note on Sorites series

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Abstract

Vagueness does not necessarily come in with vague predicates, nor need it be expressed by them[1], but undoubtedly 'vague predicates' are traditionally in the focus of main stream discussions of vagueness. In her current modal logic presentation and discussion of the Sorites paradox Susanne Bobzien[1] lists among the properties of a Sorites series a rather weak modal tolerance principle governing the 'grey zone' containing the borderline cases of the Sorites series, which later proves crucial for her solution of the Sorites paradox by use of epistemic interpreted modal operators in 1st order modal logic. We suggest (for different research interest) instead a non-modal description of the switch in the grey zone (respecting tolerance), by resort to similarity sequences, thus getting tangent to two other areas of research in the field. Let's say - any way - the Sorites paradox vanishes, the Sorites series does not.

1 Introduction

Bobzien's exposition of 'A generic solution to the Sorites paradox...' - which we would like to cite as our 'Sorites' frame of reference - starts

"A Sorites series w.r.t. some given predicate F is (i) a finite sequence of objects $\mathbf{a_1}$ to $\mathbf{a_n}$ that is ordered with respect to some dimension (e.g. height, numbers of grains), with the ordering being total and strict,1 for which (ii) the principle **POLAR** and (iii) the principle $\mathbf{MONOTONICITY}_{\Box}$ hold, and which (iv) displays tolerance.", and Bobzien states as formal properties

"2.1 $\Box\, \mathbf{Fa_1} \wedge \Box \neg\, \mathbf{Fa_n}$ **POLAR**"

...

"2.2 $\forall \mathbf{i}\ ((\Box \mathbf{Fa_i} \rightarrow \Box \mathbf{Fa_{i-1}}) \wedge (\Box \neg \mathbf{Fa_i} \rightarrow \Box \neg \mathbf{Fa_{i+1}}))$ $\mathbf{MONOTONICITY}_{\Box}$"

...

"2.3 $\exists \mathbf{i}\ (\neg \Box\, \mathbf{Fa_i} \wedge \neg \Box \neg \mathbf{Fa_i}) \leftrightarrow \neg \exists \mathbf{i}\ (\Box \mathbf{Fa_i} \wedge \Box \neg \mathbf{Fa_i})$ **BORDERLINE – AS – BUFFER**"

...

"2.4 $\forall \mathbf{i}\ \neg \Box \neg (\mathbf{Fa_i} \leftrightarrow \mathbf{Fa_{i+1}})$ $\mathbf{TOLERANCE}_{\neg\Box\neg}$"

Bobzien [1], §2, pp. 3-5

Reference to the epistemic interpretation of e.g. $\exists \mathbf{i}\ \nabla \mathbf{Fa_i}$ which assures the existence of borderline cases (agnostic point of view) allows Bobzien to account for liability to the fallacy; the fallacy diagnosed to be caused by acceptance of plausible but invalid Sorites 'induction' Conditional (SC), which Bobzien consequently replaces by a 'weakened Conditional (WC)' [1], §8, pp. 21 ff.

We agree to rejecting the Sorites 'induction', but being interested mainly in some special location of the landscape of supposedly vague predicates, in which the Sorites chose to reside, for now we leave Bobzien's exposition at this point. And, using to a different purpose[2] different means, we try another way describing and replacing Sorites 'induction' by a weaker principle in non-modal context.

2 similarity

2.1 similarity relation vs. equivalence relation

We recall elementaries from the logic of binary relations:

A binary relation $\mathbf{E}$ is an *equivalence* relation iff $\mathbf{E}$ is reflexive $\bigwedge_{\mathbf{x}} \mathbf{Exx}$, symmetric $\bigwedge_{\mathbf{xy}} (\mathbf{Exy} \rightarrow \mathbf{Eyx})$ and transitive $\bigwedge_{\mathbf{xyz}} (\mathbf{Exy} \wedge \mathbf{Eyz} \rightarrow \mathbf{Exz})$

In every 1st order language $\mathbf{L}$ containing identity, the identity relation is the strongest equivalence relation expressible in $\mathbf{L}$, which is reflected by 'substitution salva veritate' axiom schemes or inference rule, e.g. by an axiom scheme

$\bigwedge_{\mathbf{xy}}[\mathbf{x} = \mathbf{y} \rightarrow (\mathbf{Ax} \rightarrow \mathbf{Ay})]$

where $\mathbf{Ax}$ is any first order condition with free occurrence of some variable $\mathbf{x}$ not containing variable $\mathbf{y}$.

Weaker equivalence relations than identity are common e.g. in (axiomatic) basic measurement (objects measured may show equal length or mass etc.) [3]

Any equivalence relation scatters its domain into a set of mutually disjunct equivalence classes.

Just another case with similarity: A binary *similarity* (or *resemblance*, *likeness*) relation $\mathbf{S}$ should be reflexive $\bigwedge_{\mathbf{x}} \mathbf{Sxx}$, symmetric $\bigwedge_{\mathbf{xy}} (\mathbf{Sxy} \rightarrow \mathbf{Syx})$, but need not be transitive.

Thus, models for a similarity relation in this sense may include structures in which

$\neg[\bigwedge_{\mathbf{xyz}} (\mathbf{Sxy} \wedge \mathbf{Syz} \rightarrow \mathbf{Sxz})]$ is true, which is equivalent to

$\bigvee_{\mathbf{xyz}} (\mathbf{Sxy} \wedge \mathbf{Syz} \wedge \neg\mathbf{Sxz})$

Of course, the conditions of reflexivity and symmetry do not define a special similarity relation but a set of such relations, in fact a set, which contains the set of all equivalence relations as a subset - viz. the set of those similarity relations, which are transitive; in other words, this concept of similarity relation is a generalization of the concept of equivalence relation.

2.2 similarity relation vs. equivalence relation - modeling

A rather simple but instructive set of models for these axioms for S (reflexive, symmetric and maybe or not transitive, in different models, or for different instances within the same model) is given by sets of line segments of some constant length (in Euclidean space)

$\mathbf{U} \subset \{< \mathbf{x}, \mathbf{y} > | \, \mathbf{0} < \mathbf{d}(\mathbf{x}, \mathbf{y}) = \mathbf{const.}\}$ [4]

containing some, say at least 3, line segments of equal length and mutually different directions and the set $\mathbf{U}$ of line segments is closed under parallel and under linear translations.

Definition: $< \mathbf{x_1}, \mathbf{y_1} >$ is called similar to $< \mathbf{x_2}, \mathbf{y_2} >$, iff there exists a (possibly empty) set of translations such that $\mathbf{x_1} = \mathbf{x_2} = \mathbf{x} \wedge \mathbf{d}(\mathbf{y1}, \mathbf{y2}) \leq \mathbf{d} < \mathbf{x}, \mathbf{y_1} >$. Obviously this is the case, iff $\sphericalangle < \mathbf{y_1}, \mathbf{x}, \mathbf{y_2} > \leq 60°$.

It's very easy now to define models for our 'similarity relation' S, containing 3 line segments $< \mathbf{x_1}, \mathbf{y_1} >, < \mathbf{x_2}, \mathbf{y_2} >, < \mathbf{x_3}, \mathbf{y_3} >$ such that
line segment $< \mathbf{x_1}, \mathbf{y_1} >$ is similar to line segment $< \mathbf{x_2}, \mathbf{y_2} >$, and
line segment $< \mathbf{x_2}, \mathbf{y_2} >$ is similar to line segment $< \mathbf{x_3}, \mathbf{y_3} >$ and
line segment $< \mathbf{x_1}, \mathbf{y_1} >$ is **not** similar to line segment $< \mathbf{x_3}, \mathbf{y_3} >$.

In this case the set of all line segments, similar to line segment $< \mathbf{x_1}, \mathbf{y_1} >$ and the set of all line segments similar to $< \mathbf{x_3}, \mathbf{y_3} >$ (which does **not** contain $< \mathbf{x_1}, \mathbf{y_1} >$) do have a non-empty intersection, containing $< \mathbf{x_2}, \mathbf{y_2} >$, which case were excluded, if **S** were transitive.

2.3 similarity by degree ?

Of course, by varying the similarity definition $\mathbf{x_1} = \mathbf{x_2} = \mathbf{x} \wedge \mathbf{d}(\mathbf{y1}, \mathbf{y2}) \leq \mathbf{d} < \mathbf{x}, \mathbf{y_1} >$ in the second conjunct we would be able to introduce additionally 'similarity to a certain degree $\mathbf{r} \in [\mathbf{0}, \mathbf{1}] \subset \mathbb{R}$ ' , by setting
similarity $= \mathbf{1} \leftrightharpoons \sphericalangle < \mathbf{y_1}, \mathbf{x}, \mathbf{y_2} > = 0°$, giving $\mathbf{d}(\mathbf{y1}, \mathbf{y2}) = \mathbf{0}$
similarity $= \mathbf{0} \leftrightharpoons \sphericalangle < \mathbf{y_1}, \mathbf{x}, \mathbf{y_2} > = 90°$, giving $\mathbf{d}(\mathbf{y1}, \mathbf{y2}) = \mathbf{d}(\mathbf{x}, \mathbf{y_1}) \times \sqrt{\mathbf{2}}$

We suggest, that this model class for 'similarity to a certain degree' supplies a metric for T. Williamson's (T1)-(T5)[5]

But this extends to a larger topic while we focus on the current context, keeping for the time being to our simple nonmetric similarity-relation(s) $\mathbf{S}(\mathbf{x}, \mathbf{y})$

2.4 similarity relation vs. equivalence relation - use in Sorites series

The idea of dropping transitivity for achieving a plausible weak Sorites 'induction' principle seems to have been introduced first by Robert van Rooij and extensively elaborated in Pablo Cobreros et alii, 'Tolerance_Classical_Strict'(**TCS**) "Central to the discussion of this principle is the specification of the properties of the indifference relation. Arguably, a relation such as " not looking to have distinct heights" is reflexive and symmetric, but not transitive: **a** can look to have nearly the same height as **b**, **b** can look to have nearly the same height as **c**, but **a** and **c** may look to have distinct heights. In our approach, the non-transitivity of the indifference relation is a central feature of all vague predicates ..."[2] p.349 "Definition 8 *Similarity predicates* .... That is, similarity predicates are classically interpreted,... Essentially, the assumption implies that similarity relations coming with a vague predicate are crisp and extensionally determinate. This may appear to be in tension with the prospect of accounting for vague predicates, but for the theory we develop here what primarily matters is the non-transitive character of such relations." [2] p.353

Our approach to similarity relations, which need not be transitive (while being rather shortcut and independently found), seems to be in good accord with this policy, but, differing from **TCS** we confine to classical FOL so far.[6]

2.4.1 preliminary conclusion

Now, it seems, that by our roughly sketched similarity approach
($\mathbf{S}(\mathbf{a_i}, \mathbf{a_{i+1}})$ meaning '$\mathbf{a_i}$ **is similar to** $\mathbf{a_{i+1}}$ **with respect to being F**')

$$\boxed{\mathbf{F}(\mathbf{a_1}) \wedge \mathbf{S}(\mathbf{a_1}, \mathbf{a_2}) \wedge ... \wedge \mathbf{S}(\mathbf{a_{i-1}}, \mathbf{a_i}) \wedge \mathbf{S}(\mathbf{a_i}, \mathbf{a_{i+1}}) \wedge \neg\mathbf{S}\left(\mathbf{a_{i-k[k<i]}}, \mathbf{a_{i+1}}\right) \wedge \neg\mathbf{F}(\mathbf{a_{i+1}}) ...}$$

the Sorites paradox has vanished, the Sorites series remaining untouched. Really ? Of course, with respect to our 'line segments models' there is nothing surprising left with $\boxed{\neg\mathbf{S}(\mathbf{a_{i-1}}, \mathbf{a_{i+1}})}$, nothing paradox. But what about the 'grey ...' or 'borderline zone' ? Has it gone ? Then, according to Bobzien's characterisation we would retain only a PM-series (POLAR-MONOTONICITY-series), not a Sorites series, which by definition contains borderline cases. The truth of course is - if there are borderline cases, they remain, independently of what logical features we use to describe them. Our simple similarity chaining $\mathbf{S}(\mathbf{a_i}, \mathbf{a_{i+1}})$ [in the case of a Sorites series necessarily by a similarity relation, which admits non-transitive instances] might be able to detect the switch to the first clear case $\neg\mathbf{Fa_{i+1}}$ (marking the end of the grey zone), but the start of the 'grey zone' is of course not marked. The start of the 'borderline zone' (as leaving clear cases of F) would only be marked, if we had at our disposal the equivalence relation $\bigwedge_{\mathbf{x},\mathbf{y}}[\mathbf{E}(\mathbf{x}, \mathbf{y}) \leftrightarrow (\mathbf{Fx} \leftrightarrow \mathbf{Fy})]$. But in this case the 'borderline zone' were empty and the PM-series would fail to qualify as a Sorites series.

3 second thoughts

3.1 on using intuitionist weakening of double negation

First we return shortly to Bobziens exposition for citing "... S4M is a modal companion of intuitionistic sentential logic. This links the sentential part of the solution to intuitionistic theories of vagueness." [7]

With respect to intuitionist sententential logic in view of the fact that

$$\vdash_{\mathbf{I}} \mathbf{A} \rightarrow \neg\neg\mathbf{A}, \quad \text{but} \quad \nvdash_{\mathbf{I}} \neg\neg\mathbf{A} \rightarrow \mathbf{A}$$

one might be tempted to try a translation of the modallogic writing of a Sorites series

$$\Box\, \mathbf{Fa_1}, ..., \;\; \bigtriangledown \mathbf{Fa_i}, \bigtriangledown \mathbf{Fa_{i+1}}, \;\; ..., \Box\neg\; \mathbf{Fa_n}$$

by a 'Sorites induction Conditional' using intuitionist (instead of classical) material implication to the effect

$$\mathbf{Fa_1}, ..., \;\; \neg\neg(\mathbf{Fa_i}), \neg\neg(\neg\mathbf{Fa_{i+1}}), \;\; ..., \neg\; \mathbf{Fa_n}$$

because of

$$\nvdash_{\mathbf{I}} \neg\neg(\mathbf{Fa_i}) \;\; \rightarrow (\mathbf{Fa_i})$$

and

$$\nvdash_{\mathbf{I}} \neg\neg(\neg\mathbf{Fa_i}) \rightarrow (\neg\mathbf{Fa_i})$$

3.2 on outcome concerning 'given by type' vs. 'given by definition' controversy

The preliminary outcome of these notes on the controversy between Whewell and Mill on whether 'natural groups are given by type, not by definition' is still modest but, we claim, not null. As we already mentioned (see footnote), Mill, in favour of 'given by definition' in first line refers to say predicate clusters, but as a second line of defence takes resort to 'resemblances' in a comparative use (resemble ... more or less ...)[8] . From our point of view, Mills resort to resemblance (similarity, likeness, ...) will only help, if the similarity relations invoked prove to be transitive. Therefore, at this point, Mills way of argument seems somewhat question begging - and thus, as far as it goes - presumably will not decide the controversy in his sense.

Notes

[1]see e.g. my 'Differences in Individuation and Vagueness'[3]

[2]while in this note we confine to Sorites, our larger cognitive interest is in the historical struggle between William Whewell and John Stuart Mill concerning the question, on whether 'natural groups are given by type, not by definition', which amounts to a discussion on whether concepts 'incapable of definition' are and may be - by best scientific practice - 'given by type' in some epistemic situations (the reference is to 'natural history', the discussed example a taxonomic issue in botany)[6], Chapt. II, §§ 9 ff. [pp. 121 ff.]. Mill, in making his case for 'given by definition', refers to 'resemblance' and 'degrees of resemblance' in addition to 'characters' [5]Chapt. VII, §4, pp. 278 ff . Of course, taxonomic methodology discussion nowadays is on another level, nevertheless this 19th-century discussion seems to deserve a review in the light of contemporary logic.

[3]see e.g. Krantz et alii [4], Def. 2 p.15) and chapt.3, pp.71 ff.

[4] where $\mathbf{d}(\mathbf{x},\mathbf{y})$ is the Euclidean distance of space-points $\mathbf{x},\mathbf{y}$, the value of $\mathbf{const.} > \mathbf{0}$ doesn't matter

[5]'First-Order Logics for Comparative Similarity'[7], pp.461-462

[6]"... A different approach consists in preserving the tolerance principle itself but appealing to a nonclassical logic. The semantics originally proposed by van Rooij belongs to that second family: it allows us to validate the tolerance principle in its plain form, and it is non-classical. The framework rests on the interaction of three notions of truth for sentences involving vague predicates: the classical notion of truth, a notion of tolerant truth, and a dual notion of strict truth. ..."[2], p. 348

[7][1] p.1, and in §17 of her paper Bobzien discusses the relation of her modal 'agnostic' approach to intuitionistic theories of vagueness

[8] "The truth is, on the contrary, that every genus or family is framed with distinct reference to certain characters, and is composed, first and principally, of species which agree in possessing all those characters. To these are added, as a sort of appendix, such other species, generally in small number, as possess nearly all the properties selected; wanting some of them one property, some another, and which, while they agree with the rest almost as much as these agree with one another, do not resemble in an equal degree any other group. Our conception of the class continues to be grounded on the characters ; and the class might be defined, those things which either possess that set of characters, or resemble the things that do so, more than they resemble anything else." Mill[5], Chapt. VII, §4, p.282

References

[1] Susanne Bobzien, *'A generic solution to the Sorites paradox based on the normal modal logic QS4M+BF+FIN' forthcoming in*, ***The Sorites Paradox: New Essays,*** (A. Abasnezhad / O. Bueno, editors), Springer, 2019, pp.01-47.

[2] Pablo Cobreros - Paul Egré - David Ripley - Robert van Rooij, *Tolerant, Classical, Strict*, ***Journal of Philosophical Logic***, vol.41(2010), no.2, pp.347-385.

[3] Wilhelm Grafe, *Differences in Individuation and Vagueness*, ***Structure and Approximation in Physical Theories*** (A. Hartkämper, H.-J. Schmidt, editors), Plenum Press, New York, 1981, pp.113-122.

[4] David H. Krantz, R. Duncan Luce, Patrick Suppes, Amos Tversky, ***Foundations of Measurement, Volume I***, London, Academic Press, 1971.

[5] John Stuart Mill, ***System of logic, ratiocinative and inductive, vol. ii***, Longmans-Greene-Reader-Dyer, London, 1872.

[6] William Whewell, ***History of Scientific Ideas vol. ii***, John W. Parker and Son, West Strand, 1858.

[7] Timothy Williamson, *First-Order Logics for Comparative Similarity*, ***Notre Dame Journal of Formal Logic***, vol.29 (1988), no.4, pp.457-481.