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The Craig Interpolation Theorem for Propositional Logics with Strong Negation

Abstract. This paper deals with propositional calculi with strong negation (N-logics) in which the Craig interpolation theorem holds. N-logics are defined to be axiomatic strengthenings of the intuitionistic calculus enriched with a unary connective called strong negation. There exists continuum of N-logics, but the Craig interpolation theorem holds only in 14 of them.

A propositional calculus with strong negation is an extension of the intuitionistic propositional calculus by an additional logical connective $\sim$ called "strong negation". It formalizes in a way the following idea: usually for the refutation of a given assertion there exist two ways: reductio ad absurdum and construction of a counter-example. From a constructivist's point of view these two ways are not equivalent; the weak and strong negation of the above calculus correspond to them.

The smallest such logic $\tilde{I}$, called the constructive logic with strong negation was formulated independently by Nelson [12] and Markov [2] and studied by them as well as by Vorobiev [3], [4], [5], Rasiowa [6], [7], [8], [9], Vakarelov [10], [11] and others. The strong negation in this logic has constructive properties, which do not hold for the intuitionistic negation: from $\tilde{I} \vdash \sim(A \cap B)$ it follows that either $\tilde{I} \vdash \sim A$ or $\tilde{I} \vdash \sim B$, and in the corresponing predicate logic the derivability of $\sim \forall x A(x)$ implies the derivability of $\sim A(\tau)$ for a certain term τ.

An algebraic semantics for the propositional calculus with strong negation, introduced by Rasiowa [7], is based on a special kind of distributive lattices, named N-lattices (algebras of Nelson, quasi-pseudo-Boolean algebras). Vakarelov in [11] gives a construction of the so-called special N-lattices, with the help of which a number of problems, related to the extensions of constructive logic with strong negation (called in this paper N-logics) and N-lattices can be attacked successfully, reducting them to analogous problems for the superintuitionistic logics and pseudo-Boolean algebras, respectively. Thus, for example, Sendlewski in [13] announces the complete list of the critical varieties of N-lattices (called here N-varieties).

In the present paper all N-logics, in which the Craig interpolation theorem holds, are found. This property, as in a number of other cases, proves to be equivalent to the amalgamation property of the correspon-ding N-varieties. All logics, which are of interest to us, are divided into two classes, the truth of the Craig interpolation theorem (CIT) in each of them proves to be equivalent to the truth of the CIT in the corresponding superintuitionistic fragments. Thus, the problem is reduced to the analogous problem in the domain of superintuitionistic logics, which was solved by Maximova [1]. It turns out, that in the continuum of consistent N-logics, the CIT holds only in 14 of them.

§0. Propositional calculi with strong negation. N-logics and N-lattices

0.1. By recursion we define a set of formulae *For* of the language $\tilde{\mathscr{L}}$, which is an extension to the language of intuitionistic propositional calculus $\mathscr{L}$, containing logical signs $\cap$, $\cup$, $\to$, $\neg$ and a set of propositional variables $\Phi_0 = \{p_0, p_1, \ldots\}$ by adding a new, one-argument logical sign $\sim$ which will be called strong negation. By *for* we shall denote the subset of *For*, containing the formulae in which the sign $\sim$ does not enter.

Some abbreviations: $A \Rightarrow B \leftrightharpoons (A \to B) \cap (\sim B \to \sim A)$; $A \leftrightarrow B \leftrightharpoons (B \to A) \cap (A \to B)$, $A \Leftrightarrow B \leftrightharpoons (A \Rightarrow B) \cap (B \Rightarrow A)$; $0 \leftrightharpoons \neg 1$.

The set of axioms of the Vorobiev's calculus is $\tilde{A} = A_0 \cup V$, where A_0 is a set of axioms for the intuitionistic propositional calculus and V is the system of Vorobiev's axioms:

(v_1) $\sim A \to (A \to B)$
(v_2) $\sim (A \to B) \leftrightarrow A \cap \sim B$
(v_3) $\sim (A \cap B) \leftrightarrow \sim A \cup \sim B$
(v_4) $\sim (A \cup B) \leftrightarrow \sim A \cap \sim B$
(v_5) $\sim \neg A \leftrightarrow A$
(v_6) $\sim \sim A \leftrightarrow A$

Rules of inference: modus ponens and substitution. An inference (proof) and provable (derivable) formula (theorem) are defined as usual.

We shall call any set $L \subseteq For$, containing $\tilde{A}$ and closed with respect to the rules of inference, a logic with strong negation (N-logic).

The smallest logic with strong negation $\tilde{I}$ bears the name: "a constructive logics with strong negation".

NOTE. Since $\sim A \to (A \to 0) \in \tilde{I}$ and $(A \to 0) \to \neg A \in \tilde{I}$, then $\tilde{I} \vdash \sim A \to \neg A$, which explains the name "strong negation". The converse implication, as we shall see below, is equivalent to $A \cup \sim A$ and added to $\tilde{I}$ gives an N-logic which coincides in essence with the classical logic.

The Craig interpolation theorem (CIT) in a logic L reads: If $A \to B \in L$, then there exists a formula C, containing only variables, which enter simultaneously A and B, such that $A \to C \in L$ and $C \to B \in L$.

The main aim of this paper is to describe all N-logics in which the CIT holds.

NOTE. The same question can be put as to the truth of the CIT relatively to the so-called strong implication $\Rightarrow$ but, as we shall see below, it is solved trivially.

0.2. We recall the definition of an N-lattice (see [8]): An algebraic system $\mathscr{N} = (A, \cup, \cap, \rightarrow, \sim, \neg, 1)$ is called an N-lattice if:

(R_0) the relation $\prec$, where $a \prec b$ denotes $a \rightarrow b = 1$, is a quasi-ordering on A,

(R_1) the system $(A, \cup, \cap, \sim, 1)$ is a quasi-Boolean algebra, i.e. a distributive lattice with 1, in which the following identities hold: (q_1) $\sim\sim a = a$ and (q_2) $\sim(a \cup b) = \sim a \cap \sim b$

(R_2) $a \leqslant b$ iff $a \Rightarrow b = 1$ ($a \leqslant b$ denote $a \cap b = a$)

(R_3) if $c \prec a$ and $c \prec b$, then $c \prec a \cap b$

(R_4) if $a \prec c$ and $b \prec c$, then $a \cup b \prec c$

(R_5) $\sim(a \rightarrow b) \prec a \cap \sim b$

(R_6) $\sim b \cap a \prec \sim(a \rightarrow b)$

(R_7) $a \prec \sim \neg a$

(R_8) $\sim \neg a \prec a$

(R_9) $a \cap \sim a \prec b$

(R_{10}) $a \cap b \prec c$ iff $a \prec b \rightarrow c$

(R_{11}) $\neg a = a \rightarrow \sim 1$

We define the relation $\approx$: $a \approx b$ iff $a \prec b$ and $b \prec a$. N-lattices can be defined only by identities (see [6]), i.e. the class of N-lattices is a variety.

Some elementary facts in N-lattices, which we shall use are the following:

1. $\sim 1 = 0$; $\sim a \leqslant \neg a$; $0 \leqslant a$.
2. if $x \prec y$, then $a \rightarrow x \prec a \rightarrow y$; if $x \leqslant y$, then $a \rightarrow x \leqslant a \rightarrow y$.
3. if $a \prec b$, then $\neg b \leqslant \neg a$; if $a \approx b$, then $\neg b = \neg a$.
4. $\neg a = 1$ iff $a \approx 0$.
5. $a \cap \neg a \approx 0$; $a \cap \sim a \approx 0$.
6. $\neg a \rightarrow a \approx \neg\neg a$; $a \rightarrow \sim a = \neg a$; $a \rightarrow \neg a = \neg a$.
7. $a \prec \sim \neg a \leqslant \neg\neg a \leqslant \neg \sim a$; $a \leqslant \neg \sim a$.
8. $\neg\neg\neg a \leqslant \neg a$; $\neg\neg\neg a \approx \neg a$.

Additional information about N-lattices can be found in [6].

NOTE. Since from $a \prec b$ and $b \prec a$ it does not follow that $a = b$, $(A, \cup, \cap, \rightarrow, \neg, 1)$ is not a pseudo-Boolean algebra.

Examples of N-lattices:

a. Let $(B_0, \cap, \cup, \rightarrow, \neg)$ be a two-element Boolean algebra. Set $\sim a \leftrightharpoons \neg a$ in B_0 and obtain a two-element N-lattice $\mathfrak{B}_0$.

b. In the linearly ordered set $C_0 = \{0, \delta, 1\}$ $(0 < \delta < 1)$ define operations $\sim$, $\neg$ and $\rightarrow$ by the tables:

x	$\sim x$	$\neg x$
0	1	1
δ	δ	1
1	0	0

$\rightarrow$	0	δ	1
0	1	1	1
δ	1	1	1
1	0	δ	1

In this way a three-element N-lattice $\mathfrak{C}_0 = (C_0, \cap, \cup, \rightarrow, \neg, \sim, 1)$ is obtained.

$\mathfrak{B}_0$ and $\mathfrak{C}_0$ are the unique (up to the isomorphism) two- and three-element N-lattices, respectively.

0.3. The reader can find in [6] a detailed information about the filter theory in N-lattices. Here, we briefly review some definitions and facts, that will be needed later.

Let $\mathcal{N} = (A, \cup, \cap, \rightarrow, \sim, \neg, 1)$ be an N-lattice. A non-empty set $\nabla \subseteq A$ is called a special filter of the first kind (s.f.f.k.) if:

(1) $a \in \nabla$ and $b \in \nabla$ imply $a \cap b \in \nabla$,

(2) $a \in \nabla$ and $a \prec b$ imply $b \in \nabla$.

THEOREM 1. *If $Ker(h)$ is a kernel of an isomorphism of N-lattices, $h: \mathcal{N}_1 \rightarrow \mathcal{N}_2$, then $Ker(h)$ is a s.f.f.k.. $h(a) = h(b)$ is equivalent to $a \Leftrightarrow b \in Ker(h)$; the relation $\equiv$, where $a \equiv b$ if and only if $a \Leftrightarrow b \in Ker(h)$, is a congruence in $\mathcal{N}_1$, at that $\mathcal{N}_1 / \equiv \; \cong \mathcal{N}_2$.* [6]

THEOREM 2. *Let ∇ be a s. f. f. k. in a N-lattice $\mathcal{N}$. Then the relation $\equiv_\nabla$, denoting $a \Leftrightarrow b \in \nabla$ is a congruence in $\mathcal{N}$. $\mathcal{N}/\nabla$ is an N-lattice. The mapping $h: \mathcal{N} \rightarrow \mathcal{N}/\nabla$, where $h(a) = |a|$ is an epimorphism and $Ker\ (h) = \nabla$.* [6]

0.4. Rasiowa in [7] shows that the Lindenbaum algebra for $\tilde{I}$ is an N-lattice. This gives us the possiblity of examining the algebraic semantics for the N-logics.

In the usual way we define a valuation of the variables and formulae from *For* in an N-lattice, the truth of a formula for a given valuation, and the validity of a formula in a given N-lattice, and in a class of N-lattices.

To any N-logic L there corresponds a variety of N-lattices *varL*, defined by the set of identities $\{A = 1 \mid A \in L\}$. Conversely, to any class of N-lattices $\mathbf{K}$ there corresponds an N-logic $L = \{A \mid \mathbf{K} \models A = 1\}$. Therefore, $varL = var(\mathbf{K})$ — a variety generated by the class $\mathbf{K}$.

NOTE. The formula $(p \cap \sim p) \Rightarrow (q \cup \sim q)$ is derivable in $\tilde{I}$ (it is valid in all N-lattices). Therefore, if in an N-logic L the CIT holds relatively to $\Rightarrow$, then in L either $(p \cap \sim p) \Rightarrow 0$ or $1 \Rightarrow (q \cup \sim q)$ is derivable. In both cases $L \vdash p \cup \sim p$. Hence, as we shall see further, L coincides with the biggest consistent N-logic C, functionally equivalent to classical propositional logic, in which the CIT really holds.

§1. Special N-lattices and special N-logics

1.1. Let us recall the following construction of Vakarelov [11]: Let $\mathfrak{B} = (B, \cap, \cup, \rightarrow, \neg, 0, 1)$ be a pseudo-Boolean algebra (PBA), i.e. distributive lattice with 0 and 1 in which:

1. for any $x, a, b \in B$: $x \cap a \leqslant b$ iff $x \leqslant a \rightarrow b$
2. $\neg a = a \rightarrow 0$.

NOTE. We shall deal simultaneously with two different algebraic systems — pseudo-Boolean algebras and N-lattices, but we shall use dentical signs for the corresponding operations in them for simplicity. This will not lead to confusion, since we shall always know which algebraic system we are dealing with.

Set $N(B) \leftrightharpoons \{(a_1, a_2)/a_1, a_2 \in B \ \& \ a_1 \cap a_2 = 0\}$,

(r_0) $\mathbf{1} = (1, 0)$, $\mathbf{0} = (0, 1)$
(r_1) $(a_1, a_2) \cup (b_1, b_2) \leftrightharpoons (a_1 \cup b_1, a_2 \cap b_2)$
(r_2) $(a_1, a_2) \cap (b_1, b_2) \leftrightharpoons (a_1 \cap b_1, a_2 \cup b_2)$
(r_3) $(a_1, a_2) \rightarrow (b_1, b_2) \leftrightharpoons (a_1 \rightarrow b_1, a_1 \cap b_2)$
(r_4) $\neg (a_1, a_2) \leftrightharpoons (\neg a_1, a_1)$
(r_5) $\sim (a_1, a_2) \leftrightharpoons (a_2, a_1)$

It can be proved directly, that:
$(a_1, a_2) \prec (b_1, b_2)$ iff $a_1 \leqslant b_1$; $(a_1, a_2) \leqslant (b_1, b_2)$ iff $a_1 \leqslant b_1$ and $b_2 \leqslant a_2$.

PROPOSITION 3. *For any* PBA $\mathfrak{B}$ *the system*

$$N(\mathfrak{B}) = (N(B), \cup, \cap, \rightarrow, \sim, \neg, 1)$$

is an N-lattice.

PROOF. Without difficulties (R_0)-(R_{11}) may be proved. ■

Also, the following can be shown directly.

PROPOSITION 4. *Let* $\mathfrak{A} \rightarrow \mathfrak{B}$ *be a homomorphism of* PBAs. *Then* h_N: $N(\mathfrak{A}) \rightarrow N(\mathfrak{B})$, *where* $h_N\big((a, b)\big) = \big(h(a), h(b)\big)$ *is a homomorphism of N-lattices.*

We shall call N-lattices of the type $N(\mathfrak{B})$ special N-lattices over PBA $\mathfrak{B}$.

NOTE. In [11] Vakarelov gives the following intuitive interpretation of the special N-lattices: PBA $\mathfrak{B}$ can be considered as Lindenbaum algebra of an intuitionistic theory or, more simply, as a set of assertions.

Let a_1 and a_2 be assertion. We say that a_2 is a counter-example of a_1 if $a_1 \cap a_2 = 0$. Then, the set of all pairs (a_1, a_2), where $a_1, a_2 \in \mathfrak{B}$ and a_2 is a counter-example of a_1, is just $N(\mathfrak{B})$. The definitions (r_0)-(r_5) provide constructions of counter-examples of $a_1 \cup b_1$, $a_1 \cap b_1$, $a_1 \to b_1$, a_2 and $\neg a_1$ if we have already constructed counter-examples of a_1 and b_1.

1.2.

PROPOSITION 5. *Let* $\mathfrak{B} = (B, \cap, \cup, \to, \neg, 1)$ *be a* PBA *and* $N(\mathfrak{B}) = (N(B), \cap, \cup, \to, \sim, \neg, 1)$ *be the corresponding special N-lattice. Then the map* $\pi\colon N(\mathfrak{B}) \to \mathfrak{B}$, *where* $\pi((a, b)) \leftrightharpoons a$, *is a lattice homomorphism, such that* $\pi(\alpha \to \beta) = \pi(\alpha) \to \pi(\beta)$ *and* $\pi(\neg\alpha) = \neg\pi(\alpha)$.

PROOF. A direct examination of the preservation of the operations. [11] ■

We shall call the map π a projector and $\pi(N(\mathfrak{B}))$ — a projection of $N(\mathfrak{B})$ into $\mathfrak{B}$.

PROPOSITION 6. $\approx$ *is a congruence in N-lattices with respect to the operations* $\cap, \cup, \to$. [11]

Let $\mathcal{N} = (N, \cup, \cap, \to, \sim, \neg, 1)$ be an N-lattice. In the set $P(N) = N/\approx$ we define:

(t_0) $0 \leftrightharpoons |\mathbf{0}|$, $1 \leftrightharpoons |\mathbf{1}|$

(t_1) $|a| \cup |b| \leftrightharpoons |a \cup b|$

(t_2) $|a| \cap |b| \leftrightharpoons |a \cap b|$

(t_3) $|a| \to |b| = |a \to b|$

(t_4) $\neg|a| \leftrightharpoons |\neg a|$.

Proposition 5 implies the correctness of the definitions (t_0)-(t_4).

PROPOSITION 7. *The system* $P(\mathcal{N}) = (P(A), \cap, \cup, \to, \neg, 1)$ *is a* PBA. [11]

PROPOSITION 8. *The map* $h\colon \mathcal{N} \to N(P(\mathcal{N}))$, *where* $h(a) = (|a|, |\sim a|)$, *is a monomorphism of N-lattices.* [11]

This proposition implies directly:

THEOREM 9 (*representation theorem*). *Any N-lattice is isomorphically embedable into a special N-lattice.*

1.3. We shall call a variety of N-lattices (N-variety) special if it is possible to define it (as a subvariety of the variety of all N-lattices $\tilde{\mathfrak{N}}$) by a system of additional identities, which only terms from *for* enter.

NOTE. Any identity in an N-lattice can be written down in the form $A = 1$ since the identity $A = B$ $(A, B \in For)$ is equivalent to $A \Leftrightarrow B = 1$.

We shall further denote zero- and one-element of both pseudo-Boolean algebras and N-lattices accordingly by 0 and 1 and this will not lead to confusion.

PROPOSITION 10. *Let $N(\mathfrak{B})$ be a special N-lattice over the* PBA $\mathfrak{B}$ *and* $(a, b) \in N(\mathfrak{B})$. *Then* $(a, b) = 1$ *iff* $a = 1$.

The proof follows directly from the definition.

LEMMA 11. *Let $\mathfrak{B}$ be a* PBA *and* $A \in for$. *Then* $\mathfrak{B} \models A = 1$ *iff* $N(\mathfrak{B}) \models A = 1$.

PROOF. Let $A = A(r_1, \ldots, r_n)$ and $(a_1, b_1), \ldots, (a_n, b_n) \in N(\mathfrak{B})$. Then $A\big((a_1, b_1), \ldots, (a_n, b_n)\big) = \big(A(a_1, \ldots, a_n), *\big)$ which is provable by direct induction. Therefore, from Proposition 10 it follows that if $\mathfrak{B} \models A = 1$ then $N(\mathfrak{B}) \models A = 1$. Conversely, let $N(\mathfrak{B}) \models A = 1$ and $a_1, \ldots, a_n \in \mathfrak{B}$. Then $(a_1, \neg a_1), \ldots, (a_n, \neg a_n) \in N(\mathfrak{B})$ and $A\big((a_1, \neg a_1), \ldots, (a_n, \neg a_n)\big) = 1$, i.e. $\big(A(a_1, \ldots, a_n), *\big) = (1, 0)$. Hence $A(a_1, \ldots, a_n) = 1$ and, therefore, $\mathfrak{B} \models A = 1$. ■

LEMMA 12. *Let $\mathscr{N}$ be an N-lattice and $A \in for$. Then $\mathscr{N} \models A = 1$ iff $P(\mathscr{N}) \models A = 1$.*

PROOF. Let $A = A(r_1, \ldots, r_n)$, $a_1, \ldots, a_n \in \mathscr{N}$. It can be proved by trivial induction that $|A(a_1, \ldots, a_n)| = A(|a_1|, \ldots, |a_n|)$, from which the lemma easily follows. ■

Lemmas 11 and 12 imply

THEOREM 13. *Let $\mathscr{N}$ be an N-lattice and $A \in for$. Then $\mathscr{N} \models A = 1$ iff $N\big(P(\mathscr{N})\big) \models A = 1$.*

Set $s\mathscr{N} = N\big(P(\mathscr{N})\big)$ for any N-lattice.

COROLLARY 14. *Let $\mathfrak{N}$ be a special N-variety and $\mathscr{N} \in \mathfrak{N}$. Then $s\mathscr{N} \in \mathfrak{N}$.*

THEOREM 15. *An N-lattice $\mathscr{N}$ is isomorphic to a special N-lattice iff there exists an element δ, such that $\delta = \sim\delta$, i.e. iff $\mathfrak{C}_0$ is isomorphically embedded into $\mathscr{N}$.*

PROOF. Note that $\mathfrak{C}_0 = N(\mathfrak{B}_0)$. Let $\mathscr{N} \cong N(\mathfrak{B})$ for some PBA $\mathfrak{B}$. Then $\delta \leftrightarrows (0, 0)$ is the required element — as such it is unique in the special N-lattices: let $(a, b) = \sim(a, b) = (b, a)$, then $a = b$ and $a \cap b = 0$. Hence $a = b = 0$.

It follows from the representation theorem that it is unique in any N-lattice in which it exists.

Now, let such an element δ exist in $\mathcal{N}$. Note the following: if $\mathcal{N}_0$ is a subalgebra of the special N-lattice $N(\mathfrak{B})$ and $\mathcal{N}_0$ satisfies the conditions: 1°. $(0, 0) \in \mathcal{N}_0$ and 2°. $\pi(\mathcal{N}_0) = \mathfrak{B}$, then $\mathcal{N}_0 = N(\mathfrak{B})$: let $a, b \in \mathfrak{B}$ and $a \cap b = 0$. Then $a \leqslant b$. There exist elements $\bar{a}, \bar{b} \in \mathfrak{B}$ such that $(a, \bar{a}) \in \mathcal{N}_0$ and $(b, \bar{b}) \in \mathcal{N}_0$. Then, $(a, \bar{a}) \cup (0, 0) = (a, 0) \in \mathcal{N}_0$ and $\neg(b, \bar{b}) = (\neg b, b) \in \mathcal{N}_0$; hence $(a, 0) \cap (\neg b, b) = (a, b) \in \mathcal{N}_0$, and thus $\mathcal{N}_0 = N(\mathfrak{B})$.

Now, let us examine the image $h(\mathcal{N})$ of the embedding $h: \mathcal{N} \to s\mathcal{N}$. It follows from the uniqueness of δ that $h(\delta) = (0, 0)$; $\pi(h(\mathcal{N})) = P(\mathcal{N})$ since for any $a \in P(\mathcal{N})$ $(|a|, |\sim a|) \in h(\mathcal{N})$. Hence $h(\mathcal{N}) = N(P(\mathcal{N})) = s\mathcal{N}$, i.e. $\mathcal{N} \cong s\mathcal{N}$. ∎

1.4. We shall call an N-logic special (the term superintuitionistic N-logic is more informative, but longer) if it can be axiomatized by a set of axioms $A = A_1 \cup V$ where A_1 is a set of tautologies of some superintuitionistic logic and V is the system of Vorobiev's axioms.

Obviously, L is a special N-logic iff *varL* is a special N-variety.

Let L be a superintuitionistic logic. The Logic $\tilde{L}$, generated from L by adding Vorobiev's axioms will be called an N-logic generated by L, i.e. $\tilde{L} = L + V$.

Let L be an N-logic. We shall call the superintuitionistic logic $I(L)$, containing the formulae, derivable in L, which the sign $\sim$ does not enter a superintuitionistic fragment of L, i.e. $I(L) = L \cap for$.

Obviously, L is a special N-logic iff $L = \widetilde{I(L)}$. In the general case $\widetilde{I(L)} \subseteq L$.

Some notations: Let $\mathfrak{M}$ be a class of PBA. Then

$$N(\mathfrak{M}) \leftrightharpoons \{\mathcal{N} / \exists \mathfrak{A} \in \mathfrak{M}: \mathcal{N} \cong N(\mathfrak{A})\}.$$

Let $\mathfrak{N}$ be a class of N-lattices. Then

$$P(\mathfrak{N}) \leftrightharpoons \{\mathfrak{A} / \exists \mathcal{N} \in \mathfrak{N}: \mathfrak{A} \cong P(\mathcal{N})\},\ s\mathfrak{N} \leftrightharpoons N(P(\mathfrak{N})).$$

Lemma 16. *Let* $\mathfrak{A}$ *be a* PBA. *Then the map* $\nu: P(N(\mathfrak{A})) \to \mathfrak{A}$, *where* $\nu(|(a, b)|) \leftrightharpoons a$, *is an isomorphism of* PBA.

Proof. It can be verified directly that ν is homomorphism; ν is a bijection:

– injection: if $a_1 = a_2$ then $(a_1, b_1) \approx (a_2, b_2)$;

– surjection: for any $a \in \mathfrak{A}$: $(a, \neg a) \in N(\mathfrak{A})$. ∎

Lemma 17. *Let* L *be an* N*-logic and* $varL = \mathfrak{N}$. *Then*:

1. $varI(L) = var(P(\mathfrak{N}))$,
2. *if* L *is a special* N*-logic then*:
   a. $N(varI(L)) \subseteq varL$,
   b. $P(\mathfrak{N})$ *is a variety*,
   c. $\mathfrak{N} = var(s\mathfrak{N})$.

PROOF. 1. $varI(L)$ and $var(P(\mathfrak{N}))$ are defined (by Lemma 12) by one and the same set of identities $\{A = 1/A \in I(L)\}$.

2.a. Let $\mathfrak{A} \in varI(L)$. Then in $N(\mathfrak{A})$ the identities $\{A = 1/A \in I(L) \cup V\}$ hold, i.e. $N(\mathfrak{A}) \in var\widetilde{I(L)} = varL$.

b. $\mathfrak{N}$ is a special N-variety. Let $\mathfrak{A} \in varI(L)$. Then $N(\mathfrak{A}) \in varL = \mathfrak{N} \mapsto \mathfrak{A} \cong P(N(\mathfrak{A})) \in P(\mathfrak{N})$, i.e. $var(P(\mathfrak{N})) = varI(L) \subseteq P(\mathfrak{N}) \subseteq var(P(\mathfrak{N}))$.

c. $s\mathfrak{N} \subseteq \mathfrak{N}$ — by Corollary 14. Conversely, $\mathfrak{N} \subseteq var(s\mathfrak{N})$ since for any $\mathcal{N} \in \mathfrak{N}$: $\mathcal{N} \hookrightarrow s\mathcal{N} \in s\mathfrak{N}$. ■

THEOREM 18 (*completeness theorem for special N-logics*). *Let L be a special N-logic, $varL = \mathfrak{N}$ and $A \in For$. The following conditions are equivalent*:

1. $A \in L$,
2. $\mathfrak{N} \models A = 1$,
3. $s\mathfrak{N} \models A = 1$.

PROOF. $1 \leftrightarrow 2$ — trivially.

$2 \mapsto 3$ — $s\mathfrak{N} \subseteq \mathfrak{N}$ from Corollary 14.

$3 \mapsto 2$ follows from the representation theorem. ■

THEOREM 19 (*separability*). *Let L be a superintuitionistic N-logic. The following conditions are equivalent for any $A \in for$*:

1. $A \in \tilde{L}$,
2. $A \in L$.

PROOF. $2 \mapsto 1$ — $L \subseteq \tilde{L}$;

$1 \mapsto 2$ — assume that $A \notin L$. Then A is refused in some PBA $\mathfrak{A} \in varL$. By Lemma 11 A is refused in $N(\mathfrak{A}) \in var\tilde{L}$, i.e. $A \notin \tilde{L}$. ■

1.5. Let $A(r_1, \ldots, r_n) \in For$ and $q_1, \ldots, q_n$ be the first n variables from Φ_0 different from $r_1, \ldots, r_n$.

Define by recursion on A a formula $A^0(r_1, \ldots, r_n, q_1, \ldots, q_n) \in for$:

1. $A \in \Phi_0 \cup \{1\}$: $A^0 \leftrightharpoons A$
2. $(A_1 * A_2)^0 \leftrightharpoons A_1^0 * A_2^0$ for $* \in \{\cap, \cup, \rightarrow\}$; $(\neg A_1)^0 \leftrightharpoons \neg A_1^0$
3. $A \underline{\underline{\circ}} \sim A_1$: by recursion on A_1:
   a′. $(\sim 1)^0 \leftrightharpoons 0$, a″. $(\sim r_i)^0 = q_i$, $i = 1, \ldots, n$
   b. $(\sim (A' \cup A''))^0 \leftrightharpoons (\sim A')^0 \cap (\sim A'')^0$,
   c. $(\sim (A' \cap A''))^0 \leftrightharpoons (\sim A')^0 \cup (\sim A'')^0$,
   d. $(\sim (A' \rightarrow A''))^0 \leftrightharpoons A'^0 \cap (\sim A'')^0$,
   e. $(\sim \neg A')^0 \leftrightharpoons (\sim \sim A')^0 \leftrightharpoons A'^0$.

By induction on A the following can be proved:

LEMMA 20. *Let* $N(\mathfrak{A})$ *be a special* N*-lattice,* $A(r_1, \ldots, r_n) \in For$ *and* $(a_1, b_1), \ldots, (a_n, b_n) \in N(\mathfrak{A})$. *Then* $A\big((a_1, b_1), \ldots, (a_n, b_n)\big) = \big(A^0(a_1, \ldots, a_n, b_1, \ldots, b_n), (\sim A)^0(a_1, \ldots, a_n, b_1, \ldots, b_n)\big)$.

We define a map φ: $For \to for$. Let $A(r_1, \ldots, r_n) \in For$. Then:

$$\varphi(A)(r_1, \ldots, r_n, q_1, \ldots, q_n) \leftrightharpoons \neg(r_1 \cap q_1) \cap \ldots \cap \neg(r_n \cap q_n) \to$$
$$\to A^0(r_1, \ldots, r_n, q_1, \ldots, q_n),$$

where $q_1, \ldots, q_n$ are the first n variables from Φ_0 which do not enter A

LEMMA 21. *Let* $\mathfrak{A}$ *be a* PBA *and* $A(r_1, \ldots, r_n) \in For$. *Then* $\mathfrak{A} \models \varphi(A) = 1$ *iff* $N(\mathfrak{A}) \models A = 1$.

PROOF. 1) Let $N(\mathfrak{A})$ *non* $\models A = 1$, i.e. there exist $(a_1, b_1), \ldots, (a_n, b_n) \in N(\mathfrak{A})$ such that $A\big((a_1, b_1), \ldots, (a_n, b_n)\big) \neq 1$, i.e. $A^0(a_1, \ldots, a_n, b_1, \ldots, b_n) \neq 1$. Then, since $\neg(a_1 \cap b_1) \cap \ldots \cap \neg(a_n \cap b_n) = 1$ it follows that $\varphi(A)(a_1, \ldots, a_n, b_1, \ldots, b_n) \neq 1$, i.e. $\mathfrak{A}$ *non* $\models \varphi(A) = 1$.

2) Let $N(\mathfrak{A}) \models A = 1$. Assume that $\mathfrak{A}$ *non* $\models \varphi(A) = 1$. Then, there exist $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that $x = \neg(a_1 \cap b_1) \cap \ldots \cap \neg(a_n \cap b_n) \nleqslant A^0(a_1, \ldots, a_n, b_1, \ldots, b_n)$. Let $\nabla = \{y \in \mathfrak{A} \mid x \leqslant y\}$, $\overline{\mathfrak{A}} \leftrightharpoons \mathfrak{A}/\nabla$. Then $x/\nabla = 1/\nabla$, $A^0(a_1, \ldots, a_n, b_1, \ldots, b_n)/\nabla \neq 1/\nabla$. Hence $a_i \leftrightharpoons (a_i/\nabla, b_i/\nabla) \in N(\overline{\mathfrak{A}})$, $i = 1, \ldots, n$, and $A(a_1, \ldots, a_n) \neq 1$. Set $c_i \leftrightharpoons \neg(a_i \cap b_i)$, $i = 1, \ldots, n$. $x \leqslant c_i \mapsto c_i \in \nabla$, $i = 1, \ldots, n$. Set $\tilde{a}_i = c_i \cap (c_i \to a_i)$, $\tilde{b}_i \leftrightharpoons c_i \cap (c_i \to b_i)$, $i = 1, \ldots, n$. $\tilde{a}_i/\nabla = c_i/\nabla \cap (c_i/\nabla \to a_i/\nabla) = a_i/\nabla$ and analogously $\tilde{b}_i/\nabla = b_i/\nabla$. Moreover, $\tilde{a}_i \cap \tilde{b}_i = c_i \cap a_i \cap b_i = 0$. And so $\beta_i \leftrightharpoons (a_i, b_i) \in N(\mathfrak{A})$, $i = 1, \ldots, n$, and $A^0(\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_n)/\nabla = A^0(a_1, \ldots, a_n, b_1, \ldots, b_n)/\nabla \neq 1/\nabla$. Hence, $A^0(\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{b}_1, \ldots, \tilde{b}_n) \neq 1$ and, therefore, $A(\beta_1, \ldots, \beta_n) \neq 1$. So $N(\mathfrak{A})$ *non* $\models A = 1$ — a contradiction. ∎

LEMMA 22. *For any superintuitionistic logic* L: $L = I(\tilde{L})$.

PROOF. $L \subseteq I(\tilde{L})$; Let $\mathfrak{A} \in varL$. Then $N(\mathfrak{A}) \in var\tilde{L}$, where $\mathfrak{A} \cong P\big(N(\mathfrak{A})\big) \in P(var\tilde{L}) \subseteq varI(\tilde{L})$. And so $varL \subseteq varI(\tilde{L})$; therefore $I(\tilde{L}) \subseteq L$. ∎

THEOREM 23. *Let* L *be a special* N*-logic and* $A \in For$. *Then* $A \in L$ *iff* $\varphi(A) \in I(L)$.

PROOF. 1) Let $A \notin L$. Then there exists a special N-lattice $N(\mathfrak{A})$, $\mathfrak{A} \in varI(L)$, such that $N(\mathfrak{A})$ *non* $\models A = 1$ and therefore $\mathfrak{A}$ *non* $\models \varphi(A) = 1$, i.e. $\varphi(A) \notin I(L)$.

2) Let $\varphi(A) \notin I(L)$. Then there exists $\mathfrak{A} \in varI(L)$ such that $\mathfrak{A}$ *non* $\models \varphi(A) = 1$. Hence, $N(\mathfrak{A})$ *non* $\models A = 1$, but $N(\mathfrak{A}) \in varL$ and therefore $A \notin L$. ∎

COROLLARY 24. *Let L be a superintuitionistic logic, complete with respect to a class* $\mathbf{K}$ *of* PBAs. *Then* $\tilde{L}$ *is complete with respect to the class* $N(\mathbf{K})$.

PROOF. $\mathbf{K} \subseteq varL \mapsto N(\mathbf{K}) \subseteq var\tilde{L}$. Let $A \in For$ and $N(\mathbf{K}) \models A = 1$. Then $\mathbf{K} \models \varphi(A) = 1$. Hence $\varphi(A) \in L \mapsto A \in \tilde{L}$. ∎

COROLLARY 25. *If L is a decidable superintuitionistic logic then $\tilde{L}$ is a decidable N-logic.*

COROLLARY 26. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* PBAs, $\mathfrak{A} \in var(\mathfrak{B})$. *Then* $N(\mathfrak{A}) \in var(N(\mathfrak{B}))$.

PROOF. Let $A \in For$ and $N(\mathfrak{B}) \models A = 1$. Then $\mathfrak{B} \models \varphi(A) = 1 \mapsto \mathfrak{A} \models \varphi(A) = 1$ and therefore $N(\mathfrak{A}) \models A = 1$. ∎

COROLLARY 27. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* PBAs *and* $var(\mathfrak{A}) = var(\mathfrak{B})$. *Then* $var(N(\mathfrak{A})) = var(N(\mathfrak{B}))$.

THEOREM 28. *An N-variety, generated from a special N-lattice, is a special N-variety.*

PROOF. Let $\mathfrak{N} = var(N(\mathfrak{A}))$. Let $L = L(\mathfrak{A})$ and $\mathcal{N}$ be Lindenbaum algebra of logic $\tilde{L}$. Since $var(P(\mathcal{N})) = varI(\tilde{L}) = varL = var(\mathfrak{A})$ then $var(s\mathcal{N}) = var(N(\mathfrak{A})) = \mathfrak{N}$. Besides, $var(s\mathcal{N}) = var(\mathcal{N})$: $\mathcal{N} \hookrightarrow s\mathcal{N}$, $var(\mathcal{N}) \subseteq var(s\mathcal{N})$ and conversely: since $var(\mathcal{N})$ is a special N-variety then $s\mathcal{N} \in var(\mathcal{N}) \mapsto var(s\mathcal{N}) \subseteq var(\mathcal{N})$. And so $\mathfrak{N} = var(\mathcal{N}) \mapsto var\tilde{L}$ is a special N-variety. ∎

COROLLARY 29. *If $\mathcal{N}$ is a special N-lattice, then $L(\mathcal{N})$ is a special N-logic.*

§2. Normal N-lattices and normal N-logics

2.1.

LEMMA 30. *In any N-lattice the following identities are equivalent:*

1. $\neg(x \leftrightarrow \sim x) = 1$
2. $\neg\neg(x \cup \sim x) = 1$
3. $\neg\sim x \approx \neg\neg x$.

PROOF. $1 \leftrightarrow 2$: in any N-lattice $\neg(x \cup \sim x) = x \leftrightarrow \sim x$:
$\neg(x \cup \sim x) = \neg x \cap \neg\sim x = (x \to \sim x) \cap (\sim x \to \sim\sim x) = x \leftrightarrow \sim x$.

$2 \to 3$: $\neg x \cap \neg\sim x \approx 0 \mapsto \neg x \cap \neg\sim x \prec 0$, i.e. $\neg\sim x \prec \neg\neg x$ and from $\sim x \prec \neg x$ it follows that $\neg\neg x \prec \neg\sim x \mapsto \neg\neg x \approx \neg\sim x$.

$3 \to 2$: $\neg\sim x \cap \neg x \prec 0$, i.e. $\neg\sim x \cap \neg x = \neg(x \cup \sim x) \approx 0 \mapsto \neg\neg(x \cup \sim x) = 1$. ∎

DEFINITIONS. We shall call an N-lattice $\mathcal{N}$ normal if $\mathcal{N} \models \neg(x \leftrightarrow \sim x) = 1$. An N-logic L^+ will be called normal if it can be obtained from a special N-logic L by adding the axiom $\neg(p \leftrightarrow \sim p)$.

LEMMA 31. *In any* PBA $\mathfrak{A}$ *the following conditions are equivalent*:

1. $\neg a = \neg\neg b$,
2. $a \cap b = 0$ *and* $\neg a \cap \neg b = 0$

for $a,\ b \in \mathfrak{A}$.

PROOF. $1 \mapsto 2$: $b \leqslant \neg\neg b = \neg a \mapsto b \cap a = 0$; $\neg a = \neg\neg b \mapsto \neg a \cap \cap \neg b = 0$.

$2 \mapsto 1$: $a \cap b = 0 \mapsto a \leqslant \neg b \mapsto \neg\neg b \leqslant \neg a$; $\neg a \cap \neg b = 0 \to \neg a \leqslant \neg\neg b$. ∎

LEMMA 32. *Let* $\mathcal{N} = N(\mathfrak{A})$ *be a special* N*-lattice. Then* $N^+(\mathfrak{A}) \leftrightharpoons \leftrightharpoons \{(a_1, a_2)/a_1, a_2 \in \mathfrak{A} \quad \& \quad \neg a_1 = \neg\neg a_2\}$ *is a subalgebra of* $\mathcal{N}$ *(we denote* $N^+(\mathfrak{A}) \leqslant \mathcal{N}$*)*.

PROOF.

1. $(0, 1) \in N^+(\mathfrak{A})$, $\quad (1, 0) \in N^+(\mathfrak{A})$;
2. $(a_1, a_2) \in N^+(\mathfrak{A}) \mapsto \neg(a_1, a_2) = (\neg a_1, a_1) \in N^+(\mathfrak{A})$;
3. $(a_1, a_2) \in N^+(\mathfrak{A}) \mapsto \sim(a_1, a_2) = (a_2, a_1) = N^+(\mathfrak{A})$ since $\neg a_2 = \neg\neg\neg a_2 = \neg\neg a_1$.
   Let (a_1, a_2) and $(b_1, b_2) \in N^+(\mathfrak{A})$. Then:
4. $(a_1, a_2) \cap (b_1, b_2) = (a_1 \cap b_1, a_2 \cup b_2) \in N^+(\mathfrak{A})$:
   $\neg\neg(a_2 \cup b_2) = \neg(\neg a_2 \cap \neg b_2) = \neg(\neg\neg a_1 \cap \neg\neg b_1) =$
   $= \neg\neg\neg(a_1 \cap b_1) = \neg(a_1 \cap b_1)$.
5. $(a_1, a_2) \cup (b_1, b_2) = \sim\big(\sim(a_1, a_2) \cap \sim(b_1, b_2)\big) \in N^+(\mathfrak{A})$.
6. $(a_1, a_2) \to (b_1, b_2) = (a_1 \to b_1, a_1 \cap b_2) \in N^+(\mathfrak{A})$:
   $\neg\neg(a_1 \cap b_2) = \neg\neg a_1 \cap \neg\neg b_2 = \neg\neg a_1 \cap \neg b_1 = \neg(\neg a_1 \cup b_1)$;
   $\neg a_1 \cup b_1 \leqslant a_1 \to b_1 \mapsto \neg(a_1 \to b_1) \leqslant \neg(\neg a_1 \cup b_1)$ and conversely:
   $\neg\neg a_1 \cap \neg b_1 \cap (a_1 \to b_1) \leqslant \neg\neg a_1 \cap \neg a_1 = 0 \mapsto \neg\neg a_1 \cap \neg b_1 =$
   $= \neg(a_1 \to b_1)$. ∎

Let $\mathcal{N}$ be an N-lattice. Denote $n\mathcal{N} = N^+\big(P(\mathcal{N})\big)$.

THEOREM 33. *Let* L^+ *be a normal* N*-logic and* $\mathfrak{A} \in varI(L)$. *Then* $N^+(\mathfrak{A}) \in varL^+$.

PROOF. $N^+(\mathfrak{A}) \leqslant N(\mathfrak{A}) \mapsto N^+(\mathfrak{A}) \in varL$. It remains to prove that $N^+(\mathfrak{A}) \models \neg(x \leftrightarrow \sim x) = 1$, which is equivalent to $N^+(\mathfrak{A}) \models \neg(x \cup \sim x) \approx 0$. Let $(a_1, a_2) \in N^+(\mathfrak{A})$. Then $\neg\big((a_1, a_2) \cup \sim(a_1, a_2)\big) = (\neg(a_1 \cup a_2),\ a_1 \cup a_2) = = (\neg a_1 \cap \neg a_2, a_1 \cup a_2) = (0, a_1 \cup a_2) \approx 0$. ∎

THEOREM 34 (*representation theorem for normal* N*-lattices*). *If* $\mathcal{N}$ *is a normal* N*-lattice then* $\mathcal{N}$ *is isomorphically embedable into* $n\mathcal{N}$.

PROOF. h: $a \to (|a|, |\sim a|)$ is the embedding of $\mathcal{N}$ into $s\mathcal{N}$. We shall prove that $h(\mathcal{N}) \subseteq n\mathcal{N}$:

$$\neg\neg|a| = |\neg\neg a| = |\neg\sim a| = \neg|\sim a|, \text{ i.e. } (|a|, |\sim a|) \in n\mathcal{N}. \quad ∎$$

LEMMA 35. *Let $\mathfrak{A}$ be a* PBA. *The smallest subalgebra of the N-lattice $N(\mathfrak{A})$, the projection of which coincides with $\mathfrak{A}$, is $N_0(\mathfrak{A})$ generated from the set $\mathfrak{A}_0 = \{(\neg a, a) / a \in \mathfrak{A}\}$.*

PROOF. Let $\mathcal{N}$ be this smallest subalgebra. For any $a \in \mathfrak{A}$ there exists $b \in \mathfrak{A}$ such that $(a, b) \in \mathcal{N}$. Then $\neg(a, b) = (\neg a, a) \in \mathcal{N}$, i.e. $N_0(\mathfrak{A}) \subseteq \mathcal{N}$. ■

LEMMA 36. *Let $\mathfrak{A}$ be a* PBA. *Then $N_0(\mathfrak{A}) = N^+(\mathfrak{A})$.*

PROOF. For any $a \in \mathfrak{A}$: $(\neg a, a) \in N^+(\mathfrak{A}) \mapsto N_0(\mathfrak{A}) \subseteq N^+(\mathfrak{A})$. Let $(a, b) \in N^+(\mathfrak{A})$. $(\neg b, b) \in N_0(\mathfrak{A})$, i.e. $(\neg\neg a, b) \in N_0(\mathfrak{A})$. Besides, $(a, \neg a)$, $(\neg a, a) \in N_0(\mathfrak{A})$. Then $(\neg\neg a, b) \cap ((a, \neg a) \cup (\neg a, a)) = (\neg\neg a \cap (a \cup \neg a), b) = (a, b) \in N_0(\mathfrak{A})$. Therefore $N^+(\mathfrak{A}) \subseteq N_0(\mathfrak{A})$. ■

THEOREM 37. *Let $\mathcal{N}$ be a normal N-lattice. Then the embedding $h: \mathcal{N} \hookrightarrow n\mathcal{N}$ is an isomorphism, i.e. $\mathcal{N} \cong n\mathcal{N}$.*

PROOF. Since $\pi(h(\mathcal{N})) = P(\mathcal{N})$ then by Lemma 35 $N_0(P(\mathcal{N})) \subseteq h(\mathcal{N})$, and by Theorem 34 and Lemma 36 $h(\mathcal{N}) \subseteq n\mathcal{N} = N_0(P(\mathcal{N}))$, i.e. $h(\mathcal{N}) = n\mathcal{N}$. ■

Thus, every normal N-lattice has the form $N^+(\mathfrak{A})$ for some PBA $\mathfrak{A}$. An N-variety will be normal if the corresponding N-logic is normal. Let $\mathbf{K}$ be a class of PBA7. Denote $N^+(\mathbf{K}) \leftrightharpoons \{\mathcal{N} / \exists \mathfrak{A} \in \mathbf{K}: \mathcal{N} \cong N^+(\mathfrak{A})\}$. The following lemma is proved just as Lemma 14:

LEMMA 38. *Let $\mathfrak{A}$ be a* PBA, *$A \in for$. Then, $\mathfrak{A} \models A = 1$ iff $N^+(\mathfrak{A}) \models A = 1$.*

Let L be a special N-logic. By L^+ we denote the normal N-logic, generated by L.

LEMMA 39. *If L^+ is a normal N-logic then $varL^+ = N^+(varI(L))$.*

PROOF. By Theorem 33, $N^+(varI(L)) \subseteq varL^+$. Conversely, let $\mathcal{N} \in varL^+$. Then, by Lemma 38 and since $\mathcal{N} \cong n\mathcal{N}$, it follows that $P(\mathcal{N}) \in varI(L) \mapsto n\mathcal{N} \in N^+(varI(L))$, i.e. $varL^+ \subseteq N^+(varI(L))$. ■

THEOREM 40 (*separability*). *Let L be a superintuitionistic logic and $A \in for$. Then the following conditions are equivalent:*

1. $A \in L^+$
2. $A \in L$

PROOF. $2 \mapsto 1$ — trivially.

$1 \mapsto 2$: Let $A \notin L$. Then there exists $\mathfrak{A} \in varL$ such that $\mathfrak{A}$ *non* $\models A = 1 \mapsto N^+(\mathfrak{A})$ *non* $\models A = 1$, but $N^+(\mathfrak{A}) \in N^+(varL) = varL^+ \mapsto A \notin \tilde{L}^+$. ■

2.2. We define a map ψ: *For*$\to$*for*. Let $A(r_1, \ldots, r_n) \in For$ and $q_1, \ldots, q_n$ be the first n variables from Φ_0, which do not appear in A. We set

$$\psi(A)(r_1, \ldots, r_n, q_1, \ldots, q_n) \leftrightharpoons (\neg r_1 \leftrightarrow \neg\neg q_1) \cap \ldots \cap (\neg r_n \leftrightarrow \neg\neg q_n) \to \to A^0(r_1, \ldots, r_n, q_1, \ldots, q_n).$$

LEMMA 41. *Let* $\mathfrak{A}$ *be a* PBA, $A \in For$. *Then* $N^+(\mathfrak{A}) \models A = 1$ *iff* $\mathfrak{A} \models \psi(A) = 1$.

PROOF. 1) Let $\mathfrak{A} \models \psi(A) = 1$ and $(a_1, b_1), \ldots, (a_n, b_n) \in N^+(\mathfrak{A})$. Then $(\neg a_1 \leftrightarrow \neg\neg b_1) \cap \ldots \cap (\neg a_n \leftrightarrow \neg\neg b_n) = 1$, and since $\psi(A)(a_1, \ldots, a_n, b_1, \ldots, b_n) = 1$ then $A^0(a_1, \ldots, a_n, b_1, \ldots, b_n) = 1$. Hence $A\big((a_1, b_1), \ldots, (a_n, b_n)\big) = 1$.

2) Let $N^+(\mathfrak{A}) \models A = 1$. Assume that $\mathfrak{A}$ *non* $\models \psi(A) = 1$, i.e. there exist $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathfrak{A}$ such that $x = (\neg a_1 \leftrightarrow \neg\neg b_1) \cap \ldots \cap (\neg a_n \leftrightarrow \leftrightarrow \neg\neg b_n) \nleqslant A^0(a_1, \ldots, a_n, b_1, \ldots, b_n)$. Let $\nabla = \{y \in \mathfrak{A} \mid x \leqslant y\}$ and $h: \mathfrak{A} \to \to \mathfrak{A}/\nabla \leftrightharpoons \overline{\mathfrak{A}}$ be the canonic isomorphism. In $\overline{\mathfrak{A}}$: $\neg a_i/\nabla = \neg\neg b_i/\nabla$ for $i = 1, \ldots, n$ and $A^0(a_1, \ldots, a_n, b_1, \ldots, b_n)/\nabla \neq 1$. By Lemma 36 $N^+(\mathfrak{A})$ and $N^+(\overline{\mathfrak{A}})$ are algebras generated correspondingly from the sets $\mathfrak{A}_0 = \{(\neg a, a) \mid a \in \mathfrak{A}\}$ and $\overline{\mathfrak{A}}_0 = \{(\neg a/\nabla, a/\nabla) \mid a/\nabla \in \overline{\mathfrak{A}}\}$.

The map h_N: $N^+(\mathfrak{A}) \to N^+(\overline{\mathfrak{A}})$, defined with $h_N\big((a, b)\big) = \big(h(a), h(b)\big)$, is homomorphism of N-lattices (Proposition 4), h is a surjective map: $\mathfrak{A}_0 \to \overline{\mathfrak{A}}_0$; thus h_N is an epimorphism. But $(a_i/\nabla, b_i/\nabla) \in N^+(\overline{\mathfrak{A}})$ for $i = 1, \ldots, n$ and $A^0(a_1/\nabla, \ldots, a_n/\nabla, b_1/\nabla, \ldots, b_n/\nabla) \neq 1$, i.e. $A\big((a_1/\nabla, b_1/\nabla), \ldots, (a_n/\nabla, b_n/\nabla)\big) \neq 1$. Hence $N^+(\mathfrak{A})$ *non* $\models A = 1$ — a contradiction. ∎

The following corollaries can be proved analogously to the corresponding assertions from §1:

COROLLARY 42. *If* L^+ *is a normal* N*-logic and* $A \in For$ *then* $A \in L^+$ *iff* $\psi(A) \in I(L)$.

COROLLARY 43. *If* L *is a superintuitionistic logic, complete with respect to a class of* PBAs $\mathbf{K}$, *then the* N*-logic* $\tilde{L}^+$ *is complete with respect to the class* $N^+(\mathbf{K})$.

COROLLARY 44. *If* L *is a decidable superintuitionistic logic then* $\tilde{L}^+$ *is a decidable* N*-logic.*

COROLLARY 45. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* PBA*s and* $\mathfrak{A} \in var(\mathfrak{B})$. *Then* $N^+(\mathfrak{A}) \in \in var\big(N^+(\mathfrak{B})\big)$.

COROLLARY 46. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* PBA*s and* $var(\mathfrak{A}) = var(\mathfrak{B})$. *Then* $var\big(N^+(\mathfrak{A})\big) = var\big(N^+(\mathfrak{B})\big)$.

THEOREM 47. *Every* N*-logic* L *containing the formula* $\neg(p \leftrightarrow \sim p)$ *is a normal* N*-logic.*

PROOF. We shall prove that $L = \widetilde{I(L)}^+$: $\widetilde{I(L)}^+ \subseteq L$; vice versa: it is sufficient to prove that $var\widetilde{I(L)}^+ \subseteq varL$. Let $N^+(\mathfrak{A}) \in N^+(varI(L)) = varI(L)^+$ and $\mathcal{N}$ be Lindenbaum algebra for L. Then $var(P(\mathcal{N})) = varI(L)$ — by Lemma 12. Hence, $\mathfrak{A} \in var(P(\mathcal{N}))$ and by Corollary 45 $N^+(\mathfrak{A}) \in var(n\mathcal{N}) = var(\mathcal{N}) = varL$, i.e. $var\widetilde{I(L)}^+ \subseteq varL$. ∎

COROLLARY 48. *Every N-variety, in which the identity* $\neg(x \leftrightarrow \sim x) = 1$ *holds, is a normal N-variety.*

COROLLARY 49. *Every normal N-lattice generates a normal N-variety (and correspondingly a normal N-logic).*

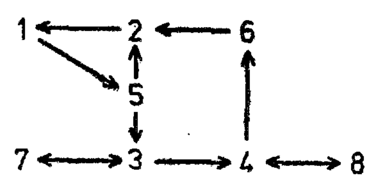
§3. Notes on the lattice of the N-logics

3.1.

LEMMA 50. *The following identities are equivalent in any N-lattice:*

1. $x \cap \sim x = 0$; 2. $x \cap \neg x = 0$; 3. $\neg x \prec \sim x$; 4. $\neg x = \sim x$;
5. $\neg x \cap \neg \sim x = 0$; 6. $x = \neg\neg x$; 7. $\neg \sim x \prec x$; 8. $\neg \sim x = x$.

PROOF. Without difficulties, by the scheme:



∎

The theorem of replacement relatively $\Leftrightarrow$ (directly proved by the completeness theorem for N-logics) and by the identities 4 and 6 it follows, that each N-logic containing any of the formulae corresponding to the identities 1—8, is essentially the classical logic. We shall denote this logic by $\boldsymbol{C}$.

$\boldsymbol{C}$ is complete with respect to the two-element N-lattice $\mathfrak{B}_0$ and, since $\mathfrak{B}_0$ is isomorphically embedable in any non-degenerate N-lattice, then the following is true:

PROPOSITION 51. *Every consistent N-logic is contained in* $\boldsymbol{C}$.

3.2. The intersection of all maximal s.f.f.k. in a given N-lattice $\mathcal{N}$ is called a radical of $\mathcal{N}$ and is denoted by $Rad_{\mathcal{N}}$.

An N-lattice is semi-simple if $Rad_{\mathcal{N}} = \{1\}$.

LEMMA 52. *For any N-lattice* $\mathcal{N}$ *the following conditions are equivalent:*

1. $\mathcal{N}$ *is semi-simple,*
2. $\mathcal{N} \models a \cup \neg a = 1$,

3. $\mathcal{N} \models a \to b = \neg a \cup b$,
4. $\mathcal{N} \models (a \to b) \to a = a$,
5. *any prime s.f.f.k. in $\mathcal{N}$ is maximal.* [15]

THEOREM 53. *Any semi-simple N-lattice $\mathcal{N}$ is isomorphic to a subalgebra of a Cartesian product $\Pi_{t\in T}\mathfrak{C}_t$, where T is a set of indices and for any $t \in T$: $\mathfrak{C}_t \cong \mathfrak{C}_0$.* [15]

Let C be a classical logic. N-logic $\tilde{C}$ is called a classical logic with strong negation and the elements of $var\tilde{C}$ — classical N-lattices. By Corollary 24 it follows that $\tilde{C}$ is complete with respect to $\mathfrak{C}_0 \cong N(\mathfrak{B}_0)$.

As Vakarelov shows in [11] the logic $\tilde{C}$ is functionally equivalent to the three-valued logic of Lukasiewicz.

By Lemma 52 it follows that $var\tilde{C}$ consists of all semi-simple N-lattices. $\tilde{C}$ is the biggest special N-logic.

PROPOSITION 54. *The logic $\tilde{C}$ is maximal in the class of the consistent N-logics, different from C.*

PROOF. Let $\tilde{C} \subseteq L \subsetneq C$. Then $varL \subseteq var\tilde{C}$; L *non* $\vdash p \cup \sim p$, and therefore there exists $\mathcal{N} \in varL$ such that $\mathcal{N}$ *non* $\models x \cup \sim x = 1$. By Theorem 53 $\mathcal{N} \cong \Pi_{t\in T}\mathfrak{C}_t$, $\forall_{t\in T}$: $\mathfrak{C}_t \cong \mathfrak{C}_0$. Let $\{\pi_t\}_{t\in T}$ be the projecting epimorphism π_t: $\mathcal{N} \to \mathfrak{C}_t$, $t \in T$. For any $t \in T$: $\pi_t(\mathcal{N}) \leqslant \mathfrak{C}_t$. If for any $t \in T$ $\pi_t(\mathcal{N}) \leqslant \mathfrak{B}_0$, then $\mathcal{N} \models x \cup \sim x = 1$. Hence, there exists $t_0 \in T$ such that $\pi_{t_0}(\mathcal{N}) \cong \mathfrak{C}_0$, i.e. $\mathfrak{C}_0$ is homomorphic image of the N-lattice $\mathcal{N}$ by π_{t_0}. Therefore, $\mathfrak{C}_0 \in varL \mapsto var\tilde{C} = var(\mathfrak{C}_0) \subseteq varL \mapsto L \subseteq \tilde{C}$, i.e. $L = \tilde{C}$. ■

3.3.

LEMMA 55. *Let the formula $A(r_1, \ldots, r_n) \underline{\circ} A_1 \to A_2$ be not derivable in $\tilde{I}$. Then there exist a special N-lattice $N(\mathfrak{A})$ and elements $(a_1, b_1), \ldots, (a_n, b_n) \in \in N(\mathfrak{A})$ for which $A_1 = 1$, $A_2 \neq 1$.*

PROOF. $A \notin \tilde{I} \mapsto \varphi(\tilde{A}) \notin I \mapsto \varphi(A)$ is refuted for some elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of some PBA $\mathfrak{A}_0$, i.e. $x = \neg(a_1 \cap b_1) \cap \ldots \cap \neg(a_n \cap b_n) \cap \cap A_1^0(a_1, \ldots, a_n) \not\leqslant A_2^0(a_1, \ldots, b_n)$. Let $\nabla = \{y \in \mathfrak{A}_0 \mid x \leqslant y\}$. Then, in $\mathfrak{A}_0/\nabla$: $x/\nabla = 1$. Hence, $a_i/\nabla \cap b_i/\nabla = 0$ for $i = 1, \ldots, n$ and $A_1^0(a_1/\nabla, \ldots, a_n/\nabla, b_1/\nabla, \ldots, b_n/\nabla) = 1$, $A_2^0(a_1/\nabla, \ldots, a_n/\nabla, b_1/\nabla, \ldots, b_n/\nabla) \neq 1$, i.e. $A_1((a_1/\nabla, b_1/\nabla), \ldots, (a_n/\nabla, b_n/\nabla)) = 1$, $A_2((a_1/\nabla, b_1/\nabla), \ldots, (a_n/\nabla, b_n/\nabla)) \neq 1$. ■

NOTE. $\tilde{C}$ (and therefore every special N-logic) is not a normal N-logic because $\mathfrak{C}_0$ *non* $\models \neg(x \leftrightarrow \sim x) = 1$ since $\neg(\delta \leftrightarrow \sim \delta) = 0$.

THEOREM 56. *Every N-logic L such that $L \nsubseteq \tilde{C}$ is normal.*

PROOF. We may assume L consistent, i.e. $L \subseteq C$, since, otherwise, the assertion is trivial.

Let $A(r_1, \dots, r_n) \in L \setminus \tilde{C}$. Then $\mathfrak{B}_0 \models A = 1$, but $\mathfrak{C}_0$ *non* $\models A = 1$, i.e. there exist elements $a_1, \dots, a_n \in \mathfrak{C}_0$, at least one of which is δ, such that $A(a_1, \dots, a_n) \neq 1$. We substitute a_i in $A(r_1, \dots, r_n)$ for all variables r_i for which $a_i \neq \delta$. We obtain a formula $B(q_1, \dots, q_k) \in L \setminus \tilde{C}$ such that $B(\delta, \dots, \delta) \neq 1$. Formula $B_1 = B \to \urcorner\big((q_1 \leftrightarrow \sim q_1) \cap \dots \cap (q_k \leftrightarrow \sim q_k)\big)$ is derivable in $\tilde{I}$: if we assume the opposite, then $\bar{B} = B \cap \big((q_1 \leftrightarrow \sim q_1) \cap \dots \cap (q_k \leftrightarrow \sim q_k)\big) \to 0$ is not derivable in $\tilde{I}$ either. Therefore, it is refused in a special N-lattice for some elements $(a_1, b_1), \dots, (a_k, b_k)$. By Lemma 55 it can be assumed that $\big(B \cap \big((q_1 \leftrightarrow \sim q_1) \cap \dots \cap (q_k \leftrightarrow \sim q_k)\big)\ \big((a_1, b_1), \dots, (a_k, b_k)\big) = 1$, i.e. $B\big((a_1, b_1), \dots, (a_k, b_k)\big) = 1$ and $a_i \leftrightarrow b_i = 1$, $i = 1, \dots, k$, i.e. $a_i = b_i = 0$, $i = 1, \dots, k$ $(a_i, b_i) = (0, 0) = \delta$, $i = 1, \dots, k$. Hence, $B(\delta, \dots, \delta) = 1$ — contradiction. And so $B_1 \in \tilde{I} \subseteq L \mapsto \urcorner\big((q_1 \leftrightarrow \sim q_1) \cap \dots \cap (q_k \leftrightarrow \sim q_k)\big) \in L$. Thus, identifying $q_1, \dots, q_k$ with p we obtain $L \vdash \urcorner(p \leftrightarrow \sim p)$. ■

3.4. Let $\boldsymbol{L}_3 = \{0, a, 1\}$ be the three-element PBA. We denote $\mathfrak{D}_0 = \{(0, 1), (0, a), (a, 0), (1, 0)\}$. $\mathfrak{D}_0 \leqslant N(\boldsymbol{L}_3)$, more exactly $\mathfrak{D}_0 \cong N^+(\boldsymbol{L}_3) \mapsto \mathfrak{D}_0 \models \urcorner(x \leftrightarrow \sim x) = 1$.

LEMMA 57. *Every consistent N-logic, different from $\boldsymbol{C}$ is contained either in $\tilde{C}$ or in $L(\mathfrak{D}_0)$.*

PROOF. Let $L \subseteq \boldsymbol{C}$, $L \neq \boldsymbol{C}$. Then, in the Lindenbaum algebra for $L - \mathscr{N}_L$, there exists an element a such that $a \cup \sim a \neq 1$. If for the element a, $a = \sim a$ is true then $\{0, a, 1\} \cong \mathfrak{C}_0 \mapsto \mathfrak{C}_0 \leqslant \mathscr{N}_L \mapsto L \subseteq \tilde{C}$. If $a \neq \sim a$ then we set $\beta \leftrightharpoons a \cup \sim a$. Then it is directly proved that $\{0, \beta, \sim\beta, 1\} \cong \mathfrak{D}_0$, i.e. $\mathfrak{D}_0 \hookrightarrow \mathscr{N}_L \mapsto L \subseteq L(\mathfrak{D}_0)$. ■

NOTE. Since $\mathfrak{D}_0 = N^+(\boldsymbol{L}_3)$ and $L(\boldsymbol{L}_3) = I + (\urcorner p \cup \urcorner\urcorner p) + \big(p \cup (p \to (q \cup \urcorner q))\big)$ then $L(\mathfrak{D}_0) = \tilde{I} + (\urcorner p \cup \urcorner\urcorner p) + \big(p \cup (p \to (q \cup \urcorner q))\big) + \urcorner(p \leftrightarrow \sim p)$.

THEOREM 58. *If an N-logic $L \subseteq \tilde{C}$ and in L the* CIT *holds then L is a special N-logic.*

PROOF. $L \subseteq \tilde{C} \mapsto L$ *non* $\vdash (p_0 \leftrightarrow \sim p_0) \to 0$ $(*)$. Let $\mathscr{N}$ be the Lindenbaum algebra of L; $\mathscr{N}$ consists of the classes of equivalent formulae from *For*, with regards to the relation $\leqslant$, where $A \leqslant B$ iff $L \vdash A \Rightarrow B$. Thus $\mathscr{N} = \{|A| \mid A \in For\}$.

We set $\Phi = \{|A| \in \mathscr{N} \mid L \vdash (p_0 \leftrightarrow \sim p_0) \to A\}$. This definition is correct and Φ is a s.f.f.k. in $\mathscr{N}$. Let $\mathscr{N}_0 \leftrightharpoons \mathscr{N}/\Phi$. All identities, which hold in the N-lattice $\mathscr{N}$, also hold in $\mathscr{N}_0$. Vice versa, let $\mathscr{N}_0 \models B(q_1, \dots, q_n) = 1$. We can consider, that the formula B is written down with variables, different from p_0. By the theorem of replacement with respect to $\Leftrightarrow$ it follows that $B(|q_1|, \dots, |q_n|) = |B(q_1, \dots, q_n)|$. And so $|B(q_1, \dots, q_n)| \in \Phi$, i.e. $L \vdash (p_0 \leftrightarrow \sim p_0) \to B(q_1, \dots, q_n)$. Then, in accordance with CIT (any closed formula is strongly equivalent in $\tilde{I}$ to 0 or 1) two cases are possible: either $L \vdash (p_0 \leftrightarrow \sim p_0) \rightarrow 0$ or $L \vdash 1 \rightarrow B(q_1, \ldots, q_n)$. Since (*) only the second possibility remains, i.e. $L \vdash B(q_1, \ldots, q_n)$, thus $\mathcal{N} \models B(q_1, \ldots, q_n) = 1$. And so $\mathcal{N} \models B = 1$ iff $\mathcal{N}_0 \models B = 1$ for any $B \in For$, therefore $var(\mathcal{N}_0) = var(\mathcal{N}) = varL$. But in $\mathcal{N}_0$: $|p_0 \leftrightarrow \sim p_0|/\Phi = 1$, i.e. $|p_0|/\Phi \approx \sim |p_0|/\Phi$ and since in any N-lattice $a \approx \sim a$ implies $a = \sim a$ then in $\mathcal{N}_0$ there exists an element $\delta = \sim \delta$. Hence, by Theorem 15 there exists a PBA $\mathfrak{A}$ such that $\mathcal{N}_0 \cong N(\mathfrak{A})$ and, by Corollary 29, $L = L(\mathcal{N}_0)$ is a special N-logic. ∎

§4. Craig interpolation theorem in N-logics and N-varieties with amalgamation property

4.1. Let **K** be a class of N-lattices. **K** has an amalgamation property if for any $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2 \in \mathbf{K}$ the following condition holds:

(A) For any pair of monomorphisms $i_1\colon \mathcal{N}_0 \rightarrow \mathcal{N}_1$ and $i_2\colon \mathcal{N}_0 \rightarrow \mathcal{N}_2$ there exist an N-lattice $\mathcal{N} \in \mathbf{K}$ and monomorphisms $\varepsilon_1\colon \mathcal{N}_1 \rightarrow \mathcal{N}$, $\varepsilon_2\colon \mathcal{N}_2 \rightarrow \mathcal{N}$ such that $\varepsilon_1 \circ i_1 = \varepsilon_2 \circ i_2$.

The triple $(\mathcal{N}, \varepsilon_1, \varepsilon_2)$ will be called a common extension of $\mathcal{N}_1$ and $\mathcal{N}_2$ over $\mathcal{N}_0$.

In [14] Czelakowski proved the equivalence of the Craig interpolation theorem in a large class of logics, the N-logics included, with the amalgamation property of the corresponding varieties of algebras.

Introducing the definitions of interpolation principles for equalities and inequalities and the over-amalgamation property of the class **K**, fully analogous to those corresponding to PBA, formulated by Maximova in [1] (taking into consideration, that the corresponding relation in N-lattices $\prec$ is a quasi-ordering) the corresponding proof from [1] can be translated without difficulties and the following, more general theorem is proved:

THEOREM 59. *For any N-logic L the following conditions are equivalent:*

1) *in L the* CIT *holds,*
2) *in $varL$ the interpolation principle of inequalities holds,*
3) *in $varL$ the interpolation principle of equalities holds,*
4) *$varL$ has the over-amalgamation property,*
5) *$varL$ has the amalgamation property,*
6) *in $varL$ the condition* (A) *holds for any fully connected N-lattices $\mathcal{N}_0, \mathcal{N}_1, \mathcal{N}_2$.*

The proof of this theorem will not be presented here, because in this paper we shall only use the $1 \leftrightarrow 5$ equivalence.

4.2. Let $\mathbf{K}$ be a class of algebras of an arbitrary signature. We shall denote the category with a set of objects, obtained from $\mathbf{K}$, identifying the isomorphical algebras and a set of morphisms obtained from the set of homomorphisms of $\mathbf{K}$ — $Hom_{\mathbf{K}}$, identifying the corresponding pairs of homomorphisms g and h, for which the diagrams

$$\begin{array}{ccc} & A \cong B & \\ g & \downarrow \quad \downarrow & h \\ & A_1 \cong B_1 & \end{array}$$

are commutative, by $\mathbf{K}^*$. Although we shall deal with concrete algebras and morphisms, we shall always keep these identifications in mind.

We shall denote by $\mathbf{K}^*_\mu$ the subcategory of $\mathbf{K}^*$ with the same set of objects and a set of morphisms, obtained from the set of monomorphisms of $\mathbf{K}$ — $Mon_{\mathbf{K}}$, by the same identification as above.

The following lemma is directly proved:

LEMMA 60. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be N-lattices and $h\colon \mathcal{N}_1 \to \mathcal{N}_2$ be a homo-(epi-, mono-)morphism of N-lattices. Then $\bar{h}\colon P(\mathcal{N}_1) \to P(\mathcal{N}_2)$ where $\bar{h}(|a|) \leftrightharpoons |h(a)|$ is a homo-(epi-, mono-)morphism of* PBAs.

THEOREM 61. *Let $\mathfrak{M}$ be a class of* PBAs. *Then*:

1) $\mathfrak{M}^* \cong N(\mathfrak{M})^*$,

2) $\mathfrak{M}^* \cong N^+(\mathfrak{M})^*$,

where $\cong$ is a categoric isomorphism.

PROOF. 1) We define a map $\mathscr{F}\colon \mathfrak{M}^* \to N(\mathfrak{M})^*$:

a) for any $\mathfrak{A} \in \mathfrak{M}$: $\mathscr{F}(\mathfrak{A}) \leftrightharpoons N(\mathfrak{A})$,

b) for any $h \in Hom(\mathfrak{A}_1, \mathfrak{A}_2)$: $\mathscr{F}(h) \leftrightharpoons h_N \in Hom\big(N(\mathfrak{A}_1),\ N(\mathfrak{A}_2)\big)$, where $h_N\big((a, b)\big) = \big(h(a),\ h(b)\big)$. By Proposition 4, h_N is a homomorphism. $\mathscr{F}$ is a functor: $\mathfrak{M}^* \to N(\mathfrak{M})^*$:

— $\mathscr{F}(Id_{\mathfrak{A}}) = Id_{N(\mathfrak{A})} = Id_{\mathscr{F}(\mathfrak{A})}$;

— let $h \in Hom(\mathfrak{A}_1, \mathfrak{A}_2)$, $g \in Hom(\mathfrak{A}_2, \mathfrak{A}_3)$. Then $\mathscr{F}(g\circ h)\big((a, b)\big) = \big(g\circ h(a),\ g\circ h(b)\big) = \mathscr{F}(g)\Big(\big(h(a),\ h(b)\big)\Big) = \mathscr{F}(g)\circ\mathscr{F}(h)\big((a, b)\big)$; $\mathscr{F}(g\circ h) = \mathscr{F}(g)\circ\mathscr{F}(h)$. $\mathscr{F}$ is a categoric isomorphism:

— $\mathscr{F}\colon \mathfrak{M} \to N(\mathfrak{M})$ is a bijection:

surjection — obviously,

injection — let $N(\mathfrak{A}_1) \cong N(\mathfrak{A}_2)$. Then $\mathfrak{A}_1 \cong P\big(N(\mathfrak{A}_1)\big) \cong P\big(N(\mathfrak{A}_2)\big) \cong \mathfrak{A}_2$.

— $\mathscr{F}\colon Hom_{\mathfrak{M}} \to Hom_{N(\mathfrak{M})}$ is a bijection:

surjection — let $h_N \in Hom\big(N(\mathfrak{A}_1),\ N(\mathfrak{A}_2)\big)$. Then

$$\begin{array}{c} \bar{h}_N\colon P\big(N(\mathfrak{A}_1)\big) \to P\big(N(\mathfrak{A}_2)\big) \\ \wr\| \qquad\qquad\qquad \wr\| \\ \mathfrak{A}_1 \qquad\qquad\qquad \mathfrak{A}_2 \end{array}$$

is a counter-image of h_N: $h_N\big((|a|,\ |b|)\big) = \big(\bar{h}_N(|a|),\ \bar{h}_N(|b|)\big)$.

injection — let $\mathscr{F}(h) = \mathscr{F}(g)$, $h \in Hom(\mathfrak{A}_1, \mathfrak{A}_2)$, $g \in Hom(\mathfrak{B}_1, \mathfrak{B}_2)$.

Then $\mathscr{F}(h) \in Hom(N(\mathfrak{A}_1),\ N(\mathfrak{A}_2))$, $\mathscr{F}(g) \in Hom(N(\mathfrak{B}_1),\ N(\mathfrak{B}_2)) \mapsto N(\mathfrak{A}_1) \cong N(\mathfrak{B}_1)$ and $N(\mathfrak{A}_2) \cong N(\mathfrak{B}_2) \mapsto \mathfrak{A}_1 \cong \mathfrak{B}_1$ and $\mathfrak{A}_2 \cong \mathfrak{B}_2$. Now $\mathscr{F}(h)((\neg a, a)) = (\neg h(a), h(a))$, $\mathscr{F}(g)((\neg a, a)) = (\neg g(a), g(a)) \mapsto (\neg h(a), h(a)) = (\neg g(a), g(a))$, thus $h(a) = g(a)$ for any $a \in \mathfrak{A}_1$. And so $g = h$.

2) We define a map $\mathscr{F}^+ : \mathfrak{M}^* \to N^+(\mathfrak{M})^*$:

a) for any $\mathfrak{A} \in \mathfrak{M}$: $\mathscr{F}^+(\mathfrak{A}) \leftrightharpoons N^+(\mathfrak{A})$,

b) for any $h \in Hom(\mathfrak{A}_1, \mathfrak{A}_2)$: $\mathscr{F}^+(h) \leftrightharpoons h_N^+ \in Hom(N^+(\mathfrak{A}_1), N^+(\mathfrak{A}_2))$, where $h_N^+((a, b)) = (h(a), h(b))$. As in 1) it is proved that $\mathscr{F}^+$ is a functor. $\mathscr{F}^+$ is a categoric isomorphism:

— $\mathscr{F}^+ : \mathfrak{M} \to N^+(\mathfrak{M})$ is a bijection:

surjection — obviously,

injection — let $N^+(\mathfrak{A}_1) \cong N^+(\mathfrak{A}_2)$. Then $\mathfrak{A}_1 \cong P(N^+(\mathfrak{A}_1)) \cong P(N^+(\mathfrak{A}_2)) \cong \mathfrak{A}_2$

— $\mathscr{F}^+ : Hom_{\mathfrak{M}} \to Hom_{N^+(\mathfrak{M})}$ is a bijection — analogously to 1). ■

COROLLARY 62. *Let* $\mathfrak{M}$ *be a class of* PBA*s. Then*:

1) $\mathfrak{M}_\mu^* = N(\mathfrak{M})_\mu^*$;

2) $\mathfrak{M}_\mu^* = N^+(\mathfrak{M})_\mu^*$.

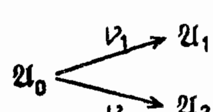
PROOF. It suffices to prove that the restrictions of functors $\mathscr{F}$ and $\mathscr{F}^+$ (by the above theorem) of $Mon_{\mathfrak{M}}$ are bijections correspondingly between $Mon_{\mathfrak{M}}$ and $Mon_{N(\mathfrak{M})}$ and between $Mon_{\mathfrak{M}}$ and $Mon_{N^+(\mathfrak{M})}$, i.e. images and counter-images of the functors $\mathscr{F}$ and $\mathscr{F}^+$ of monomorphisms are monomorphisms, too.

a) Let $h \in Mon_{\mathfrak{M}}$ and $\mathscr{F}(h) = h_N$, $h_N((a, b)) = h_N((c, d))$, i.e. $(h(a), h(b)) = (h(c), h(d))$. Then $h(a) = h(c)$, $h(b) = h(d) \mapsto a = c$, $b = d \mapsto (a, b) = (c, d) \mapsto h_N \in Mon_{N(\mathfrak{M})}$. Analogously, $h_N^+ \in Mon'\ N^+(\mathfrak{A}), N^+(\mathfrak{B}))$.

b) Let $\nu \in Mon(N(\mathfrak{A}),\ N(\mathfrak{B}))$ and $\bar{\nu} = \mathscr{F}^{-1}(\nu)$, $\bar{\nu} \in Hom(\mathfrak{A}, \mathfrak{B})$. Let $\bar{\nu}(a_1) = v(a_2)$. Then $\bar{\nu}(\neg a_1) = \bar{\nu}(\neg a_2) \mapsto \nu((\neg a_1, a_1)) = \nu((\neg a_2, a_2)) \mapsto a_1 = a_2 \mapsto \bar{\nu} \in Mon(\mathfrak{A}, \mathfrak{B})$. Analogously, if $\nu^+ \in Mon(N^+(\mathfrak{A}), N^+(\mathfrak{B}))$, then $\bar{\nu}^+ = (\mathscr{F}^+)^{-1}(\nu^+) \in Mon(\mathfrak{A}, \mathfrak{B})$. ■

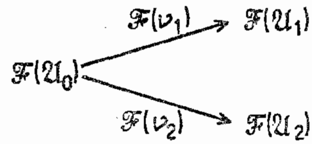
THEOREM 63. *Let* **K** *and* **S** *be classes of algebras of an arbitrary signature and* $\mathbf{K}_\mu^* = \mathbf{S}_\mu^*$. *Then* **K** *has amalgamation property iff* **S** *has amalgamation property.*

PROOF. Let **K** have the amalgamation property and

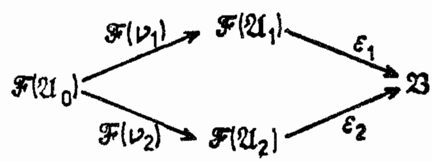


be a diagram in $\mathbf{S}_\mu^*$.

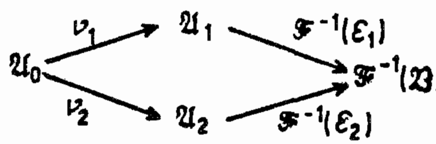
Let $\mathscr{F}$: $\mathbf{S}^*_\mu \to \mathbf{K}^*_\mu$ be a functor, effecting the isomorphism between $\mathbf{S}^*_\mu$ and $\mathbf{K}^*_\mu$. Then



is a diagram in $\mathbf{K}^*_\mu$, for which there exist $\mathfrak{B} \in \mathbf{K}$ and $\varepsilon_i \in Mon(\mathscr{F}(\mathfrak{A}_i), \mathfrak{B})$, $i = 1, 2$, such that the diagram



s commutative. Then the diagram



iis commutative, too. And so $\mathbf{S}$ has the amalgamation property, too. Vice versa analogously. ∎

COROLLARY 64. *Let $\mathfrak{M}$ be a class of* PBA*s. Then*:

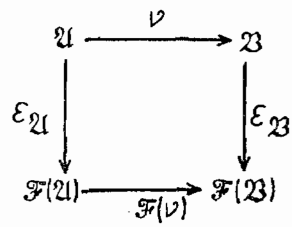
1) *$\mathfrak{M}$ has the amalgamation property iff $N(\mathfrak{M})$ has the amalgamation property.*
2) *$\mathfrak{M}$ has the amalgamation property iff $N^+(\mathfrak{M})$ has the amalgamation property.*

Hence, by Lemma 39 it follows directly:

THEOREM 65. *In an N-logic L^+ the* CIT *holds iff it holds in $I(L)$.*

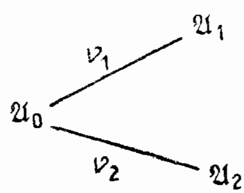
4.3. We shall call a subcategory Q of a category R a retract of R if there exists a functor $\mathscr{F}$: $R \to Q$ such that $\mathscr{F} \restriction Q = Id_Q$. $\mathscr{F}$ will be called a retraction.

THEOREM 66. *Let $\mathbf{K}$ and $\mathbf{S}$ be classes of algebras of an arbitrary signature and $\mathbf{K}^*_\mu$ be a retract of $\mathbf{S}^*_\mu$, with retraction $\mathscr{F}$ the following condition* *holds: for any $\mathfrak{A} \in S$ there exists $\varepsilon_{\mathfrak{A}} \in Mon(\mathfrak{A}, \mathscr{F}(\mathfrak{A}))$ such that for any $\nu \in Mon(\mathfrak{A}, \mathfrak{B})$ the diagram*

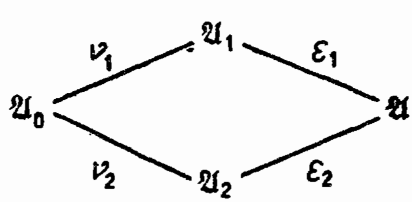


is commutative. Then the class **K** *has the amalgamation property iff class* **S** *has the amalgamation property.*

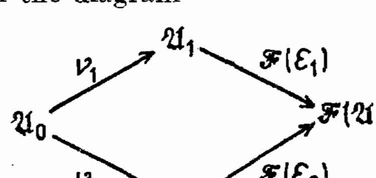
PROOF. 1) Let **S** has the amalgamation property and



be a diagram in $\mathbf{K}_{\mu}^{*}$. Then there exist $\mathfrak{A} \in \mathbf{S}$ and $\varepsilon_i \in Mon(\mathfrak{A}_i, \mathfrak{A})$, $i = 1, 2$, such that the diagram

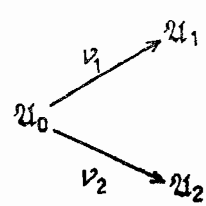


is commutative. Then the diagram

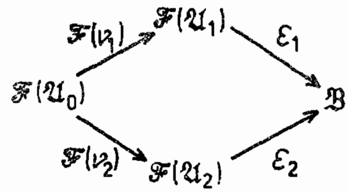


is commutative, too. And so, the class **K** has the amalgamation property too.

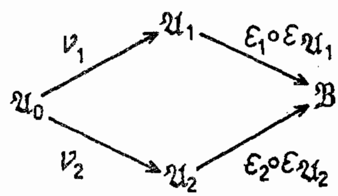
2) Let **K** have the amalgamation property and



be a diagram in $\mathbf{S}_\mu^*$. Then there exist $\mathfrak{B} \in \mathbf{K}$ and $\varepsilon_i \in Mon(\mathscr{F}(\mathfrak{A}_i), \mathfrak{B})$, $i = 1, 2$, such that the diagram



is commutative. Hence, the diagram



is commutative too:

$$\varepsilon_1 \circ \varepsilon_{\mathfrak{A}_1} \circ \nu_1 = \varepsilon_1 \circ \mathscr{F}(\nu_1) \circ \varepsilon_{\mathfrak{A}_0} = \varepsilon_2 \circ \mathscr{F}(\nu_2) \circ \varepsilon_{\mathfrak{A}_0} = \varepsilon_2 \circ \varepsilon_{\mathfrak{A}_2} \circ \nu_2.$$

And so, $\mathbf{S}$ has the amalgamation property too. ■

THEOREM 67. *Let $\mathfrak{N}$ be a class of N-lattices such that $s\mathfrak{N} = N(P(\mathfrak{N})) \subseteq \mathfrak{N}$. Then $\mathfrak{N}$ has the amalgamation property iff $s\mathfrak{N}$ has the amalgamation property.*

PROOF. We define a map $\mathscr{F}\colon \mathfrak{N}_\mu^* \to s\mathfrak{N}_\mu^*$:

a) for any $\mathscr{N} \in \mathfrak{N}$: $\mathscr{F}(\mathscr{N}) \leftrightharpoons s\mathscr{N}$,

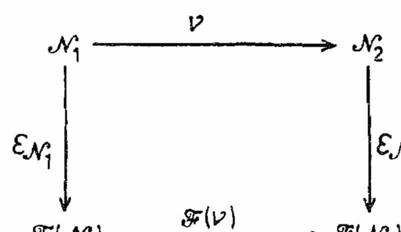
b) for any $\nu \in Mon(\mathscr{N}_1, \mathscr{N}_2)$: $\mathscr{F}(\nu) \leftrightharpoons \bar{\nu} \in Mon(s\mathscr{N}_1, s\mathscr{N}_2)$, where $\bar{\nu}((|a|, |b|)) = (|\nu(a)|, |\nu(b)|)$.

It is directly proved that the definition is correct and $\bar{\nu}$ is a monomorphism and $\mathscr{F}$ is a functor. $\mathscr{F}$ is a retraction:

1) let $\mathscr{N}_0 = s\mathscr{N} \in s\mathfrak{N}$. Then $P(\mathscr{N}_0) \cong P(\mathscr{N}) \mapsto \mathscr{F}(\mathscr{N}_0) = s\mathscr{N}_0 \cong s\mathscr{N} = \mathscr{N}_0$;

2) let $\nu \in Mon(s\mathscr{N}_1, s\mathscr{N}_2)$ and $\nu((|a|, |b|)) = (|\alpha|, |\beta|)$. Then
$\mathscr{F}(\nu)((|a|, |b|)) = \mathscr{F}(\nu)\big((|(|a|, |\neg a|)|, |(|b|, |\neg b|)|)\big)$
$= \big(|\nu((|a|, |\neg a|))|, |\nu((|b|, |\neg b|))|\big) = (|a|, |\beta|)$
$= \nu((|a|, |b|))$. And so $\mathscr{F}(\nu) = \nu$.

Now, let $\mathscr{N} \in \mathfrak{N}$. We define $\varepsilon_{\mathscr{N}}\colon \mathscr{N} \to s\mathscr{N}\colon \varepsilon_{\mathscr{N}}(a) \leftrightharpoons (|a|, |\sim a|)$. $\varepsilon_{\mathscr{N}} \in Mon(\mathscr{N}, \mathscr{F}(\mathscr{N}))$. Let $\nu \in Mon(\mathscr{N}_1, \mathscr{N}_2)$. Then the diagram



is commutative: $\varepsilon_{\mathcal{N}_2}\circ\nu(a) = (|\nu(a)|, |\nu(\sim a)|) = \mathscr{F}(\nu)\big((|a|,\ |\sim a|)\big) =$
$= \mathscr{F}(\nu)\circ\varepsilon_{\mathcal{N}_1}(a)$.

And so, by Theorem 66 $\mathfrak{N}$ has the amalgamation property iff $s\mathfrak{N}$ has the amalgamation property. ■

COROLLARY 68. *If $\mathfrak{N}$ is a class of N-lattices such that $s\mathfrak{N} \subseteq \mathfrak{N}$, $\mathfrak{N}$ has the amalgamation property iff $P(\mathfrak{N})$ has the amalgamation property.*

PROOF. By Corollary 64 $P(\mathfrak{N})$ has the amalgamation property iff $s\mathfrak{N}$ has the amalgamation property which, by Theorem 67, holds iff $\mathfrak{N}$ has the amalgamation property. ■

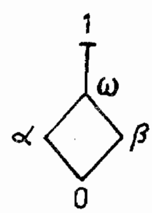
Hence, it directly follows:

THEOREM 69. *A special N-variety $\mathfrak{N}$ has the amalgamation property iff $P(\mathfrak{N})$ has the amalgamation property.*

THEOREM 70. *In a special N-logic L the* CIT *holds iff it holds in $I(L)$.*

4.4. As a consequence of the last theorems and Theorem 58 we can already show an example of a non-normal N-logic $L^0 \subseteq \tilde{C}$ which is not special, i.e. its additional axiomatization over $\tilde{I}$ cannot be translated into the intutionistic language.

Let $C_2 = B_0^2 + B_0 = \{0, \alpha, \beta, \omega, 1\}$ be a PBA. As a distributive lattice it looks as follows:



We set
$$\mathcal{N}^0 = \{\underset{0}{(0,1)},\ \underset{a}{(0,\alpha)},\ \underset{b}{(0,\omega)},\ \underset{c}{(\alpha,\beta)},\ \underset{\sim c}{(\beta,\alpha)},\ \underset{\sim a}{(\alpha,0)},\ \underset{\sim b}{(\omega,0)},\ \underset{1}{(1,0)}\}$$
$\mathcal{N}^0$ is a subalgebra of $N(C_2)$. From the table

p	$p\leftrightarrow\sim p$	$p\cup\neg p$
0	0	1
a	$\sim c$	1
b	b	1
c	b	$\sim b$
$\sim a$	$\sim c$	$\sim b$
$\sim b$	b	$\sim b$
$\sim c$	b	$\sim b$
1	0	1

it is seen that in $L^0 = L(\mathcal{N}^0)$ the CIT does not hold:
$L^0 \vdash (p \leftrightarrow \sim p) \to (q \cup \neg q)$, but L^0 *non* $\vdash (p \leftrightarrow \sim p) \to 0$ and L^0 *non* $\vdash 1 \to \to (q \cup \neg q)$. Besides, $\pi(\mathcal{N}^0) = C_2$ and in $L(C_2)$ (as Maximova proved in [1]) the CIT holds.

Since L^0 is not a normal N-logic ($\mathcal{N}^0$ *non* $\models \neg(x \leftrightarrow \sim x) = 1$) then, by Theorem 56, $L^0 \subseteq \tilde{C}$. By Lemma 20 it follows that $I\big(L(N(C_2))\big) = L(C_2)$. Hence, by Theorem 70 (by Theorem 58 $L(N(C_2))$ is a special N-logic) in $L(N(C_2))$ the CIT holds. If we assume that L^0 is a special N-logic, then $L^0 = L(\mathcal{N}^0) = L(s\mathcal{N}^0) = L(N(C_2))$, but as we have already seen, in $L(N(C_2))$ the CIT holds, while in L^0 this is not the case. Therefore, L^0 cannot be a special N-logic.

4.5. In [1] Maximova proved, that there exist exactly 7 consistent superintuitionistic logics in which the CIT holds. They are obtained from the intuitionistic logic I with additional axioms as follows:

$$L_1 = I$$

$$L_2 = I + \neg p \cup \neg\neg p$$

$$L_3 = I + p \cup (p \to (q \cup \neg q))$$

$$L_4 = L_3 + (p \to q) \cup (q \to p) \cup (p \leftrightarrow \neg q)$$

$$L_5 = L_3 + \neg p \cup \neg\neg p$$

$$L_6 = I + (p \to q) \cup (q \to p)$$

$$L_7 = C = I + p \cup \neg p.$$

THEOREM 71. *There exist exactly* 14 *consistent logics with strong negation in which the* CIT *holds and they are the following*:

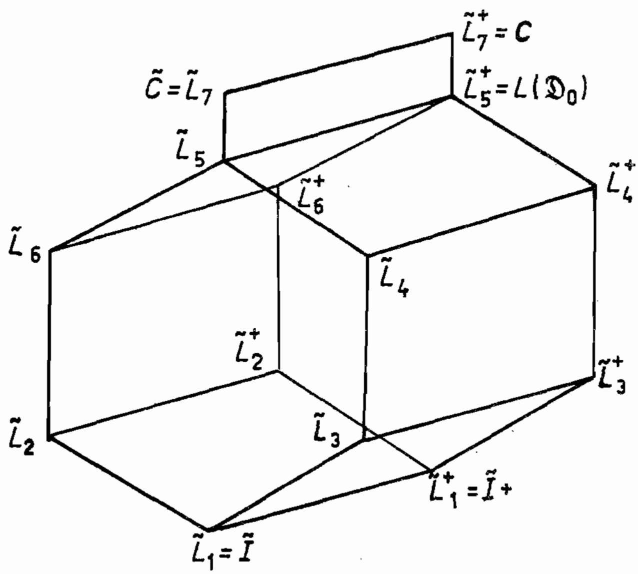
$$\tilde{L}_i = L_i + V, \quad i = 1, \ldots, 7 \text{ (where } V \text{ is the system of Vorobiev's axioms)}$$

and

$$L_i^+ = L_i + V + \neg(p \leftrightarrow \sim p), \quad i = 1, \ldots, 7.$$

PROOF. Let in a consistent N-logic L the CIT hold. If $L \subseteq \tilde{C}$ then, by Theorem 58, L is a special N-logic. By Theorem 70, $I(L)$ coincides with one of the logics L_1-L_7 and therefore, L is one of $\tilde{L}_1$-$\tilde{L}_7$. If $L \nsubseteq \tilde{C}$ then, by Theorem 56, L is a normal N-logic. Hence, by Theorem 47, $L = \overline{I(L)}^+$ and, by Lemma 25, $I(L) = I\big(\overline{I(L)}\big)$, therefore, by Theorem 65, L coincides with one of the logics $\tilde{L}_1^+$-$\tilde{L}_7^+$. ∎

The lattice of these N-logics is the following:



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