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Keeping Semantics Pure

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I. Introduction

The most familiar and fruitful treatments of many formal languages are model theoretic. Indeed, formal languages tend to be viewed rather sniffily until they have been provided with model theories which either seem to reflect our understanding of the notions being formalised or can be used in proving soundness and completeness theorems for some interesting logics. Thus Benacerraf and Putnam mocked modal logic as 'muddle logic' until Kripke provided it with a satisfactory model theory.[1]

But some common uses of model theories come at an apparent price. Consider tense logics. Tense-logical languages feature expressions that are meant to be interpreted using tensed operators like 'it was the case that _'. The usual model theory for those languages is based upon indices, which we are encouraged to think of as *times*. The special operators in tense-logical languages are then given model-relative truth conditions which are defined over the indices in the models.

It is natural to use the model theory for tense-logical languages when studying inferences formalisable within them. For instance, it is easy to find models based upon the natural numbers in which the premiss of a sequent[2] formalising the inference '*P* was the case; hence *P* is the case' holds but its conclusion does not—it is easy, that is, to find a *countermodel* to the relevant sequent. It is tempting to infer that the inference is therefore invalid.

By what right, though, do we draw conclusions about the invalidity of real tense-involving inferences from facts about models for tense-logical languages? Here is one sketch of an answer. First, the tensed locutions formalised in tense-logical languages have truth conditions which speak of times. So, second, the model theory for tense-logical languages reflects the contributions which those locutions make to the truth conditions of the sentences in which they occur. Hence, finally, suitable countermodels to sequents of tense-logical languages correspond to possible set-ups which invalidate the inferences formalised by those sequents.

The preceding line of thought is attractive although sketchy. Yet it is startlingly strong. For instance, its correctness would make standard model theoretic methods for demonstrating tense-involving invalidities immediately unavailable to presentists. And analogous arguments promise to put familiar model theoretic methods for proving facts about the validity of, say, modal inferences beyond the reach of those who want to remain agnostic about whether there are possible worlds.

While considering model theoretic methods within modal logic, Plantinga introduces a nice and currently relevant distinction between *pure* and *applied* model theoretic semantics.[3] A pure semantics merely specifies models relative to which a stipulative 'truth at' relation for a formal language's wffs is defined. One starts doing applied semantics when one starts viewing a model theory as reflecting the truth conditions of those propositions which are formalisable within the relevant formal language.

When discussing modal logics, Plantinga also remarks that 'it is not to the pure semantics as such that we must look for insight into our modal notions'.[4] Similarly, Haack writes that a 'pure semantics, by itself is not sufficient; to justify the claim of a formal system *to be a modal logic* . . . some intuitive account of the formal semantics, connecting that set-theoretical construction with the ideas of necessity and possibility . . . seems essential'.[5] And Lewis states that 'if the metalogical results [concerning models for formal modal languages] are to be at all relevant to modality, *some* quantificational analysis [of modal notions in terms of possible worlds] has to be correct'.[6]

Those comments chime with the musings which started this paper. For the remarks generalise to suggest that model theoretic facts can only ground conclusions about validity if one accepts that the relevant model theory can be used as an *applied* semantics. If that is correct then, for instance, those who use standard models for tense-logical languages in drawing conclusions about validity are committed to treating 'it was the case that _' as a quantifier over times. And those who use standard models for modal languages in drawing conclusions about validity are committed to treating 'it is possible that _' as a quantifier over possible worlds.

More generally, those who wish to use model theoretic facts in drawing conclusions about validity face a conditional challenge. If they do not want to treat the relevant model theory as an applied semantics, they need to explain how they can avoid doing so. The rest of this paper formulates some highly general techniques which can help to keep semantics pure.

II. Some constraints

One particular use of model theoretic facts, their employment in drawing conclusions about modal inferences, has recently received a fair bit of attention. A brief discussion of two of the approaches which have been suggested, by Chihara and Divers, will illustrate some demands which we can make upon proposed methods for linking model theoretic facts to conclusions about validity.

Chihara describes his aim as being 'to show, by means of a logical theorem obtained as a result of an analysis of possible worlds semantics, that one can apply systems of modal quantificational logic to evaluating modal principles and to assessing modal arguments, without incurring a commitment to the existence of possible worlds'.[7] The main idea behind his treatment, whose details are largely irrelevant here, is to treat the indices of Kripke-models for first-order modal languages as representations of how things might have been.[8]

Divers's approach is based upon Rosen's modal fictionalism.[9] The central idea of modal fictionalism is that modal statements are equivalent to ones concerning what a certain fiction states to be the case. Rosen suggests, in particular, that for a given modal proposition *P*, where Lewis would offer *P** as an analysis of *P*, *P* is equivalent to the following: according to Lewis's theory of possible worlds, *P**.[10]

Divers argues that Rosen's theory yields 'one of the benefits that is associated with talking in terms of a plurality of possible worlds—namely, doing modal logic by proxy in a purely first-order medium of inference'.[11] Suppose that the *P*s are some modal premisses and that *Q* is a modal conclusion. And assume that the *P**s and *Q** are Lewis's analyses of the *P*s and *Q*. Then, Divers argues, it follows from modal fictionalism that if 'the *P**s; hence *Q**' is a valid inference, so is the inference 'the *P*s; hence *Q*'. So when we can use model theoretic reasoning to show that 'the *P**s; hence *Q**' is valid, Divers's conclusion allows us to infer that 'the *P*s; hence *Q*' is valid.[12]

Chihara's and Divers's treatments of modal inferences are uneconomical in certain respects, and it would be nice to have methods which are more economical than theirs. In particular, Chihara restricts his attention to modal logics containing **S5** and his arguments employ the characteristic **S5** principle that whatever is possible is necessarily possible.[13] His methods for deriving facts about modal validity therefore appear to be available only to those who accept that principle. In a similar vein, Divers's arguments assume the characteristic **S4** principle, that whatever is necessary is necessarily necessary.[14]

But consider the valid inference 'necessarily, *P* and *Q*; hence *P*', which corresponds to a provable sequent of the weak modal logic **T** formulated below. Chihara's argument for the validity of that inference assumes that every possibility is necessarily possible, while Divers's assumes that necessary truths are always necessarily necessary. Yet one might reasonably hope to provide a model theoretic demonstration of the above inference's validity, and that of the other modal inferences expressed by provable sequents of **T**, without relying upon such very strong assumptions concerning iterated modalities.

Chihara's ideas also lack generality in an important way. Although his primary interest is first-order modal logic, he sometimes makes more ambitious claims for his ideas. For instance, he once states that his aim is to account 'for the use of possible worlds semantics in modal logic, in a way that does not presuppose the existence of possible worlds'.[15]

Now, model theoretic techniques based upon Kripke-models have been applied well beyond first-order modal logics. They have, for instance, been applied to counterfactual logics. But there is no apparent way to extend Chihara's techniques to that case. So an important use of possible worlds semantics within modal logic remains unaccounted for by Chihara's proposals. Also, the model theoretic insights underlying the use of Kripke-models have been very widely employed, for instance in studying deontic logics, epistemic logics and tense logics. But Chihara's ideas do not naturally generalise to cover the nonmodal domains which have been illuminated by those insights, again making his ideas too restrictive for my current purposes.

Divers's approach extends more widely than Chihara's. For it can be applied whenever one has a modal inference whose premisses and conclusion have been analysed by Lewis. Divers's methods can therefore be applied to, for instance, counterfactual inferences as well as to ones involving the standard alethic modalities. But the reach of Divers's ideas is nonetheless inadequate for our current purposes. For it is very hard to see how one could extend those ideas to cover additional areas in which we wish to draw conclusions about validity from model theoretic facts. As far as I can tell, for instance, Divers's techniques cannot be used to ground conclusions about the validity of tense-involving inferences in facts about models for tense-logical languages.[16]

All that suggests a constraint which proposed model theoretic routes to conclusions about validity would do well to satisfy: they should be topic-neutral. We don't want methods which only have a chance of applying to modal inferences, or which only have a chance of applying to tense-involving ones. We rather want methods which have a chance of applying to all of those inferences and to many more. The following section formulates a range of ideas needed for the development of some suitable techniques.

III. Some useful concepts

Suppose that we are given a formal language. A *sequent* of the language consists of a set of the language's wffs, followed by the symbol '$\vdash$', which is in turn followed by a wff of the language. The wffs to the left of a sequent's '⊢' are its *premisses*, that to the right its *conclusion*. A *logic* is a set of sequents. A sequent is *provable* in a logic just in case it is a member of the logic. If a sequent whose premiss-set is empty is provable in a logic, its conclusion is a *theorem* of the logic.

It may be stipulated that some of the language's symbols are to be interpreted as having specific meanings. Those are, I shall say, the *logical* symbols of the language. It is normally stipulated, for instance, that the symbol '&' is to interpreted as expressing conjunction. Hence the ampersand is usually a logical symbol.

The *nonlogical* symbols of a formal language are, by contrast, not assigned specific interpretations. One can rather interpret them in many ways, although the interpretations may have to accord with certain constraints. So, for instance, the predicate letters of first-order formal languages can be variously interpreted, although they must always be read as expressing predicates.

An *interpretation* of a formal language consists of a permissible assignment of meanings to its nonlogical expressions. By abstracting away from specific interpretations, we can use sequents of formal languages to express inferences having many instances. So, for instance, the sequent (p&q)⊢p expresses the inference '*P* and *Q*; hence *P*'. A many-instanced inference is *valid* just in case each of its instances is necessarily truth-preserving.[17]

Next, an *axiomatisation* consists of the following: first, a set of wffs, the axiomatisation's *axioms*; second, a set of rules which derive a wff from suitable wffs, labelled as the axiomatisation's *universal* rules; and finally, a set of rules which yield a wff from suitable wffs, labelled as the axiomatisation's *admissible* rules.[18] The reason for the distinction between universal and admissible rules should become clear in the following paragraphs.

The logic *axiomatised* by an axiomatisation consists of the set containing precisely the following sequents. First, those sequents whose premiss-set is empty, and whose conclusion can be derived from the axiomatisation's axioms through zero or more uses of its universal and admissible rules—the axiomatisation's *theorem-sequents*. And, second, those sequents whose conclusion can be derived from the sequent's premisses and, perhaps, the conclusions of some theorem-sequents, using zero or more applications of the axiomatisation's universal rules.

There is a natural difference which provides the intuitive underpinning for the distinction between universal and admissible rules. Universal rules correspond to rules of reasoning which we can always safely use in reasoning from some premisses to a conclusion. For instance, the rule 'given *P* and "if *P*, *Q*", infer *Q*' is such a rule. But admissible rules correspond to rules which can only safely be applied to claims which have certain special features. For instance, the rule 'given *P*, infer that necessarily, *P*' will only lead to true conclusions when applied to propositions which are necessary.

I want next to introduce a notion which will be central to what follows. Suppose that we are given a logic. And suppose that the logic has the following feature: each provable sequent of the logic expresses a valid inference. Then the logic is, I shall say, *informally sound*. So, for instance, suppose that L is a standard propositional language.[19] The logic consisting of all and only the sequents of L whose premisses are true at a line of a truth-table only if the sequent's conclusion is also true at that line—classical propositional logic—is informally sound.

We can sometimes use axiomatisations to argue that logics are informally sound.[20] In particular, consider an axiomatisation whose elements have the following features. First, each of the given axioms expresses a necessary truth under each interpretation of the axiomatisation's language. Second, each of the given universal rules expresses a valid inference. Finally, suppose that each of given admissible rules meets the following condition: if the ϕs are the conclusions of some of the axiomatisation's theorem-sequents, and the rule applies to the ϕs to give ψ, ψ expresses a necessary truth on each interpretation of the axiomatisation's language. It follows from the assumptions just stated that the logic axiomatised by the given axiomatisation is informally sound.[21] It follows, that is, that a sequent is provable in the logic only if it expresses a valid inference.

So, for instance, here is a slightly baroque axiomatisation of classical propositional logic **Cl** in the propositional language L (the admissible rule is redundant but included for illustrative purposes):

(**Cl**Ax) *Axioms*: each instance of the schemas $(\phi\rightarrow(\psi\rightarrow\phi))$; $((\phi\rightarrow(\psi\rightarrow\chi))\rightarrow((\phi\rightarrow\psi)\rightarrow(\phi\rightarrow\chi))$; $((\neg\psi\rightarrow\neg\phi)\rightarrow((\neg\psi\rightarrow\phi)\rightarrow\psi))$.
Universal rule: modus ponens ('infer ψ from ϕ and $\phi\rightarrow\psi$').
Admissible rule: infer $(\phi\vee\psi)$ from ϕ.

(**Cl**Ax) meets the first of the conditions which ensure that an axiomatisation axiomatises an informally sound logic—each of its axioms expresses a necessary truth under each interpretation of L.

Thus consider the first axiom schema of (**Cl**Ax), $(\phi\rightarrow(\psi\rightarrow\phi))$. Stripped of abbreviations—in L, $(\phi\rightarrow\psi)$ is short for $(\neg\phi\vee\psi)$—that schema becomes $(\neg\phi\vee(\neg\psi\vee\phi))$. Under each interpretation of L, therefore, a given instance of that schema expresses a proposition of the form 'either not-P or either not-Q or P'. But, assuming the law of excluded middle, all propositions of that form are necessarily true, because they are equivalent to propositions of the form 'either not-Q or either not-P or P'. Further arguments starting from intuitively obvious truths can be used to show that each instance of (**Cl**Ax)'s remaining axiom schemas expresses a necessity under every interpretation of L.

(**Cl**Ax)'s sole universal rule obviously expresses a valid inference. Could (**Cl**Ax)'s admissible rule apply to the conclusions of some theorem-sequents of (**Cl**Ax) to generate a wff which is interpretable as expressing a nonnecessary proposition? No it could not. We have seen, first, that (**Cl**Ax)'s axioms can only be interpreted as expressing necessary truths. Modus ponens's validity ensures, second, that applying that rule to wffs which can only be interpreted as expressing necessities produces a wff which can only be interpreted as expressing necessities. Finally, if ϕ is a wff which can only be interpreted as expressing necessities, the result of applying (**Cl**Ax)'s admissible rule to ϕ will produce a wff which is only interpretable as expressing necessities, because the inference '*P*; so *P* or *Q*' is evidently valid.

It follows that applications of (**Cl**Ax)'s admissible rule to the conclusions of theorem-sequents of (**Cl**Ax) will produce wffs which can only be interpreted as expressing necessities. For suppose that ϕ is the conclusion of a theorem-sequent of (**Cl**Ax). Then a route can be traced from (**Cl**Ax)'s axioms to ϕ, using (**Cl**Ax)'s universal and admissible rules. The three facts noted in the previous paragraph ensure that ϕ can only be interpreted as expressing necessities. But the third fact then guarantees that applying (**Cl**Ax)'s admissible rule to ϕ will produce a wff which can in turn only be interpreted as expressing necessities.[22]

Putting everything together, we see that (**Cl**Ax) meets the special conditions on axiomatisations described above. Hence **Cl** is informally sound. For further reference, note that the preceding argument for **Cl**'s informal soundness never employed the idea that the notions expressed by *L*'s logical expressions are functions defined over two special objects, the True and the False. The reasoning instead appealed to opinions which can reasonably be held regardless of one's views on how far *L*'s model theory is usable as an applied semantics. All of the arguments for informal soundness results below are meant to follow a similar pattern.

Now that I have explained the notions articulated in this section, the main business of this paper can begin. The following section describes a very simple but pleasingly general technique for generating model theoretic arguments that inferences are valid.[23] Despite the method's simplicity and generality, it seems to have been passed over in the recent literature on how model theoretic facts can ground conclusions about validity.

IV. Arguments for validity: a case study

Recall the standard propositional language *L*. The classical model theoretic semantics for *L* is based upon assignments of so-called truth values to its sentence letters and the truth tables for the operators $\neg$ and $\vee$. Suppose that no model for *L* is a countermodel to a given sequent. Then we conclude that the sequent expresses a valid inference. Or, as I shall say, we assume that the class of models for *L* is *comprehensive*: we assume that a sequent of *L* expresses an invalid inference only if there is a countermodel to the sequent.

One way of arguing for that comprehensiveness claim assumes that there are two special objects, the True and the False, and that the operators which are used to interpret the logical expressions of L are functions upon those things. Here, though, is another way of arguing for the comprehensiveness assumption, one which takes the model theory for L a little less seriously.

Consider classical propositional logic **Cl** as formulated in L. We saw in the previous section that **Cl** is informally sound. And it was also remarked that the informal soundness of **Cl** can be demonstrated without assuming that the notions formalised within L are mappings based upon the True and the False.

Now, the class of models for L is *complete* for **Cl**.[24] That is, if no model for L is a countermodel to a given sequent, the sequent is provable in **Cl**. That completeness result is again derivable without the assumption that the notions formalised within L are functions defined over the True and the False. But the completeness of the class of models for L for **Cl** immediately combines with **Cl**'s informal soundness to yield that the class of models for L is comprehensive. They combine, that is, to yield that if no model for L is a countermodel to a sequent, that sequent expresses a valid inference.

For suppose that no model for L is a countermodel to some given sequent. Then by the completeness of the class of models for L for **Cl**, the sequent is provable in **Cl**. But by **Cl**'s informal soundness, it follows that the sequent expresses a valid inference. We can, therefore, use model theoretic facts to derive conclusions about the validity of inferences expressed by sequents of L. And our doing so does not require that we believe in the True and the False.

More generally, informal soundness results combine with completeness results to generate comprehensiveness results. And the latter results justify inferences from model theoretic facts to conclusions about validity. When informal soundness and completeness results can be got using suitable assumptions, we can get conclusions about validity from model theoretic premisses without treating the relevant model theory as an applied semantics. The next two sections apply the techniques developed in this section to a range of cases that are more interesting than the one considered so far and which have recently been widely discussed, namely ones involving modal notions.

V. Propositional modal inferences

We can augment the propositional language L by adding an unary operator $\Box$, which is to be read as 'it is necessary that _'. The resulting language is L_M.[25] The standard semantics for L_M is based on Kripke-models. A Kripke-model for L_M consists of a set of indices, one of which is singled out as the distinguished index, an accessibility relation on the indices, and a mapping taking each pair of a sentence letter of L_M and one of the model's indices to a truth value. The index-relative truth values of wffs whose main connective is either $\neg$ or $\vee$ is fixed in the usual way, while a wff $\Box\phi$ is true at a given index just in case ϕ is true at each index bearing the accessibility relation to the index. A wff of L_M is true in a given Kripke-model if it is true at the model's distinguished index.

The propositional modal logic **T** can be axiomatised in L_M as follows:

(**T**Ax) *Axioms*: Each truth functional tautology is an axiom. Each instance of the following schemas is an axiom: $\Box(\phi\rightarrow\psi)\rightarrow(\Box\phi\rightarrow\Box\psi)$; $\Box\phi\rightarrow\phi$.
Universal rules: modus ponens.
Admissible rules: rule of necessitation ('infer $\Box\phi$ from ϕ').

Here is an argument for thinking that the modal logic **T** axiomatised by (**T**Ax) is informally sound.

We saw in section III that **T** will be informally sound if (**T**Ax)'s elements meet the following conditions. First, each of (**T**Ax)'s axioms expresses a necessary truth under each interpretation of L_M. Second, each of (**T**Ax)'s universal rules expresses a valid inference. Finally, each of (**T**Ax)'s admissible rules meets the following condition: if the ϕs are the conclusions of theorem-sequents of (**T**Ax), and the rule applies to the ϕs to give ψ, ψ expresses a necessary truth on each interpretation of L_M.

So let's start by considering (**T**Ax)'s axioms. It is clear that each instance of a truth functional tautology can only be interpreted as expressing necessities. And it seems likewise obvious that necessarily, the necessity of a necessary conditional's antecedent ensures the necessity of the conditional's consequent, and that necessarily, all necessities are true. Everyone should therefore accept that (**T**Ax)'s axioms can only be interpreted as expressing necessary truths.

Modus ponens evidently expresses a valid inference, so (**T**Ax)'s universal rule does its best to guarantee that **T** is informally sound. Finally, what about (**T**Ax)'s admissible rule? Could applying the rule of necessitation to the conclusion of some theorem-sequent of (**T**Ax) generate a wff which is interpretable as expressing a nonnecessary proposition? If it could not, it follows that **T** is informally sound.

Suppose that ϕ is the conclusion of a theorem-sequent of (**T**Ax). Then applying the rule of necessitation to ϕ produces $\Box\phi$. That application of the rule of necessitation has produced a wff which can only be interpreted as expressing necessities just in case ϕ is only intepretable as expressing *necessarily* necessary truths. Applying the rule of necessitation to $\Box\phi$ will, in turn, produce a wff which can only be interpreted as expressing necessities precisely if ϕ is only interpretable as expressing *necessarily necessarily* necessary truths. More generally, applications of the rule of necessitation to the conclusions of theorem-sequents of (**T**Ax) will produce wffs which can only be interpreted as expressing necessities just in case those conclusions can only be interpreted as expressing *utter* necessities; ones which are necessarily necessary, necessarily necessarily necessary and so on.

(**T**Ax)'s inclusion of the rule of necessitation as an admissible rule thus reflects the idea that certain necessities are utterly necessary. It is therefore unsurprising that some restricted assumptions concerning iterated modalities are needed to argue for the informal soundness of **T**. But the relevant assumptions are pretty weak—they are, crucially, much less strong than the principles concerning iterated modalities assumed by Chihara and Divers. In particular, the rule of necessitation will help towards **T**'s informal soundness just in case, first, it is utterly necessary that the provable sequents of **Cl** express valid inferences, and second, the principles expressed by (**T**Ax)'s modal axiom schemas are utterly necessary. Those assumptions seem reasonable ones, and they are employed in the following argument.

The preceding assumptions imply that each axiom of (**T**Ax) is only interpretable as expressing utter necessities. But modus ponens expresses an inference whose validity is utterly necessary (by the provability of the sequent $p, p \rightarrow q \vdash q$ in **Cl**), so applications of modus ponens to wffs which can only express utter necessities will generate wffs which are also only interpretable as expressing utter necessities. Finally, if a wff of L_M is only intepretable as expressing utter necessities, applying the rule of necessitation to that wff will produce a wff which can only be read as expressing utter necessities. Those three points combine to imply that applying the rule of necessitation to the conclusions of theorem-sequents of (**T**Ax) will produce wffs which can only be interpreted as expressing (utter) necessities. **T**'s informal soundness follows immediately, as noted earlier.

Did the preceding argument for **T**'s informal soundness implicitly assume that the truth conditions of ascriptions of possibility and necessity are to be given using possible worlds? Surely not. There was nothing in the argument which is especially the preserve of possible worlds theorists. Even the most implacable foe of possible worlds can readily accept, for example, that it is utterly necessary that necessary truths are true and that each provable sequent of **Cl** expresses an inference whose validity is utterly necessary.

Now, the class of Kripke-models for L_M whose accessibility relation is reflexive is complete for **T**. That completeness result can be derived whether or not one believes in possible worlds. But we have just seen that **T**'s informal soundness can also be demonstrated without employing claims about possible worlds. So we can show, without calling upon any assumptions about possible worlds, that the class of Kripke-models for L_M whose accessibility relation is reflexive is comprehensive. For if no such model is a countermodel to a given sequent, it follows that the sequent is provable in **T**. By **T**'s informal soundness, it follows that the sequent expresses a valid inference.

So, for instance, the sequent $\Box(p \rightarrow q), \Diamond\neg q \vdash \Diamond\neg p$ holds in each Kripke-model for L_M which has a reflexive accessibility relation. We can safely conclude that the inference 'necessarily, if *P* then *Q*; possibly not-*Q*; hence possibly not-*P*' is valid. And we can draw that conclusion without treating the model theoretic semantics for L_M as an applied semantics.

Those who follow Chihara and Divers in making strong assumptions about iterated modalities may claim to demonstrate the validity of additional inferences expressible using L_M, without going beyond pure semantics. For instance, those who assume that whatever is possible is necessarily possible can use axiomatisations to argue that the modal propositional logic **S5** is informally sound.[26] But the class of Kripke-models for L_M whose accessibility relation is universal—that is, in which each index is accessible from every other—is complete for **S5**. People who think that possibilities are always necessarily possible may infer that that class of models is comprehensive. They can then argue that, for instance, the sequent $\diamond\diamond p, \Box\diamond\Box(q\&r)\vdash\diamond(p\&(q\&r))$ expresses a valid inference.

So far, I have only considered modal inferences which are formalisable within the relatively simple propositional modal language L_M. Can the above ideas also be applied to the more complex case of first-order modal inferences? The next section, whose technicalities are more fiddly than those elsewhere in this paper, shows that they can be. (Some readers may prefer to pass over the following discussion and proceed straight to the final two paragraphs of the next section, which lead into the paper's discussion of model theoretic arguments for invalidity.)

VI. First-order modal inferences

Let LFE_M be a first-order modal language with names and the identity sign '='. LFE_M's logical expressions are those of L_M plus the existential and universal quantifiers and the identity sign, all of which have their customary readings. Interpretations of LFE_M result from interpreting the *n*-place predicate letters of LFE_M besides the identity sign as expressing extensional *n*-place predicates and from interpreting the names of LFE_M as referring to particular things. Finally, suppose that ϕ is a wff of LFE_M which contains one or more free variables and which does not occur as part of another wff. Take some wff ψ of LFE_M within which each of ϕ's free variables is bound by a suitable universal quantifier at ψ's front (and where, if necessary, ϕ's bound variables and quantifiers are relettered so that no quantifier in ψ using a given variable is within the scope of another quantifier employing the same variable). Then under each interpretation of LFE_M ϕ is to be read as expressing the proposition expressed by ψ.[27]

The model theory for LFE_M is built upon that for L_M. Kripke-models for LFE_M are, however, associated with first-order models which supply index-relative denotations for the nonlogical expressions of LFE_M. The standard truth conditions for the purely first-order elements of LFE_M then become index-relative truth conditions, and thus transformed they combine with the index-relative truth conditions for the logical expressions of L_M to generate index-relative truth values for the wffs of LFE_M.

There is, notoriously, a welter of first-order modal logics. And the principles within first-order modal logics are sometimes motivated using arguments which treat the previously described model theory for LFE_M as an applied semantics. So, for instance, Hughes and Cresswell suggest the following defence of the Barcan formula ('$\forall x\Box\phi x\rightarrow\Box\forall x\phi x$'): 'Even if each world has its own domain D_w of the things which exist in w there is no reason why all these D_ws can't be collected into one single domain D [over which the universal quantifier in the Barcan formula can be understood as ranging]'.[28]

It might be therefore be wondered whether the need for informal soundness results will prevent section IV's methods for establishing validities from applying to first-order modal inferences. For perhaps we can only establish informal soundness results for first-order modal logics using the idea that the truth conditions of ascriptions of possibility and necessity are to be given using possible worlds. We would then need to treat the model theory for LFE_M as an applied semantics if we were to use the techniques formulated above in studying first-order modal inferences. And that would clash with the ostensible point of those strategies.

Happily, however, things are not so; an illustrative example follows. Given a logic **S** formulated in L_M, the free first-order modal logic **Q1R** + **S** can be axiomatised in LFE_M as follows ($\phi[t'/t]$ is the wff resulting when one substitutes the term t' for each free occurrence of term t in ϕ):[29]

(**Q1R** + **S**Ax) *Axioms*: each substitution instance of a theorem of **S**; the wff $t = t$, where t is a term of LFE_M.
Universal rules: modus ponens; where t and t' are terms, infer $\phi[t'/t]$ from ϕ and $t = t'$; where t and t' are terms, infer $\Box(t = t')$ from $t = t'$; where t and t' are terms, infer $\Box(t \neq t')$ from $t \neq t'$; infer $((\phi\&\exists y(y = t))\rightarrow\psi[t/x]$ from $\phi\rightarrow\forall x\psi$.
Admissible rules: when the variable y does not appear in $\phi\rightarrow(\forall x\psi[x/y])$, infer $\phi\rightarrow(\forall x\psi[x/y])$ from $(\phi\&\exists z(z = y))\rightarrow\psi$; the rule of necessitation.

Reconsider the informally sound logic **T** formulated in L_M. A case can be made for saying that **Q1R** + **T** is informally sound, and the relevant argument does not treat the model theory for LFE_M as an applied semantics. But—typically for the move from propositional to first-order modal logic—the argument is rather more complex than the earlier argument for **T**'s informal soundness.

Let the axiomatisation of **Q1R** + **T** generated by the above schematic list of axioms and rules be called '(**Q1R** + **T**Ax)'. (**Q1R** + **T**Ax)'s inclusion of the rule of necessitation as an admissible rule means that some assumptions concerning iterated modalities are needed to argue for **Q1R** + **T**'s informal soundness. Once again, the required assumptions seem reasonable, and they are much weaker than those employed by Chihara and Divers.

The relevant assumptions are as follows. First, that the modal axiom schemas of (**T**Ax) above express principles which are utterly necessary. Second, that facts about the identity and distinctness of individuals are utterly necessary. And, finally, that certain rules and sequents which correspond to provable sequents of standard free first-order logics express inferences whose validity is utterly necessary.

An argument which refashions the previous section's reasoning concerning the rule of necessitation in (**T**Ax) leads to the conclusion that there are no interpretations of LFE_{M} on which a substitution instance of a theorem of **T** expresses a proposition which is not utterly necessary. (It is that argument which uses the first of the three assumptions noted in the previous paragraph.) That conclusion ensures that (**Q1R** + **T**Ax)'s first group of axioms are only interpretable as expressing necessities. It will also be needed when we examine (**Q1R** + **T**Ax)'s admissible rules.

Now consider (**Q1R** + **T**Ax)'s second group of axioms, the wffs $t = t$ where t is a term of LFE_{M}. The necessity of self-identity ensures that none of those axioms can be interpreted as expressing nonnecessary propositions. Indeed—and this point will be needed when we look at (**Q1R** + **T**Ax)'s admissible rules—the fact that each thing is self-identical and our earlier assumption that facts about identity are utterly necessary imply that each of (**Q1R** + **T**Ax)'s second group of axioms can only be interpreted as expressing utter necessities.

Let's next examine (**Q1R** + **T**Ax)'s universal rules. (**Q1R** + **T**Ax)'s first and last universal rules correspond to provable sequents of standard free first-order logics. They certainly express valid inferences, and I take it that they express inferences whose validity is utterly necessary.[30] The necessity of self-identity and the necessity of distinctness ensure that (**Q1R** + **T**Ax)'s third and fourth universal rules express valid inferences. The assumption that facts about identity and distinctness are utterly necessary leads to the conclusion that (**Q1R** + **T**Ax)'s third and fourth universal rules in fact express inferences whose validity is utterly necessary, which stronger conclusion will be needed when we consider (**Q1R** + **T**Ax)'s admissible rules.

Finally, let's examine (**Q1R** + **T**Ax)'s admissible rules (the reasoning which follows is a little thorny but, as far as I can tell, straightforward). The various claims concerning iterated modalities noted in the last few paragraphs are now required. Those theses imply that (**Q1R** + **T**Ax)'s axioms are only interpretable as expressing utter necessities and that (**Q1R** + **T**Ax)'s various universal rules express inferences whose validity is utterly necessary.

Now consider a wff $(\phi \& \exists z(z = y)) \rightarrow \psi$. And suppose that the wff is only interpretable as expressing utter necessities. Under each interpretation of LFE_{M}, $(\phi \& \exists z(z = y)) \rightarrow \psi$ expresses the proposition expressed by the wff $\forall y((\phi \& \exists z(z = y)) \rightarrow \psi)$. So $\forall y((\phi \& \exists z(z = y)) \rightarrow \psi)$ is only interpretable as expressing utter necessities.

Assume that the variable y does not occur in $\phi \rightarrow (\forall x\psi[x/y])$. Then y does not occur in ϕ. Hence the sequent $\forall y((\phi \& \exists z(z = y)) \rightarrow \psi) \vdash \phi \rightarrow \forall y\psi$ expresses a valid inference.[31] Indeed, I shall take it that the preceding sequent, which corresponds to ones which are provable in standard free first-order logics, expresses an inference whose validity is utterly necessary. Hence the wff $\phi \rightarrow \forall y\psi$ can only be interpreted as expressing utter necessities.

But it is clear that, under each interpretation of LFE_M, the wffs $\phi \rightarrow \forall y\psi$ and $\phi \rightarrow (\forall x\psi[x/y])$ express the same proposition. (More generally, wffs of LFE_M which differ only in the identity of their bound variables are interpretable as expressing identical ranges of propositions.) So the wff $\phi \rightarrow (\forall x\psi[x/y])$ can only be interpreted as expressing utter necessities.

Generalising, we get that applying (**Q1R** + **T**Ax)'s first admissible rule to a wff that is only interpretable as expressing utter necessities leads to another wff which can only be interpreted as expressing utter necessities. But applying the rule of necessitation to a wff which is only intepretable as expressing utter necessities also generates a wff which can only be interpreted as expressing utter necessities.

Those two points combine with the previously flagged ascriptions of utter necessity relating to (**Q1R** + **T**Ax)'s axioms and universal rules to imply that applying (**Q1R** + **T**Ax)'s admissible rules to the conclusions of theorem-sequents of (**Q1R** + **T**Ax) will produce wffs which can only be interpreted as expressing (utter) necessities. The net result of the previous arguments concerning the elements of (**Q1R** + **T**Ax) is that **Q1R** + **T** is informally sound. And the arguments can, as far as I can tell, be accepted by people who are reluctant to use the model theory for LFE_M as an applied semantics.

A completeness result is available for **Q1R** + **T**.[32] The resulting comprehensiveness result means that model theoretic facts can straightaway be used to demonstrate the validity of inferences formalisable within LFE_M—but I will spare the reader examples. As in the propositional case, those who accept strong principles concerning iterated modalities can use model theoretic facts to argue for the validity of further inferences formalisable within LFE_M. For example, people who think that possibilities are always necessarily possible can argue for the informal soundness of **Q1R** + **S5**, and a completeness result is available for **Q1R** + **S5**.

This section and the previous two have applied the methods articulated in section IV to areas of increasing interest and complexity. But the techniques apply more broadly still, because their foundations have the kind of topic-neutrality which was demanded earlier, in section II. As long as one can argue, without drawing upon problematic assumptions, that a given axiomatisation axiomatises an informally sound logic, the strategies can be applied wherever there are completeness results. So, for instance, a survey of relevant literature shows that they can be used to study inferences formalisable in tense-logical languages, in deontic and epistemic languages and in the languages of counterfactual logics.

To what extent, though, is the described method limited by its reliance upon completeness results? And is the method somehow flawed because of its need for informal soundness results? Those questions, and others which might make one doubtful of the value of the techniques described here, are considered in section X below. Having discussed a way of arguing that inferences are valid, I want next to discuss methods for arguing that inferences are invalid. The next section considers, and rejects as insufficiently general, a method which might naturally be proposed. The method parallels the way of generating arguments for validity considered in the last three sections.

VII. Arguments for invalidity: a method rejected

Here is the method for demonstrating invalidities which most closely imitates the earlier technique for demonstrating validities. Let **S** be a logic. Now suppose that **S** meets the following condition: if a given sequent isn't provable in **S**, the sequent expresses an invalid inference. Then the logic **S** is, I shall say, *informally complete*. The notion of informal completeness is the converse of the notion of informal soundness introduced earlier.

Next, assume that a class C of models for **S**'s language is *sound* for **S**. Assume, that is, that a sequent is provable in **S** only if C does not contain a countermodel to the sequent. And suppose that C contains a countermodel to a specific sequent. By C's soundness for **S**, it follows that the sequent isn't provable in **S**. But we assumed that **S** is informally complete. Hence the sequent expresses an invalid inference. Soundness results, then, combine with informal completeness results to generate arguments for invalidity.

The main drawback of the technique just described is its reliance upon informal completeness results. Reconsider the propositional modal logic **T**. Suppose that one wanted to argue that **T** is informally complete without using the model theory for L_M as an applied semantics. One needs somehow to make a case that each of the infinitely many sequents which is not provable in **T** expresses an invalid inference. How to do so?

If one thinks that the model theory for L_M can be used as an applied semantics, one might be able to argue that countermodels to sequents of L_M map onto arrangements of possible worlds which invalidate the inference expressed by those sequents. But, *ex hypothesi*, no such argument is currently permissible. I cannot see any alternative way of proceeding which is free of assumptions about how model theory relates to meaning. The above method's reliance upon informal completeness results thus makes it unacceptable for our current purposes. The next section describes a preferable way of generating arguments for invalidity.

VIII. Another way

Reconsider classical propositional logic **Cl**, formulated in the language L. **Cl**'s informal soundness and the completeness of the class of models of L for

Cl can be demonstrated without the assumption that the notions formalised within *L* are functions defined over the True and the False. We can consequently use model theoretic facts to demonstrate the validity of inferences expressed by sequents of *L* without using that assumption. Can we similarly demonstrate the invalidity of inferences expressed by sequents of *L*?

Consider the sequent $(p \vee q) \vdash p$. The inference expressed by that sequent is obviously invalid. And there are models for *L* which are countermodels to the sequent. For instance, the model *M* for *L* in which p is assigned 0 while each other sentence letter of *L* is assigned 1 is a countermodel to $(p \vee q) \vdash p$.

In fact, if we construct *M* we quickly see that *any* model for *L* in which p is assigned 0 while q is assigned 1 is a countermodel to $(p \vee q) \vdash p$. But a model for *L* is one in which p is assigned 0 while q is assigned 1 just in case the model is one in which $\neg p$ is true and q is true. So any model for *L* in which $\neg p$ is true and q is true is a countermodel to $(p \vee q) \vdash p$. That is, any model for *L* in which $\neg p$ is true and q is true is one in which both $(p \vee q)$ and $\neg p$ are true.

Now, the class of models for *L* is complete for **Cl**. But we have just seen that no model for *L* is a countermodel to the sequents $\neg p, q \vdash (p \vee q)$ and $\neg p, q \vdash \neg p$. So both of those sequents are provable in **Cl**.

Cl is informally sound. So the sequents $\neg p, q \vdash (p \vee q)$ and $\neg p, q \vdash \neg p$ express valid inferences. Finally, though, notice that we can interpret the shared premisses of those two sequents so that the premisses express compossible propositions—that is, so that they express propositions which can be true together. For instance, we can interpret p as stating that $0 \neq 0$ and q as stating that $0 = 0$.

It follows that the conclusions of the sequents $\neg p, q \vdash (p \vee q)$ and $\neg p, q \vdash \neg p$ can also be interpreted as expressing compossible propositions. That is, it follows that $(p \vee q)$ and $\neg p$ can be interpreted so that they express compossible propositions. Hence the sequent $(p \vee q) \vdash p$ expresses an invalid inference.

The preceding demonstration didn't assume that the notions formalised within *L* are functions defined over the True and the False. And it clearly shrouds a general strategy, one which recycles the way of deriving conclusions about validity described earlier. I shall now generalise the example just given.

Let **S** be a logic whose informal soundness can be argued without treating the model theory for **S**'s language as an applied semantics. And assume that a similar case can be made for saying that class *C* of models is complete for **S**. Take some sequent $\Phi \vdash \psi$ of **S**'s language. Suppose that we are able to identify a member *M* of *C* which is a countermodel to $\Phi \vdash \psi$. We can typically identify aspects of *M* which are irrelevant to its being a countermodel to $\Phi \vdash \psi$.

More precisely, we can usually trace the fact that *M* is a countermodel to $\Phi \vdash \psi$ back to the fact that certain simpler wffs, the Γs, are true in *M*. When we do so, we establish the following: that for each ϕ in Φ, no model in C is a countermodel to the sequent $\Gamma \vdash \phi$, and no model in C is a countermodel to the sequent $\Gamma \vdash \neg\psi$.

C is complete for **S**. So for each ϕ in Φ, the sequent $\Gamma \vdash \phi$ is provable in **S**, and the sequent $\Gamma \vdash \neg\psi$ is provable in **S**. But **S** is informally sound. Hence for each ϕ in Φ, the sequent $\Gamma \vdash \phi$ expresses a valid inference, and the sequent $\Gamma \vdash \neg\psi$ expresses a valid inference.

Finally, suppose that we are able to identify a way of interpreting **S**'s language on which the Γs express compossible propositions. It follows from the validities just stated that if we interpret **S**'s language in the identified way, the Φs express propositions which are compossible with the proposition expressed by $\neg\psi$. And hence the sequent $\Phi \vdash \psi$ expresses an invalid inference. The next section briefly discusses some modal applications of the above points.

IX. Some propositional modal applications

Consider the sequent $\Box$(p$\vee$q), $\Box$r$\vdash\Diamond$(p&r) of the modal propositional language L_M. It expresses the inference 'necessarily, *P* or *Q*; necessarily, *R*; hence possibly, *P* and *R*'. Suppose we are given a Kripke-model for the modal propositional language L_M whose accessibility relation is reflexive. And suppose that $\neg\Diamond$p, $\Box$q and $\Box$r are all true in the model. Then the model is a countermodel to the sequent $\Box$(p$\vee$q), $\Box$r$\vdash\Diamond$(p&r). That is, the model is such that $\Box$(p$\vee$q), $\Box$r and $\neg\Diamond$(p&r) are all true in it. It follows that the class of Kripke-models for L_M whose accessibility relation is reflexive does not contain any countermodels to the following three sequents: first, $\neg\Diamond$p, $\Box$q, $\Box$r$\vdash\Box$(p$\vee$q); second, $\neg\Diamond$p, $\Box$q, $\Box$r$\vdash\Box$r; and third, $\neg\Diamond$p, $\Box$q, $\Box$r$\vdash\neg\Diamond$(p&r)

It was remarked earlier that the class of Kripke-models for L_M whose accessibility relation is reflexive is complete for the modal propositional logic **T**. So the sequents $\neg\Diamond$p, $\Box$q, $\Box$r$\vdash\Box$(p$\vee$q); $\neg\Diamond$p, $\Box$q, $\Box$r$\vdash\Box$r; and $\neg\Diamond$p, $\Box$q, $\Box$r$\vdash\neg\Diamond$(p&r) are provable in **T**. But **T** is informally sound. Hence the preceding three sequents express valid inferences. But we can interpret the language L_M so that $\neg\Diamond$p, $\Box$q and $\Box$r—the shared premisses of the three sequents just considered—express compossible propositions. For instance, read p as stating that $0 \neq 0$, and q and r as stating that $0 = 0$. It follows that the conclusions of the three sequents listed in this paragraph are compossible. Hence the sequent $\Box$(p$\vee$q), $\Box$r$\vdash\Diamond$(p&r) expresses an invalid inference.

We saw earlier that **T**'s informal soundness can be established using only relatively weak assumptions concerning iterated modalities. The above demonstration of invalidity therefore does not require the very strong assumptions about iterated modalities which rendered Chihara's and Divers's techniques problematic. But those who make such strong assumptions can sometimes give simpler arguments for invalidity than those who do not make the assumptions. For instance, the final stage of the method just illustrated involves identifying interpretations of formal languages on which sets of wffs express compossible propositions. Those who think that the modal propositional logic **S5** is informally sound never need to consider sets containing wffs featuring iterated occurrences of $\Box$ or $\diamond$.

Many of the countermodels cited in the earlier example will look somewhat suspect to those who think that, for instance, the modal propositional logic **S5** is informally sound. For many of those countermodels will also be countermodels to, for instance, $\diamond p \vdash \Box\diamond p$, a sequent which expresses a valid inference according to supporters of **S5**. Can supporters of **S5** accept the example?

They can. The class of Kripke-models for L_M whose accessibility relation is reflexive certainly contains countermodels to the sequent $\diamond p \vdash \Box\diamond p$. It follows that there are sequents whose premisses are such that *if* those premisses can be interpreted as expressing compossible propositions *then* the sequent $\diamond p \vdash \Box\diamond p$ expresses a invalid inference. But those who think that **S5** is informally sound will deny that the premisses of those sequents can be appropriately interpreted. They can therefore happily accept the argument for invalidity which started this section.

We saw earlier that proposed strategies for deriving conclusions about validity from model theoretic facts should be topic-neutral. The method being discussed fits that bill. It can be applied whenever we can argue, without assuming that a model theory is an applied semantics, that the model theory generates a class of models which is complete for an informally sound logic. The technique can thus be used to argue for the invalidity of not only propositional modal inferences but also for that of, say, first-order modal, counterfactual, deontic, epistemic and tense-involving inferences.

The methods for deriving conclusions about validity from model theoretic facts which this paper has formulated can often help us to refrain from viewing systems of models in an applied manner. That is, the techniques allow us to see model theoretic frameworks as purely mathematical constructs while still using those frameworks to generate conclusions about validity. The techniques thereby refute the natural line of thought spelled out at the start of this paper, and which we saw to be endorsed by Haack, Lewis and Plantinga, according to which those who use model theoretic facts in drawing conclusions about validity are committed to viewing the relevant models as reflecting truth conditions.

To close, I want to consider some worries which the reader might have about the techniques which were developed earlier. The worries relate to whether the methods really allow us to use model theoretic facts to the extent to which we would like to use them and to which we may be able to use them if we regard model theories as reflecting truth conditions.

X. Some worries

The various techniques described above rely upon informal soundness results and assessments of whether wffs of formal languages can be interpreted so that they express compossible propositions. Here are some ways of arriving at such results and assessments, using model theoretic facts.

Suppose that for each model in a class of models for a language, the wffs holding in the model can be interpreted so that they express compossible propositions. Then if there is a model in our class wherein each of the Φs is true, there is a way of interpreting the relevant language so that the Φs express compossible propositions.

Next, suppose that one thinks that a certain class of models for a formal language meets the following condition: for each set of the language's wffs which can be interpreted so that they express compossible propositions, there is a model in the class for the set of wffs. Then if each provable sequent of logic **S** holds in each model in the relevant class, **S** is informally sound (using the fact that a sequent $\Phi \vdash \psi$ holds in each model in a class only if there is no model in the class for the set $\Phi \cup \{\neg\psi\}$).

If one thinks that a class of models satisfies one of the above conditions, one can straightaway infer either compossibility or informal soundness claims *from model theoretic facts*. But why might someone think that an (interesting)[33] class of models meets either of those conditions? Somebody might do so, it is natural to suppose, if he thinks that a class of models represents possible scenarios bearing on the truth values of propositions formalisable in the relevant formal language.

For instance, suppose that a person thinks that each Kripke-model for the modal propositional language L_M whose accessibility relation is an equivalence relation adequately represents a portion of logical space.[34] Then she may claim that each set of wffs holding in one of those Kripke-models can be interpreted as expressing a set of compossible propositions. She can then argue from model theoretic facts to compossibility results.

Or suppose that someone thinks that each sector of logical space is adequately represented by a Kripke-model for L_M whose accessibility relation is an equivalence relation. Then he may argue that there is such a Kripke-model for each set of the language's wffs which can be interpreted so that they express compossible propositions. He can subsequently argue from model theoretic facts to informal soundness results.

What if one has less faith in the representational powers of a given model theory? It is very hard to see how one could argue that the wffs holding in each model in a given and interesting class can express compossible propositions. And although the techniques described earlier for deriving conclusions about validity from model theoretic facts allow one to argue from model theoretic facts to informal soundness results, in the current context that seems like cheating because one needs to assume informal soundness results to do so.

It may seem, therefore, that my discussion has ignored some impressive uses of model theoretic facts, *viz*. in generating compossibility and informal soundness results. The uses are ones that can apparently be catered for by those who make suitable assumptions about the relationships between model theory and meaning. It might be thought, therefore, that people who make suitable semantic assumptions are better off than those who do not make them and who use the methods expounded in this paper. For the former can provide *model theoretic* arguments for the characteristic compossibility and informal soundness claims employed by the latter, who presumably justify those claims using prior *modal* beliefs.

I doubt that there is a real difficulty here. Suppose, for instance, that somebody holds that each Kripke-model for L_{M} whose accessibility relation is an equivalence relation adequately represents a portion of logical space. How might our character justify her belief? She will presumably need to rely upon prior modal beliefs. In particular, she will need to use prior modal beliefs which imply that a whole range of situations—those which she takes to be representable using Kripke-models for L_{M} whose accessibility relations are equivalence relations—are possible.

That example suggests that those who make suitable assumptions about the relationships between model theory and meaning are not inevitably better placed than those who do not make the assumptions and who use the methods examined in this paper. Although the former can derive the compossibility results used by the latter from model theoretic facts, it appears that to do so they will be forced to rely upon prior modal beliefs. It is thus hard to see why those who derive compossibility results from model theoretic facts, using semantic assumptions, are bound to be better off than those who simply derive informal soundness results from prior modal beliefs.

A similar point applies to the use of semantic assumptions in deriving informal soundness results from model theoretic facts. Suppose, for example, that someone thinks that every portion of logical space is adequately represented by a Kripke-model for L_{M} whose accessibility relation is an equivalence relation. How could he avoid calling on prior modal beliefs in justifying that claim? He will surely need to use prior modal beliefs which imply that every possible situation is adequately captured by one of his favoured Kripke-models.

To close, I want to consider a final worry. Completeness results evidently play an essential role in the methods presented here. The logical literature is, however, replete with so-called incompleteness results, which take a class of models and show that the set of sentences holding in each model in the class is not recursively enumerable. For instance, the class of wffs holding in all full second-order models is not recursively enumerable, nor is the class of wffs holding in all standard models of first-order Peano arithmetic.[35] It is therefore natural to suspect that we will sometimes want to move from facts concerning a class of models to conclusions about validity when no completeness result can be proved for the relevant class. In those cases, the methods presented here will be inapplicable.

In fact, the incompleteness results contained in the literature do not immediately block this paper's techniques. For the account of axiomatisations used above was extremely liberal. The account does not, for instance, imply that axiomatisable logics have recursively enumerable sets of theorems (axiomatisations were simply said to consist of 'sets' of axioms and rules). The account's permissiveness is desirable, as it is possible to identify classes of models which are complete for informally sound logics even though the set of sentences holding in every model in the relevant class is not recursively enumerable.

Thus consider a standard first-order axiomatisation of Peano arithmetic within a language containing the usual arithmetical logical vocabulary. We can add an ω-rule to the axiomatisation as a universal rule, stating that $\forall n\phi[n]$ may be inferred if we have each $\phi[n]$. The resulting axiomatisation meets the various special conditions which were earlier spelled out and whose satisfaction by an axiomatisation implies that the logic thereby axiomatised is informally sound. Yet although the set of wffs holding in each standard model of first-order arithmetic is not recursively enumerable, the class of standard models of first-order arithmetic is complete for the logic generated by the previous axiomatisation.

Although the methods formulated here are not immediately blocked by the so-called incompleteness results contained in the literature, it is nonetheless likely that the methods will sometimes break down because of a lack of suitable completeness results. For example, it is hard to see how one could identify an axiomatisation meeting the following conditions: first, the axiomatisation axiomatises the set of precisely those sequents holding in all full second-order models; and, second, the axiomatisation can be used to prove the informal soundness of that logic, without the use of any problematic assumptions. If no such axiomatisation can be provided, it follows that the methods presented here cannot straightforwardly be used in deriving conclusions about validity from facts about full second-order models.[36]

Notes

[1] Benacerraf (1996), p. 17.

[2] In what follows, a *sequent* of a formal language consists of a set of the language's wffs, followed by the symbol '$\vdash$', which is in turn followed by a wff of the language. The wffs to the left of a sequent's '$\vdash$' are said to be the sequent's *premisses* and that to the right is said to be its *conclusion*.

[3] Plantinga (1974), pp. 126–8.

[4] Plantinga (1974), p. 127.

[5] Haack (1978), p. 189.

[6] Lewis (1986), p. 20. Loux (1979), pp. 29–30 makes similar remarks. Humberstone (1996) states that he is interested in model theories because of their implications for validity (pp. 215–6). He offers a homophonic model theoretic semantics for the propositional modal logic **S4**, because he doubts that 'the only way to explain in systematic terms what we mean when we talk modally is in terms of' 'the apparatus of possible worlds' (p. 217). Humberstone seems to be assuming that those who draw conclusions about validity from facts about the standard model theory for modal languages need to view the model theory as an applied semantics. Finally, Peacocke writes that it 'is indeed true that, if we are to use the Kripkean model-theoretic semantics in assessing the validity of a modal argument without engaging in doublethink, the actual world must correspond to some world in the range of worlds recognized in the model theory, and something must correspond to the elements of the domains of nonactual possible objects recognized in the model theory' (Peacocke (2002), p. 144).

[7] Chihara (1998), p. 3.

[8] See chapter 6 of Chihara (1998).

[9] See Rosen (1990).

[10] This is a bit rough, but it will do for present purposes. For a more careful characterisation of the fiction which supposedly determines modal truth, see Rosen (1990), pp. 333–4.

[11] Divers (1999), p. 317.

[12] Divers's discussion doesn't consider inferences from model theoretic facts to conclusions stating that particular inferences are *invalid*. But his techniques can easily be adapted to yield arguments for invalidity, using the central ideas of section VIII below. In fact, those ideas allow one to transform any methods for deriving validities into techniques for establishing invalidities.

[13] See, for instance, Chihara (1998), p. 207. **S5** is axiomatised below.

[14] Divers twice assumes **S4**, on pp. 336–7 of Divers (1999).

[15] Chihara (1998), p. 207.

[16] This point and the one about Chihara's ideas made at the end of the previous paragraph aren't meant to be criticisms of Divers's and Chihara's approaches *per se*—the points are simply meant to emphasise that those positions aren't helpful in addressing the very general issue concerning the use of model theoretic facts in drawing conclusions about validity with which I'm concerned.

[17] Zero-premiss 'inferences'—as expressed by sequents whose premiss-set is empty—are treated below as valid just in case each instance of their 'conclusion' is necessary.

[18] This is a highly permissive account of axiomatisations. It does not demand, for instance, that the set of axioms is recursive. Its permissiveness becomes relevant at the end of this paper.

[19] For the record, and because it recurs a few times below, the set of *L*'s wffs is the smallest set meeting the following conditions. First, each sentence letter p, q, r, p_1, q_1, r_1 . . . is a wff. Second, if ϕ is a wff, so is $\neg\phi$. And third, if ϕ and ψ are wffs, so is $(\phi\vee\psi)$. '$(\phi\rightarrow\psi)$' is short for '$(\neg\phi\vee\psi)$', while '$(\phi\&\psi)$' is short for '$\neg(\neg\phi\vee\neg\psi)$'.

[20] I concentrate upon axiomatisations in what follows, but the described methods for deriving conclusions about validity and invalidity from model theoretic facts can be extended to deal with logics presented using, for instance, systems of natural deduction or sequent calculi.

[21] Suppose that we have a logic which is axiomatised using an axiomatisation which meets the conditions spelled out in the text. Take a sequent $\Phi\vdash\psi$ which is provable in the logic. There are two cases to consider. First, assume that $\Phi\vdash\psi$ is a theorem-sequent of the axiomatisation. Then an induction on the length of some proof of ψ shows that ψ is only interpretable as expressing necessities and that $\Phi\vdash\psi$ therefore expresses a valid inference in the sense of fn. 17. (The inductive basis is ensured by the fact that the axiomatisation's axioms can only be interpreted as expressing necessary truths, while the inductive step is guaranteed by the conditions applying to the axiomatisation's universal and admissible rules.) Second, assume that ψ can be derived from the members of Φ and some set θ of theorem-sequent conclusions, using zero or more applications of the axiomatisation's universal rules. It follows from the transitivity

of entailment and the fact that each of the universal rules expresses a valid inference that $(\Phi\cup\theta)\vdash\psi$ expresses a valid inference. But θ's members can only be interpreted as expressing necessities, as we just saw. And if $(\Phi\cup\theta)\vdash\psi$ expresses a valid inference and θ's members can only be interpreted as expressing necessary truths, $\Phi\vdash\psi$ also expresses a valid inference.

[22] In what follows, the materials for similar inductive arguments are provided whenever it is argued that an axiomatisation's admissible rules will always produce wffs which are only interpretable as expressing necessities.

[23] The main thread of the ideas expounded in the following section is, I have discovered, recoverable from the discussion in Kreisel (1967) of why completeness proofs matter.

[24] There is a small wrinkle here. Completeness results are usually proved for sets of wffs, which are axiomatised using only one type of rule, rather than for logics in my sense. The wrinkle is easily ironed out, so long as one is working with logics which meet a couple of very weak conditions.

[25] In what follows, $\diamond\phi$ is short for $\neg\Box\neg\phi$. $\diamond\phi$ is naturally read using 'it is possible that _'. I sometimes speak below as though $\diamond$ is an expression of L_M.

[26] **S5** can be axiomatised as follows. First, take each truth functional tautology as an axiom, and each instance of the schemas $\Box(\phi\rightarrow\psi)\rightarrow(\Box\phi\rightarrow\Box\psi)$, $\Box\phi\rightarrow\phi$ and $\diamond\phi\rightarrow\Box\diamond\phi$. Take modus ponens as the sole universal rule and have the rule of necessitation as the sole admissible rule.

[27] This stipulation is unproblematic; those wffs which differ merely in the identity of their bound variables express the same proposition under each interpretation of LFE_M.

[28] Hughes and Cressell (1996), p. 303.

[29] Each occurrence of a name is free. The **Q1R** + **S**s are reaxiomatisations of the **Q1R**s considered by Garson (1984) (see p. 257), in the light of this paper's distinction between universal and admissible rules. I consider a free first-order modal logic because I think that there are good reasons for thinking that first-order modal logics should be built upon free logics rather than classical ones (for discussion see Garson (1984), pp. 258–9). For simplicity's sake, I have dropped the primitive existence predicate contained in the language within which Garson axiomatises the **Q1R** + **S**s and instead used its explicit definition within those logics. I have also altered Garson's rule of universal generalisation slightly by blocking the immediate use of generalisation on names (see the second admissible rule). This change is proof-theoretically unimportant—any proof of a wff containing a name supplies proofs of those wffs which substitute new free variables for the name—but it makes arguing for **Q1R** + **T**'s informal soundness easier.

[30] I should note that my earlier stipulation regarding the construal of wffs of LFE_M with free variables means that (**Q1R** + **T**Ax)'s first universal rule does not express an inference whose validity is utterly necessary simply because modus ponens, as normally understood, does. Rather, it expresses one because sequents of the following form express inferences whose validity is utterly necessary: $\forall x\forall y\forall z\ldots\phi$, $\forall x\forall y\forall z\ldots(\phi\rightarrow\psi)\vdash\forall x\forall y\forall z\ldots\psi$ (although the validity of those inferences is essentially owed to that of modus ponens). An analogous remark applies to (**Q1R** + **T**Ax)'s second and last universal rule.

[31] For the sequent $\forall y(\phi\&\exists z(z=y)\rightarrow\psi)\vdash\forall y(\phi\rightarrow(\exists z(z=y)\rightarrow\psi))$ clearly expresses a valid inference. Yet y does not occur in ϕ, so the sequent $\forall y(\phi\rightarrow(\exists z(z=y)\rightarrow\psi))\vdash\phi\rightarrow\forall y(\exists z(z=y)\rightarrow\psi))$ expresses a valid inference. But, finally, the sequent $\phi\rightarrow\forall y(\exists z(z=y)\rightarrow\psi))\vdash\phi\rightarrow\forall y\psi$ expresses a valid inference.

[32] See Garson (1984), p. 285.

[33] Some uninteresting cases: the empty set; sets containing a single model . . .

[34] Exactly articulating the notion of 'adequate representation' being used here and in the next example is pretty difficult, so for rhetorical reasons it is not discussed further in what follows. But it is worth noting that it is a far from trivial matter for possible worlds theorists to justify the use of facts about Kripke-models in drawing conclusions about the validity of modal inferences.

[35] See, for instance, Shapiro (1991), p. 87, for the first result; the second is another consequence of Gödel's incompleteness theorems.

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