarrow [U](B \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, C, A_{i+1}, \dots, A_n))).$$

♣

Note that every instance of all three Witness rule schemata can be represented in a uniform way:

$$\frac{\varphi(\mathcal{O}p \Rightarrow A), \text{ for all } p \in \text{VAR}}{\varphi(A)}$$

for a suitable formula $\varphi(q)$. We shall refer to such a representation as $\varphi(\mathcal{O}p \Rightarrow A)/\varphi(A)$.

Note also that the antecedent of any instance of (WIT) or (WIT₁) can be \top , and therefore omitted.

In the next lemma we list some technical theorems of ML^\neq , which will be of later use.

Lemma 2.4 *The following formulae are theorems of ML^\neq :*

- (i) $\langle U \rangle (\mathcal{O}p \wedge A) \Rightarrow [U](\mathcal{O}p \Rightarrow A)$,
- (ii) $\langle U \rangle \mathcal{O}p \Rightarrow (\langle U \rangle (\mathcal{O}p \wedge A) \Leftrightarrow [U](\mathcal{O}p \Rightarrow A))$,
- (iii) $\langle U \rangle (\mathcal{O}A \wedge \mathcal{O}C) \wedge \langle U \rangle (\mathcal{O}B \wedge \mathcal{O}C) \Rightarrow \langle U \rangle (\mathcal{O}A \wedge \mathcal{O}B)$,
- (iv) $\langle U \rangle \mathcal{O}p \Rightarrow (\langle U \rangle (\mathcal{O}p \wedge \neg A) \Leftrightarrow \neg \langle U \rangle (\mathcal{O}p \wedge A))$,
- (v) $\langle U \rangle (\mathcal{O}p \wedge (A \vee B)) \Leftrightarrow (\langle U \rangle (\mathcal{O}p \wedge A) \vee \langle U \rangle (\mathcal{O}p \wedge B))$,
- (vi) $\langle U \rangle (\mathcal{O}p \wedge \langle \alpha \rangle (\mathcal{O}q_1 \wedge A_1, \dots, \mathcal{O}q_n \wedge A_n)) \Leftrightarrow$
 $\langle U \rangle (\mathcal{O}p \wedge \langle \alpha \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \wedge \langle U \rangle (\mathcal{O}q_1 \wedge A_1) \wedge \dots \wedge \langle U \rangle$
 $(\mathcal{O}q_n \wedge A_n)$,
- (vii) $\langle U \rangle \mathcal{O}p \Rightarrow (\langle U \rangle \mathcal{O}p \wedge \langle \neq \rangle (\mathcal{O}q \wedge A)) \Leftrightarrow \neg \langle U \rangle (\mathcal{O}p \wedge \mathcal{O}q) \wedge \langle U \rangle$
 $(\mathcal{O}q \wedge A))$.

Proof. Use the nonstandard semantics for ML^\neq .

The following is an important lemma for ML^\neq , which is based only on the properties of the universal modality and axiom IV.

Lemma 2.5 (Strong Replacement Theorem) *Let $\varphi(p)$ be a formula in which the variable p has unique occurrence and let $\varphi(A)$ be the result of the replacement of p by A . Then the following formula is a theorem of ML^\neq :*

$$[U](A \Leftrightarrow B) \Rightarrow (\varphi(A) \Leftrightarrow \varphi(B)).$$

Proof. The proof can be done by induction on the complexity of $\varphi(p)$. The nontrivial case is $\varphi(p) = [\alpha](P_1, \dots, P_{i-1}, \psi(p), P_{i+1}, \dots, P_n)$, where p is not in P_1, \dots, P_n and by the induction hypothesis the assertion for $\psi(p)$ is true.

The following is a sketch of the proof of this case. For notational convenience we will assume that $[\alpha]$ is a one-place modality.

1. $[U](A \Leftrightarrow B) \Rightarrow (\psi(A) \Leftrightarrow \psi(B))$ – by the induction hypothesis,

2. $[U][U](A \Leftrightarrow B) \Rightarrow [U](\psi(A) \Leftrightarrow \psi(B))$ – from 1 by the monotonicity of $[U]$,
3. $[U](A \Leftrightarrow B) \Rightarrow [U](\psi(A) \Leftrightarrow \psi(B))$ – from 2 by S5,
4. $[U](\psi(A) \Leftrightarrow \psi(B)) \Rightarrow [\alpha](\psi(A) \Leftrightarrow \psi(B))$ – by axiom IV.
5. $[\alpha](\psi(A) \Leftrightarrow \psi(B)) \Rightarrow ([\alpha]\psi(A) \Leftrightarrow [\alpha]\psi(B))$ – by the minimal modal logic K_σ ,
6. $[U](A \Leftrightarrow B) \Rightarrow ([\alpha]\psi(A) \Leftrightarrow [\alpha]\psi(B))$ – by the propositional logic from 3, 4 and 5. ♣

2.3 Theories in σ -logics

Throughout this section we consider L to be a fixed consistent σ -logic.

A set of formulae Γ is called a *theory in L* if it satisfies the following conditions:

- (t1) All theorems of L are contained in Γ ,
- (t2) Γ is closed under Modus Ponens, i.e., if $A, A \Rightarrow B \in \Gamma$ then $B \in \Gamma$,
- (t3) Γ is closed under the rules (WIT), (WIT₀) and (WIT₁) i.e., whenever all premises of an instance of any of these rules are in Γ then its conclusion is in Γ .

Obviously the set of all theorems of L is a theory of L .

A theory Γ is *consistent* if $\perp \notin \Gamma$.

Γ is said to be a *maximal theory* if it is consistent and for all A : either $A \in \Gamma$ or $\neg A \in \Gamma$.

Note that every maximal theory is consistent.

We will use without explicit mentioning the following properties of a maximal theory Γ :

- $\neg A \in \Gamma$ iff $A \notin \Gamma$,
- $A \wedge B \in \Gamma$ iff $A \in \Gamma$ and $B \in \Gamma$,
- $A \vee B \in \Gamma$ iff $A \in \Gamma$ or $B \in \Gamma$.

Let Γ be a set of formulae and A be a formula. Define:

$$\Gamma + A = \{B \in \text{FOR}_\sigma \mid A \Rightarrow B \in \Gamma\}$$

Lemma 2.6 (Deduction lemma for theories) *Let Γ be a theory in L . Then:*

- (i) $\Gamma + A$ is the smallest theory, containing Γ and A ,
- (ii) $\Gamma + A$ is consistent iff $\neg A \notin \Gamma$.

Proof. Standard. Note that $A \Rightarrow \varphi(\mathcal{O}p \Rightarrow B)/A \Rightarrow \varphi(B)$ is tautologically equivalent to an instance of a Witness rule whenever $\varphi(\mathcal{O}p \Rightarrow B)/\varphi(B)$ is such an instance. ♣

Lemma 2.7 (Lindenbaum Lemma)

- (i) Any consistent theory Γ can be extended to a maximal theory Δ .
- (ii) If Γ is a theory and $A \notin \Gamma$ then there exists a maximal theory Δ such that $\Gamma \subseteq \Delta$ and $A \notin \Delta$.
- (iii) If A is not a theorem of L then there exists a maximal theory Γ such that $A \notin \Gamma$.

Proof. (i). Let A_0, A_1, \dots be an enumeration of all formulae. We define inductively a sequence of consistent theories $\Gamma_0, \Gamma_1, \dots$ in the following way.

Define $\Gamma_0 = \Gamma$ and suppose that $\Gamma_0, \dots, \Gamma_n$ are defined and consistent. For Γ_{n+1} we consider two cases.

Case 1: If $\Gamma_n + A_n$ is consistent, then put $\Gamma_{n+1} = \Gamma_n + A_n$.

Case 2: Let $\Gamma_n + A_n$ be inconsistent. Then $\neg A_n \in \Gamma_n$. Note that there are finitely many representations of A_n as a conclusion of any of the Witness rules $\varphi_1(B_1), \dots, \varphi_k(B_k)$. We define the finite sequence $\Gamma_n^0, \dots, \Gamma_n^k$ inductively as follows. Let $\Gamma_n^0 = \Gamma_n$ and suppose that $\Gamma_n^0, \dots, \Gamma_n^i, i < k$ are defined. Then there exists a propositional variable p such that $\Gamma_n^i + \neg\varphi_i(\mathcal{O}p \Rightarrow B_i)$ is consistent. For, suppose the contrary, i.e. for any p , $\Gamma_n^i + \neg\varphi_i(\mathcal{O}p \Rightarrow B_i)$ is inconsistent. Then $\varphi_i(\mathcal{O}p \Rightarrow B_i) \in \Gamma_n^i$ for any variable p and by the corresponding Witness rule we obtain that $\varphi_i(B_i) = A_n \in \Gamma_n^i$. But $\neg A_n$ also belongs to Γ_n^i , which implies that Γ_n^i is inconsistent — a contradiction. Let p_i be the first variable such that $\Gamma_n^i + \neg\varphi_i(\mathcal{O}p_i \Rightarrow B_i)$ is consistent. Then define $\Gamma_n^{i+1} = \Gamma_n^i + \neg\varphi_i(\mathcal{O}p_i \Rightarrow B_i)$.

Finally $\Gamma_{n+1} = \Gamma_n^k$. That completes the definition of the sequence $\Gamma_0, \Gamma_1, \dots$

Now put $\Delta = \bigcup_{n=0}^{\infty} \Gamma_n$. It is straightforward to show that Δ is a maximal theory containing Γ .

Conditions (ii) and (iii) follow from (i). ♣

We denote by W_L the set of all maximal theories of L .

Lemma 2.8 (Witness Lemma) *For any $\Gamma \in W_L$ there exists a variable p such that $\mathcal{O}p \in \Gamma$.*

Proof. Suppose, for the sake of contradiction, that $\Gamma \in W_L$ and for any variable p we have $\mathcal{O}p \notin \Gamma$. Then, by the maximality of Γ for any variable p we have that $\mathcal{O}p \Rightarrow \perp \in \Gamma$ and hence, by the Basic Witness rule, $\perp \in \Gamma$ — a contradiction. ♣

2.4 Canonical models for σ -logics

Throughout this section we will assume that L is a fixed consistent σ -logic and Γ is a maximal theory of L . Following an idea very similar to the classical Henkin's construction of canonical models in first-order logic we will construct a *canonical σ -structure* $(W^\Gamma, f_1^\Gamma, \dots, f_k^\Gamma, c_1^\Gamma, \dots, c_l^\Gamma, r_1^\Gamma, \dots, r_m^\Gamma)$, and a canonical valuation on it, related to Γ .

First we define $\Sigma_\Gamma = \{p \in VAR \mid \langle U \rangle \mathcal{O}p \in \Gamma\}$.

By the witness lemma $\Sigma_\Gamma \neq \emptyset$. We define the following relation \approx in Σ_Γ :

$p \approx q$ iff $\langle U \rangle (\mathcal{O}p \wedge \mathcal{O}q) \in \Gamma$.

It is easy to see that \approx is a reflexive and symmetric relation in Σ_Γ . By 2.4(iii) we obtain that \approx is also a transitive relation. Hence \approx is an equivalence relation in Σ_Γ . Then define:

$|p| = \{q \in \Sigma_\Gamma \mid p \approx q\},$
 $W^\Gamma = \{|p| \mid p \approx q\}.$

2.4.1 Definition of relations:

Let r be a relational symbol of arity $n + 1$ from the signature σ . For $|p|, |q_1|, \dots, |q_n| \in W^\Gamma$ define

$r^\Gamma(|p|, |q_1|, \dots, |q_n|)$ iff $\langle U \rangle (\mathcal{O}p \wedge \langle r \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \in \Gamma$.

The correctness of this definition follows from the next lemma.

Lemma 2.9 *Let r be an $n + 1$ -place relational symbol from the signature σ and let for $i = 1, \dots, n$ the following hold:*

1. $p, p', q_i, q'_i \in \Sigma_\Gamma,$
2. $p \approx p', q_i \approx q'_i,$
and
3. $\langle U \rangle (\mathcal{O}p \wedge \langle r \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \in \Gamma.$

Then $\langle U \rangle (\mathcal{O}p' \wedge \langle r \rangle (\mathcal{O}q'_1, \dots, \mathcal{O}q'_n)) \in \Gamma.$

Proof. Suppose that the conditions (1)–(3) are fulfilled. Then by (1) and (2) we obtain:

$\langle U \rangle (\mathcal{O}p \wedge \mathcal{O}p') \in \Gamma, \langle U \rangle (\mathcal{O}q_i \wedge \mathcal{O}q'_i) \in \Gamma, i = 1, \dots, n.$

Then, applying 2.4(i), we obtain

$[U](\mathcal{O}p \Leftrightarrow \mathcal{O}p') \in \Gamma, [U](\mathcal{O}q_i \Leftrightarrow \mathcal{O}q'_i) \in \Gamma, i = 1, \dots, n.$

From this, applying $n + 1$ times the strong replacement lemma to (3), we obtain the result. ♣

2.4.2 Definition of functions:

Let f be a functional symbol of arity n . Before defining the function f^Γ we define an $(n+1)$ -ary relation R_f as in the definition of relations. Namely, for $|p|, |q_1|, \dots, |q_n| \in W^\Gamma$ we put

$$R_f(|p|, |q_1|, \dots, |q_n|) \text{ iff } \langle U \rangle (\mathcal{O}p \wedge \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \in \Gamma.$$

The correctness of this definition follows from 2.9. In the next lemma we will show that R_f is a total functional relation with respect to $|q_1|, \dots, |q_n|$.

Lemma 2.10 *Let f be an n -ary functional symbol from the signature σ . Then for every $|q_1|, \dots, |q_n| \in W^\Gamma$ there exists a unique $|p| \in W^\Gamma$ such that $R_f(|p|, |q_1|, \dots, |q_n|)$.*

Proof. (Existence) Let $|q_1|, \dots, |q_n| \in W^\Gamma$. Then $\langle U \rangle \mathcal{O}q_1, \dots, \langle U \rangle \mathcal{O}q_n \in \Gamma$. From this and axiom (f1) we obtain

$$(1) \quad \langle U \rangle \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n) \in \Gamma.$$

Now we will show that there exists $p \in VAR$ such that

$$(2) \quad \langle U \rangle (\mathcal{O}p \wedge \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \in \Gamma.$$

Suppose the contrary, namely that for every $p \in VAR$ we have

$$\langle U \rangle (\mathcal{O}p \wedge \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \notin \Gamma.$$

From this, by the maximality of Γ we get that for every $p \in VAR$ we have $[U](\mathcal{O}p \Rightarrow \neg \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \in \Gamma$. Then by the Extended Witness rule for Γ we get $[U](\neg \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \in \Gamma$ and from here $\neg \langle U \rangle \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n) \in \Gamma$, which contradicts (1).

Using monotonicity of $\langle U \rangle$ it follows from (2) that $\langle U \rangle \mathcal{O}p \in \Gamma$, $|p| \in W^\Gamma$ and $R_f(|p|, |q_1|, \dots, |q_n|)$ — the existence part is proved.

(Uniqueness) Suppose $R_f(|p|, |q_1|, \dots, |q_n|)$ and $R_f(|p'|, |q_1|, \dots, |q_n|)$. By the definition of R_f we obtain

$$(3) \quad \langle U \rangle (\mathcal{O}p \wedge \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \in \Gamma \text{ and}$$

$$(4) \quad \langle U \rangle (\mathcal{O}p' \wedge \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \in \Gamma$$

Applying the monotonicity of $\langle U \rangle$ and axiom (f2) to (3) and (4) we obtain

$$(3') \quad \langle U \rangle (\mathcal{O}p \wedge \mathcal{O} \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \in \Gamma \text{ and}$$

$$(4') \quad \langle U \rangle (\mathcal{O}p' \wedge \mathcal{O} \langle f \rangle (\mathcal{O}q_1, \dots, \mathcal{O}q_n)) \in \Gamma.$$

Now from (3'), (4') and Thm 2.4(iii) we obtain $\langle U \rangle (\mathcal{O}p \wedge \mathcal{O}p') \in \Gamma$, which shows that $p \approx p'$ and consequently $|p| = |p'|$. ♣

Now the definition of f^Γ is the following: we put

$$|p| = f^\Gamma(|q_1|, \dots, |q_n|) \text{ iff } R_f(|p|, |q_1|, \dots, |q_n|).$$

2.4.3 Definition of constants

Let c be a constant symbol from the signature σ . If we treat c as zero-place function we can repeat the construction for the functions given above. We will only formulate the relevant lemma.

Lemma 2.11 *Let c be a constant symbol from the signature σ . Then there exists unique $|p| \in W^\Gamma$ such that $\langle U \rangle (\mathcal{O}p \wedge \langle c \rangle) \in \Gamma$.*

Proof. Similar to the proof of 2.10, using axioms (c1) and (c2). ♣

Now we put $c^\Gamma = |p|$, where $|p|$ is the unique element of W^Γ from 2.11.

We have defined the canonical σ -structure. To define the canonical model it remains to define the canonical valuation v^Γ .

2.4.4 Definition of the canonical valuation:

For arbitrary $q \in VAR$ we put

$$v^\Gamma(q) = \{|p| \in W^\Gamma \mid \langle U \rangle (\mathcal{O}p \wedge q) \in \Gamma\}.$$

The correctness of this definition is obtained from the following lemma.

Lemma 2.12 *Let*

1. $p \approx p'$ and
2. $\langle U \rangle (\mathcal{O}p \wedge q) \in \Gamma$.

Then

$$\langle U \rangle (\mathcal{O}p' \wedge q) \in \Gamma.$$

Proof. From (1) we obtain $\langle U \rangle (\mathcal{O}p \wedge \mathcal{O}p') \in \Gamma$. Applying 2.4(i) we obtain $[U](\mathcal{O}p \Leftrightarrow \mathcal{O}p') \in \Gamma$.

Now from this, (2), and strong replacement lemma we obtain the result.

♣

2.5 The Truth Lemma

The following theorem, often called “truth lemma”, states that the canonical valuation has the same form for arbitrary formulae.

Theorem 2.13 (Truth Lemma) *Let $v = v^\Gamma$ be the canonical valuation. The following equivalence is true for any formula Q and $|p| \in W^\Gamma$:*

$$|p| \in v(Q) \text{ iff } \langle U \rangle (\mathcal{O}p \wedge Q) \in \Gamma$$

Proof. The proof will be done by induction on the complexity of Q .

- $Q \in VAR$. This case is true by the definition of v^Γ .
Suppose Q has one of the forms $\neg A, A \vee B, (\neq)A, \langle r \rangle (A_1, \dots, A_n), \langle f \rangle (A_1, \dots, A_n), \langle c \rangle$ and for A, B, A_1, \dots, A_n the assertion is true (induction hypothesis, IH).
- $Q = \neg A$:
 $|p| \in v(\neg A)$ iff $|p| \notin v(A)$ iff (by IH) $\langle U \rangle (\mathcal{O}p \wedge A) \notin \Gamma$ iff (by the maximality of Γ) $\neg \langle U \rangle (\mathcal{O}p \wedge A) \in \Gamma$ iff (by 2.4(iv)) $\langle U \rangle (\mathcal{O}p \wedge \neg A) \in \Gamma$.
- $Q = A \vee B$:
 $|p| \in v(A \vee B)$ iff $|p| \in v(A)$ or $|p| \in v(B)$ iff (by IH) $\langle U \rangle (\mathcal{O}p \wedge A) \in \Gamma$ or $\langle U \rangle (\mathcal{O}p \wedge B) \in \Gamma$ iff (by the maximality of Γ) $\langle U \rangle (\mathcal{O}p \wedge A) \vee \langle U \rangle (\mathcal{O}p \wedge B) \in \Gamma$ iff (by 2.4(v)) $\langle U \rangle (\mathcal{O}p \wedge (A \vee B)) \in \Gamma$.
- $Q = \langle r \rangle (A_1, \dots, A_n), Q = \langle f \rangle (A_1, \dots, A_n), Q = \langle c \rangle, Q = (\neq)A$. The proofs of these cases follow from the next four lemmas.

Lemma 2.14 *The following are equivalent:*

- (i) $\langle U \rangle (\mathcal{O}p \wedge \langle \alpha \rangle (A_1, \dots, A_{i-1}, A_i, A_{i+1}, \dots, A_n)) \in \Gamma$,
- (ii) *There exists $q_i \in VAR$ such that*
 $\langle U \rangle (\mathcal{O}p \wedge \langle \alpha \rangle (A_1, \dots, A_{i-1}, (\mathcal{O}q_i \wedge A_i), A_{i+1}, \dots, A_n)) \in \Gamma$.
- (iii) *There exist $|q_i| \in W^\Gamma, i = 1, \dots, n$ such that*
 $\langle U \rangle (\mathcal{O}p \wedge \langle \alpha \rangle (\mathcal{O}q_1 \wedge A_1, \dots, \mathcal{O}q_n \wedge A_n)) \in \Gamma$,
- (iv) *(For the case $\alpha = r$.) There exist $|q_i| \in W^\Gamma, i = 1, \dots, n$ such that*
 $r^\Gamma(|p|, |q_1|, \dots, |q_n|)$ and $\langle U \rangle (\mathcal{O}p \wedge A_i) \in \Gamma$ for each $i = 1, \dots, n$.

Proof. (i) \rightarrow (ii): Suppose (i) and, for the sake of contradiction, that (ii) is not fulfilled. Then by the maximality of Γ we obtain that for any $q_i \in VAR$ the following is true:

$$[U](\mathcal{O}p \Rightarrow [\alpha](\neg A_1, \dots, \neg A_{i-1}, \mathcal{O}q_i \Rightarrow \neg A_i, \neg A_{i+1}, \dots, \neg A_n)) \in \Gamma.$$

From this, by (WIT₁) we obtain

$$[U](\mathcal{O}p \Rightarrow [\alpha](\neg A_1, \dots, \neg A_{i-1}, \neg A_i, \neg A_{i+1}, \dots, \neg A_n)) \in \Gamma$$

which is equivalent to

$$\neg \langle U \rangle (\mathcal{O}p \wedge \langle \alpha \rangle (A_1, \dots, A_{i-1}, A_i, A_{i+1}, \dots, A_n)) \in \Gamma$$

which contradicts (i).

Let us note that q_i , the existence of which is claimed in (ii), belongs to Σ_Γ . This follows from the monotonicity of $\langle U \rangle$ and $\langle \alpha \rangle$, and axiom (U).

(ii) \rightarrow (i): This implication follows by the monotonicity of $\langle U \rangle$ and $\langle \alpha \rangle$.

Thus we have obtained the equivalence (i) \leftrightarrow (ii). Using the above note and applying this equivalence n -times for $i = 1, \dots, n$ we obtain the equivalence (i) \leftrightarrow (iii).

The equivalence (iii) \leftrightarrow (iv) follows by 2.4(v) and the definition of r^Γ .

♣

Lemma 2.15 *The following are equivalent:*

- (i) $\langle U \rangle (\mathcal{O}p \wedge \langle f \rangle (A_1, \dots, A_n)) \in \Gamma$,
- (ii) *there exist $|q_i| \in W^\Gamma$, $i = 1, \dots, n$, such that $|p| = f^\Gamma(|q_1|, \dots, |q_n|)$ and $\langle U \rangle (\mathcal{O}p \wedge A_i) \in \Gamma$ for any $i = 1, \dots, n$.*

Proof. The proof is similar to that of 2.14 by using the definition of f^Γ . ♣

As a particular case we obtain:

Lemma 2.16 *The following conditions are equivalent:*

- (i) $\langle U \rangle (\mathcal{O}p \wedge \langle c \rangle) \in \Gamma$,
- (ii) $|p| = c^\Gamma$.

Lemma 2.17 *The following conditions are equivalent:*

- (i) $\langle U \rangle (\mathcal{O}p \wedge \langle \neq \rangle A) \in \Gamma$,
- (ii) *There exists $|q| \in W^\Gamma$ such that $|p| \neq |q|$ and $\langle U \rangle (\mathcal{O}q \wedge A) \in \Gamma$.*

Proof. By 2.14 condition (i) is equivalent to the following one:

- (i') There exists $q \in VAR$ such that $\langle U \rangle (\mathcal{O}p \wedge \langle \neq \rangle (\mathcal{O}q \wedge A)) \in \Gamma$.

For this q we easily obtain that $|q| \in W^\Gamma$, hence, by 2.4(vii), (i') is equivalent to:

- (i'') There exists $|q| \in W^\Gamma$ such that $\neg \langle U \rangle (\mathcal{O}p \wedge \mathcal{O}q) \wedge \langle U \rangle (\mathcal{O}q \wedge A) \in \Gamma$.

Finally, (i'') is obviously equivalent to (ii). ♣

The proof of the Truth Lemma is completed. ♣

2.6 The canonical model lemma

Lemma 2.18 (Canonical Model Lemma) *Let L be an arbitrary consistent σ -logic. Then the following conditions are equivalent for any formula A :*

- (i) *A is a theorem of L ,*
- (ii) *A is true in all canonical models of L .*

Proof. (i) \rightarrow (ii) Suppose that A is a theorem of L and, for the sake of contradiction, that A is not true in some canonical model of L determined by some maximal theory Γ . Then there is a propositional variable p such

that $|p| \in W^\Gamma$ and that $|p| \notin v^\Gamma(A)$. So $|p| \in v^\Gamma(\neg A)$. By the Truth Lemma (2.13) we obtain that $\langle U \rangle (\mathcal{O}p \wedge \neg A) \in \Gamma$. By the monotonicity of $\langle U \rangle$ we get that $\langle U \rangle \neg A \in \Gamma$ and hence that $\neg[U]A \in \Gamma$. But since A is a theorem of L then $[U]A$ is a theorem of L , too, and hence $[U]A \in \Gamma$ — a contradiction.

(ii) \rightarrow (i). Suppose that A is not a theorem of L . Then by the Lindenbaum Lemma (2.7(iii)) there exists a maximal theory Γ such that $A \notin \Gamma$, so $\neg A \in \Gamma$. We shall show that A is falsified in the canonical model determined by Γ . By the Witness Lemma there is a propositional variable p such that $\mathcal{O}p \in \Gamma$. Then $\mathcal{O}p \wedge \neg A \in \Gamma$ and by S5 that $\langle U \rangle (\mathcal{O}p \wedge \neg A) \in \Gamma$. By the Truth Lemma we obtain that $|p| \in v^\Gamma(\neg A)$, hence $|p| \notin v^\Gamma(A)$, which shows that A is not true in the canonical model determined by Γ . ♣

2.7 Soundness and completeness of ML^σ .

Theorem 2.19 *The logic ML_σ is sound with respect to its algebraic and Kripke-style semantics.*

Proof. It is straightforward to check that all axioms are valid and that all rules preserve validity. (For the soundness of the Witness rule, see similar results e.g. in Passy and Tinchev 1991, Gargov and Goranko 1993, Goranko 1998.) ♣

Theorem 2.20 *The following are equivalent for any formula A in ML_σ :*

- (i) *A is a theorem of ML^σ .*
- (ii) *A is valid in all σ -structures.*
- (iii) *A is valid in all complex σ -algebras.*

Proof. (i) \rightarrow (ii): — this is the soundness theorem for ML^σ .

(ii) \rightarrow (i): Suppose A is true in all σ -structures. Then A is true in all models over all σ -structures and consequently A is true in all canonical models of ML^σ . Then by the Canonical Model Lemma A is a theorem of ML^σ .

Finally, (ii) and (iii) are equivalent by definition. ♣

3 Modal logics for classes of σ -structures.

In this section we introduce uniform methods for complete axiomatization of the modal logic $ML_{\mathcal{C}}^\sigma$ consisting of the valid formulae of a class \mathcal{C} of σ -structures for a large family of elementary classes.

3.1 Canonical definability

Theorem 3.1 (Canonical definability of universal formulae) *Let Γ be a maximal theory in a σ -logic L and let \underline{W}^Γ be the canonical structure determined by Γ . Let $\varphi(x_0, \dots, x_n)$ be an open formula in the first-order language of the signature σ and τ be the translation defined by 1.3. Then:*

- (i) *For any $|p_0|, \dots, |p_n| \in W^\Gamma$, $\varphi(|p_0|, \dots, |p_n|)$ holds in \underline{W}^Γ iff $\tau(\varphi(p_0, \dots, p_n)) \in \Gamma$.*
- (ii) *Let ψ be the closed formula $(\forall x_0) \dots (\forall x_n) \varphi(x_0, \dots, x_n)$. Then $\tau(\psi)$ is a theorem of L iff ψ holds in all canonical structures of L .*

Proof. (i) We will proceed by induction on the complexity of $\varphi(x_0, \dots, x_n)$.

The case for atomic formulae:

- If $\varphi(x_0, x_j)$ is $x_0 = x_j$ then:
 $|p_0| = |p_j|$ iff $p_0 \approx p_j$ iff $\langle U \rangle (\mathcal{O}_{p_0} \wedge \mathcal{O}_{p_j}) \in \Gamma$ iff $\tau(p_0 = p_j) \in \Gamma$.
- If $\varphi(x_0, x_1, \dots, x_n)$ is $r(x_0, x_1, \dots, x_n)$ then:
 $r^\Gamma(|p_0|, |p_1|, \dots, |p_n|)$ is true in \underline{W}^Γ iff
 $\langle U \rangle (\mathcal{O}_{p_0} \wedge \langle r \rangle (\mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_n})) \in \Gamma$ iff
 $\tau(r(p_0, p_1, \dots, p_n)) \in \Gamma$.
- $\varphi(x_0, x_1, \dots, x_n)$ is $x_0 = f(x_1, \dots, x_n)$. Then we have:
 $|p_0| = f^\Gamma(|p_1|, \dots, |p_n|)$ iff $\langle U \rangle (\mathcal{O}_{p_0} \wedge \langle f \rangle (\mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_n})) \in \Gamma$ iff
 $\tau(p_0 = f(p_1, \dots, p_n)) \in \Gamma$.
- $\varphi(x_0)$ is $x_0 = c$. Then we have:
 $|p_0| = c^\Gamma$ iff $\langle U \rangle (\mathcal{O}_{p_0} \wedge \langle c \rangle) \in \Gamma$ iff $\tau(p_0 = c) \in \Gamma$.

The case of arbitrary open formulae:

$\varphi(x_0, \dots, x_n)$ has a form $\neg\psi(x_0, \dots, x_n)$ or $\psi(x_0, \dots, x_n) \vee \theta(x_0, \dots, x_n)$ and by induction hypothesis the assertion is true for ψ and θ . Both cases are standard.

(ii) Recall that $\tau(\psi)$ is $\langle U \rangle \mathcal{O}_{p_0} \wedge \dots \wedge \langle U \rangle \mathcal{O}_{p_n} \Rightarrow \tau\varphi(p_0, \dots, p_n)$.

(\rightarrow) Suppose $\tau(\psi)$ is a theorem of L but, for the sake of contradiction, that ψ is not true in some canonical structure over some maximal theory Γ . Then for some $|p_0|, \dots, |p_n| \in W^\Gamma$ we have that $\varphi(|p_0|, \dots, |p_n|)$ does not hold. By (i) we obtain

(1) $\tau(\varphi(p_0, \dots, p_n)) \notin \Gamma$.

From $|p_0|, \dots, |p_n| \in W^\Gamma$ we obtain

(2) $\langle U \rangle \mathcal{O}_{p_i} \in \Gamma, i = 0, \dots, n$.

But (1) and (2) imply that $\tau(\psi) \notin \Gamma$, which contradicts the assumption that $\tau(\psi)$ is a theorem of L .

(\leftarrow) Suppose that $\tau(\psi)$ is not theorem of L . We shall show that ψ is not true in some canonical structure of L .

Since $\tau(\psi)$ is not theorem of L then by the Lindenbaum Lemma (2.7(iii)) there exists a maximal theory Γ such that $\tau(\psi) \notin \Gamma$. From here we obtain

(3) $\langle U \rangle \mathcal{O}p_i \in \Gamma, i = 0, \dots, n$, and

(4) $\tau(\varphi(p_0, \dots, p_n)) \notin \Gamma$.

From (3) we get

(5) $|p_0|, \dots, |p_n| \in W^\Gamma$

and from (4) we obtain

(6) $\varphi(|p_0|, \dots, |p_n|)$ is not true in the canonical structure determined by Γ .

From (5) and (6) we obtain that ψ is not true in all canonical models of L . ♣

3.2 The modal logic of a universal class of structures.

Given a universal class \mathbf{C} of σ -structures, axiomatized by a set of universal sentences Δ , we obtain the logic $ML_{\mathbf{C}}^\sigma$ of all valid formulae in \mathbf{C} by extending ML^σ with the following additional group of axioms:

Axioms for the class \mathbf{C}^* :

(ψ^*) $\tau(\psi)$, for every formula ψ from Δ .

Theorem 3.2 *For any universal class \mathbf{C} the following are equivalent for any formula A of M_σ :*

(i) A is a theorem of $ML_{\mathbf{C}}^\sigma$,

(ii) A is valid in all σ -structures of the class \mathbf{C} .

(iii) A is valid in all complex σ -algebras from \mathbf{C}^* .

Proof. (i) \rightarrow (ii): this is the soundness theorem for $ML_{\mathbf{C}}^\sigma$ which follows from the soundness theorem for ML^σ and lemma 1.4.

(ii) \rightarrow (i): Note that by the Canonical Definability Theorem, all canonical structures of $ML_{\mathbf{C}}^\sigma$ are in the class \mathbf{C} . Suppose now that A is true in all σ -structures from the class \mathbf{C} . Then A is true in all models over all σ -structures from the class \mathbf{C} and consequently A is true in all canonical models of $ML_{\mathbf{C}}^\sigma$. Then by the Canonical Model Lemma A is a theorem of $ML_{\mathbf{C}}^\sigma$.

Again, (ii) and (iii) are equivalent by definition. ♣

This theorem covers a large family of important classes of structures, such as:

- implicative lattices, pseudo-Boolean algebras, Boolean algebras (axiomatized in Goranko and Vakarelov 1998 using a method which is generalized here), Post algebras, N-lattices, modal algebras, dynamic algebras, (representable) relation algebras, (representable) cylindric algebras, etc.

- groups, abelian groups, groups of exponent p , torsion-free groups; ordered groups, etc.
- rings, commutative rings, integral domains, etc.

3.3 The modal logic of a Π_2^0 class of structures.

We shall extend the result above by providing a uniform complete axiomatization of any Π_2^0 class of structures, by means of additional *rules* of the type of Witness.

Given a Π_2^0 class \mathbf{C} of σ -structures axiomatized by a set of Π_2^0 sentences Δ , we obtain the logic $ML_{\mathbf{C}}^\sigma$ of all valid formulae in \mathbf{C} by extending ML^σ with the following additional group of rules:

Rules for the class \mathbf{C}^* :

For every formula $\psi = \forall \bar{x} \exists \bar{y} \phi(\bar{x}, \bar{y})$ from Δ , where $\bar{x} = x_1, \dots, x_k, \bar{y} = y_1, \dots, y_m$ are disjoint strings of variables and ϕ is open, we add the following rule to the axiomatic system ML^σ :

(WIT $_\psi$):

For any n -ary modality $[\alpha]$ and $i \in \{1, \dots, n\}$, if

$\vdash A \Rightarrow [\alpha](A_1, \dots, A_{i-1}, (\langle U \rangle \mathcal{O} \bar{y} \wedge \tau(\phi(\bar{x}, \bar{y}))) \Rightarrow B, A_{i+1}, \dots, A_n)$ for all $y_1, \dots, y_m \in \text{VAR}$, where $\langle U \rangle \mathcal{O} \bar{y} = \langle U \rangle \mathcal{O} y_1 \wedge \dots \wedge \langle U \rangle \mathcal{O} y_m$, then

$A \Rightarrow [\alpha](A_1, \dots, A_{i-1}, \langle U \rangle \mathcal{O} \bar{x} \Rightarrow B, A_{i+1}, \dots, A_n)$ where $\langle U \rangle \mathcal{O} \bar{x} = \langle U \rangle \mathcal{O} x_1 \wedge \dots \wedge \langle U \rangle \mathcal{O} x_k$.

Theorem 3.3 *For any Π_2^0 class \mathbf{C} the following are equivalent for any formula A of M_σ :*

- (i) A is a theorem of $ML_{\mathbf{C}}^\sigma$,
- (ii) A is valid in all σ -structures of the class \mathbf{C} .
- (ii) A is valid in all complex σ -algebras from \mathbf{C}^* .

Proof. We shall outline the major steps in the proof. For more details of similar results, see Goranko 1998.

(i) \rightarrow (ii): We only need to show the soundness of each of the rules (WIT $_\psi$).

Let $\psi = \forall \bar{x} \exists \bar{y} \phi(\bar{x}, \bar{y})$, $\langle U \rangle \mathcal{O} \bar{x}$ and $\langle U \rangle \mathcal{O} \bar{y}$ be as above and suppose

$$ML_{\mathbf{C}}^\sigma \not\models A \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, (\langle U \rangle \mathcal{O} \bar{x}) \Rightarrow B, A_{i+1}, \dots, A_n)).$$

Then, for some structure $\mathcal{S} \in \mathbf{C}$,

$$\mathcal{S} \not\models A \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, (\langle U \rangle \mathcal{O} \bar{x}) \Rightarrow B, A_{i+1}, \dots, A_n)),$$

i.e. for some valuation v and $a \in \mathcal{S}$,

$$\mathcal{S}, v, a \not\models A \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, (\langle U \rangle \mathcal{O}\bar{x}) \Rightarrow B, A_{i+1}, \dots, A_n)).$$

Then $\mathcal{S}, v, a \models A$ and $\mathcal{S}, v, a_j \not\models A_j$, $\mathcal{S}, v, a_i \not\models (\langle U \rangle \mathcal{O}\bar{x}) \Rightarrow B$ for some $a_1, \dots, a_n \in \mathcal{S}$ such that $R_\alpha(a, a_1, \dots, a_n)$ where R_α is the relation in \mathcal{S} corresponding to α .

Then $\mathcal{S}, v, a_i \models \langle U \rangle \mathcal{O}\bar{x}$ and $\mathcal{S}, v, a_i \not\models B$. Therefore, there are elements $s_1, \dots, s_k \in \mathcal{S}$ such that $v(x_1) = \{s_1\}, \dots, v(x_k) = \{s_k\}$. For these s_1, \dots, s_k there exist $t_1, \dots, t_m \in \mathcal{S}$ such that

$$\mathcal{S} \models \phi(s_1, \dots, s_k, t_1, \dots, t_m).$$

Let y_1, \dots, y_m be variables not occurring in

$$A \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, (\langle U \rangle \mathcal{O}\bar{x}) \Rightarrow B, A_{i+1}, \dots, A_n))$$

and v' be a valuation in \mathcal{S} modifying v on y_1, \dots, y_m by $v'(y_1) = \{t_1\}, \dots, v'(y_m) = \{t_m\}$. Then

$$\mathcal{S}, v' \models (\langle U \rangle \mathcal{O}\bar{y} \wedge \tau(\phi(\bar{x}, \bar{y})))$$

and

$$\mathcal{S}, v', a_i \not\models B$$

hence

$$\mathcal{S}, v', a \not\models A \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, (\langle U \rangle \mathcal{O}\bar{y} \wedge \tau(\phi(\bar{x}, \bar{y}))) \Rightarrow B, \dots, A_n)).$$

Therefore,

$$ML_{\mathcal{C}}^\sigma \not\models A \Rightarrow ([\alpha](A_1, \dots, A_{i-1}, (\langle U \rangle \mathcal{O}\bar{y} \wedge \tau(\phi(\bar{x}, \bar{y}))) \Rightarrow B, \dots, A_n)),$$

whence the soundness of the rule (WIT_ψ) .

(ii) \rightarrow (i): First, note that analogues of the Basic Witness and the Extended Witness rules are likewise derivable for each of the rules (WIT_ψ) .

We now modify the completeness proof for ML^σ by building the canonical model from theories, additionally closed under all (basic, standard and extended versions of) rules (WIT_ψ) . All lemmas apply accordingly, hence, as in the previous theorem, it suffices to show that all canonical structures of $ML_{\mathcal{C}}^\sigma$ are in \mathbf{C} . We shall demonstrate that for every canonical structure \underline{W}^Γ and axiom $\psi = \forall \bar{x} \exists \bar{y} \phi(\bar{x}, \bar{y})$ as above, $\underline{W}^\Gamma \models \psi$. Indeed, let $|x_1|, \dots, |x_k| \in W^\Gamma$, hence $\langle U \rangle \mathcal{O}x_1, \dots, \langle U \rangle \mathcal{O}x_k \in \Gamma$, and suppose $\underline{W}^\Gamma \not\models \phi(|x_1|, \dots, |x_k|, |q_1|, \dots, |q_m|)$ for any $|q_1|, \dots, |q_m| \in W^\Gamma$. Then for any $y_1, \dots, y_m \in \text{VAR}$, $\langle U \rangle \mathcal{O}\bar{y} \wedge \tau(\phi(\bar{x}, \bar{y})) \notin \Gamma$, hence $(\langle U \rangle \mathcal{O}y_1 \wedge \dots \wedge \langle U \rangle \mathcal{O}y_m \wedge \tau(\phi(\bar{x}, \bar{y}))) \Rightarrow \perp \in \Gamma$. Then, by closedness under (WIT_ψ) , $\langle U \rangle \mathcal{O}\bar{x} \Rightarrow \perp \in \Gamma$, whence $\perp \in \Gamma$ — a contradiction. ♣

3.4 Axioms vs rules.

Since every universal formula is a Π_2^0 formula, we have a choice to axiomatize a universal class either by means of axioms or rules, and it is quite easy to derive the axiom from the corresponding rule.

Same choice exists for various modally definable Π_2^0 properties. Here is a representative sample of some algebraically important Π_2^0 classes which can be modally axiomatized either way.

- *Atomless Boolean algebras* can be modally axiomatized by extending the axiomatic system for Boolean algebras with the additional axiom:

$$(\mathcal{O}A \wedge \neg\langle 0 \rangle) \Rightarrow \langle U \rangle (A \langle \wedge \rangle \top \wedge \neg A \wedge \neg\langle 0 \rangle);$$

- *Rings with division* can be modally axiomatized by extending the axiomatic system for rings with the additional axiom:

$$(\mathcal{O}A \wedge \neg\langle 0 \rangle) \Rightarrow [U](A \langle \times \rangle \top \wedge \top \langle \times \rangle A)$$

Division rings are then axiomatized as ring with division without zero divisors (a universal condition).

- *Fields* are modally axiomatized over integral domains with the additional axiom:

$$(\mathcal{O}A \wedge \neg\langle 0 \rangle) \Rightarrow [U](\langle 1 \rangle \Rightarrow A \langle \times \rangle \top)$$

The modal logic of fields is easily extended further to fields of finite characteristics, or fields with characteristics 0, ordered fields, etc.

Note however, that some modal formulae (such as $\diamond\Box p \Rightarrow \Box\diamond p$ which determines Church-Rosser's Π_2^0 property $\forall xyz\exists t(Rxy \wedge Rxz \rightarrow (Ryt \wedge Rzt))$) although canonical in modal logics without rules, cease to be such in the stronger sense relevant to logics with Witness type rules (where there are fewer maximal consistent sets), and hence added as axioms to such logics present a problem, while the corresponding rules co-operate smoothly.

This suggests that using rules for axiomatization is generally preferable, at least what concerns proving completeness. Moreover, we argue that rules behave better than axioms in the formal derivations, too, and this will be discussed in more detail elsewhere.

3.5 Some notes on the second-order expressiveness of the modal languages.

The modal languages for complex algebras, being essentially of a second-order nature, naturally allow for universal monadic second-order quantification over important types of subsets of structures, such as *substructures*. For instance, universal quantification over subgroups can be effected by using a modally definable predicate for a subgroup: $\mathbf{s}(X) = \langle U \rangle X \wedge [U](X \langle \times \rangle X^{(')} \Rightarrow X)$, and then $\mathbf{s}(X) \Rightarrow \Psi(X)$ is valid in a group iff all subgroups of the group have the property Ψ . This pattern obviously generalizes to substructures of an arbitrary structure.

Finally, the congruences in many classes of structures (such as Boolean algebras, groups, rings) etc. can be determined by specific subsets (resp. filters, normal subgroups, ideals), which sometimes are modally definable and thus enable universal quantification over congruences, hence over homomorphic images, too. For instance, a modally definable predicate for a filter in a Boolean algebra is: $\mathbf{f}(X) = [U](\langle \langle 1 \rangle \Rightarrow X \rangle \wedge (X \Rightarrow \neg \langle 0 \rangle) \wedge (X \langle \wedge \rangle X \Rightarrow X) \wedge (X \langle \vee \rangle \top \Rightarrow X))$, and for an ultrafilter: $\mathbf{u}(X) = \mathbf{f}(X) \wedge [U](\langle - \rangle X \Leftrightarrow \neg X)$.

4 Concluding remarks

This study raises a number of interesting questions of logical or algebraic importance, which will be addressed in a subsequent paper. We shall briefly mention some.

Logical issues. Traditional logical problems arise around the modal languages and logics for complex algebras. Some general questions are:

- *Expressiveness.* The modal languages, via their standard translation a la van Benthem, cover a fragment of the universal monadic second-order extension of the first-order theory of the underlying class of structures. Can that fragment be characterized in a coherent model-theoretic fashion extending Goldblatt & Thomason's results from classical modal logic?
- *Modal axiomatizability.* The axiomatization results presented here can be further extended in various ways. Some of them are based on appropriate generalizations of Sahlqvist-type syntactic forms (see Venema 1993 for a discussion and some results) or of semantic persistence conditions on formulae (see Goranko 1998) which would ensure preservation of their validity from canonical structures (in the stronger sense used in this paper) to the underlying frames, i.e. their behavior like canonical formulae in modal logics without additional

rules, and hence the successful co-operation of such axioms with the Witness-type rules employed by the axiomatic systems. Others call upon extension of well developed model-theoretic techniques (such as filtration, unraveling etc.) for proving completeness of non-canonical modal logics to the general framework studied here.

- A number of more specific logical questions arise regarding the axiomatizations proposed in this paper, one of them being: when is the Witness rule schema redundant, or at least replaceable by (finitely many) axiom schemata? This question calls upon a general study of the proof theory of Witness-type rules. Another one is about axiomatizing the *finite* structures of a given class. For instance, on partially ordered structures, Grzegorzczuk's formula

$$\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$$

where the modality \Box corresponds to the partial ordering, defines the class of structures with no infinite descending chains, hence on linear orderings or some finite partial orderings in which a bound on the width can be established (such as Boolean algebras) it defines the class of *finite* structures. The question arises when Grzegorzczuk's formula added to the modal logic of a class of partially ordered structures axiomatizes the finite structures of that class.

- *Decidability.* When is the modal logic of the complexes of a class of structures decidable? This question is closely related to the previous one. In general, the (un)decidability of the first-order theory does not imply (un)decidability of the modal logic as the two languages are incomparable, so a finer analysis of the expressiveness of the latter is necessary in order to adapt known techniques such as e.g. tiling, or word problems. On the other hand, since the full monadic second-order theory covers the modal language, decidability of the former implies decidability of the latter, so in particular, Rabin's theorem guarantees decidability of the modal logic of any class of structures interpretable in trees with at most countably many successors.
- *Complexity.* How does the complexity change when passing from the (universal) first-order theory of structures to the modal logic of their complexes?
- *The first-order theory of complex algebras.* The full first-order language for complex σ -algebras encapsulates, via the translation τ , the

full monadic second-order theory of σ -structures. What are the logical implications of that translation? A challenging program in that respect is to study the full first-order theory of complex σ -algebras.

Algebraic issues. The main one is: to what extent is the proposed modal treatment of arbitrary structures interesting from algebraic perspective and how does it contribute to the study of intrinsically algebraic properties of these structures? More specific questions include:

- What is the precise algebraic characteristic of the (non)definability of the difference operator in a class of σ -structures?
- What are the algebraic properties of the complex σ -algebras and how does the algebraic theory of the underlying σ -structures determine them? In particular, when do their substructures (up to isomorphism) form a variety? One general approach to this question comes from Goldblatt's study of varieties of complex algebras Goldblatt 1989.
- What algebraically important, essentially second-order properties of specific σ -structures are definable in the modal language of their complex σ -algebras? And how far reaching are the consequences of the capacity of the modal languages to quantify over substructures and congruences for the formal modal analyzing of the algebraic theory of structures?

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