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PROVING UNPROVABILITY IN SOME NORMAL MODAL LOGIC

The present communication suggests deductive systems for the operator $\dashv$ of unprovability in some particular propositional normal modal logics. We give thus complete syntactic characterization of these logics in the sense of Łukasiewicz: for every formula ϕ either $\vdash \phi$ or $\dashv \phi$ (but not both) is derivable. In particular, purely syntactic decision procedure is provided for the logics under considerations.

All background in modal logic, necessary for this paper can be found in the initial chapters of [1], [3] or [5].

Henceforth we shall informally read $S \dashv \phi$ as "ϕ is unprovable in S". All systems presented here will contain the "minimal" $\dashv\vdash$ system Ł consisting of the *axiom*:

F : $$\dashv \bot$$

and the rules:

Reverse substitution:

RS $$\frac{\dashv \sigma(\phi)}{\dashv \phi}$$

for any uniform substitution σ.

Modus Tollens:

$$MT \qquad \frac{\vdash \phi \to \psi,\ \dashv \psi}{\dashv \phi}$$

(In this rule $\vdash$ and $\dashv$ will always refer to one and the same logic.)

It is crucious that the rule $\dashv \Box\phi / \dashv \phi$, which is admissible for all normal modal logics, turns out redundant in many cases including all considered here. Also let us note that the rule RS can be specified (as it can be seen from the proofs below) in all considered cases as follows: it is enough to admit only $\Box$-free substitutions, i.e. such that every variable is substituted by $\Box$-free formula.

Definition. Let **S** be a modal logic and **S** $\dashv$ be a deductive system for $\dashv$, including **Ł**. An *Inference in* **S** $\dashv$ is every finite sequence $\alpha_1, \ldots, \alpha_n$ such that every α_i is either an axiom of **S** $\dashv$ or is obtained form $\alpha_1, \ldots, \alpha_{i-1}$, the axioms of **S** $\dashv$ and the theorems of **S**, accordingly using some of the rules of **S** $\dashv$. The last formula α of every such a sequences is called **S** $\dashv$-*inferable*, denoted **S** $\dashv \alpha$.

We say that **S** $\dashv$ is *correct for* **S** if only non-theorems of **S** are inferable in $S \dashv$. We call a system **S** *complete in the sense of Łukasiewicz* or **Ł**-*complete* for short (**Ł**-decidable in [2] and [7]) if for every formula ϕ, exactly one of **S** $\vdash \phi$ and **S** $\dashv \phi$ takes place.

Here we shall sketch two methods for proving **Ł**-completeness. The first one (in theorem 2 below) is syntactic, based on some uniform presentation of the formulas, and is suitable for particular cases. Similar idea is used in [7] when an **Ł**-complete system for $S5$ is presented. The second method (theorem 3) is semantic and is applicable in more general situation. Its idea is close to the approach in [6] (where an **Ł**-complete system for the intuitionistic calculus is given), but using Kripke semantics rather than algebraic one. Both the methods however, are essentially based on suitable semantic characterizations of the unprovable formulas of the logics under consideration. Other natural approach, using semantic tableau is illustrated in [2].

The propositional language which deal with consists of a countable set of propositional variables $\{p_0, p_1, \ldots\}$ and the logical signs $\top, \wedge, \neg$ and $\Box, \bot, \vee$ and $\Diamond$ are accordingly introduced. Modal *depth* of a formula is defined as usual. The formulae with depth 0 are called $\Box$-*free*. We shall use the sign $\Vdash$ to denote the validity at the root of a generated model, i.e. $<F, V> \Vdash \phi$ means $<F, V> \Vdash \phi[r(F)]$ where $r(F)$ is the root of the frame F.

Hereafter by λ we shall denote an arbitrary $\Box$-free formula.

Obviously, the system **Ł** is correct for every consistent modal logic. Moreover, the following holds:

Theorem 1. *The system* **Ł** *is* **Ł***-complete for (and only for) the two maximal normal modal logics* $Ver : \mathbf{K} + \Box\perp$ *and* $Triv : \mathbf{K} + p \leftrightarrow \Box p$.

This result corresponds to the fact that **Ł**, taken in the classical propositional language, is **L**-complete for the classical logic CL, as shown by Łukasiewicz [4].

Theorem 2.

1) The system **Ł** *extended with the axiom* $\dashv \Diamond\top$ *and the rule*

$$R_{\mathbf{KW}} \qquad \frac{\dashv \lambda,\ \dashv \Diamond\psi \vee \psi \vee \theta_1, \ldots, \dashv \Diamond\psi \vee \psi \vee \theta_k}{\dashv \lambda \vee \Box\theta_1 \vee \ldots \vee \Box\theta_k \vee \Diamond\psi}$$

is **Ł***-complete for the logic* $\mathbf{KW} = \mathbf{K} + \Box(\Box p \to p) \to \Box p$*;*

2) The system **Ł** *extended with the axiom* $\dashv \Diamond\top$ *and the rule*

$$R_{\mathbf{K}} \qquad \frac{\dashv \lambda,\ \dashv \psi \vee \theta_1, \ldots, \dashv \psi \vee \theta_k}{\dashv \lambda \vee \Box\theta_1 \vee \ldots \vee \Box\theta_k \vee \Diamond\psi}$$

is **Ł***-complete for the logic* **K**.

Sketch of the proof:

First we define *normal modal form* (NMF for short):

i) every $\Box$-free formula is in NMF;

ii) every conjunction of formulas of the kind $\lambda \vee \Box\theta_1 \vee \ldots \vee \Box\theta_k \vee \Diamond\psi$ or $\lambda \vee \Diamond\psi$, where λ is $\Box$-free and ψ and θ's are in NMF, is in NMF itself.

One can prove by an easy induction on the depth of formulae that for every formula ϕ there exists a formula θ in NMF such that $\mathbf{K} \vdash \phi \leftrightarrow \theta$.

The proof of 1) is hung on the fact that **KW** is complete with respect to all finite irreflexive transitive tree-like frames, or **KW**-trees, for short (see [5]). Now, for every **KW**-unprovable formula ϕ we attach a natural number $l(\phi)$ which is the least length of a **KW**-tree in which ϕ is refutable. An induction on $l(\phi)$, proves that for every **KW**-unprovable $\phi, \mathbf{KW} \dashv \phi$. The inductive step uses a normal form of ϕ.

2) is proved much in the same way, using the fact that **K** is complete with respect to all finite irreflexive intransitive trees (see [5]).

THEOREM 3.
1) **Ł** *extended with the rule*

$$R_{\mathbf{S4Grz}} \quad \frac{\dashv \lambda,\ \dashv \Diamond\psi \vee \theta_1, \ldots, \dashv \Diamond\psi \vee \theta_k}{\dashv \lambda \vee \Box\theta_1 \vee \ldots \vee \Box\theta_k \vee \Diamond((\lambda \vee \Box\theta_1 \vee \ldots \vee \Box\theta_k) \wedge \psi)}$$

is **Ł***-complete for the logic* $\mathbf{S4Grz} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$.
2) **Ł** *extended with the rule*

$$R_{\mathbf{T}} \quad \frac{\dashv \lambda,\ \dashv \psi \vee \theta_1, \ldots, \dashv \psi \vee \theta_k}{\dashv \lambda \vee \Box\theta_1 \vee \ldots \vee \Box\theta_k \vee \Diamond((\lambda \vee \Box\theta_1 \vee \ldots \vee \Box\theta_k) \wedge \psi)}$$

is **Ł***-complete for the logic* **T**.
3) **Ł** *extended with the axiom* $\dashv \Diamond\top$ *and the rule*

$$R_{\mathbf{K4.3W}} \quad \frac{\dashv \lambda,\ \dashv \Diamond\psi \vee \psi \vee \theta}{\dashv \lambda \vee \Box\theta \vee \Diamond\psi}$$

is **Ł***-complete for the logic* $\mathbf{K4.3W} = \mathbf{KW} + \Box(\Box p \wedge p \to q) \vee \Box(\Box q \wedge q \to p)$.
4) **Ł** *extended with the rule*

$$R_{\mathbf{S4.3Grz}} \quad \frac{\dashv \lambda,\ \dashv \Diamond\psi \vee \theta}{\dashv \lambda \vee \Box\theta \vee \Diamond((\lambda \vee \Box\theta) \wedge \psi)}$$

is **Ł***-complete for the logic* $\mathbf{S4.3Grz} = \mathbf{S4.Grz} + \Box(\Box p \to q) \vee \Box(\Box q \to p)$.
5) **Ł** *extended with the rule*

$$R_C \quad \frac{\dashv \lambda,\ \dashv \theta}{\dashv \lambda \vee \Box\theta}$$

is **Ł***-complete for the logic* $\mathbf{K} + \Diamond p \leftrightarrow \Box p$.
6) **Ł** *extended with the axiom* $\dashv \Diamond\top$ *and the rule* R_C *is* **Ł***-complete for the logic* $\mathbf{K} + \Diamond p \to \Box p$.

Sketch of the proof. 1) **S4Grz** is complete with respect to all finite reflexive and transitive trees (called within this proof simply trees), (cf. [5]). Let us give an exact definition:

i) Every frame of the kind $O = <\{x\}, \{<x, x>\}>$ is a *tree* with a *root* $r(O) = x$ and a *length* $l(O) = 0$; call such trees *trivial*; they will be freely identified with their roots.

ii) Let $T_1, \ldots, T_k$ be disjoint trees, $T_i = <W_i, R_i>$ for $i = 1, \ldots, k$ with corresponding roots $x_1, \ldots, x_k$ and let x does not belong to any of

$T_1, \ldots, T_k$. Then the frame T

$$x(T_1, \ldots, T_k) =< \{x\} \cup \bigcup_{i=1}^{k} W_i, \bigcup_{i=k}^{k} R_i \cup \{< x, x >\} \cup \{< x, y >: y \in \bigcup_{i=1}^{k} W_i\} >$$

is a *tree* with a *root* $r(T) = x$ and a *length* $l(T) = max(l(T_1), \ldots, l(T_k)) + 1$.

The elements of a tree will be called *nodes*. Every node x in a tree T appears there as a root of a smaller tree, called a *subtree of* T *generated by* x. *Leaves* of a tree are all nodes which generate trivial subtrees. An easy semantic argument shows that the rule $R_{\mathbf{S4.Grz}}$ (and hence the whole system) is correct for **S4.Grz**.

Now, let $T = < W, R >$ be an arbitrarily fixed tree with a set of nodes $W = \{x_1, \ldots, x_n\}$. We attach to these nodes different propositional variables $q(x_1), \ldots, q(x_n)$ or for short $q_1, \ldots, q_n$. This set will be referred as $var(T)$. Denote $\chi(x_i) = q_i \wedge \bigwedge\{\neg q_j : q_j \in var(T), j \neq i\}$.

Now we shall successively define for every subtree S of T formulae ϕ_S and ψ_S as follows: If S is a leaf, $S = < \{x_i\}, \{< x_i, x_i >\} >$, put $\psi_S = \phi_S = \chi(x_i) \wedge \Box\chi(x_i))$. Let $S = y(S_1, \ldots, S_k), r(S_1) = y_1, \ldots, r(S_k) = y_k$ and suppose $\phi_{S_1}, \ldots, \phi_{S_k}, \psi_{S_1}, \ldots, \psi_{S_k}$ are already defined. Then put

$$\psi_S = \chi(y) \wedge \Diamond\psi_{S_1} \wedge \ldots \wedge \Diamond\psi_{S_k} \wedge \Box((\chi(y) \wedge \Diamond\psi_{S_1} \wedge \ldots \wedge \Diamond\psi_{S_k}) \vee \phi_{S_1} \vee \ldots \vee \phi_{S_k})$$

and

$$\phi_S = \psi_S \vee \phi_{S_1} \vee \ldots \vee \phi_{S_k}.$$

Informally speaking, ψ_S is characteristic for the root of S while ϕ_S is characteristic for the whole S.

The proof goes further through the following lemmas.

LEMMA 3.1. **S4Grz** $\dashv \neg\psi_T$ *for every tree* T.

A valuation V in the tree T will be called *suitable* if for every $x_i \in T, V(q_i) = \{x_i\}$. If V is a valuation in T, we may also consider it (its restriction) as a valuation in any subtree S of T.

LEMMA 3.2. *For every trees* S *and* T *and a valuation* V *in* S, *if* $< S, V > \Vdash \psi_T$ *then there exists a valuation* V' *in* S *coinciding with* V *over* $var(T)$, *such that there exists a p-morphism* $f :< S, V' >$ *onto a model* $< T, V'' >$ *for some suitable valuation* V''.

LEMMA 3.3. *If* $< T, V > \Vdash \neg\Theta$ *then there exists a substitution* σ *such that* $\mathbf{S4Grz} \vdash \sigma(\Theta) \rightarrow \neg\psi_T$.

Now, to prove **Ł**-completeness, let $\mathbf{S4Grz} \nvdash \theta$. Then $< T, V > \Vdash \neg\theta$ for some tree-like model. Hence, by lemma 3.3, for some substitution $\sigma, \mathbf{S4Grz} \vdash \sigma(\theta) \rightarrow \neg\psi_T$. By lemma 3.1 $\mathbf{S4Grz} \dashv \neg\psi_T$, hence $\mathbf{S4Grz} \dashv \sigma(\theta)$ by MT. Therefore $\mathbf{S4Grz} \dashv \theta$ by RS.

All other statements of the theorem are proved in the same way. They are respectively based on the following facts (cf. [5]):

2) **T** is complete with respect to all finite reflexive intransitive trees.

3) **K4.3** is complete with respect to all finite irreflexive (strict) linear orderings. (It is also the logic of $< \mathbf{N}, > >$.)

4) **S4Grz** is complete with respect to all finite linear orderings. (It is also the logic of $< \mathbf{N}, \geq >$.)

5) $\mathbf{K} + \Diamond p \leftrightarrow \Box p$ is complete with respect to all finite intransitive chains in which only the last element is reflexive. (It is also the logic of infinite irreflexive intransitive chain $< \mathbf{N}, S >$ where S is the relation "next": xSy iff $y = x + 1$.)

6) $\mathbf{K} + \Diamond p \rightarrow \Box p$ is complete with respect to all finite irreflexive intransitive chains.

REMARK. In fact most of the above introduced rules are rather rule schemata, since every number k yields a rule $R(k)$. One can be easily persuaded that these schemata cannot be restricted to some fixed k. For instance, take $R_{\mathbf{K}}$. An easy induction shows that if we restrict $R_{\mathbf{K}}$ to some $R(k)$ then all $\mathbf{K} \dashv$-deducible formulae would be refuted in trees in which every node has no more than k branches. But then the **K**-unprovable formula

$$Alt_n = (\Diamond p_1 \wedge \ldots \wedge \Diamond p_n) \rightarrow \bigvee_{i \neq j} \Diamond(p_i \wedge p_j)$$

for any $n > k$ will remain $\mathbf{K} \dashv$-unprovable.

The full proofs of the above theorems as well as other related results will appear in a forthcoming paper.

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