objects are (not) ...

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Abstract

My goal in this paper is, to tentatively sketch and try defend some observations regarding the **ontological dignity of object references**, as they may be used **from within in a formalized language**.

Hence I try to explore, **what properties objects are presupposed to have**, in order **to enter the universe of discourse** of an interpreted formalized language [1].

First I review **Frege′s analysis** of the logical structure of truth value definite sentences of scientific colloquial language, to draw suggestions from his **saturated vs. unsaturated sentence components paradigm**.

Next try investigate, in how far **reference to non pure_math objects** might allow for a role as argument of a truth value function. Object kinds to be considered are: common sense objects, technical objects, humanities object kinds (social, psychical, ...), objects of art, ... , be they abstract, concrete, or in this respect mixed objects.

Then have a comment on the just referenced label **abstract objects**.

Next try to get an idea wrt the ontological significance of the fact, that **pure_math objects and functions** in some important classical cases can be **uniquely defined by means of categorical theories**. Here, in the course of my argument, I have a **little corollary** wrt the standard model of first order Peano arithmetic, reducing the epistemological significance of the existence of non standard models.

Next, wrt a special concept of a **formalized empirical theory**, I care for whether the **impure math objects** and **mixed objects** described here, and complying best with truth value function mapping, are also reliable **candidates for the ontological commitment** of such theories; and discuss an alternative, which reduces the ontological significance of the universe of discourse of the theories intended models.

In the end I **sum up** what is - from my point of view - **the status quo**, according to my respective findings.

[1] consider here formalized languages with two valued standard logic; a partially different prospect might be given from a constructivist account, based on intuitionist logic

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1 introduction

1.1 epistemologically naive uses of 'object' - defense lines

One of the most commonly debated - but as often unduly taken as unproblematic - concepts in philosophical papers is **object**[2]. In contexts, where use of any of

[2]the whole pitifulness of the field is rather looking through the well taken stock in e.g. Francesco Berto and Matteo Plebani's [3], introduction

the words **object**, **(any)thing**, **entity**, ... etc. is taken to be unproblematic, e.g. as kind of coverall term[3], authors seem to rely on :

- a first line of defense - the assumption, that some or other well accepted systematization of formal logic provides a very general but plain sailing specification of what it means to be a usable **object reference** (henceforth for short **obj.ref**.) wrt the semantics of this systematization,

- another typical line of defense - the assumption, that colloquial language used in a common sense way has implemented reference to every day objects in some plain and unquestionable manner,

- a third typical line of defense - the reference to so called 'ontologies' (more or less systematized systems of [types of] objects), as are supposed to be referred to in some or other science or humanities discipline, else are used within in some software environment, especially as implemented using "object oriented" programming languages and/or database systems[4].

All these defense lines, else some more or less unholy interminglings of them, then seem supposed to be rather innocent and epistemologically unproblematic, at least in general. This paper, however, is written under the opposite impression, viz. that neither of these defense lines, nor some other not listed here, is really resilient.

Admittedly more epistemological scrutiny in mainstream discussions is though provided for e.g.

- 'objects' in some supposedly metaphysical sense, sometimes borderlined by 'objects of ontologies' of basic physical theories,

- 'object identity' for 'enduring objects', having 'personal identity' as a special case.

Now, in order to have some benchmark - some tertium comparationis - available, for this paper I restrict considerations concerning the ontological "dignity" of **obj.ref**. to the case of **obj.ref**. from within standard formalized languages. Hence now ...

1.2 the question, what qualifies for belonging to the universe of discourse ?

From a first order logic (henceforth for short FOL) point of view:

rather any two-valued FOL-semantics is - in one way or other (translation semantics, set theoretic semantics=model theory, truth value semantics, ... and what else ...), based on Tarski's semantics architecture for formalized languages, at its core some variant of the recursive satisfaction definition for the well formed

[3]see e.g. [31]

[4]for an perhaps instructive example of the latter kind of work, see e.g. Claudine Métral et alii : "ONTOLOGY-BASED RULE COMPLIANCE CHECKING FOR SUBSURFACE OBJECTS" [29], for the central w3c-standard **RDF** referred to see e.g. [44]. The epistemic point wrt such examples of course is, that their e.g. 'subsurface objects' are already RDF modelled data, not the original objects, say e.g. a certain subway tunnel in London. On the one hand, this of course decreases the epistemic interest wrt such 'objects' and 'ontologies', on the other hand, it's a way to provide seemingly unproblematic **obj.ref**. for non pure_math objects (**n.p_m.o**). Such 'ontologies' seem already widely in use, e.g. for taking stock and administrative purposes based upon, as the example shows. The RDF standard dates from 2004, 2014, and it's reported to be a modernization, else generalization, of earlier data modeling standards, as has been e.g. the entity-relationship (ER)-model, used for database design in the nineties of the past century.

formulas (**wff**) of a standard formal language[5]. Each of these **wff** contains finitely many ($\mathbf{n_{\geq 0}} \in \mathbb{N}$) free object variables, taking in each interpretation of the language $\mathbf{n-tuples}$ of objects from the universe of discourse as arguments, to map the **wff** to some truth value; this is the sense in which each interpreted **wff** occurs as a truth value function[6], shorthand in a symbolic standard notation:

$$ {}^{\mathbf{n}}_{\mathbf{0}}\mathbf{WFF} \times \mathbf{U^n} \;\mapsto\; \{\mathbf{TRUE}, \mathbf{FALSE}\}, \quad \mathbf{n} \in \mathbb{N} \qquad \text{formula } \mathbf{FOL\_TVF} $$

Now when looking through introductory logic textbooks, else through lots of 'philosophy of logic' texts, we meet examples of the quality 'Callias is a greek', 'the square root of 2 is irrational', 'naples is rather on the same latitude as New York', 'Scott was the author of Waverley', or what the hell else. Hence we get a confused, blurred, mazy collection of **obj.ref**.

It's then clear from the outset, that for the truth value function purpose, most of the mentioned **obj.ref**. are rather badly qualified as they stand, because lot's of context information (e.g. space/time information, cultural context information, ...) need be added, or at least presupposed to be in principle available. This then, **to allow the thus referenced object to be added** as an element **to** the universe of discourse **U** for a **sound** (viz. truth value function mapping supporting) **interpretation of a formal language** over universe **U**.

I try explore this somewhat muddy face of the **obj.ref**. coin later in this paper in a separate section on reference to non pure_math objects (**n.p_m.o**).

1.3 abbreviations

as already noted, for sake of shortness I use some abbreviations throughout this text:

obj.ref.	for 'object reference(s)', i.e. for those parts of a supposedly truth value definite sentence, which purport to refer to objects
p_m.o	for pure_math object(s), viz. for objects from a structure of pure mathematics
n.p_m.o	for 'non pure_math object(s)', viz. for all other objects [7]
FOL	for 'first order logic' [8]

[5]the classical source is Tarski [35]

[6]truth value functions, taking objects as arguments, to map to a truth value - this, for short, is Frege's way of explaining the working of truth value definite sentences. Expand this in the next section.

[7]Ketland in his "Foundations ...", [25] section 1.2 "Mixed Predicates; Mixed & Impure Objects" introduces a more detailed and more technical distinction [pure_math vs. impure_math objects, mixed functions and predicates], as well narrower, because he relates this distinction only to a certain logical form, viz. application signatures $\mathbf{ZFCA}_\sigma$ in FOL, he is about. My distinction [pure_math vs. non pure_math objects] here is, while somewhat broader and less definite, rather of the same intention, and both approaches (Ketland's and mine) rather share the paradigm case of pure_math objects.

[8]A first order logic system is given by specifying a standard formal language (with functional signs and identity), including logical connectives and quantifiers for any sort of object variables (sometimes more than one sort of object variables is specified), and logical axioms and deduction rules appropriate for a two-valued semantics of this language.

HOL	for 'higher order logic' of finite order [9]
wff	for referring to some well formed formula(s) of the formal language of a **FOL** or **HOL** system
$^{\mathbf{n}}_{\mathbf{m}}$**WFF**	the set of all **wff** of a **FOL** system with −**arity** lowbound **m** and highbound **n**. lowbound **m** omitted, means only −**arity n**

2 truth value functions and objects in Frege - and related issues

2.1 Frege's logical analysis of sentences

As already indicated by accentuating the truth value function role of logical predicates (**wff** containing **n** free object variables), I start for finding an minimal, but resilient, account of 'objects' from Gottlob[10] Frege's analysis method for [truth value definite] sentences.

Frege′s first and main assumption is, that by [not necessarily unique[11]] logical analysis such sentences can always be split up in a saturated part (one or more names and/or definite descriptions - referring to objects -) forming the logical subject, and an unsaturated frame, the logical predicate clause (referring to some concept or relation), showing placeholders for the respective **obj.ref**. For this paper, I **adopt this first assumption**. Frege communicates this view in most simple terms in his letter to Marty[12], written two years before his publication of 'Grundlagen der Arithmetik ...' in 1884:

"In diesem Falle, wo das Subjekt ein Einzelnes ist, ist die Beziehung von Subjekt und Prädikat nicht ein Drittes, das zu beiden hinzukommt, sondern sie gehört zum Inhalte des Prädikates, wodurch dieses eben ungesättigt ist.Ich glaube nun nicht, dass das Bilden der Begriffe dem Urteilen vorausgehen könne, weil das ein selbständiges Bestehen des Begriffes voraussetzte, sondern ich denke den Begriff entstanden durch Zerfallen eines beurteilbaren Inhaltes.Ich glaube nicht, dass es für jeden beurteilbaren Inhalt nur eine Weise gebe, wie er zerfallen könne, oder dass eine der möglichen Weisen immer einen sachlichen Vorrang beanspruchen dürfe. In der Ungleichung 3>2 kann man ebensowohl 2 als Subjekt ansehen wie 3. In dem ersteren Falle hat man den Begriff "kleiner als 3", in dem letzteren "grösser als 2". Man kann auch wohl "3 und 2" als ein complexes Subjekt ansehen. Als Prädikat hat man dann den Beziehungsbegriff des

[9] A higher order logic system is like a first order logic system, but in addition contains higher order variables and quantifiers (for sets, predicates, and functions of each addressed type level), and hence logical axioms and deduction rules are accordingly enlarged or accustomized. In HOL systems, "identity of indiscernibles" is expressible with quantifiers of order $\mathbf{n+1}$ for objects of order $\mathbf{n}$, viz.

$$\boxed{\bigwedge_{(^{\mathbf{n+1}}\mathbf{F})}[\bigwedge_{(^{\mathbf{n}}\mathbf{x}),(^{\mathbf{n}}\mathbf{y})}(^{\mathbf{n+1}}\mathbf{F}(\,^{\mathbf{n}}\mathbf{x}\,) \rightarrow \,^{\mathbf{n+1}}\mathbf{F}(\,^{\mathbf{n}}\mathbf{y}))\,) \rightarrow \,^{\mathbf{n}}\mathbf{x} = \,^{\mathbf{n}}\mathbf{y}\,]}$$

[10] once it was a smiling annotation (by pronounciation only) for the German speaking community, that Friedrich Ludwig Gottlob Frege's third given name rather means 'praise the Lord'

[11] see below the letter to Marty [17], also cf. [12] p.199

[12] cf. [17] pp. 163-165

Grösseren zum Kleine[re]n. Allgemein stelle ich das Fallen eines Einzelnen unter einen Begriff so dar: F(x), wo x Subjekt (Argument), F( ) Prädikat (Funktion) ist, die leere Stelle in der Klammer nach F deutet das Ungesättigtsein an." [17] p. 164.

More detailed we meet Frege's generalization of the 'function' concept in 'Funktion und Begriff'[13]. Starting with arithmetical function examples he explains:
"... das eigentliche Wesen der Funktion liegt; d. h. also in dem, was in

$$\mathbf{2} \times \mathbf{x}^{\mathbf{3}} + \mathbf{x}$$

noch ausser dem **x** vorhanden ist, was wir etwa so schreiben könnten

$$\mathbf{2} \times ()^{\mathbf{3}} + ().$$

Es kommt mir darauf an, zu zeigen, dass das Argument nicht mit zur Funktion gehört, sondern mit der Funktion zusammen ein vollständiges Ganzes bildet ; denn die Funktion für sich allein ist unvollständig, ergänzungsbedürftig oder ungesättigt zu nennen. Und dadurch unterscheiden sich die Funktionen von den Zahlen von Grund aus. ..." Frege[13], p. 6

After having dealt with the notions 'function', 'argument of a function', 'value of a function for this argument', 'course-of-values of a function', Frege argues that the same schema applies to concepts as truth value functions :
"Wir sehen ... , wie eng das, was in der Logik Begriff genannt wird, zusammenhängt mit dem, was wir Funktion nennen. Ja, man wird geradezu sagen können : ein Begriff ist eine Funktion, deren Wert immer ein Wahrheitswert ist." [13], p.15 ['Begriff'/'concept' in this context is here to be understood broadly to mean n-ary [$n \geq 1$] relations].

2.1.1 historical digression 1 - the predication relation

Bertrand Russell, in his 'presidential address to the (Aristotelian) society' of 1911, dealing with "... the relations of universals and particulars"[13] is promoting as an alternative, what he calls the '(asymmetric) relation of predication'[14]. By this Russell takes a move that seems suited to thwart Frege's 'unsaturated' analysis. But as closer inspection shows, there would be a simple line of defense in favour of Frege's way of putting things: **either** the predication relation itself showed a property of being unsaturated, asking for a logical subject [say, a particular and a universal both] in subject position and the predication relation in predicate position; [15] **Or**, the unedifying alternative, we were in need to resort to a fourth logical element, viz. some relation or function operation, that conjoins these three [logical subject - predication relation - universal] and so on ad infinitum, else stopping on some stage with some unsaturated entity, some kind of frame.

[13] and in this context without explicit mention of, nor attention to, Frege's analysis of predicates as unsaturated parts of truth value definite sentences

[14] [33], pp.108f.

[15] and presumably Russell, at the time co-author of Principia Mathematica, vol. I, would have hurried to agree, that in his construct (only ad hoc sketched in [33] in order to argue for the existence of universals) the sign for the predication relation itself were the 'incomplete symbol' requested

2.2 Frege and reference to objects

Frege shows great tolerance wrt what may count as an object. In 'Funktion und Begriff' he notes:
"Wenn wir so Gegenstände ohne Einschränkung als Argumente und als Funktionswerte zugelassen haben, so fragt es sich nun, was hier Gegenstand genannt wird. Eine schulgemässe Definition halte ich für unmöglich, weil wir hier etwas haben, was wegen seiner Einfachheit eine logische Zerlegung nicht zulässt. Es ist nur möglich, auf das hinzudeuten, was gemeint ist. Hier kann nur kurz gesagt werden: Gegenstand ist Alles, was nicht Funktion ist, dessen Ausdruck also keine leere Stelle mit sich führt."[13] p.18

The viability of this liberalism may of course be rather shortlegged, when applied to non pure_math objects (**n.p_m.o**), see next section.

Unquestionably, Frege's understanding of truth-value definite sentences has as a presupposition a **second assumption**, viz. that the names and/or definite descriptions occurring in truth value definite sentences (i.e. the **obj.ref.**) are, what Kripke a century later called 'rigid designators' [16]. And it doesn't really matter, that Kripke's rigid designators are meant for the sake of interpreting modal operators to ensure identity of designated objects through a set of 'possible worlds', because Kripke semantics for modal systems is a special application of modeltheory, viz. his 'possible words' are rarely different from relational structures (what possible models of theories are called in model theory), and the modal operator interpretations, Kripke introduces, rely on relations between them. Hence[17] I feel free to use Kripke's term 'rigid designator' so far in my context here. And again, it's not by accident that Kripke in the above cited passage from 'Naming and Necessity' replays Frege's struggle with the 'identity statements' puzzle 'if true, then necessarily true' [18].

For my way of argument here, I'll have to **weaken this second assumption**, because in general it involves much too much reliance on the validity of some already established ontology [of objects], e.g. some common sense ontology, some math object Platonism, or whatsoever. That is to say, this second assumption taken as it stands, were question begging in a context of revisiting the nature of justifiable uses of 'object'. **Nevertheless we have to keep the requirement** of **obj.ref.**: being able to serve in truth value definite sentences as arguments for truth value functions, i.e. as (parts of) the logical subject of a logical predicate.

2.2.1 logical form of object references (obj.ref.) in FOL

A note on logical form seems in place, for sake of simplicity again I restrict considerations here to **FOL**:

[16]"... it was already clear ... - without any investigation of natural language - that the supposition ... that *objects* can be 'contingently identical', is false. Identity would be an internal relation even if natural language had contained no rigid designators. ..." [26], p.4 , cf. also e.g. p.21 note 21 on rigidity de iure etc.

[17]relations between models (though not in modal logic context) as kind of benchmark for rigid vs. non rigid designation are referred to in the **p_m.o** section

[18]Frege put it rather that way: **a** = **a** looks a priori true, but what about **a** = **b**, [14] pp.25 ff

Wrt an **interpreted standard 1st order language** (1-sorted, including function signs and identity) to mention here:

(α) relating to the **FOL_TVF** formula given above, one may picture Frege's analysis of truth value definite **FOL**-sentences as analysing

$$\boxed{{}^{\mathbf{0}}\mathbf{WFF} \times \mathbf{U^0} \mapsto \{\mathbf{T}, \mathbf{F}\}} \text{ in terms of } \boxed{{}^{\mathbf{n}}_{\mathbf{1}}\mathbf{WFF} \times \mathbf{U^n} \mapsto \{\mathbf{T}, \mathbf{F}\}}, \ \mathbf{n} \in \mathbb{N} - \{\mathbf{0}\}$$

(β) a **truth value function** (excluding the sentences, which are Frege's analysandum) appears hence syntactically as any open **wff** of the language, i.e., any **wff** with a least one object variable occurring free in it, i.e., not bound by a quantifier

(γ) **object references(obj.ref.)** appear syntactically as

($\gamma\alpha$) all in a **wff** by some quantifier **bound** occurences of **object variables**

($\gamma\beta$) **terms** embedded in a **wff**, with all object variables - occuring in them - bound in the embedding **wff**,

($\gamma\gamma$) **proper names** for objects, in formal languages often specified as 0-ary function symbols, else as constant symbols.

($\gamma\delta$) **definite descriptions** (the unique x, such that ...)

However, it's important to realize, that **object variables**, **occuring free in a wff**, **are not obj.ref. in this wff**. A variable free in a **wff**, is simply a **placeholder** for an **obj.ref.**, i.e., as already stated above in (β), such a **wff** denotes a truth value function.

This point,which may be called the lack of semantic definiteness of object variables occuring free in a **wff**, imop is important to understand the technical impact of the saturated/unsaturated distinction; and it's faithfully reflected by the **universal closure metatheorem**.
This metatheorem is of interest because of the observation, that

$$\Vdash \bigwedge_{\mathbf{x}} \mathbf{F}(\mathbf{x}) \rightarrow \mathbf{F}(\mathbf{y}), \text{ but } \nVdash \mathbf{F}(\mathbf{y}) \rightarrow \bigwedge_{\mathbf{x}} \mathbf{F}(\mathbf{x}),$$

the metatheorem states an metatheoretical equivalence, which is somewhat weaker, than were the (but invalid) FOL logical equivalence $\nVdash \mathbf{F}(\mathbf{y}) \leftrightarrow \bigwedge_{\mathbf{x}} \mathbf{F}(\mathbf{x})$

The metatheoretical equivalence is

$$\boxed{\Vdash \bigwedge_{\mathbf{x}} \mathbf{F}(\mathbf{x}) \text{ iff } \Vdash \mathbf{F}(\mathbf{y})} \qquad \textbf{universal closure metatheorem}$$

2.2.2 historical digression 2 - only one type of object(-reference)s

Frege allows for functions (including truth value functions = concepts) of first and maybe any finite higher order, but only one type of objects. Hence one would expect sets (Begriffsumfänge) to be objects in the same logical category as truth values, the morning star and Frege's writing desk.

Oswaldo Chateaubriand paints a clear and convincing picture of Frege's view in this point:

"Frege's ontology can be interpreted as a hierarchy of functions and objects. Objects appear exclusively at the bottom level 0 and are characterized as everything that is not a function. Functions appear at all higher levels and have as arguments and values entities of lower levels. Frege's fundamental distinction between functions and objects is that the former are incomplete and the latter complete. He also says that functions are unsaturated whereas objects are saturated. An object can be the reference of a name and a concept can have a somewhat analogous relation to a predicate. Concepts, including relations, are functions which applied to (or saturated by) appropriate arguments give as value (or result in) one of the two truth values. ... The extensions of functions are logical objects for Frege, and therefore belong at level 0.With this move he "reflected" the whole hierarchy of functions into level 0, which allowed him to formulate his logical account of arithmetic as an account of logical objects but which was also instrumental in the eventual collapse of his system. It is important to realize, however, that Frege's claim that arithmetical concepts are logical concepts is quite independent of the move through extensions. ..." [19]

This linkage to the "collapse" is even reflected by Frege's himself - only three months before his passing away. In his letter to Hönigswald of April 1925 wrt Russell's paradox he denies sets, understood as concept extensions the status of an object: "... Das Wesentliche dieses in ein Gestrüpp von Widersprüchen führenden Verfahrens ist in folgendem zusammenzufassen. Man sieht die unter F fallenden Gegenstände als ein Ganzes, einen Gegenstand an und bezeichnet es mit dem Namen "Menge der F" ("Begriffsumfang von F", "Klasse der F", "System der F" usw.). Man verwandelt hiermit ein Begriffswort "F" in einen Gegenstandsnamen (Eigennamen) "Menge der F". Dieses ist unzulässig wegen der wesentlichen Verschiedenheit von Begriff und Gegenstand, die freilich in unseren Wortsprachen sehr verdeckt ist. ... Der Ausdruck "der Umfang des Begriffes F scheint durch seine vielfache Verwendung so eingebürgert und durch die Wissenschaft beglaubigt zu sein, dass man es nicht für nötig hält, ihn genauer zu prüfen; aber die Erfahrung hat gelehrt, wie leicht man dadurch in einen Sumpf geraten kann. Auch ich gehöre zu den Leidtragenden. Als ich die Zahlenlehre wissenschaftlich begründen wollte, war mir ein solcher Ausdruck sehr gelegen. Zwar kamen mir zuweilen bei der Ausarbeitung leise Zweifel, aber ich beachtete sie nicht. So geschah es, dass nach Vollendung der Grundgesetze der Arithmetik mir der ganze Bau zusammenstürzte. ... Eine weithin sichtbare Warnungstafel muss aufgerichtet werden: niemand lasse sich einfallen, einen Begriff in einen Gegenstand zu verwandeln! Hiermit mögen die Paradoxien der Mengenlehre zunächst abgetan sein! " [16], pp.86 f.

Hence, wrt sets (concept extensions, 'Begriffsumfänge') we have two variants in Frege's work:

first variant, and from the history of science view point up to today the more important:

from 'Grundlagen' to 'Grundgesetze', 'Begriffsumfänge' [= concept extensions

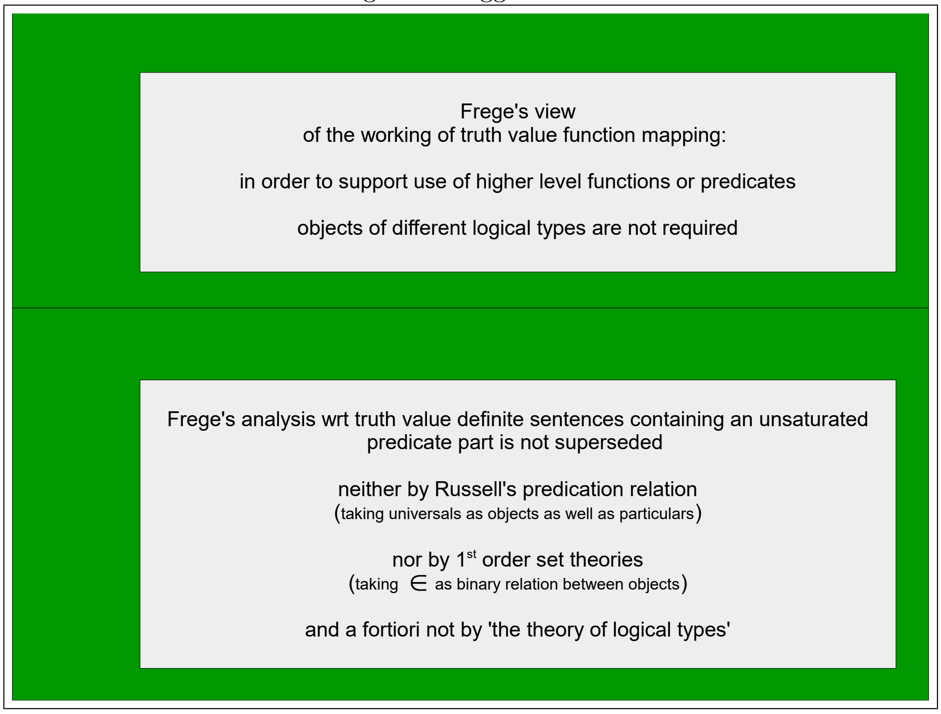
[19] Chateaubriand [5] p.297

= sets] are accepted as objects, but are never 'objects of a higher level' [as was but insinuated later in e.g. **STT** (simple theory of types)[20]], because with Frege's approach, objects do not differentiate into levels or layers (esp. something like the 'cumulative hierarchy (of objects)' is far from being considered here)[21] ;

second variant - and especially in his above cited letter of 1925 -

we get as, say, the great logician's last legacy[22], that set theory is wrong from the outset, is a path to be strictly avoided [at least as far as it were based on (even only moderate) comprehension else separation axioms]. This is a legacy which is, as far as I know, hardly reflected in the literature. Maybe, because nobody dared to oppose David Hilbert's commitment (also from 1925) "Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können."[23]

Figure 1: suggestions



Imop it's not only a historical annotation but systematic challenge for the philosophy of logic, that after seemingly many mathematicians, and admittedly Bertrand Russell himself[24] found the Simple Theory of Types **STT** (acknowledging objects of any finite type) somewhat cumbersome, the community afterwards saw the triumphal procession of first order set theories[25], the winner

[20]see e.g. Copi's book, covering simple as well as ramified type theory[7]

[21]Frege's ancestorship wrt the simple theory of types is controversely debated, see e.g. Bruno Bentzen[1] section 3 object and function types

[22]which we perhaps should not depreciate before thinking it over

[23][23], p.170, "No one shall drive us out of the paradise which Cantor has created for us." as translated by Erna Putnam and Gerald J. Massey

[24]e.g. in his in [32](1919), pp. 135 ff.

[25]some of them acknowledging classes, which are not sets, as a second sort of object

in the long run of this competition up to now, **ZF(C)**, again allows only for one sort of objects, viz. sets [of course save 'urelements' for applications]; and this for the high price of weakening comprehension to separation, and, of course the only relation except identity, viz. $\in$, is a simple two place predicate, which again, according to Frege's lasting analysis, is unsaturated.

3 non pure_math objects (n.p_m.o)

non pure_math objects (**n.p_m.o**) shade rather in the spectrum between

- objects which show (some) builtin math structure,
e.g. objects of science (e.g. masspoints, velocity vectors, ...), technical (hardware) objects, data models, software designs, music details (♪, ♩, metronomes, rythm, cadence, ...), paintings by applied techniques (golden ratio, perspective, ...), economic, finance, and administrative objects, statistic objects, traffic rules, geographic locations(latitude, longitude), ... open list ...

- and objects which don't,
e.g., objects, we're acquainted with (one's dog[26] , one's favourite songs, ...), objects we've a qualitative description only (a recommended shop, the dish of the day, ...), objects we perceive (sunsets, ...), ... open list ...

Anyway, in this section I consider them rather altogether, only observe the demarcation line to pure_math objects (**p_m.o**) as the frontier.

Well, whether or not we accept Frege's analysis of truth value definite sentences as a basically correct account (and I think, we were well advised to do so), now when trying to prove the case of mapping indicative sentences via an assigned logical structure to $\{\mathbf{T}, \mathbf{F}\}$, we may soon get prompted to address obvious difficulties:
at least outside math's and science, we face the issue that neither the n **obj.ref** nor the $n-ary$ relation predicated to hold between, really have the logical properties required to do this job, viz. to form truth value definite sentences, i.e. that the n **obj.ref** as arguments would map the $n-ary$ relation (truth value function !?) to a truth value **T** or **F**.
How we perhaps might cut corners to streamline predicates in this respect, and the respective epistemological tradeoff, I've already dealt with in two earlier papers[27]. Now, for sake of argument let's suppose the streamlining on the predicate side already done and ask, what remains to be done on the **obj.ref**. side.

3.1 resuming case study 'weight pieces'

Let's start from the predicate streamlining example from [21], section "2.2.2 basic calibration". This example is about adjusting a set of weight pieces (for

[26]while the marketing consultant of the regional pet food distributor, in his predictive 'pet food sales' data model, might like to account for 'Fido' and his fellows for some relevant details ☺ ☹
[27][20], [21]

a beam balance) of purportedly equal weight, by omitting pieces which do fail some transitivity test, to the end to make the relation 'equal weight' in the resulting subset of weight pieces truly an equivalence relation. Hence, this was an adjustment of an object set wrt the relation 'x,y have equal weight'. Now it were conceivable, that a different subset of the original set of weight pieces were to be selected, if we instead or additionally wanted, to make the relation 'x,y are made of the same iron-nickel alloy' an equivalence relation, provided some respective technical test were available for comparing the material, the pieces are made of. For suppose, some of the weight pieces were made of stainless steel including chrome, others not. Now, what were we to do, in order to have an object set fit for predication wrt the two independent equivalence relations. Seems, the only possible reaction might be a brute force reaction, viz. take the intersection of the two subsets. The trouble of course may be, by selecting more and more equivalence relations to be predicated from this object set, in the end the resulting subset might be unedifying small, might be a unit set, might even be empty. But, were there any rationale in proceeding this way? The fact, that we might end up even with the empty set, shows, that also a defense line 'individual concept' won't be convincing. Wrt (fuzzy) object sets, which are candidates for natural kinds, we will hope, that truth value definite predication may give us a non empty and more or less stable subset, but this - if and where it happens - will be an empirical result, meaning that our initial object set choice was significant, else, that respective properties (equivalence relations) hence promise some inductive potential.

For a different perspective now I turn to ...

3.2 another example

When once (decades ago) I had had the privilege (and as well the burden) to deliver the logic introductory courses , my first and absolute favourite example for a truth value definite(!?) sentence, thus an exercise for logical analysis and formalization I had created for the beginner's class[28], had been

"die Erde ist im Winter der Sonne näher als im Sommer" [in winter(time) the earth is closer to the sun than in summer(time)" "summertime" example

This example of course may be analysed at different stages of logical scrutiny. So let's have a take to try something more, than may fairly be demanded from beginners in 'formalize':

- 'winter(time)' here is indexed for the earths northern hemisphere, and may be dated either as meteorological wintertime from (December,1 to the February 28th, else 29th of the following year, depending on whether we have a 'leap year'); else as astronomical wintertime from December solstice to March equinox, which both vary in the years with their exact day dates,
- and correspondingly adjustments had to be done wrt 'summer(time)',
- the implicit year-reference of summertime, wintertime might be taken to be to one year, e.g. the current year the utterance was done; but, we would rather expect the reference to be meant as 'every (!?) year' ,

[28] having used it then but only once, I may be excused to reuse it here

- to be honest, in order to determinate our selected seasons time segment(s) really situation-independent, we would even need anchor these time segment(s) to a reliable start date; viz., astronomers might give us (theory dependent) an approximate estimate of the time coordinate for the year, to which we refer in our statement [perhaps "some time" after (last ??) big bang]. Well, presumably nobody would try seriously this way, but one would e.g. stipulate some 'now' (if only, say, the Gregorian(?) calendrical start of year 1 C.E.) as a rigid designator,

- and of course we were the more at a loss for a comparable space location anchoring, instead had to stipulate that at least one of the respective **obj.ref.**, e.g. 'the sun', were something like a rigid designator, at least for the respective seasons time segment(s) of the year(s), referred to in our statement, i.e. embedded in the larger astronomical time segment, our planetary system was created and stabilized to more or less current values and may continue to perform so for a while,

- now, even if we successfully identify in whatever accepted way the respective time segment(s) and locations, we will need additionally a value of the respective average distance of the earth to the sun - and we would have to know a lot of things and/or would have to rely on lots of reported information to get there, having started with a seemingly simple assertion in colloquial language. But this last duty - fortunately - seems manageable, at least in principle.

This now may well be enough for a first take on this example. That the ontological dignity of the indicated stipulations of rigid designators is questionable, goes - I think - without saying (any more). If we select a Fregean sentence analysis, according to which 'the sun' and 'the earth' are the **obj.ref.**, forming the logical subject, and the remaining rest of the sentence the (unsaturated) truth value function = predicate, we may note, that this kind of object reference appeals to not much more than two masspoints, related by some basic facts of "celestial kinematics". Hence this kind of object reference seems far from rigid designation, and wrt the mass point picture even counterfactual.

3.3 interim balance ?

Now I return to the general question, of whether (and if, how) we might identify minimal requirements for non pure_math objects (**n.p_m.o**), whose fulfillment would enable them to appear in the universe of discourse of a formalized (two valued) language, i.e. to be able to be referred to as arguments of a truth value function, given by an interpreted **wff** of that language.

Reconsider the confused, blurred and mazy variety of object types, now wrt typical ways, we get them to know:

- common sense objects: by deixis, by paradigm and similarity , by acquaintance, ... [in an introductory textbook in logic once[29] was given the example: try to define 'shoes']

[29] sorry, decades ago, and now don't remember author and title

- technical objects: by deixis, by paradigm and similarity, by technical specification, by software, by user's manual, ... [examples from ancient hoist to data in processor storage]

- social objects, e.g. social rules/contracts/laws/expectations/ritualized behavior/... : from everyday life experience, from theories and/or field studies in humanities, ...

- psychical objects: sometimes direct testability, often indirect confirmation by assuming, testing or proving construct validity [30], ...

In reviewing the weight pieces example above I suggested, that there won't be much use - except in special cases - in cutting object sets smaller and smaller in order to increase the number of independent predicates usable for truth value definite statements about the remaining object set.

But of course some corner cutting procedere for fuzzy object sets may lead to acceptable results, as long, as the identity of the relevant mutually independant predicates is known and their number is finite (and "small"). Which may be the case e.g. in traditional data modelling environments, at least in their synchronous view.

Just the opposite seems the case in everyday life considerations, and often enough in humanities or science [see the example above]. It seems instead, that it are just the more or less well known individuation and behaviour of established, but fuzzy object sets, which give rise to new questions, new suggestions, checking for the significance of new functions, predicates, algorithms, hypotheses. And of course, new hypotheses may lead to adjusting membership and/or description of an existing object set. Hence it seems, that it is often just the non rigidity of object designators, that helps keep research language (and as well ordinary language) alive.

But how does this comply with the requirements of applying standard logic,when object sets are fuzzy and/or predicates vague ? In a strict way, of course, it doesn't [comply]. Nevertheless, application of standard logic is sometimes found useful, as long as the participants of a respective discussion are in control of using logical consequence relations only in the relevant intersections of clear cut cases wrt objects and truth value functions[predicates]. But, thus far, relying on the validity of such logical reasoning is like dancing on an eggshell.

3.4 trying to get a bit better - by looking beyond philosopher's wheelhouse

Now again, are there perhaps any properties of objects(**n.p_m.o**), which support at least to some extent, that **obj.ref.** to them may work for truth value function mapping, and is there a kind of systematic approach for classifying such properties ?

Respective properties of objects, which come to mind rather immediately are
- that an object can (e.g. operationally) be delimited against its environment

[30] for the concept see e.g. [10], and, for psy-nerds, presumably an interesting (concept at work) study example were Sullivan et alii [34]

- that an object has a certain durability(say, for a quantifiable time segment), at least in special environments, e.g. that the objects keeps to some extent its shape, ...
- that an object is identifiable (allowing for true non trivial identity statements), e.g. that the object is recognizable (e.g. in different positions, or, at different times)

In looking for such properties of objects it may be useful to consider analoga from elsewhere, and what I suggest now is, to compare the search for respective properties of objects, with a somewhat corresponding approach in the theory of transaction processing (**TP** for short) in **IT**. Here the aim is to get an idea, of what it does mean for an organized compound of data processing steps, to form a transactional unit. So let me draw attention to - what in this field is called the **A.C.I.D** properties[31]. These, in telegram style (and mostly in my words) paraphrased at the lefthand partition of **figure 2**. To this **TP** - account of 'transactional unit', some structural analogies wrt philosopher's 'object' wheelhouse seem visible; the more, as **n.p_m.o** regularly exhibit more 'process' than 'thing' character[32]:

Hence, at the right hand side of this figure I try (of course cum grano salis only) a transposition of the transactional A.C.I.D. properties into analogous object properties.

Figure 2: A.C.I.D. transposition - does it make some sense ?

IT transactional unit - properties A.C.I.D.	are they transposable to **n.p_m.o.** objects ?
A.tomicity (of transactional units) the respective set of data processing steps, appears to concurring processes in the environment as a single step, non interruptible.	A.tomicity (objects - stipulations for referability) object is presupposed: to be either simple (meaning, objects internal structure irrelevant and/or not accounted for), else, if complex, object is uniquely decomposable into simple objects
C.onsistency (of transactional units) a valid transaction transforms its environment from a consistent state into an again consistent state,On the other hand, a failing transaction backs out all hitherto done updates.	C.onsistency (objects - stipulations for referability) object is presupposed: to be (re-)identifiable, recognizable, else object is omitted from reference
I.solation (of transactional units) a transaction need run - in a system of concurring transactions - as if there were no concurring transactions, especially it does not share any data with them	I.solation (objects - stipulations for referability) object is presupposed: to be delimitable against its environment, especially no "overlap" or "intersection" with other objects, hence to be discrete, hence to be countable
D.urability (of transactional units) requires, that results of transactions having completed successfully must not be forgotten by the system	D.urability (objects - stipulations for referability) object is presupposed: to exist in all time segments, wrt which it may be referenced

[31] cf. eg. Jim Gray and Andreas Reuter "Transaction Processing ..."[22], section 4.2.3 'flat transactions' pp. 165 ff. A transaction is here categorized as a special kind of a 'protected action'

[32] there is an instructive discussion (following A.N. Whitehead's footsteps) in the philosopy of biology of John Dupré , see e.g. his (2023)[8]

Of course, this right hand side of figure 2 forms a very provisional list only, with single purpose to propose a direction of work. The list just will need be extended, amended or replaced in whatever way seems suitable. However, one proposal is, to try these four object suitable for **obj.ref**. stipulations as the start of an open checklist, to apply when non pure_math objects (**n.p_m.o**) shall be referred to from within a formalized language; and to add to and/or modify the list, as seems suitable for the application case in question. Though, we can't expect to approach that way logically perfect **obj.ref**. The very best, we can hope, imop is, to reach for special argumentation situations **obj.ref**., stable enough to support judgment wrt a finite and manageable set of truth value definite sentences. But any change of the argumentation situation, especially the adoption of additional sentences to be considered, may necessitate to review the adequacy of agreed stipulations and accepted presuppositions.

An interesting current approach to visual object recognition from deep learning, which but is not the place to be considered here, may be found e.g. in Ayuns Luz and Abboby Alih's "Enhancing Object Detection with Object Attributes in the Detect and Describe Framework", for a commented list of examples of considered object attributes cf. [28], p.6 f.

4 note on abstract objects

Now let's have a look on the topic of abstract objects, as wrt these the situation may be quite different, as the question of their ontological dignity need be considered in another way.

The detailed, well informed and informative, and for all this helpful SEP-article on 'abstract objects' starts

"One doesn't go far in the study of what there is without encountering the view that every entity falls into one of two categories: concrete or abstract. The distinction is supposed to be of fundamental significance for metaphysics (especially for ontology), epistemology, and the philosophy of the formal sciences (especially for the philosophy of mathematics); ... " [33].

Of course, it is. And it's not by accident, that Frege's views are by the authors at many occassions used as a tertium comparationis [34]

Nevertheless I do not follow the main line of this paper, which is to focus on looking for approaches drawing some or other demarcation line between 'abstract' and 'concrete' objects, elaborating on their contrast. This for reason, that imop such an enterprise were kind of a lost game from the beginning: as both labels not only need be introduced independently from one another, but also need be sketched by incommensurable features. The more, such looking for a demarcation line imop tends to distract perception from the importance and ubiquitousness of the mixed cases. Hence my slightly different approach is, without any intention to detect or create a demarcation line, to consider samples of all kinds: of clear cut examples of abstract and of concrete objects - but as well of clear cut (!) examples of mixed cases; and, to consider these all wrt their respective suitablility to serve as targets of **obj.ref**. from within

[33] [9], p.2

[34] referring to [11] and [15]

formalized languages[35].

Examples of the latter two (concrete and mixed) I discussed already in the previous section on **n.p_m.o**; e.g. the weight pieces example imop is clearly a (wrt 'abstract vs. concrete object') mixed case[36], and this more detailed because:
(α) we do not put abstract weight pieces on an abstract scale, but
(β) we do put, say, carefully calibrated concrete weight pieces on an also carefully calibrated concrete scale, but
(γ) by doing this we instantiate/apply/realize an ideal basic measurement structure, whose core is defined in purely math else logic terms
(δ) and to serve in this function is usually the only purpose, these artefacts (weight pieces and scale) were planned, constructed and produced for.

Now I turn to clear cut cases (if any) of 'abstract objects', and consider examples of pure_math objects **p_m.o** in the next section. From these considerations it will also (hopefully) become evident, that the **p_m.o**-property, else, the property of being a clear cut case of an 'abstract object', does not entail nor exclude being referable by rigid designation; but that within the realm of abstract objects in mathematics we may find both cases, and again rather by degree than by dichotomy.

In my 'philosophical views', wrt 'abstract objects' I gave a tripartite answer[37]. First, from the provided answer boxes I selected both, 'Reject nominalism' and 'Lean against platonism', and decided to add an explanatory comment: 'abstract objects may be said to exist, but this existence is a dependent one'. Here I try to motivate this stance considering some hard core abstract objects, taken from the holy grail of standard math's

5 pure_math objects (p_m.o)

In order to get an impression, what **reference to math objects** may look like from the viewpoint of formal logic, philosophers imop should risk at least a short glance into logicians showrooms. This may help, even if logicians showcases are only partially understood. Then, for the endeavour of trying to understand **obj.ref**. in math's, a viable path might be to look at showcases promoting the labels *categorical theory* and κ – *categorical theory*

5.1 categorical theories (HOL)

It is known already from the early (typetheoretic) stages of set theoretic semantics for formalized languages, that important classical mathematical structures

[35] my access to the topic of 'abstract objects' then matches only tangential the kaleidoscope of views and opinions reported and discussed in the SEP article, viz. wrt two section headings, viz. "3.5.4 The Discernibility / Non-Duplication Criteria" only with reservations, "3.7 The Ways of Weakening Existence" match only in the headline, not really in the views reported

[36] another clear cut mixed case of course is e.g. a marriage, as are (m)any other social and/or legal compounds

[37] as a public respondent in the philsurvey 2020, https://philpeople.org/profiles/friedrich-wilhelm-grafe/views

like the theory of natural numbers [$\to \mathbb{N}$], the theory of algebraically closed ordered fields [$\to \mathbb{R}$], Euclidean geometry and some more are categorical, viz. it's known for each of these theories, that their models are mutually isomorph, will say, all models of a categorical theory exhibit exactly the same mathematical structure, or, as the case is also pronounced, their models are identical up to isomorphism. As a concequence, for such theories their objects (e.g. the natural numbers) and their relations and functions are uniquely and completely defined by some axiom set. Hence this is rather a theoretical maximum for axiomatic mathematics, and as far, also a maximum of abstract object identification. E.g., the well known (categorical) Peano axiomatization in second order logic (with full induction) defines its object set (its universe of discourse), the set of natural numbers $\to \mathbb{N}$ uniquely [38]; a more contemporary sketch of the categoricity of second order Peano Arithmetic may be found e.g. at the open logic project [6], pp. 367-370; categoricity of Euclidean geometry is discussed in [37].

Hence, categorical theories are defining implicitly but uniquely their object set, functions and predicates, are then for this reason paradigms for rigid designation (rigid **obj.ref**. from within a formalized language) in math's. And even so in physics, wrt classical mechanics, if we adhere to an extreme 'realism' wrt the ontological interpretation of theories of mathematical physics. This last consequence of 'realism' in physics is discussed by Tarski, relating the case to the strict categoricity of metrical Euclidean geometry, giving the base for classical kinematics etc. ([36], pp.96 ff.). Of course, this also were a topic of its own.

5.2 κ – categorical theories (FOL)

The situation is **different for formalized theories in 1st order logic**. Due to the Löwenheim-Skolem-Tarski theorems(downwards/upwards, cf. e.g. [43]) any consistent theory admitting infinite models has also models of differing cardinalities, and already for this reason can't be categorical[39]. Hence the interesting question is, whether such theories and which, are categorical for some cardinality, especially, for the cardinality of its intended model. This question is addressed by the FOL-notion of κ – *categoricity*. The take in figure 3 is from H. J. Keisler's Chapter 'Fundamentals of Model Theory'. Without going into any detail, this (weaker) categoricity-concept in first order logic seems though plenty good enough for providing examples of rigid designation of pure_math objects (**p_m.o**). Some of the theory examples may be comprehensible to philosophers without special math skills, a reliable base in elementary logic will be sufficient; e.g. the ω dense order example (think of the set of rational numbers $\mathbb{Q}$), or the ω_1 algebraic closed fields of characteristic 0 (think of the set of real numbers

[38] for an early report see e.g. [36]. Tarski there mentions ([36], p.91) that - besides some other theories -,the (second order) Peano arithmetic is even *strictly categorical*, meaning that in any case the isomorphism between two models is even unique (the technical term here is 'monotransformabel')

[39] nevertheless first order logic now already for many decades has grown to be the flagship of standard logic. Reasons may be manyfold, but presumably prevail: first order logic is the strongest standard logic providing compactness and (related) completeness of the logic system (Lindström 1968 [27] Theorem 2), and as already mentioned, first order logic is mainstream in contemporary axiomatized set theory. Tradeoff - of course, on the other hand there are several disadvantages - besides the loss of categoricity of formalized theories an often mentioned drawback is the Löwenheim-Skolem Paradox

IR). As well equivalence relations and Boolean algebras might in principle be in reach. And the mentioned correlation with the notion of a complete theory imop may as well help get the point.

Figure 3: $\kappa - categorical$ theories in first order logic, [24], p.66

66 KEISLER/FUNDAMENTALS OF MODEL THEORY [CH. A.2, §4

The Löwenheim–Skolem Theorem is often used in proofs that theories are complete. The simplest completeness test is the Łos–Vaught test.

4.10. Definition. A theory T is said to be *κ-categorical* if, up to isomorphism, T has exactly one model of cardinal κ.

4.11. Łos–Vaught test. *If T no finite models and T is κ-categorical for some κ, then T is complete.*

Proof. If T has two distinct complete extensions, then T must have two nonisomorphic models of cardinal κ. □

4.12. Example. Examples of ω-categorical theories are:
(i) Dense linear order without endpoints.
(ii) Equivalence relations with infinitely many classes and all classes infinite.
(iii) Atomless Boolean algebras.
(iv) Abelian groups with all elements of order p.

4.13. Example. Examples of ω_1-categorical theories are:
(i) Algebraically closed fields of given characteristic.
(ii) Uniquely divisible torsion-free abelian groups.
(iii) The theory of structures $\langle M, f\rangle$ where f is a permutation of M with no finite cycles.
(iv) Abelian groups with all elements of order p.

5.3 object references (obj.ref.) in categorical and non categorical theories

To return to the main topic of this paper, as expanded so far - **obj.ref.** for truth value functions - I consider first (for the well known example of Peano arithmetic) the topic of categoricity of theories, which is - I repeat - a theoretical maximal base for rigid **obj.ref.**, supporting truth value function mapping to $\{\mathbf{T}, \mathbf{F}\}$.

One reason, to select this example is, that while Peano arithmetic, axiomatized in second order logic (with full set theoretical semantics) is categorical, the Peano arithmetic axiomatization in first order logic is not (already not because of the Löwenheim/Skolem/Tarski theorems), but even not $\kappa - categorical$.

After considering this example I do switch to the other end of the spectrum and consider a math's example, which does not show maximal but rather minimal math structure, viz. the concatenation operation, (often only implicitly) used to define formal languages. We'll see that, in that example, the theories information about its objects there will be rather flimsy.

5.3.1 example Peano arithmetic, rigidly designating the natural numbers - either way

We define implicitly, but in a constructive way, natural numbers, starting with the 'successor function' $\mathbf{S}(\ )$, characterized by axioms as an injective (1-1) function, and with a single object $\mathbf{0}$ which is not successor of whatever object, and then we define the set of all objects, which are the result of applying the successor function $\mathbf{S}()$ finitely many times, starting from the initial object $\mathbf{0}$. This set $\mathbb{N} = \{\mathbf{0}, \mathbf{S0}, \mathbf{SS0}, ...\}$ known so far by these two axioms, then need be a subset of the universe of discourse $\mathbf{U}$ in any model $\mathbf{M_{PA}} = \langle\ \mathbf{U},\ \langle \mathbf{0}, \mathbf{S} \rangle\ \rangle$ of these two axioms:

$(PA_\alpha\ )$	$\mathbf{0}$ is not a successor	$\Vdash_{\mathbf{PA}} \bigwedge_{\mathbf{x}}\ \mathbf{0} \neq \mathbf{Sx}$,
$(PA_{\alpha 1})$	but all others are	$\Vdash_{\mathbf{PA}} \bigwedge_{\mathbf{y}}(\mathbf{y} \neq \mathbf{0} \rightarrow \bigvee_{\mathbf{z}}(\mathbf{y} = \mathbf{Sz}) \mid (PA_\alpha), (PA_{I\_FOL})$[40]
$(PA_\beta\ )$	$\mathbf{S}(\ )$ injective	$\Vdash_{\mathbf{PA}} \bigwedge_{\mathbf{xy}}(\mathbf{Sx} = \mathbf{Sy} \rightarrow \mathbf{x} = \mathbf{y})$[41]

Now the difference between the Peano axiomatization being *categorical* in second order logic,while being not in first order logic, is given by the respective implementation of the induction principle in these axiomatizations:

in 1st order implementation the induction principle is expressable only by an axiom schema, providing an axiom for each of the countably many unary truth value function $\mathbf{\Phi}(\ )$ of the $\{\mathbf{0}, \mathbf{S}()\}$ – **language**, stating that

$$(PA_{I\_FOL})\ [\mathbf{\Phi}(\mathbf{0}) \wedge \bigwedge_{\mathbf{x}}(\mathbf{\Phi}(\mathbf{x}) \rightarrow \mathbf{\Phi}(\mathbf{S}(\mathbf{x})))] \rightarrow \bigwedge_{\mathbf{x}} \mathbf{\Phi}(\mathbf{x})$$

But it's well known, that this axiomatization, as it stands, is even not $\aleph_0$ – *categorical*, as there exist countable non-standard models, for an overview see e.g. [39]. But this leads in rather high level modeltheoretic considerations, not really suitable nor needed in my rather elementary logical context here, but gives a valuable suggestion wrt the structure of these non standard models.

More easy access to the topic is then given in C.C. Chang and H.J. Keisler's 'model theory' textbook by a rather easy to understand ...

"Example 2.2.1. Let T be Peano arithmetic and let $\mathbf{\Sigma}(\mathbf{x})$ be the set

$$\{\mathbf{0} \neq \mathbf{x}, \mathbf{S0} \neq \mathbf{x}, \mathbf{SS0} \neq \mathbf{x}, ...\} \quad .$$

An element is said to be *nonstandard* iff it realizes $\mathbf{\Sigma}(\mathbf{x})$. The standard model of T omits $\mathbf{\Sigma}(\mathbf{x})$, while all non standard models realize $\mathbf{\Sigma}(\mathbf{x})$" [4] p. 77

The point of this assertion is of course: between two models of first order Peano arithmetic, none of them containing nonstandard elements, an isomorphism is trivially constructed. The more, this observation gives rise to a simple switch of perspective in order to reinstate the (most) rigid designation for natural numbers and successor function as well for first order Peano arithmetic. To show this,

[40] proof most trivial, but $(PA_{\alpha 1})$ will be used to simplify the proof sketch below

[41] axioms introducing addition and multiplication are omitted here, as these are conservative extensions only, do not restrict the class of models of Peano arithmetic, but use the model structure already given by the successor function, as defined by (PA_α), (PA_β)

we proceed next by extending the $\{\mathbf{0}, \mathbf{S}()\}$ – **language** of Peano arithmetic to $\{\mathbf{0}, \mathbf{S}(), <\}$, and state in this extended language in a well known way the complete strict order relation 'less than' induced via the successor function:

(PA_γ)	predecessor less	$\Vdash_{\mathbf{PA}} \bigwedge_{\mathbf{x}} \ \mathbf{x} < \mathbf{Sx}$
(PA_δ)	irreflexive	$\Vdash_{\mathbf{PA}} \bigwedge_{\mathbf{x}} \ \neg \ \mathbf{x} < \mathbf{x}$
(PA_ϵ)	transitive	$\Vdash_{\mathbf{PA}} \bigwedge_{\mathbf{xyz}} \ [(\mathbf{x} < \mathbf{y} \wedge \mathbf{y} < \mathbf{z}) \rightarrow \mathbf{x} < \mathbf{z}]$
(PA_ζ)	semiconnected	$\Vdash_{\mathbf{PA}} \bigwedge_{\mathbf{xy}}[\ \mathbf{x} < \mathbf{y} \ \vee \ \mathbf{y} < \mathbf{x} \ \vee \ \mathbf{x} = \mathbf{y}]$

Then we are in a position to state - tying up to Chang-Keisler's 'Example 2.2.1" - a

little corollary

The standard model $\mathbb{N} = \{\mathbf{0}, \mathbf{S0}, \mathbf{SS0}, ...\}$ of Peano arithmetic is, up to isomorphism, the only model of first order Peano arithmetic, whose universe of discourse $\mathbf{U}$ is well ordered by the $' < '$ relation, which is induced via the successor function $'\mathbf{S}(\,)'$.

proof sketch

be $\mathbf{M_{PA}} = \langle\ \mathbf{U},\ \langle\ \mathbf{0}, \mathbf{S}, < \rangle\ \rangle$ a model of $PA_{\alpha,...,\zeta}$, the universe of discourse $\mathbf{U}$ well ordered by $' < '$. And be $\mathbf{NSE} \ = \ \{\ \mathbf{x} \in \mathbf{U} \mid \mathbf{x \text{ realizes } \Sigma}\ \}$ the set of all non standard elements in $\mathbf{U}$.

From (PA_α), (PA_β) we know $\mathbb{N} = \{\mathbf{0}, \mathbf{S0}, \mathbf{SS0}, ...\} \ \subseteq \ \mathbf{U}$ and, by construction from "Example 2.2.1" both, $\mathbb{N} \cap \mathbf{NSE} = \lambda$ (*the empty set*), and $\mathbb{N} \cup \mathbf{NSE} = \mathbf{U}$.

Then we show, that the subset $\mathbf{NSE}$ cannot have a smallest element wrt $' < '$, hence need be empty, as $\mathbf{U}$ is well ordered by $' < '$.
Now suppose $\mathbf{NSE}$ not empty, then there need be a least element $\mathbf{u} \in \mathbf{NSE}$, as $\mathbf{NSE} \subset \mathbf{U}$, and $\mathbf{U}$ well ordered by $' < '$. By definition, no element $\mathbf{u} \in \mathbf{NSE}$ is equal to $\mathbf{0}$. But then each $\mathbf{u} \in \mathbf{NSE}$ by $(PA_{\alpha 1})$ need have a predecessor, say $\mathbf{v}$, hence $\mathbf{v} < \mathbf{u}$ by (PA_γ), and $\mathbf{v} \neq \mathbf{u}$ by (PA_δ). Hence $\mathbf{u}$ is not the least element in $\mathbf{NSE}$, contradicting the assumption. Hence $\mathbf{NSE}$ need be empty.

Hence $\mathbf{U} = \mathbb{N}$, therefore $\mathbf{M_{PA}} = \langle\ \mathbf{U},\ \langle\ \mathbf{0}, \mathbf{S}, < \rangle\ \rangle \ = \langle\ \mathbb{N},\ \langle\ \mathbf{0}, \mathbf{S}, < \rangle\ \rangle$.

But the $\mathbf{M_{PA}} = \langle\ \mathbf{U},\ \langle\ \mathbf{0}, \mathbf{S}, < \rangle\ \rangle$ was any model of first order Peano arithmetic, well ordered by the strict order relation induced by $\mathbf{S}(\,)$. Hence this model is unique in its kind ($\mathbf{U}$ well ordered by $' < '$) up to isomorphism. $\dashv$

but note, this well ordering property is not fully expressable in **FOL**:

it's rather easy to see, why we are not able to fully express the property of being (strictly) well ordered by $' < '$ in **FOL** itself, in this respect it's quite the same as with the induction axiom schema (PA_{I_FOL}), viz.:
In order to express this well ordering property, we'd have to introduce quite analogously an special axiom schema, giving one axiom for each **FOL** unary truth value function $\mathbf{\Phi}(\,)$ of language $\{\mathbf{0}, \mathbf{S}(), <\}$ and stating, that if the extension of $\mathbf{\Phi}(\,)$ is not empty, it has a least element, viz.

$$(WO_{FOL}) \qquad \bigvee_{\mathbf{x}} \mathbf{\Phi}(\mathbf{x}) \ \rightarrow \ \bigvee_{\mathbf{x}} [\ \mathbf{\Phi}(\mathbf{x}) \ \wedge \ \bigwedge_{\mathbf{y}}[\mathbf{\Phi}(\mathbf{y}) \rightarrow (\mathbf{x} < \mathbf{y} \vee \mathbf{x} = \mathbf{y})]\ ]$$

And the trouble with this axiom schema is quite as it is with the **FOL** induction axiom schema (PA_{I_FOL}), both do not exhaust the full powerset $\wp(\mathbf{U})$ of the universe of discourse **U** in the model, while such axiom schemas of course exhaust the set of finite and cofinite subsets of **U**.

in second order implementation the induction principle is expressable using besides quantifiers for objects also quantifiers for sets of objects

The range of these quantifiers is the full power set $\wp(\mathbf{U}) = \mathbf{2^U}$ of the universe of discourse $\mathbf{U}$ [$\supseteq \mathbb{N}$]
By this no axiom schema is necessary, hence a single axiom can be used, which may be written

$$(PA_{I\_SOL})\ \bigwedge_{\mathbf{M}\subseteq\{\mathbf{x}|\mathbf{x}=\mathbf{x}\}}[\ (\mathbf{0} \in \mathbf{M} \wedge \bigwedge_{\mathbf{x}}(\mathbf{x} \in \mathbf{M} \rightarrow \mathbf{S}(\mathbf{x}) \in \mathbf{M}) \rightarrow \mathbf{M} = \{\mathbf{x} \mid \mathbf{x} = \mathbf{x}\}\ ]$$

So let's just chew the cud, to note explicitly what does help in second order logic to achieve categoricity of the Peano axiomatization of arithmetic and is not available in first order logic: Via the quantifiers, having as their range the (full) powerset $\wp(\mathbf{U})$ of the universe of discourse **U** (which make the underlying logic system a second order system) in the implementation of the induction principle we are able to state, that if we include in a subset M of **U** the **0** and its successors (i.e. $\mathbf{M} = \mathbb{N}$), than this subset contains already all objects in the universe of discourse, hence then $\mathbf{M} = \mathbb{N} = \mathbf{U}$.

We understand, that the seemingly simple base given by the Peano arithmetic axiomatization, be it provided in first order axiomatization (in as far, as the first order axiomatization describes the standard model), or in second order axiomatization, suffices to introduce the arithmetic operations, and establish theorems like the 'fundamental theorem of arithmetics'[42], and much more.
Of course, the classification of some models of first order Peano arithmetic as non standard models depends on 'what is accepted as the standard'. And yes, there are at least two acceptable answers, the first points to the mathematical tradition and the respective intuition of professional mathematicians, the other points to the strict categoricity of the theory in second order logic. And with both answers, the rigid designation of the numerals is considered unquestionable. And as I tried to show above, there is to some extent also an acceptable answer in first order Peano arithmetic.

5.3.2 counter example free monoid, non rigid designation (of the elements of 'the word set') in a p_m.o structure

Another well known example of math's abstract objects is taken from the other end of the 'math's rigid designation' spectrum. Let's turn for this to the pure_math objects referred to from a mini-theory of rather say anything and nothing, viz. the theory of the 'free monoid'. The use of this algebraic mini-structure is presumably manifold, but it prominently figures in the theory of (grammars for) formal languages[43]. Such grammars select a proper subset of

[42] cf. [41]
[43] cf. [38]

the set of all possible finite concatenations ... see below ...

Formally, the free monoid is a semigroup with one (binary) associative group operation, maybe commutative or not, and a neutral element wrt this operation, the operation defined on the set, which contains a non empty set as a proper subset and its closure wrt the group operation.

It's a basic requisite for creating formal grammars for e.g. the usually context independent grammar rules for the languages of formal logic (e.g. the languages of first order or second order logic), and had been used in the past in the creation of transformational grammars for natural languages (e.g. Chomsky grammars), where of course context sensitive grammar rules were required then in the transformational rules component.

In this grammar definition use, the non empty 'starter' set $\mathbf{\Sigma}$ is intuitively seen as and hence called 'the alphabet' (regardless of whether the elements of $\mathbf{\Sigma}$ are taken to be signs in some intuitively acceptable sense), the semi group operation is then intuitively spelled out as 'concatenation operation', forming strings by concatenation starting with elements of the alphabet. [44]. And, for his intended use, the associative group operation is postulated to be non commutative. The neutral element of the semigroup operation is then spelled out as 'the empty string', the symbol chosen to denote the empty string in writing down the axioms for the free monoid, is then often simply the set theoretic symbol for the null-set. The functional closure of the alphabet $\mathbf{\Sigma}$ wrt the concatenation operation $()^\frown()$ [i.e., the smallest set, which contains the elements of the alphabet, and all finite concatenations of them] is then called the word set $\mathbf{\Sigma}^*$ of alphabet $\mathbf{\Sigma}$.

We may write down this mini-theory $\boxed{\mathbf{F-MON}}$
$< \ \mathbf{U}, \ < \mathbf{\Sigma}, \ ()^\frown(), \ \lambda > \ >$

$$\mathbf{\Sigma} = \{\mathbf{a_1}, ..., \mathbf{a_n}, \mathbf{a_{n+1}}, ...\}, \qquad \bigwedge_{\mathbf{x,y,z}}(\mathbf{x}^\frown\mathbf{y})^\frown\mathbf{z} = \mathbf{x}^\frown(\mathbf{y}^\frown\mathbf{z}), \qquad \bigwedge_{\mathbf{x}}(\mathbf{x}^\frown\lambda = \lambda^\frown\mathbf{x} = \mathbf{x})$$

$$\mathbf{\Sigma}^* = \{\bigcap \mathbf{M} \subseteq \mathbf{U} \mid \mathbf{\Sigma} \subseteq \mathbf{M} \wedge \bigwedge_{\mathbf{x,y}}[(\mathbf{x} \in \mathbf{M} \wedge \mathbf{y} \in \mathbf{M}) \rightarrow \mathbf{x}^\frown\mathbf{y} \in \mathbf{M}\}]$$

and may add a definition for stringlength function $\mathbf{lg}()$

$$\mathbf{lg}(\lambda) = \mathbf{0}, \qquad \bigwedge_{\mathbf{x}}(\mathbf{x} \in \mathbf{\Sigma} \rightarrow \mathbf{lg}(\mathbf{x}) = \mathbf{1}), \qquad \bigwedge_{\mathbf{x,y}} \mathbf{lg}(\mathbf{x}^\frown\mathbf{y}) = \mathbf{lg}(\mathbf{x}) + \mathbf{lg}(\mathbf{y})$$

The wordset of course is given by an inductive definition, and this far reminds of the Peano example. But, there is no functional dependence between the elements of the alphabet or strings of them requested, the number of alphabet elements may be freely chosen some finite number, or may be allowed infinitely many. Obviously, as long, as the alphabet is a countable set, the word set constructed from it is as well countable.

The grammar rules, which may be defined on the word set, are now simply asymmetric binary relations (intuitively understood as string replacement rules), context sensitive rules specify replacements dependent of the context of the string, which were to be replaced. But such grammar rules are an addon, not part of the definition of the free monoid.

Now, what about **obj.ref**. for truth value functions wrt the free monoid ?

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[44]cf. [40]

The short version is: truth values for sentences wrt free monoid facts are either not related to any special objects in the word set, else, and only if the elements of the alphabet are explicitly specified (e.g. individually named, listed as value of some known function, ...), these facts are about the sequence of the alphabet elements in the concatenations(i.e. strings), and their length. This is all wrt object reference supplied with the free monoid, and hence nothing more can be used for mapping a truth value function of the monoid language to a truth value, without adding further information.

To make this lack of **obj.ref**. rigidity a showcase, I sketch an ad hoc constructed and presumably **not intended model** of the mini-theory $\mathbf{F-MON}$.

For this model the concatenation operation will be interpreted als concatenation of light signals (non commutative), between a finite number of stars, arriving with a certain minimal intensity, within a radius of, say, 400 light years from our sun[45].

”starlight” example

The ”alphabet” of this free monoid shall contain a finite (and small) number of basic light signals

$\mathbf{\Sigma} = \{<\mathbf{Sol},\mathbf{Sirius}>,<\mathbf{Canopus},\mathbf{Sirius}>,<\mathbf{AlphaCentauri},\mathbf{Canopus}>,<\mathbf{Canopus},\mathbf{Sol}>,<\mathbf{AlphaCentauri},\mathbf{Sirius}>,<\mathbf{Sol},\mathbf{AlphaCentauri}>,<\mathbf{Sirius},\mathbf{Sol}>\}$

hence $<\mathbf{Sol},\mathbf{Canopus}> \notin \mathbf{\Sigma}$, while a concatenated signal $<\mathbf{Sol},...>^\frown ...^\frown <...,\mathbf{Canopus}>$ exists in $\mathbf{\Sigma}^*$.

The ’empty string’ then is to be taken as the ’0-signal of light (in either direction)’, e.g.

$<\mathbf{Sol},\mathbf{Sirius}>^\frown \lambda = \lambda^\frown <\mathbf{Sol},\mathbf{Sirius}> = <\mathbf{Sol},\mathbf{Sirius}>$

The length of a concatenated light signal then is the number of concatenations of basic signals in $\mathbf{\Sigma}$

n.p_m.o comment

As far as in this model the elements of $\mathbf{\Sigma}$, hence of $\mathbf{\Sigma}^*$, are **n.p_m.o**, all the respective concerns wrt required **obj.ref**. stipulations from **section 3** above apply. And it may be worth a minute or two, to check the applicability of the **obj.ref**. **A.C.I.D**.-properties wrt the elements of $\mathbf{\Sigma}^* \subseteq U$

More relaxed comments **returning to intended models**

Of course, as already mentioned, having started with the free monoid definition, types of grammar rules etc. may be defined. But what about the objects in the word set ? Rather nothing is fixed, as long as there is not more specific information wrt the alphabet, and even then, as long as there is not more specific information wrt the grammar rules, viz. the respective asymmetric relations on

[45] my sincere excuses to astronomers for this example’s poverty. I understand, that this example may not really make astronomical sense, but my point here is only an (epistemo)logical. Hence in case ... please take the example - in an obvious way - as an example of a (set of) directed graph(s), instead of the 4 stars take 4 nodes whatever, instead of light signals take directed edges between, $\mathbf{\Sigma}^*$ is then the set of paths in the graph.

the word set. Hence, the axioms for the free monoid sketch a widely differing variety of models (rigid designation far away), and per se this axiomatization gives only a minimum of information as to the objects, to which the concatenation operation may be applied. In this context belongs, that the finite strings in $\mathbf{A}^*$ are not uniquely decomposable. A decomposition procedure need be based on aditional information, which is usually given by context independent grammar rules. Such decomposition then is not applicable to all objects in the word set, but only for those, allowed by the (typically recursive) grammar rules.

Even if the word set is given by specifying the alphabet, rather nothing about the objects in the word set is known except concatenation properties. These will be relevant only lots of stages of language description later. E.g., if in case the word set is used for the description of a language of standard logic, the concatenation properties of the elements of the word sets play a role for e.g. induction proofs in the logic system of the language, after a lot of decisive information has been added by the formation rules, defining the correctly built expressions of the language (the well formed formulas = **wff**s), and after logical axioms and derivation rules are specified.

6 objects of empirical theories

After having dealt with objects of theories of pure mathematics I turn to considering a special concept of a formalized empirical theory, promoted and discussed by Jeffrey Ketland in his 'Foundations of Applied Mathematics I'[25]), especially considering the 'mixed' objects (and functions) wrt their representational role for physical objects.

6.1 Ketland on impure objects, and mixed objects, functions, and predicates

Ketland's account of empirical theories as 'applied mathematics' imop may fairly be appreciated as a reliable and general proposal and reference for epistemic reconstruction of logical structure of empirical theories. It's special importance for my discussion here lies, besides this paradigm status of his exposition, in the fact, that in this exposition special attention is given to respective objects kinds and **obj.ref**. kinds (Ketland handles the latter as logical types).

Ketland gives a rather rigorous account of the coordination of non pure_math objects (**n.p_m.o**) with pure_math objects (**p_m.o**) in formalized empirical theories[46]. Here's a very short introductory sketch:

a (formalized) empirical theory, understood as a piece of applied math's, is described as an application specific extension of the first order set theory **ZF(C)** [47], allowing for application specific 'urelements'=atoms, shorthand **ZFCA**, and further characterized (as usual within modeltheoretic semantics) by an application specific sequence of n-ary functions and predicates σ, hence $\mathbf{ZFCA}\sigma$, including constants, truth values, and, in a way (details see below) identity .

The first order logic (**FOL**), he uses, is many sorted, 4 sorts of variables correspond to 4 "logical types", viz. 'atom','boole', 'set' and 'global'. Global

[46]For a general orientation wrt intentions and formalism, his [25] section 1and 3 are recommended, section 2 supplies highly instructive presented examples

[47]i.e. Zermelo-Fraenkel set theory with axiom of choice or equivalent

variables allow also for values of the three other logical types.

Figure 4: Ketland's objects of empirical theories

considered object kinds

cited from Jeffrey Ketland's "Foundations of applied mathematics I", section 1.4 "A Classification"

General consideration of the kinds of mathematical objects appearing in applied mathematics suggests the following seven-fold classification of basic impure & mixed mathematical objects:[11]

IMPURE MATHEMATICAL OBJECTS	MIXED MATHEMATICAL OBJECTS
1. Sets (of atoms)	4. Mixed relations
2. Relations (of atoms)	5. Charts (co-ordinate systems)
3. Structures (on atoms)	6. Scales (mass, length, duration, etc.)
	7. Fields (scalar, spinor, tensor, etc.)

11 The point of the qualification "basic" is that further impure and mixed objects can then be defined by standard mathematical constructions: pairs, products, etc.

Ketland introduces the label 'impure mathematical objects' for set theoretical constructs built exclusively over 'atoms' (sets, relations, functions); objects, functions and relations allowing for arguments of different types, he calls mixed. His impure and mixed objects together correspond to my cover all term **n.p_m.o** [as far as these occur as **obj.ref**. in formalized empirical theories].

Convincing, as it is, my concerns start, where and as far as these logically typed objects were considered to be of high ontological dignity, mostly, my impression, from the indispensability argument. No, I won't ever debate the indispensability of math's and math's objects and techniques in science, especially not debate it in physics, but simply have another impression of their ontological relevance. Ketland himself refers to Quine's ontological commitment criterion wrt an example ([25], section 5), as far as I see, without arguing for or against.

6.2 Migration of ontological commitment proposal

Elsewhere, in parentheses of other work intention, I offered a proposal to migrate the load of the 'ontological commitment of a formalized empirical theory'[48] from pointing to the universe of discourse as a whole, to but instead considering the values of the **pnf**-matrices of their theorems. If this course were adopted, the ontological significance of the objects in the universe of discourse were significantly limited accordingly.

[48]W.V.O. Quine in [30]

The short form of this proposal is:

for any theorem $\mathbf{\Phi}$ of the formalized theory **Th** (in first order logic) and model of **Th**, $< \mathbf{U}, \mathbf{R}_\alpha >$,
1. select a **prenex normal form Γ of Φ**, viz. $\mathbf{pnf(\Phi)} = \mathbf{\Gamma} = \mathbf{QP_\Gamma} {}^\frown \mathbf{M_\Gamma}$, where $\mathbf{QP_\Gamma}$ is the quantifier Prefix, and $\mathbf{M_\Gamma}$ the quantifier free Matrix of the selected prenex normal form $\mathbf{\Gamma}$ of $\mathbf{\Phi}$.
2. consider for $\mathbf{M_\Gamma}$ its $\mathbf{V_{M_\Gamma}}$ value in the model $< \mathbf{U}, \mathbf{R}_\alpha >$
3. the ontological commitment of theory **Th** then is the union of the sets $\mathbf{V_{M_\Gamma}}$ of the **pnf** $\mathbf{M_\Gamma}$ of all theorems $\mathbf{\Phi}$ of theory **Th**. And of course, the ontological commitment of the theory hence remains relative to the considered model of the theory.
For motivation, background and technical details of this construct see my papers [18] and [19].

Now, to get this my rather abstract proposal somewhat down to earth, I'll try my best, to relate my proposal to a seemingly simple example for a theorem $\mathbf{\Gamma}$ from celestial kinematics:

I consider a single statement from celestial mechanics, stating that the earth-sun Lagrange points $\mathbf{L_4}$ and $\mathbf{L_5}$ keep their respective equilateral triangle positions with earth and sun during a given time interval [49].
All - maybe - errors or misperceptions in this example being mine, nevertheless I hope I'll succeed here in illustrating my point:

$\mathbf{T} = [\mathbf{t_0}, \mathbf{t_k}]$ considered time interval
$\mathbf{Obj} = \{\mathbf{Sol}, \mathbf{Terra}, \mathbf{L_4}(\mathbf{Sol}, \mathbf{Terra}), \mathbf{L_5}(\mathbf{Sol}, \mathbf{Terra})\}$ set of considered **n.p_m.o**
pos an astrometrical localization function giving the position of these **n.p_m.o** for every $\mathbf{t} \in \mathbf{T}$

Now the statement $\boxed{\mathbf{L_4} - \mathbf{L_5}\ \mathit{for}\ \mathbf{Sol} - \mathbf{Terra}}$ in a provisional formalization:

$$\bigwedge_{\mathbf{t} \in \mathbf{T}} \ \bigwedge_{\mathbf{xyzu}}$$
$$[[\mathbf{x} = \mathbf{pos}(\mathbf{Sol}, \mathbf{t}) \wedge \mathbf{y} = \mathbf{pos}(\mathbf{Terra}, \mathbf{t}) \wedge \mathbf{z} = \mathbf{pos}(\mathbf{L_4}(\mathbf{Sol}, \mathbf{Terra}), \mathbf{t}) \wedge \mathbf{u} = \mathbf{pos}(\mathbf{L_5}(\mathbf{Sol}, \mathbf{Terra}), \mathbf{t})]$$
$$\rightarrow \quad [\sphericalangle \mathbf{xyz} = \sphericalangle \mathbf{yzx} = \sphericalangle \mathbf{zxy} \ \wedge \ \sphericalangle \mathbf{xyu} = \sphericalangle \mathbf{yux} = \sphericalangle \mathbf{uxy}]]$$

[49] for details I refer to the wikipedia page 'Lagrange points'[42]

Figure 5: Lagrange points showcase from [42], wikimedia commons, thanks to user cmglee

wikimedia commons , downloaded 2024-02-06 , https://commons.wikimedia.org/wiki/Category:Lagrange_points

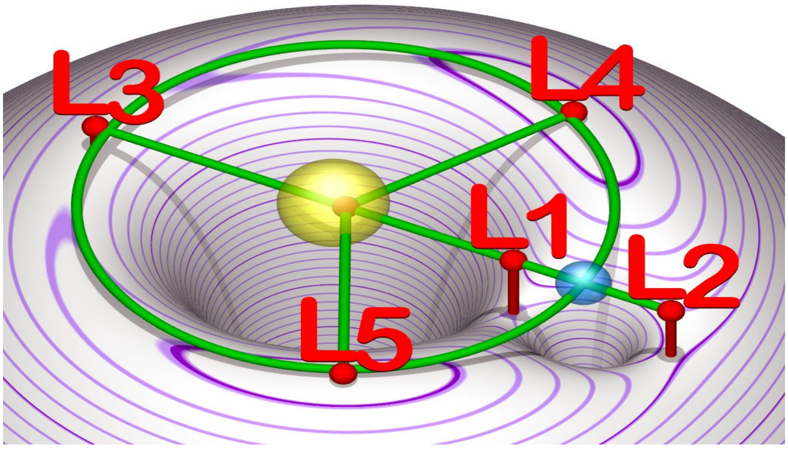

Image showing the relationship between the five Lagrangian points (red) of a planet (blue) orbiting a star (yellow), and the effective potential (gravitational + rotational) in the plane containing the orbit (grey surface with purple contours of equal potential). The potential was computed in POV-Ray using for q = 0.1 and z = 0.[1]

User:cmglee - Own work

Then my example for an empirical theorem $\boxed{\mathbf{L_4} - \mathbf{L_5}\ for\ \mathbf{Sol} - \mathbf{Terra}}$ might be read in plain words:

at any time both triangles,formed by sun,earth and the resp.equilibrium point,are equiangular

'mixed objects' in Ketland's [25] logical type-ing, and of course **n.p_m.o** in my somewhat broader **obj.ref**. classification, are then
- the $\mathbf{t} \in \mathbf{T}$ i.e., the points in time (real valued wrt a physical frame of reference),
- and my four objects focussed on, viz. $\mathbf{Obj} = \{\mathbf{Sol}, \mathbf{Terra}, \mathbf{L_4}(\mathbf{Sol}, \mathbf{Terra}), \mathbf{L_5}(\mathbf{Sol}, \mathbf{Terra})\}$
- and, a fortiori, the elements of $\mathbf{T} \times \mathbf{Obj}$
- the angles $\sphericalangle \mathbf{abc}$ referred to

'mixed functions and predicates' in Ketlands logical type-ing imop were
- the localization function **pos()**
- and, I think, as well the identity-relation $' = '$ between the angles[50].

Theorem $\mathbf{L_4} - \mathbf{L_5}\ for\ \mathbf{Sol} - \mathbf{Terra}\ [= \mathbf{\Phi}]$ is already in prenex normal form $[\mathbf{\Phi} = \mathbf{\Gamma}]$ with prefix $\mathbf{QP_\Gamma} = \lfloor\ \bigwedge_{\mathbf{t} \in \mathbf{T}}\ \bigwedge_{\mathbf{xyzu}}\ \rfloor$, the matrix $\mathbf{M_\Gamma}$ is the whole fomula righthand of these universal quanifiers for $\mathbf{t} \in \mathbf{T}$ and for $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{u}$. The value $\mathbf{V_{M_\Gamma}}$ of the matrix $\mathbf{M_\Gamma}$ is then an infinite set of object sequences of

[50]Ketland assigns to 'identity' the type declaration global, meaning that terms built from every sort of variables [atom, set, boole, global] are allowed as arguments. Imop, as the respective angles are already mixed objects, 'identity' for these arguments works as a mixed relation. For details cf. [25], section 3.2, Definition 3 (Sorted Variables), Definition 5(Application Signature)

$\mathbf{card} = |[\mathbf{t_0}, \mathbf{t_k}]|$, each containing a point in time-reference and the respective four object localizations.[51]

And wrt this example my proposal, else thesis is:
only this infinite sequence of timed localizations - being the value, on which the truth of the theorem $\boxed{\mathbf{L_4} - \mathbf{L_5}\ \mathit{for}\ \mathbf{Sol} - \mathbf{Terra}}$ in this model exlusively depends - may be allowed to claim some ontological commitment. The theorem is not ontologically committed to the (many sorted, else mixed) universe of discourse as a whole. Imop, but even this just agreed ontological relevance of the value of the matrix of the theorems **pnf** is only indirect, in that the timed localization sequences do only provide virtual, else theoretical, measurement point suggestions, i.e. suggest only points to get in touch with the real physical phenomena, the theorem is about. And of course, a statement, which (in addition) included beyond the kinematic also the dynamic aspect, as to the equilibrium of forces (gravitational, inert, ...) which is supposed to be the cause of the location of the Lagrange points $\mathbf{L_4}, \mathbf{L_5}$, the matrix of its **pnf** imop would show more object variables occurring free, hence an enlarged ontological commitment, while - in a sense - wrt the same physical phenomena.

other comments on the example
First, from my point of view, the usual **n.p_m.o** comments (see above my note on the "starlight" example) apply.
Next, as shown in figure 5 (cited from wikimedia commons), the prediction of Lagrange points is not peculiar to the Sol-Terra constellation, but is a general prediction for similar star-planet configurations, just more generally "... For any combination of two orbital bodies, there are five Lagrange points, L1 to L5, all in the orbital plane of the two large bodies. ..."[42]. This may be pictured by replacing the **obj.ref**. **Sol** and **Terra** by new object variables, say **w** and **v** occuring free, hence getting a binary truth value function, a binary predicate:

$$\begin{aligned}&\bigwedge_{\mathbf{t}\in\mathbf{T}} \bigwedge_{\mathbf{xyzu}}\\&[[\mathbf{x} = \mathbf{pos}(\mathbf{w}, \mathbf{t}) \wedge \mathbf{y} = \mathbf{pos}(\mathbf{v}, \mathbf{t}) \wedge \mathbf{z} = \mathbf{pos}(\mathbf{L_4}(\mathbf{w}, \mathbf{v}), \mathbf{t}) \wedge \mathbf{u} = \mathbf{pos}(\mathbf{L_5}(\mathbf{w}, \mathbf{v}), \mathbf{t})]\\&\ \rightarrow\ [\sphericalangle\mathbf{xyz} = \sphericalangle\mathbf{yzx} = \sphericalangle\mathbf{zxy}\ \wedge\ \sphericalangle\mathbf{xyu} = \sphericalangle\mathbf{yux}) = \sphericalangle\mathbf{uxy}]]\end{aligned}$$

and thus we were to look for **obj.ref**. replacing these placeholders **w**, **v**, which then would map this truth value function to truth value **T**.

In the course of this then, re-examining wrt the A.C.I.D. properties of such **obj.ref**. for this truth value function, we may wonder (and astronomers presumably will know), in how far e.g. gas giants as Saturn and Jupiter, with many moons - else moonlets, may be viewed as providing candidates for relevant two celestial body systems.

[51] But of course as far only, as the position values returned by the localization function **pos** for **n.p_m.o** pairs of $\mathbf{T} \times \mathbf{Obj}$ are based on (mutually independent) measurements, these values may be considered empirically significant in a prominent way.

7 summary

Now, wrt **obj.ref**. - not only from within formalized languages - rarely nothing is, what it is purported to be; nor, what it seems to be at first glance. And where but it is, viz. most rigid designation of objects by **obj.ref**. in strictly categorical (math's) theories in higher order logic, the respective objects are abstract objects comme il faut; hence, as I have already argued, their existence is a dependent one, and in these cases dependent on theory construction. In other words, while the elements of categorical structures are logical perfect objects par excellence, referred to by most rigid designators, this not debated fact does imply rather nothing to support e.g. a so called 'Platonist view' of say numbers existing as transcendental objects.

In the non pure_math cases considered, **obj.ref**. as they stand are not already well suited for truth value function mapping. Hence In these cases it depends on the type of **obj.ref**. and its usage environment, whether we are willing or even able to supply in some suitable way (environment- and knowledge-depending) enough additional object information for our otherwise **logically imperfect obj.ref**.. Else we may be presupposing, that this information could be given principally in some suitable way, to turn our **logically imperfect obj.ref**. into - else to take them counterfactually as - **logically perfect obj.ref**.[52]; to the end, that the enriched, else stipulated as enriched **obj.ref**. were taken to be suitable as (parts of) the logical subject of some or other hopefully truth value definite sentences.

Again, such additional information

- may be added only to some extent, and may then be claimed but not added for the longing rest (cf. discussion of the "summertime" example above) ;

- may even be refused to be given, by more or less unqualified appeal to every day experience, else common sense

- may include, in the case of software environment ontologies, introducing more parameter columns into the data model for database layout, in order to remove situation dependency of object description, most easy example may be introducing columns for managing local time, these columns to be used in connection with respective web services providing UTC and other 'official times';

Hence, at least in any case of reference to objects in colloquial language, as well as in ontologies specified by software environments, and in some way also in the case of formalized empirical theories, we have to understand, that this reference is something, which is in part only stipulated for sake of argument. And while for many purposes this way to handle 'objects' may do very well, we have to acknowledge that in general the ontological dignity of such reference is and remains more or less opaque, hence questionable.

However, the 'mixed' objects of Ketlands approach seem to fit very well for

[52]this is e.g. the case with with 'ontologies' providing (RDF) modeled data objects, I mentioned in the introduction

the truth value function mapping, due to the imop carefully top down designed logical reconstruction of empirical theories as applications of ZF(C) set theory. But this remarkable epistemic achievement has a price, and from my point of view this price is, that this way of theory reconstruction shows rather blatantly the gap between the 'mixed objects and/or functions', serving the truth value function mapping role very well, and the ontological commitment of the theory contouring the physical phenomena, the theory is about. As I've tried to make plausible in the last section.

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