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The Basic Algebra of Game Equivalences

Abstract. We give a complete axiomatization of the identities of the basic game algebra valid with respect to the abstract game board semantics. We also show that the additional conditions of termination and determinacy of game boards do not introduce new valid identities.

En route we introduce a simple translation of game terms into plain modal logic and thus translate, while preserving validity both ways, game identities into modal formulae.

The completeness proof is based on reduction of game terms to a certain 'minimal canonical form', by using only the axiomatic identities, and on showing that the equivalence of two minimal canonical terms can be established from these identities.

Keywords: game operations, game algebra, game identities, axiomatization, completeness, modal logic

1. Introduction

The relationships between logic and games go back to the ancient Greek philosophy, and have been explored in modern times by a number of logicians and computer scientists (see [1] and [3] for details and further references). Parikh has initiated a formal logical study of games by introducing the Game Logic in [5]. Recently this idea has been advanced in [1] and systematically developed for coalition games in [4].

In particular, an algebraic approach to the study of the games structure and equivalence between games emerged from [5] and was further developed in [1], where the problem of establishing the complete axiomatization of the valid identities of the basic game algebra was raised. Here we give a solution to that problem.

The paper is organized as follows. In Section 2 we introduce the syntax and semantics of the basic algebra of games in terms of abstract game boards and in Section 3 we give an axiomatization of its valid identities. In Section 4 we define canonical forms of game terms and show that every game term is provably equivalent to a minimal canonical one. In Section 5 we introduce a translation of game terms and identities to plain modal logic and show that it preserves validity of game identities. The converse preservation of validity

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is proved in Section 6 where we also establish other technical results used in the completeness proof presented in Section 7. The last Section 8 includes some further results and concluding remarks, where we show that restriction of the semantics to determined and terminating games does not introduce new valid identities. We also discuss the complexity of the validity problem and the relations between game algebras and logics.

2. Basic algebra of games

We consider two-player games of a most general type. The **game language** GL consists of:

- a set of **atomic games** $\mathcal{G}_{at} = \{g_a\}_{a\in A}$;
- **game operations:** $\vee, ^d, \circ$.

For technical convenience, we include an '*idle*' atomic game $\iota = g_0$ in $\mathcal{G}_{at}$.

DEFINITION 1. **Game terms:**

- Every atomic game is a game term.
- If G, H are game terms then G^d, $G \vee H$ and $G \circ H$ are game terms.

Besides, we define $G \wedge H := (G^d \vee H^d)^d$.

Intuitively, the operations $^d, \vee, \wedge, \circ$ mean respectively *dualization* (swapping the two players' roles), *choice of first player, choice of second player,* and *composition* of games.

The algebra of game terms will be denoted by $\mathcal{GA}$. Atomic games and their duals will be called **literals.**

Models for GL are **game boards***:* $\langle S, \{\rho_a^i\}_{a\in A; i=1,2}\rangle$ where S is a set of **states** and $\rho_a^i \subseteq S \times P(S)$ are **atomic forcing relations** satisfying the following **forcing conditions:**

- *upwards monotonicity (MON):* for any $s \in S$ and $X \subseteq Y \subseteq S$, if $s\rho_a^i X$ then $s\rho_a^i Y$;
- *consistency of the powers (CON):* for any $s \in S, X \subseteq S$, if $s\rho_a^1 X$ then not $s\rho_a^2(S - X)$ and (hence) likewise with 1 and 2 swapped.

We also consider the following *optional* conditions:

- *termination of the games (FIN):* for any $s \in S$, $s\rho_a^i S$. This is of a less imperative nature, since some games may go on forever and never reach an outcome state. Game boards satisfying that condition will be called **terminating** and the class of terminating game boards will be denoted by **FIN**.

- *determinacy (DET):* $s\rho^2_a(S-X)$ iff not $s\rho^1_a X$. Game boards satisfying this condition will be called **determined** and the class of determined game boards will be denoted by **DET**.

The forcing relations ρ^i_ι of the idle game ι have a fixed interpretation: $s\rho^i_\iota X$ iff $s \in X$. Compositions of idle literals (ι or ι^d) will be called *idle game terms.*

Given a game board, the atomic forcing relations are extended to forcing relations $\{\rho^i_G\}_{G\in\mathcal{G};i=1,2}$ for all game terms, following the recursive definitions given in [1]:

- $s\rho^1_{G^d}X$ iff $s\rho^2_G X$;
- $s\rho^2_{G^d}X$ iff $s\rho^1_G X$;
- $s\rho^1_{G_1\vee G_2}X$ iff $s\rho^1_{G_1}X$ or $s\rho^1_{G_2}X$;
- $s\rho^2_{G_1\vee G_2}X$ iff $s\rho^2_{G_1}X$ and $s\rho^2_{G_2}X$;
- $s\rho^1_{G_1\circ G_2}X$ iff there exists Z such that $s\rho^1_{G_1}Z$ and $z\rho^1_{G_2}X$ for each $z \in Z$;
- $s\rho^2_{G_1\circ G_2}X$ iff there exists Z such that $s\rho^2_{G_1}Z$ and $z\rho^2_{G_2}X$ for each $z \in Z$.

The meaning[1] of $s\rho^i_G X$ is: "Player i has a strategy to play the game G so that if an outcome state is attained, it is in X."

Proposition 2. *Each forcing condition propagates over all forcing relations.*

Proof. Routine check. Note that the cases for (FIN) and (DET) use (MON). ∎

It is easy to see that all idle terms have the same forcing relations as ι.

3. Axiomatization of the algebra of games

3.1. Inclusions and identities of game terms

Definition 3. Let G_1 and G_2 be game terms and B a game board.

- G_1 is **i-included in G_2 on B** for $i = 1, 2$, denoted $G_1 \subseteq_i G_2$, if $\rho^i_{G_1} \subseteq \rho^i_{G_2}$.
- G_1 is **included in G_2 on B**, denoted $B \models G_1 \preceq G_2$ if $G_1 \subseteq_1 G_2$ and $G_2 \subseteq_2 G_1$ on B.

[1] This is the 'partial correctness' style of interpreting forcing relations. Alternatively, they can be interpreted like 'total correctness' statements: "Player i has a strategy to play the game G so that an outcome state is attained and it is in X."

- G_1 and G_2 are **equivalent on** B, denoted $B \models G_1 = G_2$ if they are assigned the same forcing relations in B.
- Further, G_1 **is included in** G_2, denoted $G_1 \preceq G_2$ if $B \models G_1 \preceq G_2$ for every game board B. Then we also say that $G_1 \preceq G_2$ is a **valid term inclusion**, denoted by $\models G_1 \preceq G_2$.
- Respectively, G_1 and G_2 are **equivalent**, denoted $G_1 \sim G_2$ if they are equivalent on every game board, i.e. $G_1 = G_2$ is a **valid term identity**, also denoted by $\models G_1 = G_2$.

Analogous notation will be used for validity in a *class of* game boards, e.g. **DET** $\models G_1 = G_2$ will mean that $G_1 = G_2$ is valid in every determined game board.

Note that $G_1 \sim G_2$ iff $G_1 \preceq G_2$ and $G_2 \preceq G_1$. Actually, $\preceq$ can be reduced to $\sim$ in the well-known lattice-theoretic fashion:

Proposition 4. $G_1 \preceq G_2$ *iff* $G_1 \vee G_2 \sim G_2$ *iff* $G_1 \wedge G_2 \sim G_1$.

3.2. The axioms of the algebra of games

The main goal of this paper is to make precise and confirm the conjecture of [1] that the following term equivalences provide a complete axiomatization of the game algebra:

1. Double dualization: $G \sim G^{dd}$;
2. The usual identities for $\vee$ in distributive lattices: idempotency, commutativity, associativity.
3. Absorption: $G_1 \vee (G_1 \wedge G_2) \sim G_1$.
4. Distributivity: $G_1 \vee (G_2 \wedge G_3) \sim (G_1 \vee G_2) \wedge (G_1 \vee G_3)$.
5. Associativity of $\circ$.
6. Distribution of ^{d} over $\circ$: $(G_1 \circ G_2)^d \sim G_1^d \circ G_2^d$.
7. Left-distribution for $\vee$ and $\circ$: $(G_1 \vee G_2) \circ G_3 = (G_1 \circ G_3) \vee (G_2 \circ G_3)$.
8. Right-distributive inclusion: $G_1 \circ G_2 \preceq G_1 \circ (G_2 \vee G_3)$. (According to Prop. 4, this is equivalent to an identity).
9. The extras for ι: multiplicative unit: $G \circ \iota \sim \iota \circ G \sim G$ and self-duality: $\iota \sim \iota^d$.

We denote the set of all these identities by $\mathbf{GA}^{\iota}$, and the set of those not involving ι by **GA**.

Note that the respective identities for $\wedge$, as well as the dual absorption, distributivity, left-distribution for $\wedge$ and $\circ$, right-distributive inclusion $G_1 \circ$

$(G_2 \wedge G_3) \preceq G_1 \circ G_2$, and the two De Morgan's laws for $\vee, \wedge$ and ^{d} easily follow from the definition of $\wedge$ and $\mathbf{GA}^\iota$ in the equational logic for the algebra of games, which includes the standard set of derivation rules reflecting the fact that $\sim$ is a congruence in the algebra of games.

PROPOSITION 5. *All identities in* $\mathbf{GA}^\iota$ *are valid.*

PROOF. Routine verification. ■

THEOREM 6. *Every valid term identity of the game algebra can be derived from* $\mathbf{GA}^\iota$ *in the standard equational logic.*

The proof of this theorem will be presented in the last section, and meanwhile we will build up the necessary machinery and will obtain auxiliary results for it.

REMARK. *We note that* ι *can be omitted from the language together with its axioms, and the remaining axiom system* **GA** *will remain complete for the reduced language. The proof of this follows the same line as the one presented here, with a little technical and notational overhead due to the absence of* ι.

4. Canonization of game terms

DEFINITION 7. **Canonical game terms** are defined recursively as follows:

- ι is a canonical term.
- Let $\{G_{ik} | k \in K_i, i \in I\}$ be a finite non-empty family of canonical terms and $\{g_{ik} | k \in K_i, i \in I\}$ be a family of literals such that g_{ik} can be an idle literal only if G_{ik} is an idle term. Then $\bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ is a canonical term.

REMARK. *Any (or all) index sets* I, K_i *above can be singletons. Nevertheless, the respective disjunctions/conjunctions remain in place.*

The only essential use of the idle term ι here is to facilitate this canonical presentation of game terms and to provide a convenient base for structural induction on canonical terms. Its use can be circumvented at the cost of minor technical complications, though.

PROPOSITION 8. *Every game term* G *is equivalent,* provably in $\mathbf{GA}^\iota$, *to a canonical game term.*

PROOF. First we prove by induction on canonical terms that the dual of a canonical term is equivalent to a canonical term. The case of ι is trivial since $\iota^d \sim \iota$. Let $G = H^d$ where $H = \bigvee_{i\in I}\bigwedge_{k\in K_i} h_{ik} \circ H_{ik}$ and the claim holds for the canonical terms H_{ik}. Then $H^d \sim \bigwedge_{i\in I}\bigvee_{k\in K_i} h^d_{ik} \circ H^d_{ik}$ which, using the distributive laws for $\vee$ and $\wedge$ and using the inductive hypothesis for the H_{ik}'s, converts into an equivalent canonical term.

Now, we prove the main claim by induction on the length of arbitrary terms. The atomic case: $g \sim \bigvee\bigwedge g \circ \iota$. The case of duals was done above. The case $G = G_1 \vee G_2$ is almost trivial, using $\iota \sim \iota \circ \iota$, if necessary.

The remaining case $G = G_1 \circ G_2$ is treated by induction on G_1, assuming that G_2 is canonical. If G_1 is a literal, $G_1 \circ G_2$ can be written as $\bigvee\bigwedge G_1 \circ G_2$ where $\bigvee$ and $\bigwedge$ are over singletons, so it is canonical. The inductive step for $G_1 = \bigvee_{i\in I}\bigwedge_{k\in K_i} g_{ik} \circ G_{ik}$ is enabled by the left-distributive laws for $\circ$, pushing it inside the $\bigvee$ and $\bigwedge$, followed by the associativity of $\circ$ which eventually reduces the case to all $G_{ik} \circ G_2$ which are covered by the inductive hypothesis. ∎

This proof also outlines an algorithm for canonizing game terms which can be easily made precise.

REMARK. *Canonical game terms impose a periodic structure on games: every game is a composition of one or several* rounds, *each consisting of:*

- *a choice of player I,*
- *followed by a choice of player II,*
- *followed by an atomic game by one of the players (depending of the sign of the literal).*

Of course, some of these choices may be vacuous, when only one disjunct or conjunct is available to choose from, but still the 'ritual' is strictly followed.

DEFINITION 9. Two canonical terms G, H are **isomorphic,** denoted $G \simeq H$, if one can be obtained from the other by means of successive permutations of conjuncts (resp. disjuncts) within the same $\bigwedge$'s (resp. $\bigvee$'s) in subterms.

In other words, isomorphic terms are the same, up to the order of the conjuncts and disjuncts. Term isomorphism is the intermediate syntactic notion between identity and semantic equivalence $\sim$, which we will eventually prove equivalent to the latter. In fact, isomorphism of terms can be replaced by genuine identity at the cost of introducing a linear ordering on literals and terms and applying it to order the $\bigwedge$'s and $\bigvee$'s in the definition of canonical terms.

PROPOSITION 10. *Isomorphic terms are equivalent,* provably in $\mathbf{GA}^\iota$.

PROOF. Easy. ■

DEFINITION 11. We define recursively **embedding** of canonical terms, denoted by $\rightarrowtail$ as follows:

- $\iota \rightarrowtail \iota$;
- Auxiliary notions: if g, h are literals and G, H are canonical terms, $g \circ G$ **embeds** into $h \circ H$ iff $g = h$ and $G \rightarrowtail H$; a conjunction $\bigwedge_{k \in K} g_k \circ G_k$ **embeds** into a conjunction $\bigwedge_{m \in M} h_m \circ H_m$ if for every $m \in M$ there is some $k \in K$ such that $g_k \circ G_k \rightarrowtail h_m \circ H_m$.
- Let $G = \bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ and $H = \bigvee_{j \in J} \bigwedge_{m \in M_j} h_{jm} \circ H_{jm}$. Then $G \rightarrowtail H$ iff every disjunct of G embeds into some disjunct of H.

PROPOSITION 12. *If G, H are canonical terms and $G \rightarrowtail H$ then $G \preceq H$ is* provable in $\mathbf{GA}^\iota$. *To be precise, then $\mathbf{GA}^\iota \vdash G \vee H = H$.*

PROOF. Double induction on G and H, using, inter alia, the right-distributive inclusions. ■

Thus, embedding of terms is the syntactic counterpart of inclusion.

DEFINITION 13. **Minimal canonical terms:**

- ι is a minimal canonical term.
- Let $G = \bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ be a canonical term where all G_{ik} are minimal canonical. Then G is minimal canonical if:

  1. ι^d does not occur in G.
  2. None of g_{ik} is ι unless G_{ik} is ι.
  3. No conjunct occurring in a conjunction $\bigwedge_{k \in K} g_{ik} \circ G_{ik}$ embeds into another conjunct from the same conjunction.
  4. No disjunct in G embeds into another disjunct of G.

Thus, minimal canonical terms are systematically 'minimized' canonical terms.

PROPOSITION 14. *Every term G can be reduced to an equivalent minimal canonical term* $\mathbf{c}(G)$ *and this can be done* provably in $\mathbf{GA}^\iota$.

PROOF. First, transform G to a canonical term, which then can be pruned down to an equivalent minimal canonical one, using the identities in $\mathbf{GA}^\iota$ (e.g. right-distribution inclusions and the absorption laws for 3 and 4). ■

From now on, our strategy towards proving Theorem 6 will be to show that *two minimal canonical terms are equivalent iff they are isomorphic.* Presumably, that can be done entirely within game semantics. Instead, we introduce and use for the purpose a simple translation of game terms and identities to modal logic, which will make our task essentially easier.

5. Translation of the algebra of games to modal logic

Here we introduce a translation of GL into plain modal logic. This translation, naturally, resembles Parikh's translation of the language GL^* extending GL with game iteration $(*)$ into μ-calculus, but is simpler, computationally lighter and easier to use. In particular, we will use it in the next section to construct counter-models to invalid game equivalences, since Kripke models are rather more transparent, flexible and easier to deal with than game boards.

To begin with, we consider the **modal language** ML comprising:

- a set of *atomic variables* $V = \mathcal{V} \cup \{q\}$ where $\mathcal{V} = \{p_a\}_{a \in A}$ and $q \notin \mathcal{V}$ is an *auxiliary variable.*
- the usual modal connectives: $\vee, \wedge, \neg, \Box, \Diamond$, where $\Diamond$ will be regarded as an abbreviation for $\neg\Box\neg$.

Some terminology and notation:

- *Substitution* $\varphi(\psi/q)$: ψ is substituted for all occurrences of the variable q in φ.
- A *dual of a modal formula* φ *with respect to the variable* q *is* $\varphi_q^d = \neg\varphi(\neg q)$. Note that $(\varphi_q^d)_q^d \equiv \varphi$.
- Furthermore, we will often treat modal formulae as set operators in the standard sense, and thus, given a formula $\varphi(q)$, Kripke model $M = \langle S, R, V\rangle$ and a set $X \subseteq S$ we will allow ourselves the sloppiness of writing $\varphi(X)$ assuming its natural meaning, viz. that q has been evaluated as X.

5.1. The translation:

All duals of modal formulas used in the translation will be with respect to q, so we can safely omit the subscript. Likewise, all substitutions will be of the type $\varphi(\psi/q)$, which hereafter we will simply write as $\varphi(\psi)$.

With every game term G we associate a modal formula $m(G)$ as follows:

- $m(\iota) = q$;

- $m(g_a) = \Diamond\Box(p_a \to q)$ for any non-idle atomic game g_a, $a \in A$;
- $m(G_1 \vee G_2) = m(G_1) \vee m(G_2)$;
- $m(G^d) = (m(G))^d$, also denoted by $m^d(G)$.
- $m(G_1 \circ G_2) = m(G_1)(m(G_2))$.

Note that:

- Every formula $m(G)$, being positive in q, is monotone in q.
- $m(g_a^d)$ is equivalent to $\Box\Diamond(p_a \wedge q)$. Hereafter we will simply consider these equal.
- $m(G_1 \wedge G_2) = m(G_1) \wedge m(G_2)$.
- $m^d(G_1 \circ G_2) = m^d(G_1)(m^d(G_2))$.

EXAMPLE.

- $m(g_1 \circ (g_2 \vee g_3)) = \Diamond\Box(p_1 \to (\Diamond\Box(p_2 \to q) \vee \Diamond\Box(p_3 \to q)))$;
- $m((g_2 \vee g_3) \circ g_1) = \Diamond\Box(p_2 \to \Diamond\Box(p_1 \to q)) \vee \Diamond\Box(p_3 \to \Diamond\Box(p_1 \to q)) = m((g_2 \circ g_1) \vee (g_3 \circ g_1))$.
- $m(((g_1 \circ g_2) \vee g_1)^d \circ g_3)) = \Box\Diamond(p_1 \wedge \Box\Diamond(p_2 \wedge \Diamond\Box(p_3 \to q))) \wedge \Box\Diamond(p_1 \wedge \Diamond\Box(p_3 \to q))$.

5.2. Preservation of validity

The main result regarding this translation is:

THEOREM 15. *For any game terms G, H, if the game inclusion $G \preceq H$ is valid on all determined game boards then $\models m(G) \to m(H)$.*

PROOF. By contraposition, suppose $M, u \not\models m(G) \to m(H)$ for some model M with a domain S and state $u \in S$. Then we define a game board $B_M = \langle S, \{\rho_a^i\}_{a \in A; i=1,2} \rangle$ as follows. For every $X \subseteq S$ and $s \in S$:

$$s\rho_a^1 X \text{ iff } M, s \models m(g_a)(X),$$

and

$$s\rho_a^2 X \text{ iff } M, s \models m^d(g_a)(X).$$

LEMMA 16. *B_M is a determined game board.*

PROOF OF THE LEMMA. The condition *MON* is immediate from the monotonicity of $m(G)$ in q. For *CON* and *DET*, notice that $M, s \not\models m(g_a)(X)$ iff $M, s \models \neg m(g_a)(X)$ i.e. $M, s \models m^d(g_a)(\neg X)$.

LEMMA 17. *For every $s \in S$, $X \subseteq S$ and term D :*

$$s\rho_D^1 X \textit{ iff } M, s \models m(D)(X),$$
$$s\rho_D^2 X \textit{ iff } M, s \models m^d(D)(X).$$

Proof of the lemma.

Structural induction on D. For atomic games this holds by definition. The cases $D = D_1^d$ and $D = D_1 \vee D_2$ are straightforward. Let $D = D_1 \circ D_2$ and suppose $s\rho^1_D X$. Then $s\rho^1_{D_1} Z$ for some $Z \subseteq S$ such that $z\rho^1_{D_2} X$ for each $z \in Z$. Then $M, s \models m(D_1)(Z)$ and $Z \subseteq V(m(D_2)(X))$, so by monotonicity $M, s \models m(D_1)(m(D_2)(X))$, i.e. $M, s \models m(D_1 \circ D_2)(X)$.

The case of $s\rho^2_D X$ is quite analogous, modulo the duality, but we'll do it nevertheless: let $s\rho^2_{D_1} Z$ for some $Z \subseteq S$ such that $z\rho^2_{D_2} X$ for each $z \in Z$. Then $M, s \models m^d(D_1)(Z)$ and $Z \subseteq V(m^d(D_2)(X))$, so by monotonicity $M, s \models m^d(D_1)(m^d(D_2)(X))$, i.e. $M, s \models m^d(D_1 \circ D_2)(X)$.

Conversely, let $M, s \models m(D_1 \circ D_2)(X)$, hence $M, s \models m(D_1)(m(D_2)(X))$. Then $Z = V(m(D_2)(X))$ is such that $s\rho^1_{D_1} Z$ and $z\rho^1_{D_2} X$ for each $z \in Z$, by the inductive hypothesis. Therefore, $s\rho^1_{D_1 \circ D_2} X$. Likewise for $s\rho^2_{D_1 \circ D_2} X$.

This completes the induction and the proof of the lemma.

Finally, recall that $M, u \models m(G)$ and $M, u \not\models m(H)$. Let $X = V(q)$. Then $u\rho^1_G X$ while $\neg u\rho^1_H X$, so $B_M \not\models G \preceq H$. ■

Corollary 18. *For any game terms G, H, if $\boldsymbol{DET} \models G = H$ then $\models m(G) \leftrightarrow m(H)$.*

6. Some technical results

First, some useful remarks.

- Since **K** is complete for the class of irreflexive tree-like Kripke models, every non-valid translation of a game inclusion or identity can be refuted in a model rooted at a state s without predecessors. Note that any re-evaluation of variables at s in such a model will not affect the truth or falsity at s of any $m(G)$, except $m(\iota)$, when the truth of q is altered, because all occurrences of other variables in these formulae are in the scope of modal operators.
- Let $F_* = \langle S_*, R_* \rangle$ where $S_* = \{*, y, z\}, R_* = \{(*, y), (y, z), (z, z)\}$. Then the Kripke model $M_+ = \langle S_*, R_*, V_+ \rangle$, where $V_+(q) = \{*, z\}$ satisfies *all* $m(G)$ at its root $*$, while the model $M_- = \langle S_*, R_*, V_- \rangle$, where $V_-(q) = \varnothing$ and $V_-(p_a) = \{z\}$ for all $a \in A$, falsifies all $m(G)$ at $*$. These models can be freely grafted on an irreflexive leaf of any model (taking care of q, if necessary).

Here's our main technical lemma:

Lemma 19. *Let G and H be minimal canonical terms. The following are equivalent:*

- $G \not\equiv H$.
- (♠) *There is a disjunct* $\bigwedge_{k \in K} g_{ik} \circ G_{ik}$ *in* G *such that every disjunct in* H *contains a conjunct* $h_{jm_j} \circ H_{jm_j}$ *not including any of the conjuncts* $g_{ik} \circ G_{ik}$ *for* $k \in K$.
- *There is a finite (tree-like) Kripke model* M *and a state* $s \in M$ *such that:* $M, s \vDash m(G)$; $M, s \nvDash m(H)$; *and* s *has no predecessors in* M.

Proof. We prove all equivalences by double induction on the structure of G and H. The case when both of them are ι is vacuous, so suppose otherwise and let $G = \bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$, $H = \bigvee_{j \in J} \bigwedge_{m \in M_j} h_{jm} \circ H_{jm}$ where the claim holds for all pairs of G_{ik}'s and H_{jm}'s. If one of G and H is ι we represent it as $\bigvee \bigwedge \iota \circ \iota$.

1) Let $G \not\equiv H$. Then there is a game board $B = \langle S, \{\rho^i_a\}_{a \in A; i=1,2} \rangle$ such that either $G \subsetneq_1 H$ or $H \subsetneq_2 G$ on B.

1.1) Suppose $G \subsetneq_1 H$. Then there is a state s and a disjunct (the choice of player I) $\bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ such that every conjunct $g_{ik} \circ G_{ik}$ enables him to achieve some outcome X from s which he cannot force on H, so every disjunct $\bigwedge_{m \in M_j} h_{jm} \circ H_{jm}$ in H contains a conjunct $h_{jm_i} \circ H_{jm_i}$ which lacks the power for player I to force an outcome X. Thus, none of the terms $h_{jm_i} \circ H_{jm_i}$, $j \in J$, includes any of $g_{ik} \circ G_{ik}$, $k \in K_i$.

1.2) Suppose $H \subsetneq_2 G$. Then player II can force some outcome X in H which she cannot force in G, so every disjunct $\bigwedge_{m \in M_j} h_{jm} \circ H_{jm}$ (possible choice of I) in H contains a conjunct (the reply of II) $h_{jm_i} \circ H_{jm_i}$ which contains (s, X) in the forcing relation for II, while this is not the case for G, so some disjunct $\bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ is such that no term $g_{ik} \circ G_{ik}$ in it contains (s, X) in its forcing relation for II, hence none of $g_{ik} \circ G_{ik}$, $k \in K_i$ is included in any of $h_{jm_i} \circ H_{jm_i}$, $j \in J$.

Thus, in either case (♠) holds.

2) Suppose (♠). Note that there can be at most one idle term amongst all $\{g_{ik} \circ G_{ik} | k \in K_i\}$ and $\{h_{jm_i} \circ H_{jm_i} | j \in J\}$.

We will build a Kripke model M which will satisfy all $\{m(g_{ik} \circ G_{ik}) | k \in K_i\}$, and hence $m(G)$, while none of $\{m(h_{jm_i} \circ H_{jm_i}) | j \in J\}$, hence it will falsify $m(H)$. M will be rooted at some state s with no predecessors, which is needed for the inductive hypothesis because models like this will be grafted at their roots on larger models as the induction goes on.

Depending on the signs of the literals $g_{ik}, k \in K_i$ and $h_{jm_i}, j \in J$, the set of all these terms splits into the following subsets:

- $T_{\dot{A}} = \{t_\alpha \circ D_\alpha | \alpha \in \mathbf{A}\}$ whose translations must be true at s;
- $T_B = \{t^d_\beta \circ D_\beta | \beta \in \mathbf{B}\}$ whose translations must be true at s;

- $T_\Gamma = \{t_\gamma \circ D_\gamma | \gamma \in \mathbf{\Gamma}\}$ whose translations must be false at s;
- $T_\Delta = \{t_\delta^d \circ D_\delta | \delta \in \mathbf{\Delta}\}$ whose translations must be false at s.
- Possibly, $T_\iota = \{\iota \circ \iota\}$.

The terms $t_\alpha, t_\beta, t_\gamma, t_\delta$ above are *non-idle atoms.* Let $p_\alpha, p_\beta, p_\gamma, p_\delta$ be their corresponding variables in the modal translation. Thus, we have to satisfy at s simultaneously the following sets of formulae:

- $F_\mathbf{A} = \{\Diamond\Box(p_\alpha \to m(D_\alpha)) | \alpha \in \mathbf{A}\}$,
- $F_\mathbf{B} = \{\Box\Diamond(p_\beta \wedge m(D_\beta)) | \beta \in \mathbf{B}\}$,
- $F_\mathbf{\Gamma} = \{\Box\Diamond(p_\gamma \wedge \neg m(D_\gamma)) | \gamma \in \mathbf{\Gamma}\}$,
- $F_\mathbf{\Delta} = \{\Diamond\Box(p_\delta \to \neg m(D_\delta)) | \delta \in \mathbf{\Delta}\}$.
- Possibly, $F_\iota = \{q\}$ or $F_\iota = \{\neg q\}$, depending on whether there is an idle term in $\{g_{ik} \circ G_{ik} | k \in K_i\}$ or $\{h_{jm_i} \circ H_{jm_i} | j \in J\}$ respectively.

We build the model $M = \langle W, R, V\rangle$ as follows:

$W = \{s\} \cup (\mathbf{A} \cup \mathbf{\Delta}) \cup ((\mathbf{A} \cup \mathbf{\Delta}) \times (\mathbf{B} \cup \mathbf{\Gamma})) \cup W'$, where $\mathbf{A}, \mathbf{B}, \mathbf{\Gamma}, \mathbf{\Delta}$ are the index sets above, which will form the 'carcass' of the model, and W' will be sub-models satisfying/falsifying the $m(D)'s$, which will be grafted on the carcass accordingly (see further).

- For better readability, in what follows the elements of a product $\mathbf{X} \times \mathbf{Y}$ will be denoted as x_y, for $x \in X, y \in Y$.

$R = \{(s, x) | x \in \mathbf{A} \cup \mathbf{\Delta}\} \cup \{(x, x_y) | x \in \mathbf{A} \cup \mathbf{\Delta},\ y \in \mathbf{B} \cup \mathbf{\Gamma}\} \cup R'$ where R' will be the union of the inherited relations from the grafted sub-models.

The rest of the model and the valuation V will be defined as follows:

- Every state α_β, for $\alpha \in \mathbf{A}$, $\beta \in \mathbf{B}$ must satisfy $p_\alpha \to m(D_\alpha)$ and $p_\beta \wedge m(D_\beta)$. For that, we set p_β true at α_β and graft a copy of M_+ at α_β.
- Every state α_γ, for $\alpha \in \mathbf{A}$, $\gamma \in \mathbf{\Gamma}$ must satisfy $p_\alpha \to m(D_\alpha)$ and $p_\gamma \wedge \neg m(D_\gamma)$. If $\alpha \neq \gamma$ we set p_α false and p_γ true at α_γ and graft a copy of M_- at α_γ. If $t_\alpha = t_\gamma$ then $D_\alpha \not\preceq D_\gamma$ (for, otherwise, $t_\alpha \circ D_\alpha \preceq t_\gamma \circ D_\gamma$, which contradicts (♠)), hence by the inductive hypothesis there is a model $M^{\alpha\gamma}$ rooted at some u such that $M^{\alpha\gamma}, u \models m(D_\alpha)$ while $M^{\alpha\gamma}, u \not\models m(D_\gamma)$. Then we set p_α true and graft a copy of $M^{\alpha\gamma}$ at α_γ.
- Every state δ_β must satisfy $p_\beta \wedge m(D_\beta)$ and $(p_\delta \to \neg m(D_\delta)$. This case is treated analogously to the previous one.
- Every state δ_γ must satisfy $p_\gamma \wedge \neg m(D_\gamma)$ and $p_\delta \to \neg m(D_\delta)$. For that we set p_γ true and graft a copy of M_- at δ_γ.

- Finally, $s \in V(q)$ iff $q \in F_\iota$.

This completes the description of M. It is immediate from the construction that M, s will satisfy all formulae in $F_{\mathbf{A}} \cup F_{\mathbf{B}} \cup F_{\mathbf{\Gamma}} \cup F_{\mathbf{\Delta}}$ and hence $M, s \vDash m(G)$, while $M, s \nvDash m(H)$.

3) If $M, s \vDash m(G)$, while $M, s \nvDash m(H)$ then, by Theorem 15, $G \not\preceq H$.

This completes the circle of equivalences and the induction step. ∎

Corollary 20. *For any game terms G, H:*

1. $\models m(G) \to m(H)$ *iff* $G \preceq H$ *is a valid game inclusion.*
2. $\models m(G) \leftrightarrow m(H)$ *iff* $G = H$ *is a valid game identity.*

Proof. One direction of (1) is by Th. 15. For the other, suppose $\models m(G) \to m(H)$. We can assume that G, H are minimal canonical, again due to Th. 15, so $G \preceq H$ by Lemma 19. (2) follows immediately from (1). ∎

Corollary 21. $\models G = H$ *iff* $\boldsymbol{DET} \models G = H$.

Given a Kripke model M and a state s, $T(M, s, u)$ will denote the model obtained from M by adding two new states u, v such that uRv and vRs.

Lemma 22. *Let G, H be any terms and g, h be non-idle literals. Then $g \circ G \preceq h \circ H$ iff $g = h$ and $G \preceq H$.*

Proof. One direction is obvious. The other we prove by contraposition, assuming $g \neq h$ or $G \not\preceq H$ and using the modal translation.

Case 1: $g \neq h$. We falsify $m(g \circ G) \to m(h \circ H)$ at the root of a model constructed by cases as follows:

1.1) $g = g_a$, $h = g_b$, $a \neq b$ for some atoms g_a, g_b. Take a copy of $T(M_-, *, u)$ and set p_a false and p_b true at $*$.

1.2) $g = g_a$, $h = g_b^d$. Take a copy of F_* and set p_a and p_b false at z.

1.3) $g = g_a^d$, $h = g_b$. Take a copy of $T(M_+, *, u)$, add a new successor s to v and graft a copy of M_- at s, setting p_a true at $*$ and p_b true at s, so $\Box\Diamond(p_a \wedge m(G))$ is true at u, while $p_b \to m(H)$ is false at s, hence $\Box(p_a \to m(H))$ is false at v, so $\Diamond\Box(p_a \to m(H))$ is false at u.

1.4) $g = g_a^d$, $h = g_b^d$, $a \neq b$. Take a copy of $T(M_+, *, u)$ and set p_a true and p_b false at $*$.

Case 2: $g = h$ and $G \not\preceq H$. We can assume that G and H are minimal canonical. Let $M, s \nvDash m(G) \to m(H)$ (by Lemma 19) and suppose $g = g_a$

or $g = g_a^d$. Then setting p_a true at s will falsify $m(g \circ G) \to m(h \circ H)$ at u in $T(M, s, u)$.

Thus, in each case we have shown that $g \circ G \not\preceq h \circ H$. ■

LEMMA 23. 1. *$g \circ G \preceq \iota \circ \iota$ iff g is an idle literal and $G \preceq \iota$.*
2. *$\iota \circ \iota \preceq g \circ G$ iff g is an idle literal and $\iota \preceq G$.*

PROOF of the non-trivial directions:
If g is non-idle then $m(g \circ G) \to m(\iota \circ \iota)$ is falsified at the root of $T(M_+, *, u)$ by setting q to false at u. Thus, if $g \circ G \preceq \iota \circ \iota$ then g is idle and $\models m(g \circ G) \to m(\iota \circ \iota)$, hence $\models m(G) \to m(\iota)$, so $G \preceq \iota$.
Likewise, if g is non-idle then $m(\iota \circ \iota) \to m(g \circ G)$ is falsified at the root of $T(M_-, *, u)$ by setting q to true at u. Thus, if $\iota \circ \iota \preceq g \circ G$ then g is idle and $\models m(\iota) \to m(G)$, so $\iota \preceq G$. ■

7. Proof of the completeness of $\mathbf{GA}^\iota$

LEMMA 24. *If G, H are minimal canonical terms then $G \preceq H$ iff $G \rightarrowtail H$.*

PROOF. If $G \rightarrowtail H$ then $G \preceq H$ is straightforward. For the other direction we proceed by double induction on the structure of both terms. The case when both of them are ι is trivial, so suppose otherwise and let $G = \bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$, $H = \bigvee_{j \in J} \bigwedge_{m \in M_j} h_{jm} \circ H_{jm}$ be minimal canonical terms (again, representing ι as $\bigvee \bigwedge \iota \circ \iota$) such that $G \preceq H$ and the claim holds for all G_{ik}'s and H_{jm}'s, i.e. if one of these is included into another then that inclusion is embedding.

Now, suppose G is not embedded into H. Then there is a disjunct $\bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$ in G such that every disjunct $\bigwedge_{m \in M_j} h_{jm} \circ H_{jm}$ in H contains a conjunct $h_{jm_i} \circ H_{jm_i}$ in which none of $g_{ik} \circ G_{ik}, k \in K_i$ is embedded. But that means, by the inductive hypothesis and Lemmas 22 and 23, that none of these terms is *included* in any of the $h_{jm_i} \circ H_{jm_i}$, for $j \in J$. This is precisely the condition (♠) of Lemma 19. Therefore $G \not\preceq H$.

This completes the inductive step and the proof of the lemma. ■

PROPOSITION 25. *If G, H are minimal canonical terms such that $G \rightarrowtail H$ and $H \rightarrowtail G$ then $G \simeq H$.*

PROOF. Again, double induction on G, H. The case when both of them are ι is trivial. Suppose $G = \bigvee_{i \in I} \bigwedge_{k \in K_i} g_{ik} \circ G_{ik}$, $H = \bigvee_{j \in J} \bigwedge_{m \in M_j} h_{jm} \circ H_{jm}$ be minimal canonical terms such that $G \rightarrowtail H$ and $H \rightarrowtail G$ and the claim holds for all G_{ik}'s and H_{jm}'s.

Take any disjunct D from G. It embeds into some disjunct D' from H, which in turns embeds into some D'' from G, so D is embedded into D'', hence D and D'' must coincide because G is minimal canonical. Therefore, $D \rightarrowtail D'$ and $D' \rightarrowtail D$, so for every conjunct C in D there is a conjunct C' in D' embedded into C; and there is a conjunct C'' in D embedded into C', hence C'' is embedded into C. Again by minimal canonicity of G, that implies that C and C'' coincide, hence $C \rightarrowtail C'$ and $C' \rightarrowtail C$. Let $C = t \circ T$ and $C' = t' \circ T'$ for some literals t and t' and minimal canonical terms T, T' for which the inductive hypothesis holds. Therefore, $C \preceq C'$ and $C' \preceq C$, hence by Lemmas 22 and 23, $t = t'$, $T \preceq T'$ and $T' \preceq T$, so $T \rightarrowtail T'$ and $T' \rightarrowtail T$ by Lemma 24, hence $T \simeq T'$ by the inductive hypothesis. Therefore $C \simeq C'$.

Thus, every conjunct from D is isomorphic to a conjunct from D' and vice versa. This is a bijection because of the minimal canonicity of G and H. Hence, every disjunct from G is isomorphic to a disjunct from H and vice versa. Again, this is a bijection due to the minimal canonicity of G and H. Therefore, $G \simeq H$. ∎

Corollary 26. *The minimal canonical terms G and H are equivalent iff they are isomorphic.*

Proof. $G \sim H$ iff ($G \preceq H$ and $H \preceq G$) iff ($G \rightarrowtail H$ and $H \rightarrowtail G$) iff $G \simeq H$. ∎

Proof of Theorem 6. Let $G \sim H$ and $c(G), c(H)$ be minimal canonical terms obtained from G and H by reduction within $\mathbf{GA}^{\iota}$. Then $c(G) \sim c(H)$, hence $c(G) \simeq c(H)$ by Corollary 26. Since each of the equivalences $G \sim c(G)$, $c(G) \simeq c(H)$, $c(H) \sim H$ is derivable in $\mathbf{GA}^{\iota}$, so is $G \sim H$. ∎

8. Concluding remarks

8.1. Valid identities and game board conditions

On the one hand, it can be easily verified that all axiomatic identities, and hence all valid ones, remain valid if the condition for consistency of powers *(CON)* is omitted.[2]

On the other hand, as Corollary 21 shows, *determinacy* of game boards does not add new valid identities of game terms. This result can be strengthened: *termination* can be added, too, without introducing new valid identities.

[2]This observation is essentially due to Yde Venema, who raised the question.

PROPOSITION 27. $\models G = H$ *iff* ***(DET***$\cap$***FIN)*** $\models G = H$.

PROOF. It is sufficient to modify the proof of Lemma 19 by showing that whenever $G \not\preceq H$ for minimal canonical terms G and H, the counter-model for $m(G) \to m(H)$ can be constructed in such a way that the game board determined by it as in the proof of Theorem 15 is terminating as well, i.e. $s\rho_a^i S$ holds for each atomic game g_a (and hence for every game term). These conditions impose the following requirements on the Kripke model:

- Termination for ι. It holds trivially.
- $s\rho_a^1 S$ iff $M, s \models \Diamond\Box(p_a \to \top)$ i.e. $M, s \models \Diamond\top$. This is satisfied by the current construction.
- $s\rho_a^2 S$ iff $M, s \models \Box\Diamond(p_a \wedge \top)$ i.e. $M, s \models \Box\Diamond p_a$ for each non-idle $a \in A$. To satisfy this condition we extend the construction in the proof of Lemma 19 as follows: for each $\alpha \in \mathbf{A}$ we add one more successor, α' to α, graft a copy of M_+ at α', and set all p_a to be true at α'. Likewise, for each $\delta \in \mathbf{\Delta}$ we add one more successor, δ' to Δ, graft a copy of M_- at δ', and set all p_a to be true at δ'. That will preserve the truth at s of all formulas from $F_{\mathbf{A}} \cup F_{\mathbf{B}} \cup F_{\mathbf{\Gamma}} \cup F_{\mathbf{\Delta}}$, while forcing all $\Box\Diamond p_a$ to be true at s.

Thus, every invalid inclusion $G \preceq H$ can be falsified in a determined and terminating game board. ■

8.2. On the complexity of the validity of game identities

While the translation m of *canonical* terms to formulae of modal logic is polynomial in the size of the terms, because only literals occur on the left of compositions, in general that translation can be *exponential* in the size of the terms (e.g. consider the translation of $(g_{11} \vee g_{21}) \circ (g_{12} \vee g_{22}) \circ \ldots \circ (g_{1n} \vee g_{2n})$). However, as Venema has noted, the number of *different subformulae* in the resulting translation is still polynomial in the size of the term, which can be shown by a simple induction on terms. Thus, the complexity of the validity of game identities is not greater than the complexity of the validity in the basic modal logic **K**. Thus, we obtain the following.

PROPOSITION 28. *The validity problem for identities of game terms is in* PSPACE.

8.3. From game algebras to game logics.

Clearly, the game algebra of a fixed language of game terms can be regarded as a fragment of the corresponding game logic. In particular, every valid game identity $G = H$ corresponds to a pair of valid formulas $\langle G \rangle q \leftrightarrow \langle H \rangle q$ and $[G]q \leftrightarrow [H]q$ (in the notation of [5] and [3]) and vice versa.

The dual-free fragment of game logic (with tests) was axiomatized and proved complete in [5], by modifying appropriately the completeness proof for PDL, while the iteration-free game logic, corresponding to the game language considered here (with additional tests), has been axiomatized and proved complete in [4] using an adaptation of Parikh's proof, combined with the method of canonical models for neighbourhood semantics of modal logic. That proof, however, does not imply the present completeness result because it does not show if the derivation of every valid formula of the type $\langle G \rangle q \leftrightarrow \langle H \rangle q$ or $[G]q \leftrightarrow [H]q$ can be translated into equational logic. On the other hand, the modal translation introduced here readily extends to the iteration-free game logic, and accordingly the method of proving completeness applied here can be modified to an alternative completeness proof for that logic, by extending the notion of canonical forms to all formulas. We note that this method can also be adapted to prove completeness of modal logic itself. In fact, that was essentially done quite a while ago in [2], where the use of normal forms in modal logic was promoted.

8.4. Representing game algebras

Meanwhile, Venema has strengthened in [6] the completeness result presented here by proving a representation theorem for abstract game algebras defined by the set of identities **GA** into game boards.

As the completeness of the full Game Logic introduced in [5] is still open, it is interesting to see if the method applied here or Venema's algebraic approach can be extended to the game language with iteration and thus provide a handle to solving that problem, too.

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