



PAPER

Persistence and nonpersistence as complementary models of identical quantum particles

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**Abstract**

Classical mechanics is based on the notion that matter consists of persistent particles that can be reidentified (or tracked) across time. However, the mathematical symmetrization procedures (due to Dirac (1926 *Proc. R. Soc. A* **112** 661) and Heisenberg (1926 *Z. Phys.* **38** 411) and Feynman (1965 *Quantum Mechanics and Path Integrals* 1st edn (New York: McGraw-Hill))) used to describe identical particles within the quantum formalism are widely interpreted as implying that identical quantum particles are not persistent (so that the concept of ‘the *same* particle’ is not meaningful) or are persistent but not reidentifiable. However, it has not proved possible to rigorously reconcile these interpretations with the fact that identical particles are routinely assumed to be reidentifiable in particular circumstances—for example, a track in a bubble chamber is interpreted as a sequence of bubbles generated by one and the same particle (Mirman 1973 *Il Nuovo Cimento* **18B** 110; de Muynck 1975 *Int. J. Theor. Phys.* **14** 327; Dieks and Lubberdink 2011 *Found. Phys.* **41** 1051; Jantzen 2011 *Phil. Sci.* **78** 39). Moreover, these interpretations do not account for the mathematical form of the symmetrization procedures, leaving open theoretical possibilities other than bosonic and fermionic behavior, such as paraparticles (Messiah and Greenberg 1964 *Phys. Rev.* **136**), which however do not appear to be realized in nature. Here we propose that the quantum mechanical behavior of identical particles is a manifestation of a novel kind of complementarity, a complementarity of persistence and nonpersistence. Accordingly, identical ‘particles’ are neither persistent nor nonpersistent; rather, these terms are to be understood as descriptors of different *models* of the same experimental data. We prove the viability of this viewpoint by showing how Feynman’s and Dirac’s symmetrization procedures arise through a synthesis of a quantum treatment of persistence and nonpersistence models of identical particle-like events, and by showing how reidentifiability emerges in a context-dependent manner. Finally, by drawing on a reconstruction of Feynman’s formulation of quantum theory (Goyal *et al* 2010 *Phys. Rev. A* **81** 022109), we construct a precise parallel between the proposed persistence–nonpersistence complementary and Bohr’s wave–particle complementarity for individual particles, and detail their conceptual similarities and dissimilarities.

1. Introduction

We ordinarily conceive of the everyday physical world as consisting of *objects* that bear *properties* and that *persist* through time. Developed early in life through our continual sensorimotor interaction with the physical world, this conception organizes our experience of the external world into a coherent, predictive model. In particular, persistence underwrites our ability to say that the object one is seeing now is *the same as* a specific object that one saw elsewhere at an earlier time. In practice, objects’ gradual motion and slowly-varying characteristic properties (such as shape and color) provide the perceptual handles that enable their reidentification.

Classical physics incorporates these key notions—objects, properties, persistence, and reidentifiability—into its abstract conceptual framework at a fundamental level. Persistence is reflected in the assumption that objects can be *labeled*. Additionally, classical mechanics posits that objects localized to point-like regions of space—*particles*—are

the fundamental constituents of matter. Much as in everyday experience, these particles can be reidentified by continuous tracking of their motion, and by measurement of their characteristic intrinsic properties (such as mass and charge). In this framework, two particles may be entirely *identical* in their intrinsic properties—a situation that does not arise in everyday experience—yet remain reidentifiable by their distinct trajectories.

It is, however, widely accepted that the quantum treatment of assemblies of identical particles brings into question the assumptions of persistence and reidentifiability. This challenge was first brought to light through Bose's derivation of Planck's blackbody radiation formula [1], in which the calculation of the number of ways in which a given number of photons can be arranged amongst cells in phase space only takes into account the *number* of photons in each cell. Thus, unlike Boltzmann's corresponding calculation for gas molecules, no account is taken of *which* photon is in which cell.

Bose's counting procedure admits two quite distinct interpretations. First, that the photons are *not persistent*, so that the very notion of 'which photon is in which cell' is meaningless. Second, that the photons, although persistent, are *not reidentifiable* by any observer. The first view was taken by many contemporary physicists. For example, at the 1927 Solvay Conference, Langevin suggested that the novel quantum statistics pointed to a suppression of the 'individuality of the constituents of the system' [2, p 453]. More pointedly, in his 1950 Dublin lectures [3], Schroedinger states: 'If I observe a particle here and now, and observe a similar one a moment later at a place very near the former place, not only cannot I be sure whether it is 'the same', but this statement has no absolute meaning.' The second view is based on the *symmetrization procedure* that was put forward by Heisenberg and Dirac [4–6] as a way of incorporating Bose's novel counting procedure and Pauli's exclusion principle into the nascent quantum formalism. According to Dirac [7, section 54], what is special about identical particles is that they are 'indistinguishable'—that is, not reidentifiable—in the sense that observations provide no information about which particle is which.

However, both of these interpretations are at odds with assumptions that are routinely made in the interpretation of primary experimental data. For example, in an experiment in which we say that electrons are liberated at a filament, diffracted through a crystal lattice, and then impact a phosphorescent screen, we presume that each scintillation on the screen is due to the same electron that was emitted by the filament, *even though there are many other electrons in the laboratory*. The correctness of the diffraction pattern calculated on the assumption of persistence demonstrates that the assumption is at least *approximately* valid in this instance. Yet, according to Dirac's symmetrization procedure, this electron is in a symmetrized state with all the other electrons (irrespective of its ostensible isolation from them), which implies that each has the same reduced state, and so is equally likely to be found at any electron location (see, for instance, [8–12]). Similarly, the notion of 'particle tracks' (say, in a bubble chamber), which is a prerequisite to the processing of primary data in particle physical experiments, implicitly assumes object persistence—a sequence of bubbles is deemed to have been generated by the *same* particle, thus constituting a 'track'—even when another particle identical to it lies simultaneously in the detector's field of view.

Additionally, neither of the above-mentioned interpretations provide a basis for accounting in detail for the quantum rules employed in the treatment of identical particles. For example, although the nonpersistence view naturally accounts for Bose's photon-counting procedure, it provides no clue as to the origin of Pauli's exclusion principle, which (in a modification of Bose's procedure) was implemented by Fermi as a single-occupancy limit on each phase-space cell [13]. Similarly, although Dirac's non-reidentifiability view explains why a system initially placed in a symmetric or antisymmetric state will remain in the same type of state under temporal evolution, it does not explain why a system cannot be in a nonsymmetric state (specifically, in a linear combination of symmetric and antisymmetric states) in the first place. Dirac's view also leaves open the theoretical possibility that a system of three or more particles could exhibit so-called parastatistical behavior, a possibility for which no experimental evidence has been found (see [7, section 54] and [14])¹. Experimental

¹ In the paper in which he originally addressed the subject, Dirac gave an argument that purported to *derive* the fact that a system of two identical particles can occupy only symmetric or antisymmetric states [4, section 3]. The fact that this argument does not generalize to three or more particles was pointed out by Wigner [19, 20]. In apparent reference to this result, Dirac subsequently speaks of the restriction to (anti-)symmetrical states as an *empirical* fact ('The invariance and permanence of the symmetry properties of the states means that for some particular kind of particle it is quite possible for only symmetrical or only antisymmetrical states to occur in nature. Whether this is the case cannot be decided by any general theoretical considerations, but can be settled only by reference to special experimentally determined facts about the particles in question.' [6, section 62, p 201]). In the third edition (1947) of [6], he adds: 'other more complicated kinds of symmetry are possible mathematically, but do not apply to any known particles' [21, section 54, p 211]. These difficulties notwithstanding, variants of Dirac's original (1926) argument are frequently encountered in textbooks of quantum theory. As pointed out, for instance, in [14, 22], these arguments are unsatisfactory in that they make implicit assumptions of an abstract nature. For example, a common argument is that a state $\psi(x_1, x_2)$ of two identical particles must be an eigenstate of the permutation operator P which describes a 'swap' of the two particles, from which it follows that only symmetric or antisymmetric states are possible. However, it is not clear *why* physically realizable states should be restricted to eigenstates of P (this does not follow from the requirement that expectation values be the same for the original and permuted state; such a requirement can be satisfied by restricting observables, A , to those that are permutation invariant, $[A, P] = 0$, without imposing any restriction on the allowable states [14]); and, even if such a restriction is granted, the argument does not generalize to three or more particles unless one rules out the possibility of describing the physical state of a system by means of Hilbert subspaces of dimension higher than one (admitting this possibility leads to so-called paraparticles) [14, 19, 20]. Another common argument is to require that the probability distribution $|\psi(x_2, x_1)|^2$ associated with two 'swapped' particles be the same as $|\psi(x_1, x_2)|^2$, and to argue that this implies $\psi(x_2, x_1) = e^{i\phi} \psi(x_1, x_2)$, and hence $e^{i\phi} = \pm 1$, a condition only satisfied by symmetric or antisymmetric states. However, such an argument tacitly assumes that ϕ is independent of x_1, x_2 [22].

searches for specific hypothetical deviations ('quons', for example [15]) from bosonic and fermionic behavior have been carried out, but no such deviations have yet been reported (see, for example, [16–18]).

The above considerations suggest that neither interpretation is satisfactory, and that a more thoroughgoing revision of our conceptual picture is necessary if we are to pinpoint the essential idea that underlies the behavior of identical particles and rigorously account for the empirical success of Dirac's symmetrization procedure. Now, as we have noted, identical particles *sometimes* behave as if persistent (for example, two electrons moving along distinct particle 'tracks'), and sometimes as if nonpersistent (as in Bose's photon-counting procedure). This suggests that, rather than trying to account for this behavioral diversity on the basis of persistence *or* nonpersistence alone, we instead attempt to combine *both* of these pictures in a more even-handed way.

In this paper, we develop a novel understanding of the quantum mechanical treatment of identical particles along these lines. We adopt an operational approach in which the raw data consists of *identical localized events*. To be concrete, one can think of observing a fixed number of localized light-flashes of the same color at successive times. At this stage, there are no 'identical particles' as such, only identical *events*. We construct two distinct models of these events, namely a *persistence* model and a *nonpersistence* model. These differ in whether or not it is assumed that successive events are generated by individual persistent entities ('particles'). We then show that these models can each be described within the Feynman formulation of quantum theory and be *synthesized* to derive the Feynman's form of the symmetrization procedure [23]. As we show elsewhere [24, 25] and summarize here, this procedure can be transformed into a state-based symmetrization procedure which is empirically adequate yet differs from Dirac's procedure in form and meaning, in particular allowing for the natural emergence of reidentifiability in special cases.

We then show that the persistence and nonpersistence models, and the manner of their synthesis, satisfy the key characteristics of Bohr's concept of complementarity. Specifically, we show that these models are mutually exclusive, but that they can be synthesized to generate an empirically-adequate predictive model. On this basis, we propose that the quantal behavior of identical particles reflects a *complementarity of persistence and nonpersistence*, analogous to the way in which the behavior of an individual electron reflects a complementarity of particle and wave.

Finally, we exhibit a precise parallel between our proposed persistence–nonpersistence complementarity and Bohr's wave-particle complementarity. In particular, we show that the Feynman amplitude sum rule can be viewed as a synthesis of the wave and particle models of elementary constituents of matter in a manner that formally mirrors the way in which a symmetrization procedure arises through a synthesis of persistence and nonpersistence models of identical localized events. These two examples thereby illustrate how complementarity can be turned into a precise methodology for synthesizing mutually-exclusive models.

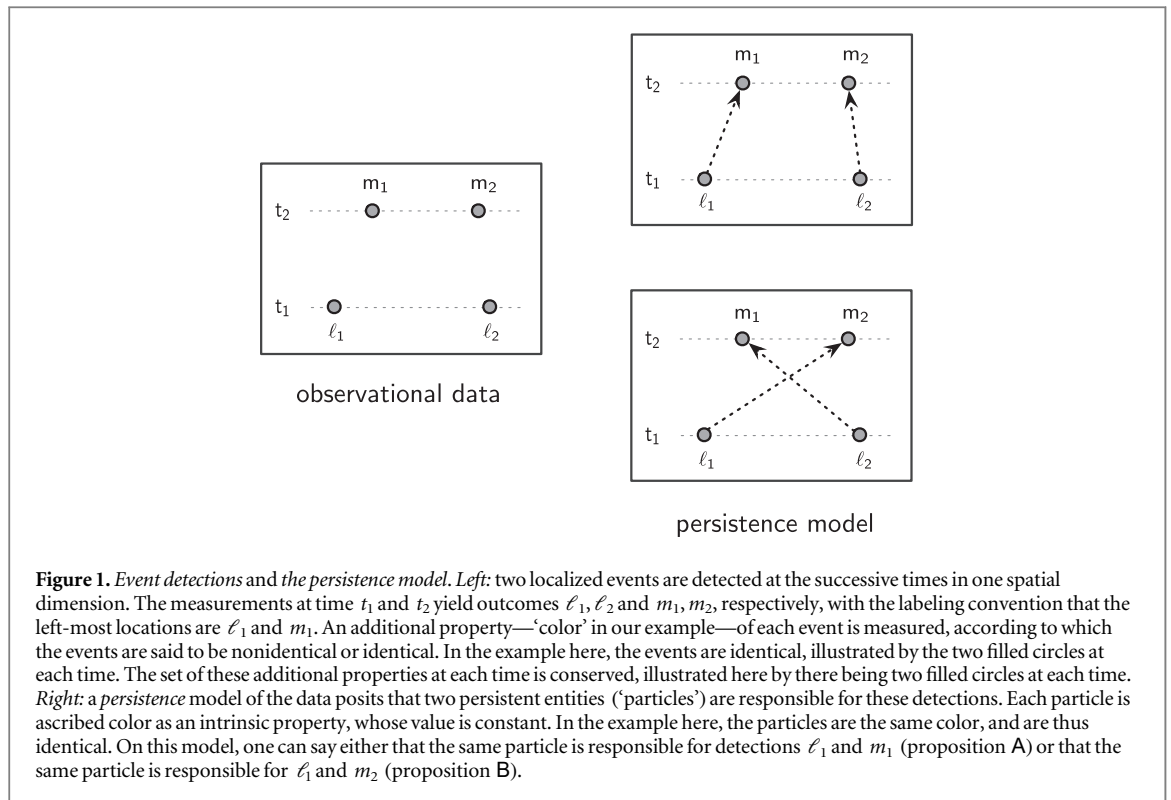
The remainder of this paper is organized as follows. In section 2, we define the concepts of persistence and nonpersistence in an operational manner. In section 3, we outline the derivation of the symmetrization procedure. In section 4, we describe our complementarity interpretation of identical particles in light of this derivation, and establish a detailed parallel with Bohr's wave-particle complementarity. We conclude in section 5 with a discussion of the broader context and some open questions such as the interpretation of the quantum field theoretic formalism in light of our complementarity interpretation of the quantum mechanical treatment of identical particles.

2. Operational framework

Our discussion will be based on the fundamental notions of *persistence* and *nonpersistence*. In order to place these notions on a clear footing, we begin by stepping back from the familiar theoretical frameworks of classical and quantum mechanics, and instead adopting an operational perspective. Such a perspective is helpful in identifying assumptions that are of limited validity in physical domains remote from everyday experience, but that may be too entrenched in our customary patterns of thought to be clearly and consistently perceived.

1. *Position measurements of localized events and their properties.* Consider a situation where a position measurement, implemented by a fine grid of detectors that tile a region of space, is performed at discrete times t_1, t_2, \dots (see figure 1). Here and subsequently, we restrict consideration to one spatial dimension for simplicity. Suppose that only two detectors fire at each time. We can speak of each such detection as a *localized event* (a 'flash'). Suppose further that these detectors are capable not only of registering a localized event, but also of measuring some additional properties of this event. For concreteness, we henceforth imagine that there is just one additional property, namely *color*; thus, at each time, one observes *two colored flashes*.

Let us further suppose that observation shows that these additional properties are conserved, in the sense that the total number of flashes of each color seen at each time is the same. For example, at each time, one obtains a blue flash and a red flash. If it should be the case that both of the localized events have the same additional property values (for example, both the flashes are blue), we shall say they are *identical*. That is the situation that



concerns us here. Finally, let us suppose that the system is isolated. Operationally, this can be established by carrying out repeated trials, and showing that the probability over the locations of the two detections at t_2 are determined by the locations at t_1 , a condition we refer to as *closure* [26].

On the basis of these observations, we say that the measurements are being performed on a ‘system’, namely a persistent object that is such that (i) it yields two localized events at each time t_b , and (ii) the property-values (colors) of these events is conserved.

2.1. Persistence model

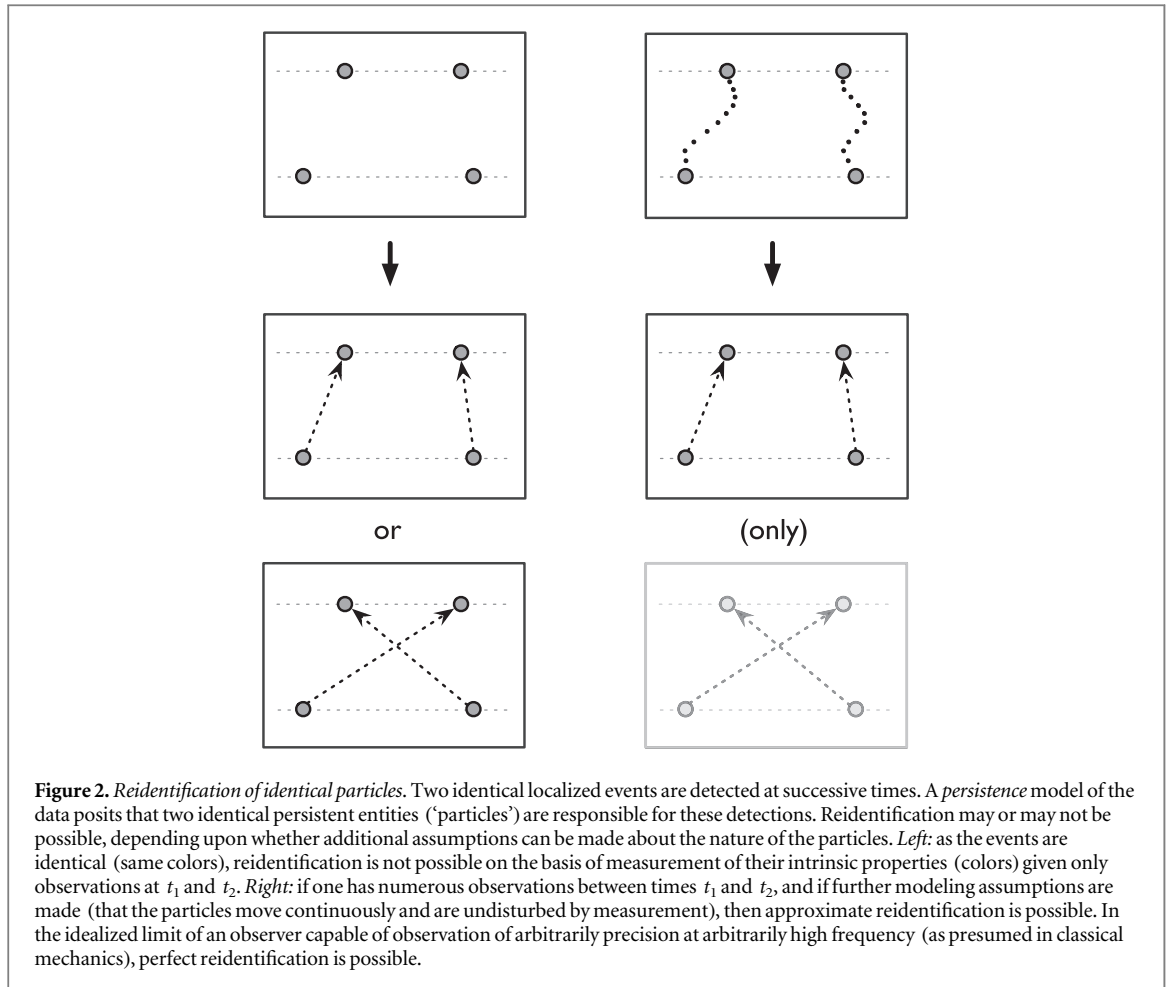
We now construct a model of data in which two identical localized events are registered at each time. Let us assume that there exist individual entities that persist in between these detections, and that these entities can (informally) be said to *cause* these detections. We shall refer to these entities as *particles* on the understanding that this word describes the localized, particle-like way that these entities manifest themselves upon detection, rather than implying anything about their nature *between* such detections. We ascribe *intrinsic* and *extrinsic* properties to these particles. The former are the same as the additional properties measured of the localized events, and their values are assumed to be constant. In this case, the flashes are the same color, so we say that there are two particles of the same color, and we assume that the color of each particle is constant. As the colors of two particles are the same, we shall say that they are *identical*.

On the basis of this *persistence model*, one can now meaningfully say that *same* particle is responsible for detections at two different times. Suppose that events are detected at locations ℓ_1 and ℓ_2 at time t_1 , and at m_1 and m_2 at t_2 . Here we adopt the labeling convention that ℓ_1, m_1 are the left-most locations at each time. According to the persistence model, even though the two events at each time are identical, one can say that one or the other of the following propositions is true:

- A = ‘the same particle is responsible for the detection ℓ_1 at t_1 and the detection at m_1 at t_2 ,’ or
 B = ‘the same particle is responsible for the detection ℓ_1 at t_1 and the detection at m_2 at t_2 .’

1. *Reidentifiability.* If it is possible for an *observer* to determine which of the above propositions is true, we shall say that it is possible to *reidentify* each particle (see figure 2). That is, reidentification is the observational counterpart of the theoretical notion of persistence. As the particles are identical (same color), reidentification on the basis of measurement of their intrinsic properties is impossible, so that the possibility of reidentification will hinge upon additional theoretical and operational considerations.

In particular, if one makes the additional theoretical assumption that particle motion is continuous, and further assume that it is possible for an observer to make non-disturbing position measurements of arbitrary



precision at arbitrarily high frequency, then reidentification (at arbitrarily high confidence level) of identical particles is possible on the basis of the measurement records.

2. *Persistence grounds particle labels.* If persistence is assumed, then either proposition A or B, as given above, is true. This provides the basis for particle labeling. Specifically, let us label '1' the particle that was at ℓ_1 at t_1 , and label the other particle '2'. Then, the *configuration* of the system at t_1 is given by the ordered pair (ℓ_1, ℓ_2) , and the configuration at time t_2 is either (m_1, m_2) or (m_2, m_1) . Each of the latter two ordered pairs thus reflects not only the observed particle positions at t_2 , but also—as a result of the theoretical assumption of persistence—some information about the observed positions at the earlier (reference) time, t_1 . For example, the configuration (m_1, m_2) specifies not only the information that there are two identical particles at locations m_1 and m_2 (which is what is observed at t_2), but also that the particle that is now at m_1 was earlier at ℓ_1 .

To say that reidentification is, in principle, possible, means that there exists an observer who can determine which of these configurations— (m_1, m_2) or (m_2, m_1) —is in fact the case. But, for an observer who is not capable of reidentification, there is a gap between the theoretical level of description of the system—the configuration—and the information available to that observer. For example, at t_2 , the theoretical description might be the configuration (m_1, m_2) , but the observer would be incapable of distinguishing this from (m_2, m_1) .

3. *Connection to classical mechanics.* In classical mechanics, persistence is assumed, and an ideal observer is capable of reidentifying identical particles (provided they cannot coincide) by tracking their continuous trajectories with arbitrary precision but without causing disturbance. Therefore, the theoretical and observational descriptions coincide in the configuration $(\mathbf{r}_1, \mathbf{r}_2)$ of two particles.

2.2. Nonpersistence model

We can, however, construct a second model—a *nonpersistence model*—of the identical localized events in which one does *not* presume that there are individual persistent entities that underlie the individual localized detections. Rather, the two localized events at each time are regarded as a manifestation of a single abstract 'system' that persists. One is thus left with the bare data of localized events $\{\ell_1, \ell_2\}$ at t_1 and $\{m_1, m_2\}$ at t_2 . Repeated trials of the experiment would yield a conditional probability distribution $\Pr(\{m_1, m_2\} | \{\ell_1, \ell_2\})$.

In such a model, the only persistent object is an abstract ‘system’ which yields two localized detection events at each measurement time. As this object is not analyzed into two separate persistent objects, the fundamental techniques that one ordinarily employs in constructing particle-based models are unavailable. For instance, in classical mechanics, one can start by positing that individual objects move uniformly if isolated, and then build up a model of a system of how two such objects interact with one another by imposing constraints in the form of conservation laws; but such a model-building strategy hinges on an analysis of the system into persistent individuals which is unavailable in the nonpersistence model.

3. Derivation of a symmetrization procedure

As indicated in the Introduction, the quantum treatment of identical particles brings the assumption of persistence into question. Without the assumption that identical localized events are underpinned by individual persistent entities, the possibility of creating a model of the events based on analysis into persistent entities is blocked.

On the other hand, we have observational evidence that persistence is at least approximately valid in certain situations, for example in the case of identical localized events in a bubble chamber. In order to construct a predictive model, we incorporate *both* of these pieces of observational evidence by formulating *two* models of these events—one that assumes non-persistence and the other that assumes persistence—and then posit a connection between them.

3.1. Synthesis of persistence and nonpersistence

To be specific, consider again the above experiment involving position measurements at times t_1 and t_2 . We now construct two theoretical models of this situation within the Feynman quantum formalism (see figure 3). In the *persistence* model, irrespective of whether or not reidentification is possible, we can say that either one or the other transition connects the observed data—a *direct* transition, where the particle that was at ℓ_1 at t_1 is found at m_1 at t_2 ; or an *indirect* transition from ℓ_1 to m_2 . Let us denote the amplitudes of these transitions α_{12} and α_{21} , respectively.

The second model, the *nonpersistence* model, does *not* presume that there are persistent entities that underlie the observed localized detections. Accordingly, the only amplitude that one can associate with the given data in this model is the transition amplitude, α , from the initial data $\{\ell_1, \ell_2\}$ at t_1 to the final data $\{m_1, m_2\}$ at t_2 .

The connection between these two models takes the form of the *operational indistinguishability postulate* (OIP), which posits that the amplitudes in the persistence model determine the amplitude in the nonpersistence model. In the case of measurements at two successive times under consideration here

$$\alpha = H(\alpha_{12}, \alpha_{21}), \quad (1)$$

where H is an unknown continuous complex-valued function to be determined.

The OIP also applies to the case where one has observations at *three* successive times, t_1, t_2 and t_3 . In that case, the persistence model has two possible transitions between times t_1 and t_2 , and two possible transitions between times t_2 and t_3 , so that there are four possible transitions overall from times t_1 to t_3 via t_2 . Let the amplitudes of these transitions be denoted $\gamma_{11}, \gamma_{12}, \gamma_{21}$, and γ_{22} . Then the generalization of equation (1) reads

$$\gamma = G(\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}), \quad (2)$$

where γ is the transition amplitude in the nonpersistence model, and G is a function to be determined.

We now incorporate the fact that there are situations in which we commonly say that the particle observed now is the ‘same’ as one previously observed via an *isolation condition*. This condition stipulates that, in the limiting case that isolation obtain for one or more of the identical particles in a given system, they can be treated as a *persistent subsystem* for the purpose of making predictions. For example, if the electron in a hydrogen atom is effectively isolated from all other electrons in a given system, then we can infer that *the same* electron is responsible for the successive electron-detections in the atom. One can accordingly apply the quantum formalism to this electron as if it were a (persistent) system. Formally, for the case of two identical events, the isolation condition requires that the transition probability $|H(\alpha_{12}, \alpha_{21})|^2$ is the same as the probability of the persistence-model transition that has non-zero probability. For example, if the direct transition is the one with non-zero probability, then

$$|H(\alpha_{12}, 0)|^2 = |\alpha_{12}|^2. \quad (3)$$

From the assumptions above, Feynman’s symmetrization procedure can be derived [24]. The key idea behind the derivation is the recognition that the amplitude for a particular process in the nonpersistence model can sometimes be computed in two different ways, and, in these instances, consistency of the assumptions implies that these calculational paths must agree. Each such call for consistency leads to a functional equation.

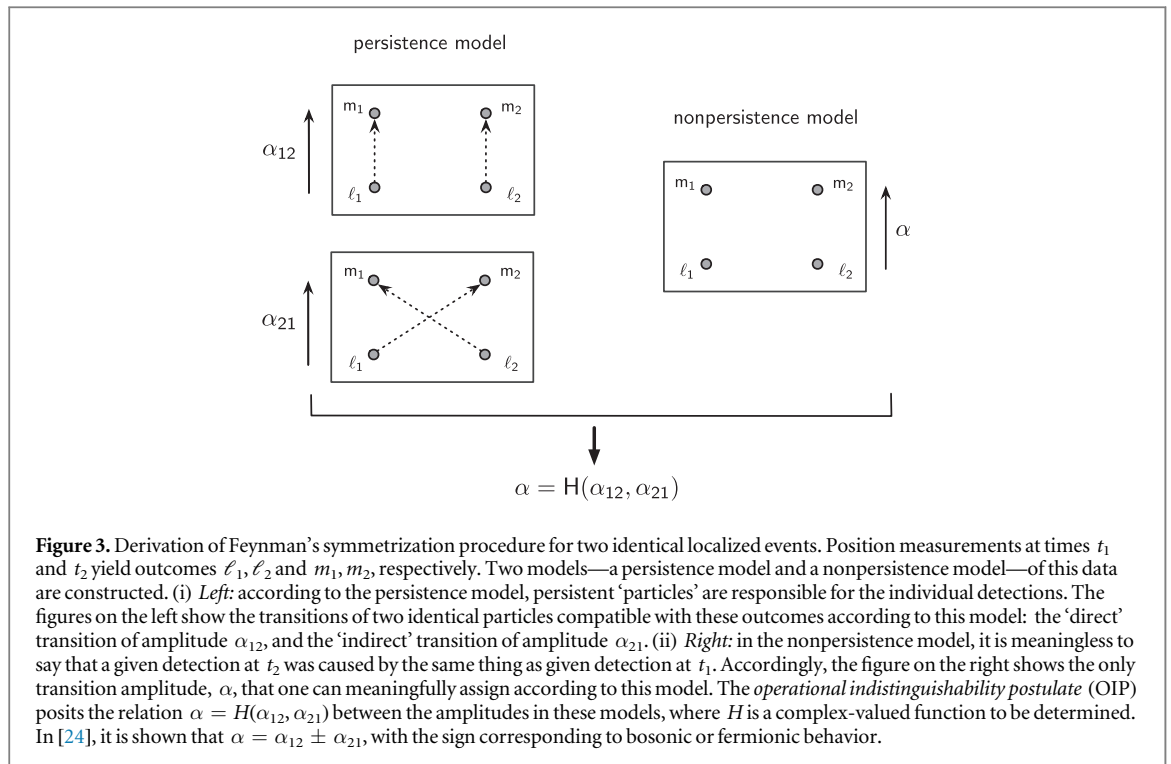


Figure 3. Derivation of Feynman’s symmetrization procedure for two identical localized events. Position measurements at times t_1 and t_2 yield outcomes ℓ_1, ℓ_2 and m_1, m_2 , respectively. Two models—a persistence model and a nonpersistence model—of this data are constructed. (i) *Left*: according to the persistence model, persistent ‘particles’ are responsible for the individual detections. The figures on the left show the transitions of two identical particles compatible with these outcomes according to this model: the ‘direct’ transition of amplitude α_{12} , and the ‘indirect’ transition of amplitude α_{21} . (ii) *Right*: in the nonpersistence model, it is meaningless to say that a given detection at t_2 was caused by the same thing as given detection at t_1 . Accordingly, the figure on the right shows the only transition amplitude, α , that one can meaningfully assign according to this model. The *operational indistinguishability postulate* (OIP) posits the relation $\alpha = H(\alpha_{12}, \alpha_{21})$ between the amplitudes in these models, where H is a complex-valued function to be determined. In [24], it is shown that $\alpha = \alpha_{12} \pm \alpha_{21}$, with the sign corresponding to bosonic or fermionic behavior.

For example, one obtains

$$G(\alpha_{12}\beta_{12}, \alpha_{12}\beta_{21}, \alpha_{21}\beta_{12}, \alpha_{21}\beta_{21}) = H(\alpha_{12}, \alpha_{21}) H(\beta_{12}, \beta_{21}). \tag{4}$$

Solution of these functional equations yields

$$\alpha = \alpha_{12} \pm \alpha_{21}, \tag{5}$$

where the \pm sign corresponds to bosonic or fermionic behavior. This is Feynman’s symmetrization rule for two particles. The above derivation generalizes naturally to N identical particles.

3.2. Probabilistic reidentifiability

If one of the transition probabilities $|\alpha_{12}|^2$ or $|\alpha_{21}|^2$ is much smaller than the other, then the transition probability $|H(\alpha_{12}, \alpha_{21})|^2$ approximates to the largest of these probabilities. In that case, one can treat the observational data as a probabilistic version of the persistence model, so that one has *probabilistic reidentifiability* of the particles. Such a situation obtains, for example, for two electrons in the field of view of a particle chamber, where we can roughly say that each electron travels along its own ‘track’, even though there is a finite probability (as computed using the persistence model) of the electrons ‘swapping’ between tracks.

3.3. State representation of the symmetrization procedure

The amplitude-based symmetrization procedure given above can be re-expressed in terms of states and observables, a more familiar arena for the description of quantum phenomena. Such a re-expression also facilitates a direct comparison with Dirac’s symmetrization procedure. To illustrate the key ideas, it suffices to consider two particles moving in one dimension. In that case, one can [24] re-express equation (1) in terms of states as

$$\psi_{\text{ID}}(x_1, x_2) = \psi(x_1, x_2) \pm \psi(x_2, x_1) \quad x_1 \leq x_2. \tag{6}$$

Although this relation resembles the Dirac symmetrization postulate, it has a quite different meaning. First, the states that occur on the left and right hand sides belong to different models. The function ψ_{ID} on the left hand side is the state of the system as described within the nonpersistence model. This state is defined over location-space (or reduced configuration space), namely $x_1 \leq x_2$, and is normalized over that space. Thus, in the expression $\psi_{\text{ID}}(x_1, x_2)$, x_i is to be interpreted as the x -value of *location* i . Consequently, $|\psi_{\text{ID}}(x_1, x_2)|^2 dx_1 dx_2$ is the probability of detection of two identical particle-like events, one in the interval $[x_1, x_1 + dx_1]$ and the other in $[x_2, x_2 + dx_2]$.

In contrast, the function ψ on the right hand side of equation (7) is the state of the system as represented in the persistence model. It is defined (and normalized) over the full configuration space $(x_1, x_2) \in R^2$. Thus, in the

expression $\psi(x_1, x_2)$, x_i represents the x -value of the location of *particle* i . Consequently, $|\psi(x_1, x_2)|^2 dx_1 dx_2$ is the probability of detecting particle 1 in interval $[x_1, x_1 + dx_1]$ and particle 2 in interval $[x_2, x_2 + dx_2]$.

The above relation is thus to be understood as establishing a *formal connection* between these two models, a connection which can be used to take states from the persistence model over to the nonpersistence model.

For convenience, one can formally extend $\psi_{\text{ID}}(x_1, x_2)$ to the entire configuration space, to obtain state $\tilde{\psi}_{\text{ID}}(x_1, x_2)$, in terms of which one can rewrite the above equation as

$$\tilde{\psi}_{\text{ID}}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi(x_1, x_2) \pm \psi(x_2, x_1)], \quad (7)$$

where now (x_1, x_2) ranges over \mathbb{R}^2 . Such a formal extension is useful in the sense that now $\tilde{\psi}_{\text{ID}}$ and ψ are both defined over \mathbb{R}^2 and both formally live in the same space (a tensor product of two labeled copies of a one-particle Hilbert space). However, the formal extension makes the reading of the labels more complicated: although x_1, x_2 on the right hand side are particle labels as before, x_1 on the left is the location of the leftmost event whenever $x_1 \leq x_2$, but the location of the rightmost event whenever $x_1 > x_2$.

If the two particles (as viewed in the persistence model) are confined to disjoint regions on the left and right sides, then α reduces to α_{12} . Further, owing to the isolation between the particles, the surviving amplitude can be written as a product of two amplitudes, one related to each particle. Thus, in the state formulation, $\psi_{\text{ID}}(x_1, x_2) = \phi_a(x_1)\phi_b(x_2)$, where ϕ_a, ϕ_b can be viewed as one-particle states of labeled particles which have no common support. Under these circumstances, the observable x_1 in the nonpersistence model, which by default is the x -location of the left-most event, gains the *additional* meaning of the x -location of particle 1. Thus, in this limiting case, one recovers reidentifiability, and one has justification to model each of the particles as a distinct entity without regard for the other.

3.3.1. Comparison with Dirac's interpretation of the symmetrization procedure

We are now in a position to contrast the understanding of identical particles developed above with the Dirac's interpretation of his symmetrization procedure. According to Dirac, 'if a system in atomic physics contains a number of particles of the same kind, e.g. a number of electrons, the particles are absolutely indistinguishable from one another' [7, section 54]. That is, in our terminology, the system can be described within the persistence model (so that the identical particles are modeled as persistent entities), but no observations are capable of reidentifying these particles.

In order to formalize this idea, Dirac proposed that a system of identical particles be described within a labeled tensor product of single-particle subspaces by states that are symmetric or antisymmetric in the subspace labels, and that observables be restricted to those that are symmetric in these labels. According to Dirac's interpretation, in a state $\psi(x_1, x_2)$ describing two identical particles, i is a *particle* label, so that x_i is the location of the i th particle. Accordingly, the states ψ_{ID} and ψ in equation (7) are interpreted as states in the *same* (persistence) model, and equation (7) itself is interpreted as a way of (anti-) symmetrizing the state ψ (itself obtained by solving for the eigenstates of a measurement operator symmetric in the particle labels) in order to generate a physically legitimate state, ψ_{ID} .

However, as we have detailed in the introduction (see, in particular, footnote 1), Dirac's understanding of identical particles (as persistent but not reidentifiable) does *not* lead to his proposed formalization without the use of additional assumptions that lack physical justification. This naturally raises the question of whether Dirac's formal procedure can be interpreted in a different manner. As we have shown above, this is indeed possible. In essence, we view equation (7) not as a means of 'symmetrizing' a state within a single (persistence) model, but rather as establishing a connection between the states of the same system as described in two *different* models—a persistence model and a non-persistence model. The fact that we are able to *derive* equation (7) by formalizing this interpretation speaks in its favor.

In this connection, we briefly note that the coherency of Dirac's interpretation—specifically, the view that the labels in a symmetrized state refer to particles (a view recently dubbed *factorism* [12])—has previously been brought into question by several other authors [10–12, 27, 28]. As we have pointed out in the Introduction, a major difficulty with Dirac's particle-label view is that, in a system of two electrons, *both* of the electrons have same reduced state, even if they are associated with widely-separated hydrogen atoms. However, this implies that each electron is as likely to be found in the vicinity of one atom as the other [10, 11], which is difficult to reconcile with our usual presumption that, owing to their isolation, one can treat each of these electrons as a distinct system confined to its respective atom. Given these and related difficulties, Dieks and Lubberdink conclude [10] that 'the quantum mechanical symmetrization postulates do not pertain to particles, as we know them from classical physics, but rather to indices that have a merely formal significance.' However, such a claim leaves the challenge of formulating an alternative understanding of these indices which, for instance, is capable of rendering intelligible the usual procedures for interpreting measurement operators. For example, if one applies the symmetrization procedure to the electrons in a helium atom, the measurement operator $(x_1 - x_2)^2$ is

ordinarily interpreted as representing a measurement of the squared-distance between the two electrons; but it is unclear how one would justify such an interpretation if the indices 1 and 2 have a ‘merely formal significance’.

However, according to the interpretation that we have developed above, the indices *do* have a physical significance, the hitherto unexpected complication being that the physical referent of an index depends upon whether the state in question is viewed as a state within the persistence model (in which case an index is a particle label) or the nonpersistence model (in which case an index is a location label). This interpretation has a number of ramifications—for instance for the understanding of the meaning of measurement operators and for the proper understanding of the entanglement of identical particles—which will be discussed elsewhere.

4. Complementarity of persistence and nonpersistence

As formulated by Bohr, *complementarity* expresses a view about the kind of theoretical model that one can formulate about microphysical phenomena. As such, it constitutes a major pillar of Bohr’s interpretation of quantum theory. Bohr gave a detailed account of the notion of complementarity in his 1927 Como lecture [29], a notion which he subsequently developed in a series of papers over the course of the next thirty years. As originally described in [29] (and subsequently elaborated upon in other papers, such as those reprinted in [30]), complementarity maintains that, in order to account for data observed in a given experiment on a microphysical system (such as an electron diffraction experiment), one must draw upon elements from two apparently incompatible classical pictures².

We note that, after his engagement with the EPR paper in 1935, Bohr emphasized a different complementarity [31], namely that between information gained about the same microscopic object in different experimental arrangements. Although this notion of complementarity has been widely influential (it can, for instance, be formalized in terms of quantum state tomography via mutually unbiased bases [32], and has served as a basis for some reconstructions of quantum theory [26, 33]), the notion of complementarity that concerns us here is the original one described in [29]. Its key constituent ideas (which we shall subsequently illustrate) can be abstractly summarized as follows:

1. *Need for two incompatible models.* A theoretical understanding of experiments on microscopic systems in general requires the use of concepts drawn from *two* distinct models of the type that are characteristic of classical physics. These models are incompatible in the sense that some of core assumptions about the nature of physical reality made in one model are in conflict with some of the core assumptions made in the other.
2. *Synthesis of models.* At least some of the models’ incompatibility arises from abstract assumptions which are not *directly* refutable via observation. Accordingly, there is sufficient latitude to combine key features of the two models into a new predictive calculus. The resulting synthetic model is not of the classical type, and has a symbolic or abstract (non-visualizable) character. Nevertheless, it provides a predictive framework for behavior that in some sense interpolates between that permitted by the original models.
3. *Limiting cases.* Whereas a quantitative understanding of the phenomena observed in general requires the use of the synthetic model, certain physical situations can be understood using only one of the original models. Correspondingly, in certain limiting cases, the behavior predicted by the synthetic model approximates to that predicted by one of the original models.

4.1. Wave and particle

We illustrate the above-mentioned key features of complementarity via Bohr’s paradigmatic example of the wave and particle models of the electron.

First, according to the particle model, the electron is a point-like entity that has a definite position at each time. In contrast, the wave model treats the electron as a delocalized wave-like object³. Each of these models paints a clear, visualizable conception of the electron, but are manifestly incompatible—an electron cannot

² The following are representative quotations from [29]: 1. ‘The very nature of quantum theory thus forces us to regard the space-time coordination and the claim of causality, the union of which characterizes the classical theories, as complementary but exclusive features of the description[...].’ (p 580, para. 3). 2. ‘The two views of the nature of light are rather to be considered as different attempts at an interpretation of experimental evidence in which the limitation of the classical concepts is expressed in complementary ways.’ (p 581, para. 1).

³ In classical physics, one often regards a ‘wave’ (such as a water wave, or a wave on a string) as underpinned by local disturbances over a region of space. In such cases, ‘wave’ is merely a collective noun that refers to a set of synchronized local disturbances, and so has no fundamental existence in and of itself. However, in the wave model of an electron, such an underpinning is *not* presumed, so that ‘wave’ is taken to refer to an unanalyzed, delocalized object.

simultaneously be a localized point-like object as well as a delocalized object. Yet, the need for both models is made plausible by the existence of experimental phenomena (such as diffraction) associated with electrons where detections are point-like whereas the distribution of detections is wave-like.

Second, the particle model's assumption that an isolated electron is continuously localized is not directly testable—the observational data consists only of point-like detections (modeled as *interactions* between the electron and the detector) at *discrete* times. Hence, although the models themselves are at odds if their core assumptions are regarded as strictly true, there may be sufficient latitude to synthesize key features of these models whilst remaining consistent with observations. Such a synthesis was first proposed by de Broglie in his composite wave-particle model of the electron, a proposal that culminated in Schrodinger's wave mechanics. In the latter synthesis, the wave aspect is reflected in an evolving wavefunction, while the particle aspect is reflected in the projection (or Born) rule, enabling the model to incorporate behavior that is characteristic of each of the original models. Unlike the wave or particle models, the synthetic model is noncommittal as to the nature of the electron, and accordingly has a relatively abstract character.

Third, there are limiting situations in which the behavior of an electron is well-modeled by the particle picture. Such a situation obtains, for example, when an electron is subject to potentials that are sufficiently slowly varying in space and time, and to position measurements that are sufficiently coarse. Additionally, in such limiting situations, the synthetic model approximates to a statistical particle model.

4.2. Persistence and nonpersistence

Wave-particle complementarity suggests that the assumption of *continuous localization*, a notion extrapolated from everyday perception, does not enjoy absolute validity in the microscopic realm. We propose that the assumption of *persistence* is similarly restricted in its validity. Furthermore, we propose that the understanding of identical particle-like events requires a synthesis of complementary persistence and nonpersistence models, analogous to the way that localization ('particle') and delocalization ('wave') models need to be combined in order to understand individual microscopic events.

Our proposal is based on the following considerations. First, through an operational analysis of an experiment in which identical particle-like events are registered at each instant (section 2), we have seen that one can construct *two* distinct models. One of these assumes that successive detections are underpinned by *persistent* underlying entities ('particles'), whilst the other assumes that no such entities exist, but rather that the events at each instant are the manifestation of a *single* abstract 'system'.

Second, we have demonstrated that these models can be synthesized to derive a quantum symmetrization procedure for describing the behavior of a system of identical particle-like events. This derivation not only places a formally *ad hoc* formal procedure on a clear operational and logical foundation, but also naturally resolves the difficulty in reconciling the assertion that identical particles are nonpersistent or not reidentifiable with their manifest reidentifiability in particular situations (section 3.2).

Third, we assert that persistence and nonpersistence are *complementary* descriptions of identical particle-like events on the grounds that the persistence and nonpersistence models satisfy the three key features of complementarity, as follows.

First, these two models are *mutually exclusive* in the sense that they make contradictory assumptions about whether or not successive individual detections are underpinned by individual persistent entities. Consequently, the statement that 'this detection was caused by the same object as a previous detection' is meaningful in the first (persistence) model but not the second.

Second, despite their contradictory nature, these models can be *synthesized*. That this is possible is due to the fact that the both models make claims about the nature of the entity of entities that exist in between the detections, claims that can (by their very nature) only be *indirectly* probed via experiment. Insofar as the synthetic model combines models that make conflicting claims as to the nature of the entity (or entities) that underpin the detections, the synthetic model cannot, in general, be interpreted as positing that the detections are or are not underpinned by persistent entities, and is, in that sense, abstract in comparison with the original models. As we have seen, the synthesis is formally realized by using the persistence model to calculate amplitudes that are then combined to generate an amplitude in the nonpersistence model. The resulting synthetic model allows for behavior intermediate between that allowed by the two original models.

Third, in the limiting case where an electron is isolated (as judged within the persistence model), it can be regarded as if persistent, and treated entirely within the persistence model. Correspondingly, in this limiting case, the synthetic model yields the same predictions as the persistence model.

4.3. Parallel between wave–particle and persistence–nonpersistence complementarities

The above considerations suggest that there is a close conceptual parallel between wave–particle and the proposed persistence–nonpersistence complementarities. We now construct a more precise formal parallel between them.

One of the difficulties in clearly understanding Bohr’s formulation of complementarity is that it refers to classical models, namely the ‘wave’ and ‘particle’ model, which are not precisely defined, and whose synthesis is far from transparent. For this reason, a derivation of quantum theory in which the quantum formalism can be interpreted as arising from the synthesis of two apparently contradictory pictures would be most useful. Our previous operational derivation of Feynman’s formulation of quantum theory [26] supports such an interpretation.

We begin by noting that, in Feynman’s formulation of quantum theory, one can interpret the Feynman amplitude sum rule as a succinct formal encapsulation of Bohr’s notion of wave–particle complementarity. Specifically, in the context of an electron double-slit experiment, there are two models of the electron (see table 1, first column). In the ‘particle’ model, the electron is treated as a localized, particle-like entity that traverses one slit or the other in its passage from the source to a given point on the screen. In the ‘wave’ model, the electron is treated as a delocalized object about which all one can say is that it *passes through the slits*. The amplitude sum rule posits a relationship between the transition amplitudes in these two models.

In our previous reconstruction of Feynman’s formulation, we have shown that the Feynman sum rule can be derived using the postulate that (roughly speaking) the amplitudes of the two possible paths in the particle-model determine the amplitude of the ‘path’ in the wave-model. A benefit of this reconstruction is that the two models being synthesized are precisely defined in operational terms. For example, the ‘particle’ model consists solely in being able to assert that a system passed through one slit or the other in its passage from source to screen—no additional properties (such as mass, energy or momentum) or associated classical mechanical equations of motion are implicitly included in the model. Thus, we shall now interpret this derivation as showing, in precise, operational terms, how Bohr’s wave–particle complementarity can be viewed as the *basis* of a constructive derivation of the Feynman sum rule, in the same way that we have shown that the proposed persistence–nonpersistence complementarity provides the basis of a derivation of Feynman’s symmetrization procedure.

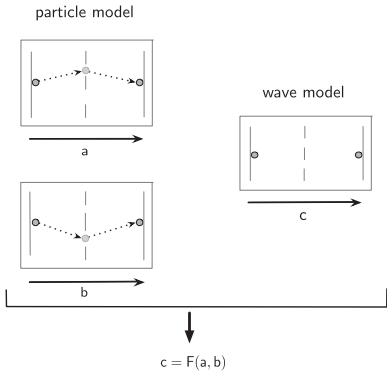
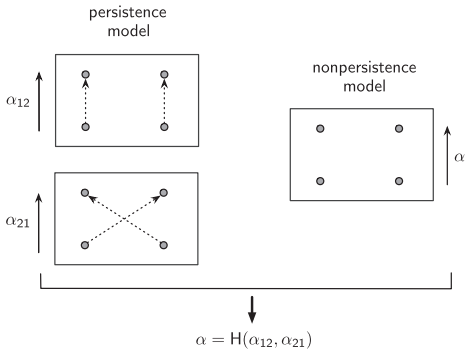
As shown in the table 1, the derivation of the Feynman sum rule allows us to exhibit a precise formal parallel between the two complementarities. In each case, two models are synthesized. One of these models permits an *analysis* of the situation into abstract *parts*—‘the electron passes from the source to screen via one slit *or* the other’ or ‘the identical particles that underpin the particle-like flashes make a direct *or* an indirect transition from one time to the next’. This analysis allows two distinct amplitudes to be defined and, in principle, calculated by making use of the Dirac–Feynman amplitude-action quantization rule [34] and an appropriate classical model of the situation. In contrast, in the other model, no such analysis is possible—all one can say is that ‘the electron passes from the source to screen via the slits’ or that ‘two identical particle-like events occur at each of two successive times’. One can associate a transition amplitude with such an unanalyzed process, but its *calculation* appears impossible if one remains within the compass of this model because no corresponding classical model exists. Although the two models in each case are contradictory in their assumptions, they can be synthesized: if one posits that the two amplitudes in each analytic model determines the amplitude in the corresponding non-analytic model, it is possible to derive the form of the functional form of the relationship between the amplitudes.

An important distinction between the two complementarities, however, deserves to be noted. As shown in [26], the Feynman sum rule can be regarded as arising from a connection between two distinct experimental arrangements—one in which there are which-way detectors at each slit, the other in which there is a single large detector covering both slits [26]. That is different from the interpretation we are adopting here, namely that the sum rule is a connection between two models (‘particle’ and ‘wave’) of the *same* arrangement in which there is a single large detector. However, in the derivation of the Feynman symmetrization procedure for identical particles described here, only one of these two options is available: the symmetrization procedure arises from the connection between two distinct models of the *same* experimental arrangement. Thus, the symmetrization procedure seems to express the notion of complementarity at a deeper level, namely as a synthesis of mutually-exclusive models of the *same* situation.

5. Discussion

It is widely believed that identical particles differ from nonidentical particles in that the former lack persistence or reidentifiability. We have pointed out the deficiencies of such a view, and proposed instead that the specialness of identical particles lies in the fact that *both* persistence and nonpersistence models must be employed in order to cover their full range of behavior. We have proved the viability of this viewpoint by

Table 1. Parallel between the complementarity interpretations of (i) amplitude sum rule (exemplified by double slit) and (ii) symmetrization procedure (for two identical ‘particles’). *Case (i):* given detections at a source and screen, the ‘particle’ model posits that a particle-like object traversed either one slit or the other. The ‘wave’ model eschews such an assumption, treating the electron as a delocalized object (‘wave’). These apparently contradictory models can nevertheless be synthesized by positing a functional relationship, $c = F(a, b)$, between the transitions amplitudes in these models. As shown in [26], the unknown function can be obtained within a broader derivation of Feynman’s rules, yielding $c = a + b$ (Feynman’s amplitude sum rule). In the limiting case where ‘interference’ can be neglected, a statistical classical model ($p_c = p_a + p_b$) is recovered. *Case (ii):* the persistence model assumes that detections are underpinned by persistent entities, an assumption eschewed by the nonpersistence model. Model synthesis is enabled by the assumption that the relation $\alpha = H(\alpha_{12}, \alpha_{21})$ holds between the amplitudes in these models. Within the Feynman framework, H can be solved [24] to yield the Feynman form of the symmetrization procedure $\alpha = \alpha_{12} \pm \alpha_{21}$, where the sign corresponds to fermionic or bosonic behavior. The situation is well-described by the persistence model if isolation obtains (so that the magnitude of either α_{12} or α_{21} is negligible); in this limit, the synthetic model predictions reduce to that of the persistence model.

	Amplitude sum rule	Symmetrization procedure
Two mutually-incompatible models	‘Particle’ and ‘Wave’ models	Persistence and nonpersistence models
Synthesis of models		
Limiting cases	When ‘interference’ can be neglected, a classical particle model applies; correspondingly, the synthetic model approximates to a statistical classical model ($p_c = p_a + p_b$).	If the particles are isolated from one another as viewed in the persistence model (either α_{12} or α_{21} can be neglected), the system is well-described by the persistence model alone; correspondingly, in the synthetic model, $ \alpha $ reduces to $ \alpha_{12} $ or $ \alpha_{21} $.

showing how the Feynman and Dirac symmetrization procedures that are employed to treat systems of identical particles can be systematically derived through a synthesis of the persistence and nonpersistence models. We have also indicated how reidentifiability emerges in a context-dependent manner.

We have further shown that the persistence and nonpersistence models, and the manner of their synthesis, satisfy the key characteristics of Bohr’s concept of complementarity. On this basis, we have proposed that the quantal behavior of identical particles reflects a *complementarity* of persistence and nonpersistence, analogous to the way in which the behavior of an individual electron is rendered intelligible through Bohr’s wave–particle complementarity. Finally, we have constructed a precise parallel between these two complementarities, which brings their conceptual similarities and dissimilarities into sharper focus.

We conclude with a few brief remarks on which we expect to elaborate elsewhere.

1. *Relationship between the two complementarities.* The parallel between the persistence–nonpersistence and wave–particle complementarities raises the question of whether there is a single broader perspective from which both complementarities can be seen to emerge, and indeed whether other related complementarities exist. We leave this question open, apart from noting the presence in each of an essential tension between ‘whole’ and ‘part’. That is, in each complementarity, one model permits analysis into ‘parts’ (either due to the assumption of persistence or the assumption of continuous localization) whereas the other describes the situation as an unanalyzed whole.

2. *Relation between the quantum mechanical and quantum field theoretic models of identical particle-like events.* According to the thesis put forward here, the symmetrization procedures used in the description of identical particle-like events are a bridge between the quantum mechanical descriptions of two different models of identical particle-like events. This bridge allows one to compute evolving states (or to compute transition amplitudes in the Feynman picture) in the persistence model, and then to combine these in specific ways (as specified by the Dirac or Feynman symmetrization procedure) to yield evolving states (or transition amplitudes) in a nonpersistence model.

As the nonpersistence model regards all the events recorded at each instant as manifestations of an abstract system, the number of these events is a state-determined property (rather than an intrinsic property) of this

system. Accordingly, a natural generalization of the nonpersistence model considered above would allow a variable number of events to be detected at each instant, and correspondingly allow state-transformations in which the number of events can change.

One can regard the quantum field theoretic treatment of identical particle-like events as such a generalization. It is commonly assumed that, in the quantum field theoretic framework, there are no persistent entities (apart from the abstract system itself), only ‘excitations’. From our standpoint, this is correct only as long as the (anti-) commutation relations amongst the creation and annihilation operators are not considered. As these relations ensure that the states in this field theoretic model agree with those of the quantum mechanical model for constant event (or particle) number, the persistence model (and the complementarity between persistence and nonpersistence) implicitly enters once the (anti-) commutation relations are imposed. This leaves open the question of how one interprets superpositions of states with differing particle numbers from the complementarity point of view.

3. *Connection to everyday experience.* We ordinarily assume that the *appearances* perceived in the present moment are underpinned by *objects* that:

- (i) *persist* in the time between these appearances; and
- (ii) assume *forms* that coincide with those of these appearances, not only at the moment of perception but also during the intervening intervals.

As an extrapolation from what is directly perceived, these assumptions constitute a ‘theory’ developed very early in life⁴, and are written into the foundation of classical physics. The above complementarities bring this extrapolation into question, at two distinct levels:

- (i) *wave–particle complementarity* brings into question the assumption that an object takes the *same form* between observations as it does during its appearances.
- (ii) *persistence–nonpersistence complementarity* brings into question the more basic idea that an object exists between observations and underpins them.

Nevertheless, the synthesis of complementary models (wave and particle models; or persistence and nonpersistence models) yields a theory that fits the observations. Thus, on the one hand, one can regard complementarity as pointing to the limitations of our ordinary models of the appearances; but, on the other, as offering a constructive path to transcend these limitations.

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⁴ As elucidated by Piaget [35], the concept of object persistence is *learned* by infants, and develops through a series of distinct identifiable stages over the first eighteen or so months of life. Prior to the development of this concept, infants sometimes behave as if an object removed from sight no longer exists.

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