DOMINIC GREGORY

COMPLETENESS AND DECIDABILITY RESULTS FOR SOME PROPOSITIONAL MODAL LOGICS CONTAINING "ACTUALLY" OPERATORS

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ABSTRACT. The addition of "actually" operators to modal languages allows us to capture important inferential behaviours which cannot be adequately captured in logics formulated in simpler languages. Previous work on modal logics containing "actually" operators has concentrated entirely upon extensions of **KT5** and has employed a particular model-theoretic treatment of them. This paper proves completeness and decidability results for a range of normal and nonnormal but quasi-normal propositional modal logics containing "actually" operators, the weakest of which are conservative extensions of **K**, using a novel generalisation of the standard semantics.

KEY WORDS: modal logic, possible worlds semantics, "actually" operators, rigidifiers.

0. INTRODUCTION

The main aim of this paper is to present and sketch proofs of the results declared in its title. Section 1 motivates the introduction of "actually" operators. "Actually" operators are identified in a way which does not rely upon contentious views about the meaning of "actually". This contrasts with a standard account of what such operators are, in ways explained in Section 2. More generally, Section 2 briefly discusses previous work on "actually" operators and describes how the results derived here differ from previous results in the literature.

Section 3 sketches completeness proofs for a bunch of propositional modal logics containing "actually" operators, using a novel semantics. The logics are conservative extensions of **K**, **KD**, **KT**, **KTB**, **KT4** and **KT5**.[1] Each of the logics has an important property: each is *informally sound* precisely if the logic which it conservatively extends is. (The notion of informal soundness is introduced in Section 1.) Section 4 sketches proofs that the logics discussed in Section 3 are decidable.

Section 5 sketches completeness and decidability proofs for another batch of propositional modal logics containing "actually" operators. The logics are nonnormal extensions of those discussed in Section 3. The logics are not informally sound, but they do have another important property:

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each is *informally safe* precisely if the logic which it extends is informally safe. (The notion of informal safety is introduced in Section 2.) The sixth and final section briefly indicates how to derive some further completeness results for the logics discussed in Sections 3–5, and presents some results concerning further logics with "actually" operators.

1. "ACTUALLY" OPERATORS: WHAT THEY ARE, AND WHY WE NEED THEM

Given a formal language J, an *interpretation of* J is a way of using J's wff to express propositions, by assigning meanings to J's expressions and semantic significance to the ways of constructing wff from J's expressions. It may be stipulated that interpretations of J must respect certain constraints. Normally, for instance, interpretations of formal languages containing & must interpret it as meaning "_ and. . .".

Suppose that **S** is a modal logic containing the classical propositional calculus, formulated in the standard modal propositional language L.[2] We stipulate that interpretations of L must interpret $\neg$ as meaning "it is not the case that _", & as meaning "_ and. . .", $\vee$ as meaning "_ or. . .", $\Box$ as meaning "it is necessary that _" and $\Diamond$ as meaning "it is possible that _".

Suppose that if the sequent $\phi_1, \ldots, \phi_n \vdash \psi$ is provable in **S**, then for any interpretation of L, the propositions assigned to $\phi_1, \ldots, \phi_n$ entail the proposition assigned to ψ. (For $n = 0$, the consequent is equated with ψ's being assigned only necessary truths.) Then **S** is *informally sound*.

Good modal logics – at least one variety – are ones which are informally sound. Interpretations of provable sequents of informally sound logics result only in valid arguments. If one identifies an ordinary modal argument as expressed by an interpreted provable sequent of an informally sound modal logic, one can straightaway conclude that it is valid; just the sort of thing that we want modal logics for.

For instance, assume that $\Box p \vdash \Box(p \vee q)$ is provable in **S**. We can interpret p as meaning "$2 + 2 = 4$" and q as meaning "$2 + 2 = 5$". **S**'s informal soundness implies that the proposition thereby assigned to $\Box p$ entails the proposition assigned to $\Box(p \vee q)$. The argument "necessarily, $2+2 = 4$; so necessarily, either $2+2 = 4$ or $2+2 = 5$" is therefore valid.

How can we prove that a modal logic is informally sound? Before considering one way of doing so, some supplementary notions must be introduced. A *logic* is a set of sequents whose wff are formulated in a single language, the language of the logic. A sequent $\phi_1, \ldots, \phi_n \vdash \psi$ is *provable in logic* **S** ($\phi_1, \ldots, \phi_n \vdash_{\mathbf{S}} \psi$) precisely if $\phi_1, \ldots, \phi_n \vdash \psi \in \mathbf{S}$. A wff ψ is a *theorem of* **S** ($\vdash_{\mathbf{S}} \psi$) just in case $\phi_1, \ldots, \phi_n \vdash_{\mathbf{S}} \psi$, for $n = 0$. An *axiomatisation* A consists of the following: first, a set of axioms; second, a set of rules which apply to some formulae to yield a formula, labelled as A's *universal rules*; and finally, a set of rules which apply to some formulae to yield a formula, labelled as A's *admissible rules*.

A sequent $\phi_1, \ldots, \phi_n \vdash \psi$ is *provable using* A just in case there is a finite sequence of wff in which each formula is either an axiom of A or one of $\phi_1, \ldots, \phi_n$ or follows from earlier formulae in the sequence by an application of one of A's universal rules. An axiomatisation A *axiomatises* the logic containing precisely those sequents $\phi_1, \ldots, \phi_n \vdash \psi$ such that: (1) $\phi_1, \ldots, \phi_n \vdash \psi$ is provable using A; or (2) there are some wff ψ_i such that, for $m = 0$, $\phi_1, \ldots, \phi_m \vdash \psi_i$ is provable using A, and ψ results from the ψ_is by an application of one of A's admissible rules.

Suppose that logic **S** is axiomatised in such a way that each of the axioms is interpretable only as expressing necessary truths. (The axioms are *informally sound*.) Suppose that each of the axiomatisation's universal rules meets the following condition: if the rule applies to $\phi_1, \ldots, \phi_n$ to give ψ, each interpretation of L assigns to $\phi_1, \ldots, \phi_n$ propositions which entail the proposition assigned to ψ. (The universal rules are *informally sound*.) Finally, suppose that each of axiomatisation's admissible rules meets the following condition: if $\phi_1, \ldots, \phi_n \vdash_{\mathbf{S}} \psi_i$, for $n = 0$, and the rule applies to the ψ_is to give ψ, each interpretation of L assigns a necessary truth to ψ. (The admissible rules are *informally sound*.) Then a simple inductive argument shows that **S** is informally sound.

We want modal logics to be informally sound. But we also want them to reflect the distinctive inferential properties of modal locutions. Preserving informal soundness while capturing additional inferential behaviours tends to require the introduction of axioms and vocabulary and the imposition of constraints upon interpretations.

For instance, suppose that "water" rigidly designates whatever stuff is actually C. And suppose that H_2O might not have been C. Consider some possible scenario in which H_2O is not C. Then in the envisaged possible circumstances, if H_2O is the stuff which is actually C, it would not have been the case that water was C. On the assumption that H_2O is actually C, therefore, it follows that water might not have been C.

The following argument can, with a little violence to the English language, be extracted from the above:

(A) It might have been that if H_2O is actually C, then water would not have been C; but H_2O is actually C; so water might not have been C.

Suppose that our resources are limited to L. How should we formalise (A)?

(A)'s first premiss appears to consist of a conditional within the scope of the initial "possibly _". We might regard that appearance as misleading. For instance, we might paraphrase (A)'s first premiss using "if H_2O is actually C, then water might not have been C", and then formalise it using $p \to \Diamond q$. That strategy will, however, lead to tears if followed elsewhere.

Consider the sentence "there might have been something which, if it actually exists, would not have been self-identical". The best formalisation of that sentence, using the strategy just mooted, is $\Diamond\exists x(\mathrm{E}x \to \Diamond x \neq x)$. But that sentence is necessarily false, while the sentence which it formalises is true, expressing the claim that there might have existed something which does not actually exist. Accordingly, I suggest that we trust (A)'s appearance.

So the only half-decent formalisation is: $\Diamond(p \to q),\ p \vdash \Diamond q$. That has, however, a glaring fault; no logic in which the above sequent is provable is informally sound. For instance, the sequent can be interpreted as expressing the following invalid argument:

(B) It might have been that if there were 9 planets, some number would not have been self-identical; but there are 9 planets; so possibly, some number is not self-identical.

Identifying the difference between (A) and (B) is easy: the antecedent of (A)'s first premiss is in the indicative, whereas the antecedent of (B)'s first premiss is not. A brief reflection on the standard strategy for formalising ordinary arguments in languages like L suggests that any differences owed to that difference will not easily be captured in L.

That standard strategy involves treating nonindicative moods as resulting from the application of a modal operator to a proposition expressible as *the proposition that* P, where P is a sentence whose main verb is in the indicative mood. (An analogous strategy is used in tense logic, where tenses are regarded as resulting from the application of a tense operator to a proposition expressible using only present tenses.) For instance, "H_2O might have been C" would standardly be formalised using $\Diamond p$, where p is interpreted as expressing the proposition *that* H_2O *is* C. But how are we then to reflect the difference between, say, "H_2O might have been C" and "it might have been that H_2O is actually C"? We – surely! – cannot.

We have a choice: we can either turn a blind eye to arguments like (A), or we must enrich L. Suppose we plump for the latter and add an unary operator A to L. The resulting language is LA.[3] A is to be interpreted as expressing an operator having the following effect: when attached to LA formula ϕ, the proposition assigned to ϕ is not to be read as operated upon by any modal operator within whose scope $A\phi$ falls.[4] When expressing the proposition assigned to LA-sentence ψ in which $A\phi$ occurs, the main verb of the sentence expressing the proposition assigned to ϕ must always be kept in the indicative mood, even if $A\phi$ is in the scope of a modal operator.[5]

For instance, suppose that we are formalising “H_2O might have been C” and “it might have been that H_2O *is actually* C”. The first would be formalised using $\Diamond p$, where p is assigned the proposition that H_2O is C; that proposition is then operated upon by the operator standardly assigned to $\Diamond$. But the second would be formalised using $\Diamond Ap$, where p is assigned the same proposition; the application of the operator assigned to A prevents the operator assigned to $\Diamond$ from applying to that proposition. Or suppose that p is assigned the proposition that there are no tangerines. Then the proposition assigned to $\Box(p \rightarrow \Diamond Ap)$ is the proposition that necessarily, if there were no tangerines then it might have been that there actually *are* no tangerines.

The inferential behaviour reflected by (A)’s validity is easily captured in axiomatisations formulated in LA. They need only include as axioms all instances of the following schema: $\Diamond(A\phi \rightarrow \psi) \rightarrow (A\phi \rightarrow \Diamond\psi)$. Each such axiom is informally sound. What other axioms involving A may axiomatisations safely include? Each instance of the following schemata is unproblematic: $\neg A\phi \leftrightarrow A\neg\phi$; $A(\phi \rightarrow \psi) \rightarrow (A\phi \rightarrow A\psi)$; $A(\phi \leftrightarrow A\phi)$.

And suppose that for any interpretation of LA, the proposition assigned to ϕ is necessary. Then for any interpretation of LA, it is necessary that the proposition assigned to ϕ is actually true.[6] The following is therefore an informally sound admissible rule when added to informally sound axioms and universal rules: if ϕ is a theorem, so is Aϕ.

What, then, are “actually” operators, and why do we need them? An operator O is an “actually” operator if O is to be interpreted in the same way as A above. One reason why we need them is because we cannot satisfactorily reflect specific logical behaviours in logics formulated in L without sacrificing informal soundness.

2. BRIEF DISCUSSION OF SOME PREVIOUS WORK ON “ACTUALLY” OPERATORS

“Actually” operators have been around for a while. A number of writers have proved completeness theorems for propositional extensions of **KT5** containing them.[7] The decidability of certain such logics has also been proved.[8] Hazen has derived completeness results for first-order extensions of **KT5** which contain “actually” operators and Hodes has proved results about the expressive powers of first-order languages in which such operators are given a certain model-theoretic treatment.[9] I do not know of completeness and decidability results for any propositional modal logics containing "actually" operators not extending **KT5**.

"Actually" operators are not always identified as in Section 1. *A* standard approach invokes a controversial interpretation of "actually" and saddles "actually" operators with a related model-theoretic treatment. For instance, Davies and Humberstone write:

> In [Crossley and Humberstone [2]] reasons are given for enriching the conventional language of modal logic with an operator "*A*" (read "actually") whose function is to effect (loosely speaking) a reference to a single world (within a model) designated as the actual world.[10]

Why treat "actually" operators in the way suggested? The idea is, presumably, that doing so reflects the function of "actually" within natural language: those sentences to which it applies are to be evaluated with respect to *the actual world*. The standard treatment of "actually" operators thus assumes a contentious treatment of "actually" as it figures in natural language.[11] The way of identifying them suggested in Section 1 is therefore preferable.

I have a more selfish reason for taking issue with the above account of "actually" operators: the semantical treatment of such operators employed below does not treat them in the same way. The divergence is in some ways unimportant, however. As Section 6 shows, the treatment used below makes it straightforward to prove completeness results for more standard semantical systems. There are two advantages to the treatment used here: it makes for straightforward completeness proofs and, more importantly, for simple decidability proofs.[12]

The standard treatment has had an unfortunate side-effect. It has led writers to advocate axioms which are not clearly informally sound. For instance, Crossley and Humberstone have $Ap \rightarrow \Box Ap$ as an axiom.[13] Why do they take it as axiomatic? Because it falls straight out of the standard treatment of "actually" operators remarked above:

> [The validity of $(Ap \rightarrow \Box Ap)$] arises from the fact that it is one and the same world that counts as the actual world, for every world in the model. So if something is true in that world it is certainly going to be true in every other world that it is true in that world.[14]

I do not advocate the standard account of "actually" operators, so the above argument is not available to me. Each axiom of the form $\Diamond(A\phi \rightarrow \psi) \rightarrow (A\phi \rightarrow \Diamond\psi)$ appears, however, to be informally sound; yet each instance of $A\phi \rightarrow \Box A\phi$ is provable in any logic extending a normal modal logic formulated in LA having each instance of $\Diamond(A\phi \rightarrow \psi) \rightarrow (A\phi \rightarrow \Diamond\psi)$ as a theorem (see Lemma 3 below).

There is a division in the literature between those whose logics have all instances of $\phi \leftrightarrow A\phi$ as theorems and those whose logics do not. For instance, Crossley and Humberstone favour logics of the latter kind.[15] But Hazen describes their reasons as "conceptually weak".[16]

Construe "actually" as validating arguments sharing the form of (A) above. Then no logic having each instance of $\phi \leftrightarrow A\phi$ as a theorem is informally sound. For instance, any such logic will have $p \leftrightarrow Ap$ as a theorem. If that theorem can be interpreted as expressing something which is not necessary, the logic is informally unsound. Suppose, for contradiction, that the theorem expresses a necessary truth, however interpreted. Then, for instance, no matter what, there would have been tangerines precisely if there are actually tangerines.

Now, it might have been that if there had been tangerines, some apple would not have been self-identical.[17] So, by the assumption of the preceding paragraph, it might have been that if there are actually tangerines, some apple would not have been self-identical. But there are actually tangerines. So some apple might not have been self-identical (we are, remember, reading "actually" so that it validates arguments like (A)). Absurdity! $p \leftrightarrow Ap$ therefore does not express a necessary truth on all interpretations.[18]

If we are after informal soundness, our loyalties should be with Crossley and Humberstone. And what else could we want? Consider the following argument: "these are the good times; so these are actually the good times". We know that if the premiss of that argument is actually true, so is its conclusion. Arguments having that feature are *safe*.

One thing which we might want of a logic is that its interpreted provable sequents express only safe arguments. That is, we might want our logics to be *informally safe*.[19] If informal safety is what we are after, the provability of all instances of $\phi \leftrightarrow A\phi$ is not a problem. Accordingly, those who desire informal safety rather than informal soundness should join Hazen.

Should we want informally safe or informally sound logics? There is surely no right answer; different strokes for different folks. But even those desiring only informal safety need "actually" operators. For instance, the arguments in Section 1 show that we cannot, while avoiding informal unsafety, prove the sequent of L which best formalises (A). We can, however, provide a formalisation of (A) in LA whose provability does not result in informal unsafety; viz. $\Diamond(Ap \to q), Ap \vdash \Diamond q$.

The next section proves completeness results for some propositional modal logics containing "actually" operators. The logics share a certain feature: they are conservative extensions of familiar propositional modal logics and each one is informally sound precisely if the familiar logic which it extends is.

3. First set of completeness results

Below, $\perp$ abbreviates $(p\&\neg p)$. Given a logic **S** formulated in LA, $\mathbf{S}+A$ is the logic resulting when all instances of axiom schemata Ax1–Ax4, the universal rule MP and the admissible rules RN and RA are added to an axiomatisation of **S**:

Ax1: $A(\phi \leftrightarrow A\phi)$,
Ax2: $\neg A\phi \leftrightarrow A\neg\phi$,
Ax3: $A(\phi \rightarrow \psi) \rightarrow (A\phi \rightarrow A\psi)$,
Ax4: $A\phi \rightarrow [\Diamond(A\phi \rightarrow \psi) \rightarrow \Diamond\psi]$,
MP: ψ is provable from ϕ and $\phi \rightarrow \psi$,
RA: if ϕ is a theorem, so is $A\phi$,
RN: if ϕ is a theorem, so is $\Box\phi$.

As pointed out in Section 2, each instance of Ax1–4 is informally sound. Throughout the rest of this paper, it is assumed that **S** is one of **K**, **KD**, **KT**, **KTB**, **KT4** or **KT5**.[20] Those logics are informally sound only if MP and RN are.[21] So if **S** is informally sound, $\mathbf{S}+A$ is too.

Also, **S** has $\neg\perp$ among its theorems. And, by truth-functional logic, if $\vdash_{\mathbf{S}+A} \phi \rightarrow \psi$, $\vdash_{\mathbf{S}+A} (\phi_1\& \ldots \&\psi\& \ldots \&\phi_n) \rightarrow \theta$, then $\vdash_{\mathbf{S}+A} (\phi_1\& \ldots \&\phi\& \ldots \&\phi_n) \rightarrow \theta$ (this fact is often needed below for proving the consistency of various sets of wff; see, for instance, Theorem 2).

LEMMA 1. $\vdash_{\mathbf{S}+A} \neg A\perp$.

LEMMA 2. $\vdash_{\mathbf{S}+A} A\phi \leftrightarrow AA\phi$.

LEMMA 3. $\vdash_{\mathbf{S}+A} A\phi \rightarrow \Box A\phi$.[22]

Proof. $\vdash_{\mathbf{S}+A} \Diamond\neg A\phi \rightarrow \Diamond(A\phi \rightarrow \perp)$. By Ax4, $\vdash_{\mathbf{S}+A} A\phi \rightarrow (\Diamond\neg A\phi \rightarrow \Diamond\perp)$. So $\vdash_{\mathbf{S}+A} A\phi \rightarrow (\Box\neg\perp \rightarrow \neg\Diamond\neg A\phi)$. But by RN, $\vdash_{\mathbf{S}+A} \Box\neg\perp$. So $\vdash_{\mathbf{S}+A} A\phi \rightarrow \neg\Diamond\neg A\phi$, i.e. $\vdash_{\mathbf{S}+A} A\phi \rightarrow \Box A\phi$. $\Box$

A sequence $M = \langle W, w^*, R, @, P\rangle$ is an *@Kripke model* iff it meets the following conditions:

@1: W is a set. The members of W are M's *indices*.
@2: $w^* \in W$. (w^* is M's *distinguished index*.)
@3: R is a subset of $W \times W$. (R is M's *accessibility relation*.)
@4: @ maps each $w \in W$ onto a $y \in W$ such that (i) $@(y) = y$; and (ii) if wRz, then $@(w) = @(z)$.
@5: P maps each propositional variable π onto a subset of W.

(Notice that $\{@(w) : w \in W\}$ may have more than one member.) A sequence $M = \langle W, w^*, R, P \rangle$ is a *sub@Kripke model* iff it meets conditions @1–@3, and P maps each sentence letter π and formula $A\phi$ onto a subset of W. If $M = \langle W, w^*, R, @, P \rangle$ is an @Kripke model, the sub@Kripke model $M' = \langle W, w^*, R, P' \rangle$ is *based on M* if P' extends P.

Given an @Kripke model $M = \langle W, w^*, R, @, P \rangle$, the definition of "$\phi$ *is true at index* w *in* M" – $(M, w) \vDash \phi$ – is as follows:

P1: For each propositional variable π of LA, $(M, w) \vDash \pi$ iff $w \in P(\pi)$.
P2: For each wff $\neg\phi$ of LA, $(M, w) \vDash \neg\phi$ iff not-$(M, w) \vDash \phi$.
P3: For each wff $(\phi\&\psi)$ of LA, $(M, w) \vDash (\phi\&\psi)$ iff $(M, w) \vDash \phi$ and $(M, w) \vDash \psi$.
P4: For each wff $\Box\phi$ of LA, $(M, w) \vDash \Box\phi$ iff for every w' such that wRw', $(M, w') \vDash \phi$.
P5: For each wff $A\phi$ of LA, $(M, w) \vDash A\phi$ iff $(M, @(w)) \vDash \phi$.

The definition of truth at an index in a sub@Kripke model $M = \langle W, w^*, R, P' \rangle$ is given by P1–P4, and the condition that $(M, w) \vDash A\phi$ iff $w \in P'(A\phi)$. ϕ is true in (sub)@Kripke model M – $M \vDash \phi$ – iff $(M, w^*) \vDash \phi$. And ϕ is valid in class C of (sub)@Kripke models – $C \vDash \phi$ – iff for every $M \in C$, $M \vDash \phi$.

Where "M" ranges over the class of @Kripke models, the following soundness results are easily verified:

$\vdash_{\mathbf{K}+A} \phi$ only if $\{M : M = M\} \vDash \phi$.
$\vdash_{\mathbf{KD}+A} \phi$ only if $\{M : M$'s accessibility relation is serial$\} \vDash \phi$.
$\vdash_{\mathbf{KT}+A} \phi$ only if $\{M : M$'s accessibility relation is reflexive$\} \vDash \phi$.
$\vdash_{\mathbf{KTB}+A} \phi$ only if $\{M : M$'s accessibility relation is reflexive and symmetric$\} \vDash \phi$.
$\vdash_{\mathbf{KT4}+A} \phi$ only if $\{M : M$'s accessibility relation is reflexive and transitive$\} \vDash \phi$.
$\vdash_{\mathbf{KT5}+A} \phi$ only if $\{M : M$'s accessibility relation is an equivalence relation$\} \vDash \phi$.

If the converses of the above soundness results hold, the logics are *complete* for the relevant classes of models.

Where "M" ranges over the class of sub@Kripke models based on some @Kripke model, the following familiar soundness and completeness results also hold for the various **S**:[23]

$\vdash_{\mathbf{K}} \phi$ iff $\{M : M = M\} \vDash \phi$.
$\vdash_{\mathbf{KD}} \phi$ iff $\{M : M$'s accessibility relation is serial$\} \vDash \phi$.

$\vdash_{\mathbf{KT}} \phi$ iff $\{M : M$'s accessibility relation is reflexive$\} \vDash \phi$.
$\vdash_{\mathbf{KTB}} \phi$ iff $\{M : M$'s accessibility relation is reflexive and symmetric$\}$ $\vDash \phi$.
$\vdash_{\mathbf{KT4}} \phi$ iff $\{M : M$'s accessibility relation is reflexive and transitive$\}$ $\vDash \phi$.
$\vdash_{\mathbf{KT5}} \phi$ iff $\{M : M$'s accessibility relation is an equivalence relation$\}$ $\vDash \phi$.

Throughout the rest of this section, W is the set of all maximal $\mathbf{S} + A$-consistent sets of wff. Because $\mathbf{S} + A$ contains the classical propositional calculus, the usual results about maximal consistent sets of wff apply. For instance, Lindenbaum's lemma applies; that is, if Γ is an $\mathbf{S} + A$-consistent set of wff, there is a maximal $\mathbf{S} + A$-consistent set w of wff such that $\Gamma \subseteq w$.

THEOREM 1. *For any $w \in W$, there is a unique $y \in W$ such that for all ϕ, $\phi \in y$ iff $A\phi \in w$. Also, for all ϕ, $A\phi \in y$ iff $A\phi \in w$.*

Proof. By w's maximality and Lemma 1, $\neg A\bot \in w$. So by w's consistency, $A\bot \notin w$. It is then easy to show that $\Gamma = \{\phi : A\phi \in w\}$ is $\mathbf{S} + A$-consistent. But Ax2 and w's maximality obviously entail Γ's maximality. And any y in W meeting the above conditions would have to have Γ as a subset; so Γ is unique in meeting those conditions. To prove the second part of the theorem, note that w's maximality and Lemma 2 entail that $A\phi \in w$ iff $AA\phi \in w$, which completes the proof. □

For all $w, y \in W$, the relation R is defined thus: wRy iff $\{\phi : \Box\phi \in w\} \subseteq y$.

LEMMA 4. *For $w, y \in W$, if wRy, then for all ϕ, $A\phi \in w$ iff $A\phi \in y$.*

Proof. The left-to-right direction follows immediately from w's maximality, Lemma 3 and the fact that wRy. For the other direction, suppose that $A\phi \in y$ and $A\phi \notin w$. Then, by w's maximality, $\neg A\phi \in w$ and so $A\neg\phi \in w$. But by Lemma 3 and w's maximality, $\Box A\neg\phi \in w$. And so, as wRy, $A\neg\phi \in y$. But therefore, by y's maximality, $\neg A\phi \in y$, contradicting y's consistency. □

THEOREM 2. *For $w \in W$, if $\neg\Box\psi \in w$, then there exists $y \in W$ such that $\neg\psi \in y$ and wRy.*

Proof. Let $\Gamma = \{\phi : \Box\phi \in w\} \cup \{\neg\psi\}$. Suppose that Γ is $\mathbf{S} + A$-inconsistent; that, for instance, there is $\Box\phi \in w$ such that $\vdash_{\mathbf{S}+A} \phi \rightarrow \psi$. Then $\vdash_{\mathbf{S}+A} \Box(\phi \rightarrow \psi)$. So $\vdash_{\mathbf{S}+A} \Box\phi \rightarrow \Box\psi$. But then, by w's maximality, $\Box\psi \in w$. Contradiction! So Γ is $\mathbf{S} + A$-consistent. There is therefore a

$y \in W$ such that $\Gamma \subseteq y$. But it is obvious that if $\Box\phi \in w$, then $\phi \in y$; and so wRy, which completes the proof. $\Box$

For each $w \in W$, let $@(w)$ be the unique $y \in W$ satisfying the conditions of Theorem 1. Note that for $w \in W$, $\phi \in @(w)$ iff $A\phi \in w$ iff $A\phi \in @(w)$ (the last follows by the second part of Theorem 1). So $@(@(w)) = @(w)$. And note that by Lemma 4, if $w, y \in W$ and wRy, $@(w) = @(y)$. Define P thus: for each LA-propositional variable π, $P(\pi) = \{w : w \in W$ and $\pi \in w\}$. And suppose that $w^* \in W$. Then $M = \langle W, w^*, R, @, P\rangle$ is an @Kripke model; it is an $\mathbf{S} + A$ *canonical @Kripke model*.

THEOREM 3. *For* $w \in W$, $\phi \in w$ *iff* $(M, w) \vDash \phi$.

Proof. Proceeds by induction on the complexity of ϕ. The cases in which ϕ is either a propositional variable or has a main connective which is truth-functional are trivial. The case where ϕ is $\Box\psi$ is easy, given Theorem 2. The case where ϕ is $A\psi$ is also easy: $A\psi \in w$ iff (see the proof of Theorem 1) $\psi \in @(w)$ iff (by inductive hypothesis) $(M, @(w)) \vDash \psi$ iff $(M, w) \vDash A\psi$. $\Box$

It was noted earlier that all of the usual results involving maximal consistent sets apply to W. In particular, the following holds: $\vdash_{\mathbf{S}+A} \phi$ iff for any $w \in W$, $\phi \in w$. Where $w \in W$, let $M_w = \langle W, w, R, @, P\rangle$. The following is easily verified, using Theorem 3: $\phi \in w$ iff $M_w \vDash \phi$. So $\vdash_{\mathbf{S}+A} \phi$ iff for any $w \in W$, $M_w \vDash \phi$. That is, $\vdash_{\mathbf{S}+A} \phi$ iff $\{M_w : w \in W\} \vDash \phi$.

If **S** is **KD**, it is easily verified that for any M_w, R is serial. If **S** is **KT**, then for any M_w, R is reflexive. If **S** is **KTB**, then for any M_w, R is symmetric and reflexive. If **S** is **KT4**, then for any M_w, R is transitive and reflexive. Finally, if **S** is **KT5**, then for any M_w, R is an equivalence relation.

The remarks in the last two paragraphs lead immediately to:

THE FIRST SET OF COMPLETENESS RESULTS. *For each* $\mathbf{S} + A$, *the converse of the soundness result stated earlier in this section holds.*

The following lemma can be used to show that each $\mathbf{S} + A$ conservatively extends **S** in relation to the A-free wff of LA; that is, if ϕ does not contain A (ϕ is a wff of both L and LA), $\vdash_{\mathbf{S}+A} \phi$ iff $\vdash_{\mathbf{S}} \phi$:[24]

LEMMA 5. *Suppose that* M *is an @Kripke model and that* M' *is a sub@Kripke model based on* M. *Then for any* ϕ *which is a wff of both* L *and* LA, *and for any* $w \in W$, $(M, w) \vDash \phi$ *iff* $(M', w) \vDash \phi$.

Proof. A trivial induction on the length of ϕ. □

The interesting parts of the conservative extension results follow from the soundness results for the $\mathbf{S}+A$, Lemma 5 and the completeness results for the $\mathbf{S}$.

4. FIRST SET OF DECIDABILITY RESULTS

The following proofs employ the technique of *mini-canonical models*, which is essentially equivalent to the use of filtrations.[25] The technique provides a way of showing that each $\mathbf{S}+A$ has the finite model property – that is, for each of the logics there is a class of finite (@Kripke) models for which the logic is sound and complete. As each $\mathbf{S}+A$ is finitely axiomatisable, it follows that each $\mathbf{S}+A$ is decidable.[26]

Given a set of wff, Φ:

(1) $\text{sub}(\Phi) = \{\psi$: there is some $\phi \in \Phi$ such that ψ is a well-formed subformula of $\phi\}$,
(2) $\neg(\Phi) = \{\neg\phi : \phi \in \Phi\}$,
(3) $A(\Phi) = \{A\phi : \phi \in \Phi\}$,
(4) $\Phi^* = \text{sub}(\Phi) \cup \neg(\text{sub}(\Phi)) \cup A(\text{sub}(\Phi)) \cup A(\neg(\text{sub}(\Phi)))$. (If Φ is finite, so is Φ^*.)

A set Γ of wff is $A\Phi$-maximal iff (i) $\Gamma \subseteq \Phi^*$; (ii) if $\phi \in \text{sub}(\Phi)$, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$; (iii) if $\phi \in \text{sub}(\Phi)$, either $A\phi \in \Gamma$ or $A\neg\phi \in \Gamma$. Assume that Γ is $A\Phi$-maximal and $\mathbf{S}+A$-consistent. The following facts are easily proven: if $\phi \in \text{sub}(\Phi)$, $\phi \in \Gamma$ iff $\neg\phi \notin \Gamma$; if $\phi\&\psi \in \text{sub}(\Phi)$, $\phi\&\psi \in \Gamma$ iff $\phi \in \Gamma$ and $\psi \in \Gamma$; and if $\phi, \psi \in \Phi^*$, $\vdash_{\mathbf{S}+A} \phi \to \psi$ and $\phi \in \Gamma$, then $\psi \in \Gamma$. The first two of those facts are used in proving the trivial parts of Theorem 10 below; the final one is used at a number of points in the following.

LEMMA 6. *If Φ is a consistent set of wff, then either $\Phi \cup \{A\phi\}$ is consistent or $\Phi \cup \{A\neg\phi\}$ is consistent.*

Proof. If both are inconsistent then there are $\phi_1, \ldots, \phi_n \in \Phi$ such that $\vdash_{\mathbf{S}+A} (\phi_1\& \ldots \&\phi_n) \to \neg A\phi$ and $\vdash_{\mathbf{S}+A} (\phi_1\& \ldots \&\phi_n) \to \neg A\neg\phi$. But $\vdash_{\mathbf{S}+A} (\neg A\phi\&\neg A\neg\phi) \to \perp$. □

THEOREM 4. *Suppose that Φ is finite. If $\Delta \subseteq \Phi^*$ and Δ is $\mathbf{S}+A$-consistent, then there is an $A\Phi$-maximal and $\mathbf{S}+A$-consistent set Γ such that $\Delta \subseteq \Gamma$.*

Proof. As Φ is finite, sub(Φ)’s members can be listed as $\phi_1, \ldots, \phi_n$. Let $\Delta_0 = \Delta$. Then, given Δ_i, define Δ_{i+1} using the following conditions: if $\Delta_i \cup \{\phi_{i+1}\}$ is consistent, $\Delta_{i+1} = \Delta_i \cup \{\phi_{i+1}\}$; otherwise $\Delta_{i+1} = \Delta_i \cup \{\neg\phi_{i+1}\}$. (It is easily shown that if $\Delta_i \cup \{\phi_{i+1}\}$ is not consistent, then $\Delta_i \cup \{\neg\phi_{i+1}\}$ is.) Next, for $i \geq n$, if $\Delta_i \cup \{A\phi_{i+1}\}$ is consistent, $\Delta_{i+1} = \Delta_i \cup \{A\phi_{i+1}\}$; otherwise $\Delta_{i+1} = \Delta_i \cup \{A\neg\phi_{i+1}\}$. (Lemma 6 justifies that definition.) The final set in the series, Δ_{2n-1}, is clearly $A\Phi$-maximal and $\mathbf{S} + A$-consistent; and $\Delta = \Delta_0 \subseteq \Delta_{2n-1}$, which completes the proof. □

Throughout the rest of this section Φ is a finite set of wff, and $W_{\mathbf{S}+A\Phi}$ is the set of all $A\Phi$-maximal $\mathbf{S} + A$-consistent sets of wff. Note that $W_{\mathbf{S}+A\Phi}$ is therefore finite.

THEOREM 5. *For any $w \in W_{\mathbf{S}+A\Phi}$, there is some $y \in W_{\mathbf{S}+A\Phi}$ such that for any $\phi \in$ sub(Φ)$\cup\neg$sub(Φ), $\phi \in y$ iff $A\phi \in w$. Also, for any ϕ, $A\phi \in w$ iff $A\phi \in y$.*

Proof. Let $\Gamma = \{\phi : A\phi \in w\} \cup \{A\phi : A\phi \in w\}$ $(= \alpha \cup \beta)$. Suppose that there is $\phi_1 \in \alpha$ and $A\phi_2 \in \beta$ such that $\vdash_{\mathbf{S}+A} (\phi_1 \& A\phi_2) \to \bot$. It is easily shown, using Lemma 2, that w is therefore $\mathbf{S} + A$-inconsistent, contrary to hypothesis. So Γ is $\mathbf{S} + A$-consistent.

Now, suppose that $\phi \in$ sub(Φ) and that $\phi \notin \Gamma$. Either $A\phi \in w$ or $A\neg\phi \in w$. So clearly $A\neg\phi \in w$; therefore, $\neg\phi \in \Gamma$. Next, suppose that $\phi \in$ sub(Φ) and that $A\phi \notin \Gamma$. Then $A\phi \notin w$. So $A\neg\phi \in w$. And hence $A\neg\phi \in \Gamma$. Γ is therefore $A\Phi$-maximal as well as $\mathbf{S} + A$-consistent.

Suppose that $\phi \in \Gamma$. Suppose that $\phi \in$ sub(Φ). Then $A\neg\phi \notin w$, by Γ’s $\mathbf{S} + A$-consistency. So, by w’s $A\Phi$-maximality, $A\phi \in w$. Suppose that $\phi = \neg\psi \in \neg$(sub(Φ)). Then $A\psi \notin w$, by Γ’s $\mathbf{S} + A$-consistency. So $A\neg\psi = A\phi \in w$. As it is obvious that if $A\phi \in w$, $\phi \in \Gamma$, we get that for any $\phi \in$ sub(Φ) $\cup \neg$(sub(Φ)), $\phi \in \Gamma$ iff $A\phi \in w$.

Finally, we need to show that for any ϕ, $A\phi \in w$ iff $A\phi \in \Gamma$. Only the right-to-left direction is nontrivial. So suppose that $A\phi \in \Gamma$. Then either $A\phi \in w$, or $AA\phi \in w$. But in the latter case, $A\phi \in$ sub(Φ); and by Lemma 2, $A\phi \in w$, completing the proof. □

For all $w, y \in W_{\mathbf{S}+A\Phi}$, define “$R_1$”, “$R_2$”, “$R_3$” and “$R_4$” thus: wR_1y iff (a) $\{\phi : \Box\phi \in w\} \subseteq y$, and (b) $\{A\phi : A\phi \in y\} \subseteq w$; wR_2y iff (a) $\{\phi : \Box\phi \in w\} \subseteq y$, (b) $\{\phi : \Box\phi \in y\} \subseteq w$, and (c) $\{A\phi : A\phi \in y\} \subseteq w$; wR_3y iff (a) $\{\Box\phi : \Box\phi \in w\} \subseteq y$, and (b) $\{A\phi : A\phi \in y\} \subseteq w$; and wR_4y iff (a) $\{\Box\phi : \Box\phi \in w\} = \{\Box\phi : \Box\phi \in y\}$, and (b) $\{A\phi : A\phi \in y\} \subseteq w$.

LEMMA 7. *For all $w, y \in W_{\mathbf{S}+A\Phi}$ and $i = 1, 2, 3$ or 4, if wR_iy then for any ϕ, $A\phi \in w$ iff $A\phi \in y$.*

Proof. The right-to-left part is trivial. Suppose that $A\phi \in w$. Then $\phi \in \text{sub}(\Phi) \cup \neg\text{sub}(\Phi)$. Suppose that $\phi \in \text{sub}(\Phi)$. Then by y's $A\Phi$-maximality, either $A\phi \in y$ or $A\neg\phi \in y$. But if the latter holds, then both $A\phi \in w$ and $A\neg\phi \in w$. The other case is essentially the same. □

THEOREM 6. *Let* **S** = **K**, **KD** *or* **KT**. *Then for* $w \in W_{\mathbf{S}+A\Phi}$, *if* $\neg\Box\psi \in w$, *then there is some* $y \in W_{\mathbf{S}+A\Phi}$ *such that* wR_1y *and* $\neg\psi \in w$.

Proof. Let $\Gamma = \{\phi : \Box\phi \in w\} \cup \{A\phi : A\phi \in w\} \cup \{\neg\psi\}$. Then $\Gamma \subseteq \Phi^*(\neg\Box\psi \in w$; so $\neg\Box\psi \in \Phi^*$; hence $\Box\psi \in \text{sub}(\Phi)$; therefore $\neg\psi \in \neg(\text{sub}(\Phi)) \subseteq \Phi^*)$. It is easily shown that Γ's $\mathbf{S} + A$-inconsistency would entail the $\mathbf{S} + A$-inconsistency of w. Γ is thus $\mathbf{S} + A$-consistent. But by Theorem 4, there is a $y \in W_{\mathbf{S}+A\Phi}$ such that $\Gamma \subseteq y$. Clearly, if $\Box\phi \in w$, $\phi \in y$. Suppose that $A\phi \in y$. Then $\phi \in \text{sub}(\Phi) \cup \neg\text{sub}(\Phi)$. Suppose that $\phi \in \text{sub}(\Phi)$. Then either $A\phi \in w$ or $A\neg\phi \in w$. But if $A\neg\phi \in w$, $A\neg\phi \in \Gamma \subseteq y$, entailing the $\mathbf{S} + A$-inconsistency of y. So $A\phi \in w$. The case where $\phi \in \neg\text{sub}(\Phi)$ is essentially the same. So wR_1y, which completes the proof. □

THEOREM 7. *For* $w \in W_{\mathbf{KTB}+A\Phi}$, *if* $\neg\Box\psi \in w$, *then there is some* $y \in W_{\mathbf{KTB}+A\Phi}$ *such that* wR_2y *and* $\neg\psi \in w$.

Proof. Let $\Gamma = \{\phi : \Box\phi \in w\} \cup \{\neg\Box\phi : \neg\phi \in w$ and $\neg\Box\phi \in w\} \cup \{A\phi : A\phi \in w\} \cup \{\neg\psi\}$. $\Gamma \subseteq \Phi^*$. Suppose that y is $A\Phi$-maximal $\mathbf{KTB} + A$-consistent and that $\Gamma \subseteq y$. We show that wR_2y. If $\Box\phi \in w$, $\phi \in y$. Suppose that $\Box\phi \in y$. Then $\Box\phi \in \text{sub}(\Phi)$; so either $\Box\phi \in w$ or $\neg\Box\phi \in w$. In the former case, $\phi \in w$, as $\mathbf{KTB} + A$ includes **KT**. So suppose that $\neg\Box\phi \in w$ and $\phi \notin w$. Then $\neg\phi \in w$, as $\phi \in \text{sub}(\Phi)$. But then $\neg\Box\phi \in \Gamma \subseteq y$, entailing y's inconsistency. So $\phi \in w$.

To conclude the proof, we must show that Γ is $\mathbf{KTB} + A$-consistent; Theorem 4 and the preceding do the rest. To this end, suppose that $\vdash_{\mathbf{KTB}+A} (\phi_1 \& \neg\Box\phi_2 \& A\phi_3) \rightarrow \psi$, where $\Box\phi_1 \in w$, $\neg\Box\phi_2 \in w$, $\neg\phi_2 \in w$ and $A\phi_3 \in w$. Then $\vdash_{\mathbf{KTB}+A}(\Box\phi_1 \& \Box\neg\Box\phi_2 \& \Box A\phi_3) \rightarrow \Box\psi$. But we then easily get that w is $\mathbf{KTB} + A$-inconsistent. So Γ is $\mathbf{KTB} + A$-consistent, which completes the proof. □

THEOREM 8. *For* $w \in W_{\mathbf{KT4}+A\Phi}$, *if* $\neg\Box\psi \in w$, *then there is some* $y \in W_{\mathbf{KT4}+A\Phi}$ *such that* wR_3y *and* $\neg\psi \in w$.

Proof. Let $\Gamma = \{\Box\phi : \Box\phi \in w\} \cup \{A\phi : A\phi \in w\}$. The rest of the proof proceeds along similar lines to that of Theorem 7. □

THEOREM 9. *For* $w \in W_{\mathbf{KT5}+A\Phi}$, *if* $\neg\Box\psi \in w$, *then there is some* $y \in W_{\mathbf{KT5}+A\Phi}$ *such that* wR_4y *and* $\neg\psi \in w$.

Proof. Let $\Gamma = \{\Box\phi : \Box\phi \in w\} \cup \{\neg\Box\phi : \neg\Box\phi \in w\} \cup \{A\phi : A\phi \in w\}$. The rest of the proof proceeds along similar lines to that of Theorem 7. □

For $w \in W_{\mathbf{S}+A\Phi}$, let $@(w) = \{\phi : A\phi \in w\} \cup \{A\phi : A\phi \in w\}$. By the second part of Theorem 6, $@(@(w)) = @(w)$. Also, by Lemma 7, for $i = 1, 2, 3$ or 4, if $wR_i y$, then $@(w) = @(y)$. For each LA-propositional variable π, let $P_{\mathbf{S}}(\pi) = \{w : w \in W_{\mathbf{S}\Phi}$ and $\pi \in w\}$. Where $i = 1$, let $\mathbf{S} = \mathbf{K}, \mathbf{KD}$ or $\mathbf{KT}$; where $i = 2$, let $\mathbf{S} = \mathbf{KTB}$; where $i = 3$, let $S = \mathbf{KT4}$; and where $i = 4$, let $\mathbf{S} = \mathbf{KT5}$. Where $w \in W_{\mathbf{S}+A\Phi}$, let $M_{i\Phi}(w) = \langle W_{\mathbf{S}+A\Phi}, w, R_i, @, P_{\mathbf{S}}\rangle$. $M_{i\Phi}(w)$ is an @Kripke model.

THEOREM 10. *For $w, y \in W_{\mathbf{S}+A\Phi}$, if $\phi \in \mathrm{sub}(\Phi) \cup \neg(\mathrm{sub}(\Phi))$, then $\phi \in y$ iff $(M_{i\Phi}(w), y) \vDash \phi$.*

Proof. The cases where ϕ is a propositional variable or has a truth-functional main connective are trivial. The cases where $\phi = \Box\psi$ are easy, given Theorems 6–9. Finally, the case where $\phi = A\psi$ is simple: $A\psi \in y$ iff $A\psi \in @(y)$ (by the proof of the second part of Theorem 5) iff $\psi \in @(y)$ (by the proof of the first part of Theorem 5) iff (by inductive hypothesis) $(M_{i\Phi}(w), @(y)) \vDash \psi$ iff $(M_{i\Phi}(w), y) \vDash A\psi$. □

Suppose not-$\vdash_{\mathbf{S}+A} \phi$. Then $\{\neg\phi\}$ is an $\mathbf{S} + A$-consistent set of wff. By Theorem 4, there is a $w \in W_{\mathbf{S}+A\{\neg\phi\}}$ such that $\neg\phi \in w$. By Theorem 10, $(M_{i\{\neg\phi\}}(w), w) \vDash \neg\phi$, and so $M_{i\{\neg\phi\}}(w) \vDash \neg\phi$. It is easily verified that where $\mathbf{S} = \mathbf{KD}$ or $\mathbf{KT}$, R_1 is respectively serial and reflexive; where $\mathbf{S} = \mathbf{KTB}$, R_2 is reflexive and symmetric; where $\mathbf{S} = \mathbf{KT4}$, R_3 is reflexive and transitive; and where $\mathbf{S} = \mathbf{KT5}$, R_4 is an equivalence relation. But each $M_{i\{\neg\phi\}}(w)$ is finite. By the soundness results stated in Section 3, therefore, the various $\mathbf{S} + A$ have the finite model property, which gives:

THE FIRST SET OF DECIDABILITY RESULTS. *$\mathbf{K} + A$, $\mathbf{KD} + A$, $\mathbf{KT} + A$, $\mathbf{KTB} + A$, $\mathbf{KT4} + A$ and $\mathbf{KT5} + A$ are all decidable.*

5. Second Group of Completeness and Decidability Results

By the first set of decidability results, we can axiomatise a logic by taking all of $\mathbf{S} + A$’s theorems plus all instances of the following axiom schema AxA′ as axioms, and having MP as the sole (universal) rule:

AxA′ : $\phi \leftrightarrow A\phi$.

Call the logic thereby axiomatised, $\mathbf{S} + A[A']$.[27] While none of the various $\mathbf{S} + A[A']$ is informally sound, for reasons explained in Section 2, each instance of AxA′ is informally safe. Informal soundness implies informal safety; so if $\mathbf{S} + A$ is informally sound, $\mathbf{S} + A[A']$ is informally safe. In the rest of this section, W' is understood to be the set of maximal $\mathbf{S} + A[A']$-consistent sets of wff.

THEOREM 11. *Suppose that $M = \langle W, w^*, R, @, P\rangle$ is an $\mathbf{S}+A$ canonical @Kripke model. Then for any $w \in W$, $@(w) \in W'$. And for any $w' \in W'$, there is $w \in W$ such that $w' = @(w)$. That is $\{@(w) : w \in W\} = W'$.*

Proof. For $w \in W$, $@(w) = \{\phi : A\phi \in w\}$ (see Section 3). By the proof of Theorem 1, $A\phi \in @(w)$ iff $A\phi \in w$; so $\phi \in @(w)$ iff $A\phi \in @(w)$. But by Theorem 1, $@(w)$ is maximal. Therefore, for any ϕ, $(\phi \leftrightarrow A\phi) \in @(w)$. Each instance of AxA′ is therefore in $@(w)$, as also are all of $\mathbf{S} + A$'s theorems; so all of $\mathbf{S} + A[A']$'s theorems are in $@(w)$. $@(w)$'s $\mathbf{S}+A[A']$-consistency is then easily proved, using $@(w)$'s maximality. So $@(w) \in W'$.

For the second part, suppose that $w' \in W'$. Then $w' \in W$. Let $\Gamma = \{A\phi : \phi \in w\prime\}$. Γ's $\mathbf{S} + A$-consistency is easily proved. By Lindenbaum's lemma, there is therefore a $w \in W$ such that $\Gamma \subseteq w$. Suppose that $A\phi \in w$ but $\phi \notin w'$. By the maximality of w', $\neg\phi \in w'$. So $A\neg\phi \in \Gamma$, and hence $A\neg\phi \in w$. But then w is $\mathbf{S}+A$-inconsistent, which is absurd. But as clearly $w' \subseteq \{\phi : A\phi \in w\}$, we get that $@(w) = w'$, completing the proof. □

Let $M = \langle W, w^*, R, @, P\rangle$ be an $\mathbf{S}+A$ canonical @Kripke model. Define "$M \vDash' \phi$" thus: $M \vDash' \phi$ iff for any $w \in W$, $(M, @(w)) \vDash \phi$. Then Theorems 3 and 11 entail that $\phi \in w'$, for all $w' \in W'$, iff $M \vDash' \phi$. And as $\vdash_{\mathbf{S}+A[A']} \phi$ iff for any $w' \in W'$, $\phi \in w'$, we get:

LEMMA 8. *For any ϕ, $\vdash_{\mathbf{S}+A[A']} \phi$ iff $M \vDash' \phi$, where M is an $\mathbf{S} + A$-canonical @Kripke model.*

Let $M = \langle W, w^*, R, @, P\rangle$ be an $\mathbf{S}+A$ canonical @Kripke model. Where $w \in W$, any @Kripke model $M' = \langle W, @(w), \mathrm{R}, @, P\rangle$ is a *centred @Kripke model based on M*. Lemma 8 entails that $\vdash_{\mathbf{S}+A[A']} \phi$ iff for each centred @Kripke model M' based on M, $M' \vDash \phi$. Notice that if M is a serial @Kripke model, then M' is a serial centred @Kripke model; if M is a reflexive @Kripke model, M' is a reflexive centred @Kripke model; and so on. Given the observations about the accessibility relations in $\mathbf{S}+A$ canonical models made prior to the first set of completeness results, that leads immediately to:

THE SECOND SET OF COMPLETENESS RESULTS. $\mathbf{K}+A[A']$ *is complete for the class of centred @Kripke models;* $\mathbf{KD} + A[A']$ *is complete* *for the class of serial centred @Kripke models;* **KT** $+ A[A']$ *is complete for the class of reflexive centred @Kripke models;* **KTB** $+ A[A']$ *is complete for the class of symmetric and reflexive centred @Kripke models;* **KT4** $+ A[A']$ *is complete for the class of transitive and reflexive centred @Kripke models; and* **KT5** $+ A[A']$ *is complete for the class of centred @Kripke models whose accessibility relation is an equivalence relation.*

It is easy to verify that the converse of each of the above completeness results holds; that is, that each of the above logics is sound for the relevant class of centred @Kripke models.[28]

The following lemma provides a neat characterisation of the theorems belonging to the various $\mathbf{S} + A[A']$:

LEMMA 9. *For any* ϕ, $\vdash_{\mathbf{S}+A[A']} \phi$ *iff* $\vdash_{\mathbf{S}+A} A\phi$.

Proof. Let $M = \langle W, w^*, R, @, P\rangle$ be an $\mathbf{S} + A$-canonical model. Then $\vdash_{\mathbf{S}+A[A']} \phi$ iff (by Lemma 8) $M \vDash' \phi$ iff for every $w \in W$, $(M, w) \vDash A\phi$, iff (by Theorem 3) $\vdash_{\mathbf{S}+A} A\phi$. □

Lemma 9 and the first set of decidability results also give:

THE SECOND SET OF DECIDABILITY RESULTS. **K** $+ A[A']$, **KD** $+ A[A']$, **KT** $+ A[A']$, **KTB** $+ A[A']$, **KT4** $+ A[A']$ *and* **KT5** $+ A[A']$ *are decidable.*

Notice that Lemma 9, the fact that $\mathbf{S} + A$ conservatively extends **S** in relation to the A-free wff of LA, and the fairly easily proven fact that, for any ϕ which is a wff of both L and LA, $\vdash_{\mathbf{S}+A} \phi$ iff $\vdash_{\mathbf{S}+A} A\phi$, entail that $\mathbf{S} + A[A']$ also conservatively extends **S**.[29]

6. Some Final Results and Remarks

1. Suppose that $M = \langle W, w^*, R, @, P\rangle$ is an @Kripke model. Then let M_{w^*} be the @Kripke model resulting when one appropriately restricts each element of M to those indices in W which are R-descendants of either w^* or $@(w^*)$ (counting both w^* and $@(w^*)$ as degenerate R-descendants of themselves). It is obvious that $M \vDash \phi$ iff $M_{w^*} \vDash \phi$. But for any $w \in W$, each R-descendant y of w is such that $@(y) = @(w)$, and $@(w) = @(@(w))$; so there is, as it were, a single actual world in M_{w^*}. Note also that R is serial only if the restriction of R figuring in M_{w^*} is serial; R is reflexive only if the restriction of R figuring in M_{w^*} is reflexive; and so on.

It follows that the various $\mathbf{S} + A$ and $\mathbf{S} + A[A']$ are complete for more familiar classes of models than the classes of @Kripke models used above. Consider, for instance, $\mathbf{KT4} + A[A']$. Suppose that $M = \langle W, w^*, R, @, P\rangle$ is a transitive and reflexive centred @Kripke model. Let a *standardised @Kripke model* be one which contains a single actual world. Then M_{w^*} is a standardised transitive and reflexive @Kripke model. But $M_{w^*} \vDash \neg\phi$ iff $M \vDash \neg\phi$. The second set of completeness proofs immediately entails that $\mathbf{KT4} + A[A']$ is complete for the class of standardised transitive and reflexive @Kripke models.

The advantages in using the wider class of @Kripke models employed above are, however, obvious. For instance, suppose that one is trying to prove the completeness of $\mathbf{KT4} + A$. One might hope to use canonical models, as they provide a simple and elegant way of proving completeness results elsewhere. But the set of all maximal and $\mathbf{KT4} + A$-consistent sets of LA's wff cannot form the basis of a standardised transitive and reflexive @Kripke canonical model; for there may be maximal and $\mathbf{KT4}+A$-consistent sets w and y for which $A\phi \in w$ but $\neg A\phi \in y$. Similar remarks apply to the proofs of decidability.

2. Notice that none of the various $\mathbf{S}+A$ studied above has each instance of $\Box\phi \rightarrow A\phi$ as a theorem. (Each $\mathbf{S} + A[A']$ which extends **KT** obviously has each instance as a theorem, and each $\mathbf{S} + A[A']$ having each instance as a theorem obviously extends **KT**.) Let $\mathbf{S} + AA^*$ be the logic resulting from the addition of each instance of the following axiom schema AxA* to the earlier axiomatisation of $\mathbf{S} + A$:

AxA*: $\Box\phi \rightarrow A\phi$.

An @Kripke model $M = \langle W, w^*, R, @, P\rangle$ is a *super @Kripke model* iff for any $w \in W$, $wR@(w)$. Where "$\mathcal{M}$" ranges over the class of super @Kripke models, the following soundness and completeness results are fairly easily proven:

$\vdash_{\mathbf{K}+AA^*} \phi$ iff $\{M : M = \mathcal{M}\} \vDash \phi$.

$\vdash_{\mathbf{KT}+AA^*} \phi$ iff $\{\mathcal{M} : \mathcal{M}$'s accessibility relation is reflexive$\} \vDash \phi$.

$\vdash_{\mathbf{KTB}+AA^*} \phi$ iff $\{\mathcal{M} : \mathcal{M}$'s accessibility relation is reflexive and symmetric$\} \vDash \phi$.

$\vdash_{\mathbf{KT4}+AA^*} \phi$ iff $\{\mathcal{M} : \mathcal{M}$'s accessibility relation is reflexive and transitive$\} \vDash \phi$.

$\vdash_{\mathbf{KT5}+AA^*} \phi$ iff $\{\mathcal{M} : \mathcal{M}$'s accessibility relation is an equivalence relation$\} \vDash \phi$.

Each of the above logics is finitely axiomatisable, has the finite model property, and is thus decidable.[30]

Now consider the logics $\mathbf{S} + \{\Box\phi \rightarrow \Box^n\Diamond\phi : n \in \text{Nat}\}$.[31] Where "$M$" ranges over the class of sub@Kripke models based on some super @Kripke model, the following soundness and completeness results hold:[32]

$\vdash_{\mathbf{K}+\{\Box\phi\rightarrow\Box n\Diamond\phi:n\in\text{Nat}\}} \phi$ iff $\{M : M = M\} \vDash \phi$.
$\vdash_{\mathbf{KT}+\{\Box\phi\rightarrow\Box n\Diamond\phi:n\in\text{Nat}\}} \phi$ iff $\{M : M$'s accessibility relation is reflexive$\}$ $\vDash \phi$.
$\vdash_{\mathbf{KTB}+\{\Box\phi\rightarrow\Box n\Diamond\phi:n\in\text{Nat}\}} \phi$ iff $\{M : M$'s accessibility relation is reflexive and symmetric$\} \vDash \phi$.
$\vdash_{\mathbf{KT4}+\{\Box\phi\rightarrow\Box n\Diamond\phi:n\in\text{Nat}\}} \phi$ iff $\{M : M$'s accessibility relation is reflexive and transitive$\} \vDash \phi$.
$\vdash_{\mathbf{KT5}+\{\Box\phi\rightarrow\Box n\Diamond\phi:n\in\text{Nat}\}} \phi$ iff $\{M : M$'s accessibility relation is an equivalence relation$\} \vDash \phi$.

It follows that $\mathbf{S} + AA^*$ conservatively extends $\mathbf{S} + \{\Box\phi \rightarrow \Box^n\Diamond\phi : n \in \text{Nat}\}$ in relation to the A-free wff of LA, by the soundness results for the $\mathbf{S} + AA^*$, Lemma 5 and the completeness results for the $\mathbf{S} + \{\Box\phi \rightarrow \Box^n\Diamond\phi : n \in \text{Nat}\}$.[33]

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Notes

[1] More traditionally known as **K**, **D**, **T**, **B**, **S4** and **S5**.

[2] For the record, the class of L's wff is defined thus: it is the smallest class WFF of strings containing each propositional variable of L; each string $\neg\phi$, $\Box\phi$, where $\phi \in$ WFF; and each string $(\phi\&\psi)$, where $\phi, \psi \in$ WFF. (L's propositional variables are $p, q, r, p_1, \ldots$) $(\phi \vee \psi)$ abbreviates $\neg(\neg\phi\&\neg\psi)$; $(\phi \rightarrow \psi)$ abbreviates $(\neg\phi \vee \psi)$; $(\phi \leftrightarrow \psi)$ abbreviates $((\phi \rightarrow \psi)\&(\phi \rightarrow \psi))$; and $\Diamond\phi$ abbreviates $\neg\Box\neg\phi$. When they do not matter, inner and outer brackets are left out below. I also frequently commit the peccadillo of speaking as if $\vee$, $\rightarrow$, $\leftrightarrow$ and $\Diamond$ are symbols of L which are to be interpreted in the standard ways, for ease of exposition.

[3] For the record, the class of LA's wff is defined thus: it is the smallest class WFF of strings containing each propositional variable of L; each string $\neg\phi$, $\Box\phi$, $A\phi$ where $\phi \in$ WFF; and each string $(\phi\&\psi)$, where $\phi, \psi \in$ WFF.

[4] How to generalise this interpretation to the first-order case? The simplest strategy is to regard open formulae as expressing singular propositions relative to an interpretation and a variable assignment.

[5] Humberstone [10] discusses the link between "actually" operators and the indicative mood at length. He expresses his points using possible worlds semantics, but the underlying themes are clearly the same as those discussed above. For instance, Humberstone writes on p. 104 that "[The 'actually' operator A] functions... as an inhibitor; semantically, it protects what is in its scope by saying 'There's no free world-variable here'."

[6] There is some sleight of hand here. When "actually" is read as validating (A), the inference from "necessarily P" to "necessarily actually P" does not owe its validity to the (false) principle "necessarily P iff necessarily actually P". Rather, it comes from the validity of "necessarily P; therefore actually P" and "actually P; therefore necessarily actually P". For more discussion of the latter inference, see Section 2; see also Lemma 3.

[7] See, for instance, [2] and [6].

[8] See [2].

[9] See respectively Hazen [7] and Hodes [8]. Hazen [6] presents results related to those proved by Hodes.

[10] [3], p. 1. See also [6], p. 40.

[11] For instance, Forbes takes issue with it: see his [4], p. 92.

[12] The only decidability proofs which I have met with in the literature are those in Crossley and Humberstone [2] for extensions of **KT5**. Their proofs are much more complex than those given below, and go via normal forms.

[13] For instance, Melia [11] states that $(Ap \rightarrow \Box Ap)$ "is intuitively false" (p. 49). Forbes [5] describes that wff as "unintuitive but valid" (p. 61).

[14] [2], p. 17. They appear to pin some of the blame on **KT5**, which is unfair. Each instance of $A\phi \rightarrow \Box A\phi$ is a theorem of all of the logics considered below, even the conservative extensions of **K**.

[15] [2], pp. 14–15.

[16] [7], p. 502.

[17] Reading the conditional as material implication and assuming that there might have been no tangerines.

[18] Why such a roundabout route to that conclusion? Because the sorts of arguments usually used – for instance, that P may obviously hold in some possible world while not holding in the actual world – employ just the kind of semantic assumptions that I am foregoing. The argument in the text shows that one can rely upon uncontentious resources, yet end up with the conclusions reached using those more theory laden means.

[19] Note that informal soundness entails informal safety.

[20] Let **N** be the schema $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$; let **T** be the schema $\Box\phi \rightarrow \phi$; let **B** be the schema $\Box\Diamond\phi \rightarrow \phi$; let **4** be the schema $\Box\phi \rightarrow \Box\Box\phi$; and let **5** be the schema $\Diamond\phi \rightarrow \Box\Diamond\phi$. Then **K** is axiomatisable in LA by taking each truth-functional tautology and and each instance of **N** as axioms; having MP as the sole universal rule; and having RN as the sole admissible rule. The other logics mentioned are axiomatisable by adding each instance of the schemata whose name follows "**K**" in the logic's name as an axiom to the preceding axiomatisation of **K**.

[21] The last of those two claims is not, strictly speaking, right: the various **S** $+ A$ contain additional theorems to those which the various **S** contain, so more is involved in claiming that RN is informally sound in the context of the various **S** $+ A$ than is involved in making

a similar claim about the various **S**. Nonetheless, I take it that each Ax1–4 is not only necessary, but necessarily necessary, and necessarily necessarily necessary, etc.; so that RN will lead us to theorems interpretable as expressing nonnecessary propositions only if there are already such theorems within the various **S**.

22 It is easy to show that each instance of Ax4 is a theorem of a normal modal logic formulated in LA if each instance of $A\phi \rightarrow \Box A\phi$ is a theorem of the logic.

23 The class of sub@Kripke models based upon some @Kripke model is, in fact, the class of sub@Kripke models. The above completeness results are easily proved by proving similar completeness results relating to the class as latterly described. And those latter results follow from similar completeness results for the counterparts of the **S** formulated in the A-free language L, given the following: (1) a one-one mapping f from the propositional variables of L onto the class of LA's propositional variables and wff of the form $A\phi$ (this provides a way of moving back-and-forth between Kripke models for L and sub@Kripke models for LA); and (2) the fact, where $\pi_1, \ldots, \pi_n$ are the propositional variables occurring in the L wff ϕ, that any proof of ϕ in the counterpart of **S** formulated in L can be transformed into a proof in **S** of the wff of LA which results when each occurrence of π_i in ϕ is replaced by an occurrence of $f(\pi_i)$.

24 One logic is usually said to conservatively extend another iff the first is formulated in an extension of the language of the second, but the logics coincide on the theorems from the language of the second logic. Let $\mathbf{S}(L)$ be the obvious counterpart of **S** formulated in L. Then as **S** conservatively extends $\mathbf{S}(L)$ in the standard sense, $\mathbf{S} + A$ conservatively extends $\mathbf{S}(L)$ in the standard sense. Similar remarks apply to all of the results below which are described as "conservative extension" results.

25 See [9], Chapter 8.

26 For an explanation of why this entails $\mathbf{S} + A$'s decidability, see [9], Chapter 8, or [1], pp. 62–63.

27 The bracketing of "A'" is to reflect the fact that these logics are not generated simply by adding each instance of AxA$'$ to the earlier axiomatisations of $\mathbf{S} + A$. The notation is owed to Segerberg: see [12], p. 177.

28 Each of the $\mathbf{S} + A[A']$ is nonnormal, as RN fails in them all. Each $\mathbf{S} + A[A']$ is, however, *quasi-normal*; each one extends **K** ([12], p. 172). Chapter 3 of Segerberg [12] introduced the notion of quasi-normality (in the terms of his discussion, centred @Kripke models are @Kripke models based upon frames containing a unique distinguished element w satisfying the condition that $@(w) = w$).

29 For any @Kripke model $M = \langle W, w^*, R, @, P\rangle$, define $@'$ thus: for any $w \in W$, $@'(w) = w^*$. To prove that for any wff ϕ of L and LA, $\vdash_{\mathbf{S}+A} A\phi$ only if $\vdash_{\mathbf{S}+A} \phi$, note that Lemma 5 and the fact that there is a sub@Kripke model based on both M and $\langle W, w^*, R, @', P\rangle$ can be used to show that, for any wff ϕ of L and LA, $M \models \phi$ iff $\langle W, w^*, R, @', P\rangle \models \phi$. And it is obvious that $\langle W, w^*, R, @', P\rangle \models \phi$ iff $\langle W, w^*, R, @', P\rangle \models A\phi$. It is then easy to show, using the soundness and completeness results for the $\mathbf{S} + A$, that $\vdash_{\mathbf{S}+A} A\phi$ only if $\vdash_{\mathbf{S}+A} \phi$.

30 Mini-canonical models can be used to prove that those logics have the finite model property.

31 That is, the logics axiomatisable by adding, for each natural number n, each instance of each schema $\Box\phi \rightarrow \Box^n\Diamond\phi$ to the earlier axiomatisations of the various **S**.

32 The soundness parts of the above results are easy to prove. Where $M = \langle W, w^*, R, P\rangle$ is a sub@Kripke model, let $\mathrm{ch}(w) = \{w' : w'$ is an R-descendant of w or w is an R-descendant of $w'\}$. The class of sub@Kripke models based upon some super @Kripke

model is, in fact, the class of sub@Kripke models in which for each index w, there is an index y such that for every $w' \in \mathrm{ch}(w)$, $w'Ry$. The completeness parts of the above results are easily proved by proving similar completeness results relating to the class as latterly described. And those latter results follow from similar completeness results for the counterparts of the $\mathbf{S} + \{\Box\phi \rightarrow \Box^n\Diamond\phi : n \in \mathrm{Nat}\}$ formulated in the A-free language L, given the following: (1) a one-one mapping f from the propositional variables of L onto the class of LA's propositional variables and wff of the form $A\phi$ (this provides a way of moving back-and-forth between Kripke models for L and sub@Kripke models for LA); and (2) the fact, where $\pi_1, \ldots, \pi_n$ are the propositional variables occurring in the wff ϕ of L, that any proof of ϕ in the counterpart of **S** formulated in L can be transformed into a proof in **S** of the wff of LA which results when each occurrence of π_i in ϕ is replaced by an occurrence of $f(\pi_i)$.

[33] I owe the conjecture that this result might be provable to the anonymous referee.

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Trinity College
Cambridge
CB2 ITQ
England