Quantification in the interpretational theory of validity

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Abstract

According to the interpretational theory of logical validity (IR), logical validity is preservation of truth in all interpretations compatible with the intended meaning of logical expressions. IR suffers from a seemingly defeating objection, the so-called cardinality problem: any instance of the statement 'There are n things' is true under all interpretations, since it can be written down using only logical expressions that are not to be reinterpreted; yet 'There are n things' is not logically true. I argue that the cardinality problem is indeed a serious problem for IR, when understood in terms of 'asymmetry of information'. I then argue that IR can be rehabilitated by making quantifiers context-sensitive: what we do not reinterpret is the Kaplanian character of a quantifier, rather than its content. 'There are n things' is false in a context where fewer than n things are relevant, so it is not logically true in IR. I finally discuss some objections and ramifications of my account: I discuss how to make space for the possibility of an explicitly absolutely general quantifier in my framework, how terms can be logical even though context-sensitive, and how to recapture classical logic within my framework.

Keywords: logical validity, context-sensitivity, logical notions, quantification

1 The interpretational theory of logical validity

This paper is about logical validity and not about a more generic notion of validity. I will assume throughout the paper that logical validity is a notion that has been discussed at least since Aristotle. I will also assume that this notion has been made precise in the last century or so through model theory, where it is defined as preservation of truth in every model, where models are set-theoretic constructions. However, philosophers disagree on what we are modelling through models because they disagree on what logical validity is.

In his book 'The concept of logical consequence', Etchemendy distinguished between a representational and an interpretational theory of logical validity (1990). According to the interpretational theory (IR), logical validity is preservation of truth across all acceptable interpretations, where an interpretation is acceptable only if compatible with the intended meaning of logical expressions.[1] According to the representational theory (RE), on the other hand, logical validity is preservation of truth in all possible worlds. IR and RE are not the only options: a hybrid view is often preferred, where validity is preservation of truth in a world and an interpretation (Hanson 1997, Shapiro 1998, Sher 2001).

IR and RE are not technical definitions, but rather explications of a pre-theoretic notion, so there is no technical notion of interpretation or representation attached to them. Yet, it is natural to interpret the models of model-theory as depicting interpretations in IR and possible worlds in RE.[2] The question: 'What is logical validity?' can be recast as the question of what models are models of. Consider a model where something satisfies 'is grass' exactly when it is snow and 'is blue' exactly when it is white. IR suggests that the model is depicting an interpretation where 'grass' means snow and 'blue' white, and the fact that 'Grass is blue' is true in the model is mirroring the fact that 'Grass is blue' is actually true, so-and-so interpreted. RE suggests that the model is depicting a world where grass is blue, and the fact that 'Grass is blue' is true in the model is mirroring the fact that, under its English reading, the sentence would be true had that world been actual. In IR we keep the world fixed and change the interpretation of the language; in RE an interpretation is prefixed, and we simply 'apply it' to different possibilities (1990, 21).

2 The cardinality problem

Here is an objection against IR. Take the following schema, where n is any positive integer[3]:

(CN) There are at least n things

Logic students know that CN can be expressed using only connectives, existential quantification and identity. People argue that CN instances come out logically true, under IR. They reason as follows: according to IR, a sentence is logically true when true in all reinterpretations compatible with the intended meaning of logical expressions. By 'intended meaning' of an expression we mean the meaning the expression has as

[1] Some think we should say 'in all interpretations of the non-logical vocabulary'. I find this misleading because it suggests that we are not interpreting the logical expressions at all. Yet, to interpret a language is to interpret all of its sentences. Meaning is compositional, so to interpret a sentence one needs to interpret all of its constituents, some of which are logical expressions. To assume that we are interpreting only the non-logical vocabulary is to ignore the definition of truth in a model, which does interpret (often homophonically) the logical expressions.

[2] These models might not necessarily validate classical logic: there is a model-theory for free logic, Kleene logic etc.

[3] In classical logic an exception is made for $n = 1$, since the domain is assumed to be non-empty.

we are using it. The meaning of logical expressions ought to be kept fixed.[4] $\exists$, $\neg$ and $=$ are all logical expressions, and since any instance of CN is composed of these logical expressions alone, it has only one acceptable interpretation: that there are n things. Thus, it is logically true if true in the intended interpretation; that is, it is logically true if true. It is true because there are n things, so it is logically true.[5]

This result is considered a defeating objection for IR because CN instances are not logical truths. Why, though? In the literature there is no unique reasoning behind the claim that CN instances are not logical truths, nor is there a specific methodology that people follow when they make this claim. The best way to show that the CN schema is not valid is by showing that there is an informational asymmetry between its instances. Consider the following:

(C30)	There are at least 30 things
(C31)	There are at least 31 things

Intuitively, C30 follows from C31, but not vice-versa. There seems to be an asymmetry in information between these sentences because what C30 says is 'contained' in what C31 says but not conversely. One can correctly conclude C30 solely in virtue of the information provided by C31, but not vice-versa. Yet, if both C30 and C31 were logical truths, they would trivially co-entail each other. They don't, so they are not logical truths.

Model-theory secures the desired result because different models have different domains, some with fewer than n things for any positive n. Also, whenever a model has $n+1$ things in the domain it has n things, so C(n+1) implies C(n) for any n. The challenge for IR, though, is how to make sense of such counterexamples if we take models to depict interpretations, since the argument above suggests that there is no interpretation compatible with the meaning of logical expressions that corresponds to any model with a finite domain.

Some might complain that we need to make clearer what we mean by 'information-asymmetry'. A way to do it is through the notion of 'analytic containment': in virtue of what the existential quantifier, conjunction, negation and identity mean, there being 30 things is contained in there being 31 things, and to deny C30 on the hypothesis that C31 is a contradiction *in terms*. Someone who understands the meaning of logical expressions understands sentences like C30 and C31, since logical expressions are all they are made out of. And someone who understands what C31 means must conclude that C30 is true, if C31 is. Yet, the reverse does not seem to be unintelligible in the same way: even if one did conclude C31 after understanding and assuming C30, one would not do so because of what the logical expressions (and thus C30) mean.

Philosophers disagree on the definition of 'analytic', or on whether the notion makes sense, at all. Under Frege's prominent definition of 'analytic', analytic containment is

[4] There is a discussion in the literature about what exactly 'keeping them fixed' amounts to. I have some suggestions later in this paper. For an insightful discussion, see Sagi 2018.

[5] This argument was famously made by Etchemendy (1990) against Tarski's account of logical validity in 1936a and 1936b. Etchemendy's argument sparked a heated discussion in the literature, with many like Sher (1996), Ray (1996) and Gómez-Torrente et al (1996) attacking Etchemendy's exegesis of Tarski's text. Here, I am not interested in the historical discussion on Tarski's account. The argument is interesting and worth discussing in its own right, for its relevance to IR.

perfectly useless as an argument against the validity of the inference from C30 to C31. Frege (1884) defines an analytic statement as something that can be converted in a logical truth by swapping synonyms with synonyms, so all logical truths are trivially analytic. Under Frege's theory, to first establish whether ⌜If C30 then C31⌝ is analytic, one would need to establish if it is a logical truth, since there is no interesting swapping to be done, as the statement is purely made out of logical terms. Yet, ⌜If C30 then C31⌝ is logically true only if the argument from C30 to C31 is logically valid.

Arguably, if one is sceptical about the notion of analyticity, or if someone is sympathetic to the Fregean definition, they should not reject altogether the idea that there is an asymmetry in information between C30 and C31. Rather, they should try to spell it out in different terms. One other way to spell out an asymmetry in information is to appeal to warrant in hypothetical reasoning. One seems warranted to infer C30 from C31; yet, if we start from C30, C31 does not seem to be guaranteed in the same way. In this sense, if C31 is true then C30 *must* be true on that basis alone, but not vice-versa. C31 cannot be logically true, for a logical truth comes with a 'self-warrant' for its own truth, so to speak: regardless of where we start, we are always warranted to infer it. Yet, by starting from C30, we do not seem to be warranted to infer C31. Warrant in hypothetical reasoning can be useful to clarify what we mean by 'informational asymmetry', and does not rely on the notion of analyticity, nor is it rendered useless by the Fregean definition.

3 Attempted solutions to the cardinality problem

The cardinality problem prompted different reactions:

1. Some give up on a purely interpretational theory, and add a representational element to address the cardinality problem (Hanson 1997, Shapiro 1998, Sher 2001).
2. Some bite the bullet and argue that CN instances are logically true, and we need to dispel any impression to the contrary (Williamson 1999).
3. Some consider changing the set of logical expressions to stop the argument in its tracks (Etchemendy 1990, Quine 1986).

3.1 First and second strategy

Reaction (1) is not really relevant to our discussion, since it is an admission of defeat rather than a solution within the IR account. What is interesting to note is that mixing RE with IR does not straightforwardly fix the issue. 'There are n things' for any n is a mathematical truth, and mathematical truths are metaphysically necessary, so true in all possible worlds. Calling for a special, stricter sense of 'logical' possibility is quite unhelpful, and might be convincing only if someone has already assumed that CN instances are not logically true, which begs the question. For arguably we understand something to be logically necessary when it is logically true. Hanson acknowledges this problem, and he looks at 'sub-worlds' of possible worlds to find a representational counterexample to CN instances. He asks us to consider 'just the Washington Monument and the White House'. If we focus on these two things alone, we have an actual case where it is false that there are three things (1997, 388). I will discuss later how we can frame Hanson's reasoning inside a purely interpretational account where we make quantifiers context-sensitive. This shows that there was no need to depart from IR in the first place.

People who bite the bullet and admit that CN are logically true tend to focus on the simplicity and elegance of the interpretational theory. They argue that these features make it worth it to depart from the commonly accepted logical truths. After all, we are only adding validities, not subtracting them, so classical logic is safe. Williamson argues that the target of the logical enquiry are 'general principles involving the selected logical constants, just as the target of a physical inquiry may be general principles involving only terms from a language for physics.' (2017, 331). Generality is all we are after. 'Adding a second dimension of necessity, a priority, or analyticity needlessly complicates the picture, mixing together questions that our fundamental terminology should hold carefully apart so that it can represent their interrelations perspicuously.' (2017, 328). Thus, according to him issues of asymmetry in information are probably misguided, since logical validity has not much to do with analyticity or even a priority.

My issue with the 'bite the bullet' strategy is that logical validity has quite a venerable pedigree, and information-containment, formality and a priority are among its commonly accepted features. This makes the departure from a classical and traditional notion of logical validity theoretically quite expensive. By following an abductive methodology, if we actually weight this theoretical departure from the traditional notion against the simplicity and elegance of the interpretational theory, it is not so clear-cut that the departure is worth doing. The best possible strategy would be to keep the simplicity of the interpretational account and do not depart from a traditional notion of logical validity as analytically truth-preserving, a priori and and as a form of information-containment. I argue that IR paired with a context-sensitive account of quantification achieves both results, and should therefore be preferred.

3.2 Third strategy: changing the logical expressions

The third strategy relies on tampering with logical expressions. People who follow this strategy usually either deny that identity is logical or deny that the existential and universal quantifiers are logical. Quine (1986) held the first view. Before we dive into the details, my general complaint about this strategy is that it does not really address the point of the cardinality problem. Even someone who thinks that there is a definite set of 'real' logical expressions will agree that sometimes we simply treat an expression as logical, for the purpose at hand. For example, the factivity of knowledge is a truth of the logic for knowledge. This fact does not require us to say that the knowledge operator is a logical expression, but only that it is treated as such, in the context of epistemic logic. If so, then the cardinality problem can be recast as follows: in the logic of quantification of identity where we treat quantifiers and identity as logical expressions, we should not expect CN instances to come out logically true because there is an asymmetry in information between its instances. The third strategy simply cannot address the cardinality problem, when put in this way.

The most popular version of the third strategy is to deny that universal and existential quantifiers are logical. This strategy itself divides into two sub-strategies, both discussed and rejected by Etchemendy (1990, 65-9). In the first sub-strategy, 'something' is treated as a variable (i.e. non-logical) term. Yet, its range of admissible reinterpretations is somehow restricted to generalised existential quantifiers like 'some-dogs', 'some-humans' etc. As Etchemendy later points out, this strategy looks quite *ad hoc*. If 'something' is not logical, it seems that we should be able to reinterpret it as any other (unary) quantifier, like 'everything'. If we cannot, then we are restricting the range of legitimate interpretations. In the interpretational view, this must be because of the logical form of sentences and the meaning of logical expressions. Yet, the existential quantifier is not logical, so it cannot be the culprit and we cannot account for the restriction.

The second strategy Etchemendy suggests is a more refined version of the first one. The existential quantifier, like any other quantifier, is composed of a determiner – 'some' – and a noun-phrase – 'thing'. Maybe what is logical is the determiner alone, and not the whole quantifier. If so, then the noun-phrase can change arbitrarily, causing a shift in the range of quantification. This strategy is not directly applicable to the syntax of standard first-order logic where there are no generalised quantifiers, yet it can be implemented by exploiting the equivalence between 'Some ϕ is ψ' and 'Something is ϕ and ψ'. Call L a language of FOL and L^+ the result of adding generalised quantification to L. A determiner in L^+ works like a binary function which takes two noun-phrases and gives out a sentence (Barwise and Cooper 1981). In the syntax, we will have formulae like $\exists(\phi, \psi)$, where ϕ is the noun-phrase that restricts the range of quantification. The idea is to uniformly translate formulae of the form $\exists x \psi$ of L in L^+. We then reinterpret 'thing' in L^+, to then translate the sentences back into the original L, via the equivalence of $\exists(\phi, \psi)$ with $\exists x(\phi \wedge \psi)$.[6]

Etchemendy claims that the proposal under-generates because it makes existential generalisation invalid. If we interpret 'some-thing' as 'some-dog', the generalisation from 'Socrates is a philosopher' to 'Something is a philosopher' fails because nothing is a dog and a philosopher. However, I see more serious issues with the proposal. Often in the literature, when we translate L into the language of generalised quantifiers, we translate 'Something is Socrates' as $\exists(x = x, x = \text{ Socrates})$. Yet, if we follow this natural translation, the solution does not work anymore since – *modulo* the logicality of identity – '$x = x$' has no non-trivial reinterpretations. This shows that enriching the language with generalised quantifiers is not really what makes the strategy work. The real engine of the solution is suggested by Etchemendy in a note: we need to translate existential sentences via a restriction to an unspecified, non-logical predicate U, which is reinterpreted arbitrarily (1990, 166). However, no grounds have been given for translating 'Something is F' in the language of generalised quantifiers as $\exists(Ux, Fx)$ rather than $\exists(x = x, Fx)$. Even if we insisted on not translating 'thing' as $x = x$, we could avoid using 'thing' altogether and instead use a new word explicitly for $x = x$, like 'self-identical'. Under Etchemendy's proposal 'There are two self-identicals' would still be logically true, since it should be translated as $\exists(x = x, \exists(y = y, x \neq y))$. Even more convincingly, Etchemendy's proposal does not work for the language of generalised quantifiers L^+, for then $\exists(x = x, \exists(y = y, x \neq y))$ would be logically true.

[6] This strategy will not work for any quantifier: 'Most dogs are four-legged' is not equivalent to 'Most things, if dogs, they are four-legged'.

The fact that we need to stipulate that U is non-logical for the strategy to work is also quite suspicious, since logicality is about the semantic value of a term, not about how we introduce it. Arguably, then, it seems to me that U is logical if self-identity is, since they mean the same thing. Yet the second is treated as logical in Etchemendy's theory, while the first is not.

4 The context-sensitive account of the quantifiers

4.1 The proposal

The existential and universal quantifier ∃ and ∀ in the object-language should roughly be taken to mean 'something' and 'everything' respectively. A very common idea is that 'something' and 'everything' are context-sensitive. An expression is context-sensitive when an aspect of its meaning cannot be captured through linguistic means alone, but must be supplied by some contextual, non-linguistic parameter. Indexical expressions are the paradigmatic examples: I find a piece of paper on the ground, it says: 'I am Greek'. I cannot tell what the content of the writing is, unless I am told who wrote the sentence. The claim I wish to defend is that ∀ and ∃ in first-order logic ought to be taken as context-sensitive, as well, roughly like 'everything' and 'something' in natural language. If so, then we have a solution to the cardinality problem: every instance $\mathrm{C}(n)$ of CN is made false by a context in which less than n things are relevant, so they are not logically true, after all.

Even though the discussion on the semantics of quantifiers in natural language is admittedly quite nuanced, a great deal of literature backs up the claim that 'something' and 'everything' are context-sensitive. I am at a dinner party and – sadly – we ran out of wine. When I say: 'Every bottle is empty', I do not mean to say that every bottle in the universe is empty, just the ones we bought for the dinner party. If I then go out to the shop to buy more, and I say: 'Every bottle is sold out', I am now talking about the bottles in the shop, not the empty ones we just drank. Different contexts, different meanings, hence context-sensitivity.[7] The proposal itself is compatible with different theories of context sensitivity. The way in which quantifiers can be context-sensitive, however, is limited by the expressive ability of the language of first-order logic which we are interpreting: the contextual work must be done in the meta-language, since there is no contextual parameter in the syntax of first-order logic.

My proposal to treat ∃ and ∀ in first-order logic as context-sensitive is natural, given the intuitive semantic identification of ∀ with its natural language counterpart, and the likely context-sensitivity of the latter. Suppose we are dealing with a formal language with the syntactic expressions 'H' and 'Y', which we paraphrase as 'here'

[7] For a discussion, see Stanley and Gendler Szabó 2000. There are ways to counter the idea that 'something' is context-sensitive. According to the pragmatic approach, 'something' is always absolutely general, but what one conveys at a pragmatic level might not be what is said on a semantic level. According to the grammatical solution quantified sentences are often elliptical: there is an unarticulated syntactic portion of the sentence that is not uttered, which restricts the domain in some non-contextual way. Both these solutions are discussed and countered by Stanley and Gendler Szabó (2000). Another theory worth mentioning is speech-act pluralism, according to which an utterance of a quantified sentence does not express any specific proposition: there are many equally good candidates for what it is said, and in all of them quantifiers are not context-sensitive (Cappelen and Lepore 2008).

and 'you'. It would be quite odd to paraphrase them in such a way, while refusing to treat 'Y' and 'H' as context-sensitive.[8]

The meaning of a logical expression ought to be kept constant across all interpretations. This is correct but misleading for context-sensitive expressions because only the so-called *linguistic* part of their meaning ought to be kept fixed. In Kaplanian terms, we save the character, not the content: we keep constant the linguistic rule that fixes the content, not the content itself. For example, in a logic for the first person 'I', the character of 'I' is the way we fix its referent when given a context; the content of 'I', given a context, is the subject of the context (Kaplan 1989). It would be silly to assume that, when treated as a logical expression, 'I' ought to keep its content fixed: that it ought to refer to the same person across all interpretations. Rather, we let its referent vary arbitrarily, yet we do not change its character: in every interpretation its content is always the subject of the context. In quite the same manner, what must be kept fixed across interpretations is only the character, not the content of $\forall$. The character of $\forall$ is a function from contexts to the content of $\forall$ in a context. The character is fixed: in all interpretations $\forall$ ranges over everything relevant at the context. Yet, its content is not fixed: in a context where all and only dogs are salient, the content of $\forall$ is that of 'every dog', while in a context where all and only numbers are salient, it is that of 'every number'.

The fallacy in the argument for the validity of CN in section 2 is diagnosed as a confusion about contexts: 'There are n things' is true under the interpretation it has as we are using it, i.e. it is true in the meta-theory for any n. Yet, this only shows that CN instances are true when interpreted under the context of the meta-theory, not under any context. The fact that we must keep its interpretation fixed means that we need to keep its character fixed, but the contextual ingredient to meaning can change, triggering a shift in content.[9]

The context-sensitive theory is still a version of IR because it is impossible to interpret context-sensitive expressions without specifying a context. The relevant notion of truth in the definition of validity is still 'truth in an interpretation': it is not that the formula is true 'in a possibility' – a context. Rather, only in a context does a sentence mean something which can be said to be true or not. It is not that the sentence is true in an interpretation and context; rather, the context is part of the interpretation. In the example above, 'I am Greek' means different things in the mouth of different people. We can assess the sentence only if we are contextually given a subject. If the subject of the context is Pericles the sentence is true, if it is Caesar it is false.

One might argue that validity will itself be context sensitive if quantifiers are context-sensitive. Truth in an interpretation is context sensitive, since an interpretation specifies a context, and the same sentence can be true and false at different

[8]If quantifiers in natural language are not context-sensitive, my proposal admittedly loses much of its grip. However, my theory might still be useful, given its high effectiveness in dealing with the cardinality problem, and its adherence to the logician's common use of variable domain semantics.

[9]Here I am not assuming that there is only one possible context of the meta-theory, for that will depend on one's stance on how to interpret the meta-theory itself. For example, suppose that the meta-theory is ZFC. If there is such a thing as *the* intended interpretation of ZFC, then that is the unique context of the meta-theory. Yet, if interpretations of ZFC are indefinitely extensible, there is more than one context we can be in as we speak in ZFC-ese. In this latter case, whatever context we end up in as we speak ZFC-ese, we are assured by the axiom of infinity that there are infinitely many things, so there are n things for any n, anyway.

contexts. However, validity itself is defined as preservation of truth in all interpretations, and therefore it implies preservation of truth in all contexts, so it is not context-sensitive. However, Georgi (2015) argues that in languages with demonstratives such as 'this' and 'that', if we agree that the reference of these expressions can vary within the same context, validity and logical truth obtain or fail to obtain only relative to a context. In a context where all occurrences of 'this' refer to the same thing 'This is this' is logically true, but it is not logically true in a context where the two occurrences refer to different things. If quantifiers are 'promiscuous' in a similar way, and they shift range within the same context of utterance, then one could claim that we need a notion of logical consequence relative to a context, as well. I assume that one can extend the framework developed by Georgi to implement this theory.[10]

4.2 Formal system

To make my proposal more precise, I now provide a set-based semantics that implements context-sensitivity. The interpretations I obtain are equivalent to models of free logic in the usual set-based model-theory.

The meta-theory I am using is Zermelo Fraenkel set-theory with choice. The language we are interpreting is a first order language L with countably many predicates $P_1, .., P_n, ..$ of arbitrary arity, the indentity predicate $=$, variables $v_1, .., v_n, ..$, constants $a_1, .., a_n, ..$, the connectives $\neg$ and $\wedge$, and the quantifier $\exists$ ($\rightarrow$, $\vee$, $\leftrightarrow$ and $\forall$ are meta-abbreviations). We use $\alpha_1, ..\alpha_m, ..$ as meta-variables for variables or constants. Where P_n has arity m, a well-formed formula is defined inductively, as: $P_n(\alpha_1, .., \alpha_m) \mid \alpha_m = \alpha_n \mid \neg\phi \mid \phi \wedge \psi \mid \exists v \phi$.

To interpret our language we start with a universe. A universe U is any set. To provide a suitable notion of logical validity for which a common calculus for first-order logic is complete, U needs to be at least countably infinite.[11] Given a suitably large universe U, we define an interpretation I based on U as a triple $\langle U, c_I, [\![]\!]_I \rangle$. c_I is any subset of U and $[\![]\!]_I$ is a function defined as follows:

(a) For any P or arity n, $[\![P]\!]_I$ is a set of n-tuples from U

(b) For any a $[\![a]\!]_I \in U$

We are identifying a context with the set of things relevant at that context because that is all we need contexts to specify. c_I might not be U: when I say that everything is packed, all and only the objects in my backpack are relevant, but there are other irrelevant things in the universe which are not inside the backpack, like me and the backpack.

Where $\mathfrak{a}$ is a variable assignment (i.e. a function from the set of variables to U), x is any element of U and v and w are variables, define $\mathfrak{a}^x_v$ as the result of switching to x the value of v:

$$\mathfrak{a}^x_v(w) = \begin{cases} x & \text{if } w = v \\ \mathfrak{a}(w) & \text{otherwise.} \end{cases}$$

[10] I thank an anonymous reviewer for pointing me to Georgi's paper.

[11] Arithmetical counterexamples are enough to prove completeness (Hilbert and Bernays 1939). This is also a consequence of the downward Löowenheim–Skolem theorem.

We extend $[\![]\!]$ as follows: $[\![\alpha]\!]_{I,\mathfrak{a}} = \mathfrak{a}(\alpha)$ if α is a variable, $[\![\alpha]\!]_I$ if it is a constant.

We define a valuation function V from all formulae, variable assignments based on U and interpretations I based on U to $\{1, 0\}$ inductively as the unique function that satisfies the following conditions:

(V1) $V(P(\alpha_1, .., \alpha_n), I, \mathfrak{a}) = 1$ iff $\langle [\![\alpha_1]\!]_{I,\mathfrak{a}}, .., [\![\alpha_n]\!]_{I,\mathfrak{a}} \rangle \in [\![P]\!]_I$

(V2) $V(\alpha_n = \alpha_m, I, \mathfrak{a}) = 1$ iff $[\![\alpha_n]\!]_{I,\mathfrak{a}} = [\![\alpha_m]\!]_{I,\mathfrak{a}}$

(V3) $V(\neg\phi, I, \mathfrak{a}) = 1$ iff $V(\phi, I, \mathfrak{a}) = 0$

(V4) $V(\phi \wedge \psi, I, \mathfrak{a}) = 1$ iff $V(\phi, I, \mathfrak{a}) = V(\psi, I, \mathfrak{a}) = 1$

(V5) $V(\exists v\phi, I, \mathfrak{a}) = 1$ iff for some $x \in c_I$ $V(\phi, I, \mathfrak{a}^x_v) = 1$

We obtain the usual models (of free logic) restricted to a given set U. To obtain all the familiar models of free logic, we would need U to be the universe of sets: we can use proper-class theory to achieve this.

I now show how we can interpret (V5) as deriving the truth-value of a quantified formula explicitly from the character of the quantifier, as the context-sensitive solution demands. The character of $\exists$ given a universe U (written $[\![\exists]\!]_U$) is a function that takes each element $c_I \in \wp U$ to the content of $\exists$ in I. The content of $\exists$ in I is the set of $x \in \wp U$ that have an element in common with c_I:

$$[\![\exists]\!]_U(c_I) = \{x \in \wp U : x \cap c_I \neq \varnothing\}$$

(V5) is equivalent to the following clause:

(V5*) $V(\exists v\phi, I, \mathfrak{a}) = 1$ iff $\{x : V(\phi, I, \mathfrak{a}^x_v) = 1\} \in [\![\exists]\!]_U(c_I)$

Note that (V5*) makes sense only because, when we apply it to any $\exists v\phi$, we have already inductively defined the set of true and false sentences under any I and $\mathfrak{a}$, for any ψ of complexity less than $\exists v\phi$; one such ψ is ϕ.

4.3 Free Logic

Logical validity is not defined by quantifying across frames with different universes because there is no ground for using different universes in an interpretational theory, but only different contexts. Rather, we fix one universe U, which models the actual universe. Then, where I ranges over all interpretations based on U and $\mathfrak{a}$ over assignments based on U, logical validity $\vDash$ is defined as follows:

($\vDash$) $\Gamma \vDash \phi \leftrightarrow_{df} \forall I \forall \mathfrak{a}(\forall\psi(\psi \in \Gamma \to V(\psi, I, \mathfrak{a}) = 1) \to V(\phi, I, \mathfrak{a}) = 1)$

The resulting logic is an inclusive free logic. Free logic is the most natural logic for IR when we pair it with my context-sensitive proposal. Existential generalisation and universal instantiation both fail. Adapting an example from Gauker 1997, suppose that in a context c all that is relevant are some wooden figurines on a table. Under c, 'Everything is made of wood' is true even though 'Socrates is made of wood' is false. Assuming compositionality, the meaning of a sentence is determined by the meaning of its constituents. The meaning of 'Socrates is human' is determined by the meaning of 'Socrates' and 'is human', neither of which are context-sensitive, so 'Socrates is human' cannot be context-sensitive, either. Thus, when we evaluate it under an interpretation that specifies a context c, c is inert.[12] I find a piece of paper on the ground, it says: 'Socrates is human'. *Modulo* the assumption that the author intended to write in English, under any context the sentence means that Socrates is human, which is something true regardless. Existential generalisation fails, as well: 'Socrates is human' is true in an interpretation I where everything relevant is in my closet, even though – I swear – nothing is human in my closet, so 'Something is human' is false in I.

One might claim that 'Socrates is human' cannot be true because, if uttered, Socrates would become relevant. I would argue that something can be true even when it cannot be uttered in any given context. 'This sentence is not uttered' is such an example. Paraphrasing Lewis (1996), truths can be 'elusive', sometimes they can be only expressed *sottovoce*. Moreover, it is not clear that if we utter sentences about x in a context, we make x relevant. While packing I can truthfully say: 'I have everything in my backpack'. Clearly, everything relevant in that context is in the backpack, yet I am not inside the backpack, nor is my backpack inside itself. So, even though 'I have everything in my backpack' is true in this interpretation that specifies a context where all that is relevant is in my backpack, 'Everything is inside something' is false, so interpreted. In my semantics we would say that 'I' and 'backpack' are part of the universe U, even though they are not part of what is relevant at the context c_I.

The logic we obtain from the previous section is also 'inclusive' because there is an empty context where nothing is relevant. Under any interpretation that specifies that context, $\exists x\phi$ is false and $\forall x\phi$ is true, for any ϕ. The system can easily be changed if we think that in any context something is relevant, by setting c_I to be non-empty, for any I. As I specify in section 5.3, we need discard the empty context if we wish to recapture classical logic.

4.4 Advantages of the context-sensitive solution

I discuss in 5.2 how quantifiers are still logical expressions albeit context-sensitive. Because we are not relying on a shift in what counts as a logical expression, my proposal has none of the shortcomings of the third strategy discussed above. My solution also works when we simply treat quantifiers as logical in the scope of a logic of quantification, similarly to how we treat the knowledge operator as logical in the scope of epistemic logic.

Contrary to Etchemendy's first proposal, the fact that in no interpretation does 'everything' have the content of 'some dog' is explained: we must keep the character fixed. Contrary to his second proposal, the fact that in an interpretation 'Something is F' has the content of 'Something is U and F' is also explained: U is what is relevant in the context of that interpretation. However, we are not interpreting 'something' as 'some U': rather, the content of 'something' coincides with that of 'some U' under this

[12] We are disregarding the tense in the verb because we are dealing with an extensional language. What happens to the interpretational theory in modal and temporal languages is beyond the scope of this paper.

interpretation, but the two still differ in meaning, in the same way that ‘I’ and the name ‘Caesar’ do in a context where Caesar is the subject.

Hanson’s counterexample to CN can be accommodated by setting ‘being the Washington Monument or the White House’ as the relevance condition for a context. The reasoning, however, is quite different: we are not imagining a possible situation where there are only two things, and ask what would be true ‘in it’. Rather, we claim that, even after a linguistic interpretation is specified, we still do not know the content of ‘There are at least three things’, so we cannot say whether the sentence is true or not. For that we need a context; when we consider what the sentence means in a context where all that is salient are the Washington Monument and the White House, we realise that, so interpreted, the sentence is simply false. Since we can interpret Hanson’s additional representational cases as interpretations equipped with a context, there was no need to abandon the interpretational theory in the first place.

Contrary to the second strategy, there is no need to bite the bullet and argue that CN instances are logical truths after all, because we do have interpretations that falsify them. We can keep a traditional notion of logical validity as formal and analytically truth-preserving without departing from the interpretational account.

The context-sensitive version of IR also confirms both the intuitions that CN instances are in some sense contingent and in some sense as necessary as mathematics. In the context of me talking about what is in my closet, ‘There are n things’ expresses something contingent because there could have been nothing in my closet. In the context of a mathematical theory, however, ‘There are n things’ expresses something whose content is the same as a necessary truth of mathematics. Yet, Regardless of what instances of CN mean in a context, there is always a strict entailment between C31 and C30.[13]

In the context-sensitive version of IR we are able to make sense of the asymmetry in information between C30 and C31. To interpret these sentences, one needs a context; regardless of the context given, if C31 is true so interpreted, so is C30. By shifting context we change the information these sentences convey: in a context where dogs are all that is salient, C30 conveys that there are 30 dogs and C31 that there are 31 dogs; in any case, the piece of information conveyed by C31 contains that conveyed by C30. If we frame asymmetry in terms of analytic containment, we would say that, in every context, C31 guarantees the truth of C30 in virtue of what the terms mean in that context (their content), even though C30 and C31 say different things in different contexts. If we frame it in terms of warrant, we would say that we are warranted to infer C30 from C31 in every context because of what the sentences mean in that context; however, the reverse does not hold, nor are we guaranteed to infer C30 from scratch, regardless of what it means in a context because we do not know whether 30 things are always relevant to any context.

[13]In two-dimensional semantics, we would say that C31 implies C30 on the diagonal; i.e. the diagonal intension of C31 is a subset of the diagonal intension of C30 (Stalnaker 1978).

5 Ramifications and objections

5.1 Meta-context and generality

Consider the following revenge-like cardinality problem. Some claim that there is an absolutely general context (Williamson 2003). The context-sensitive theory of interpretations is consistent with this view: that there is a most general context does not imply that the most general context is the only context. Suppose that a friend of absolute generality adds to the language a quantifier which is explicitly absolutely general. Arguably, the resulting expression will not be context-sensitive because in any context it 'points back' at the most general context. Suppose we substitute such a quantifier to 'there are' in C30, obtaining the resulting sentence:

(A30) There (absolutely) are at least 30 things.

*A*30 is a truth of the logic of the absolutely general quantifier iff it is true in any (linguistic) interpretation and context, iff it is true in all interpretations that specify the most general context, iff it is true in the most general context. It is true in the most general context, so it is a truth of the logic of the absolutely general quantifier. The same reasoning applies to A(n), for any n.

It is not clear, however, that A30 is subject to the same issues C30 had. A30 does not mean that there are 30 things, it means that there absolutely are 30 things. Suppose equip the language with an arithmetical quantifier 'there is in arithmetic', and suppose we treat it as a logical expression. We should expect 'There are in arithmetic 30 things' to be a logical truth of the logic of the arithmetical quantifier. Similarly, we should expect that A30 is a logical truth of the logic of an explicitly absolutely general quantifier.

A representationalist might complain, claiming that A30 is still false in a world where my pen and my notebook are all there is. In this case, the interpretationalist should bite the bullet. When said in a context where the socks in my closet are all that is relevant, 'There are at least 30 things' says something contingent: there could have been nothing in the closet. On the other hand, A30 is as contingent as mathematics, and thus cannot have been false in any metaphysically possible world. The asymmetry problem also does not apply because we do not need a context to pin down what A30 means: it always means that there absolutely are 30 things, which is something I know simply by knowing what absolute quantification means. Similarly, to understand what 'There are in arithmetic 30 things' means, I need to know what 'in arithmetic' means, and if I know that, I must know that there are 30 things in arithmetic. Since I can always prove that there are 30 things in the most general context, I am always warranted to infer the truth of A30, under any interpretation. So, I am also warranted to infer any instance of A(n) from any other, as well.[14]

[14] For a similar discussion about an (explicitly) absolutely general quantifier, see Williamson 1999.

5.2 Logicality in context

One might doubt that quantifiers can be logical albeit context-sensitive. The discussion on logicality for context-sensitive expressions is perhaps not explicitly well-developed in the literature, but it fits nicely within the already existing literature on logicality. In this section, I focus on invariance theories. Invariance theories of logicality claim that an expression is logical when its meaning is invariant under arbitrary transformations of the universe, usually bijections (Bonnay 2014).

That universal and existential quantifiers come out logical despite context-sensitivity should not surprise us. What might trigger non-logicality, if anything, would be the fact that they change content across contexts. Yet, in standard semantic systems there already are different models with different domains; clearly, the content of $\forall$ in a model with domain D is not that of a model with domain D' where $D \neq D'$. As Sher claims, 'while non-logical terms are defined within models, logical terms are defined by fixed functions over models' (1996, 675). We do not expect the content of the quantifier in a model to be invariant, but rather its characteristic fixed function over models (which is structurally similar to a character). For example, the operation across models that corresponds to $\exists$ is the (class-sized) function that take any model $\mathfrak{M}$ with domain x to the set of non-empty elements of $\wp x$. This function is invariant under arbitrary bijections (McGee 1996, Sher 1991).

Albeit the discussion is scarce, some invariance theorists have defended the logicality of some contexts-sensitive expressions. For example, in his PhD thesis, MacFarlane develops a sophisticated account of logical expressions using a categorial semantics. In section 6.3 he suggests that the type for objects O is context-sensitive, since 'what counts as a thing in some stretch of discourse is determined largely by context and conversational convention' (2000, 200). MacFarlane goes on to say that 'we must add the notion of a context and replace the old notion of "semantic value" with two new notions, content and character', where the character is a function from contexts to contents (2000, 201). He then redefines invariance as invariance of character. A similar account is developed by Woods (2017); Woods defines the character and the content of an expression within a possible world semantics, and proposes an invariance test which is similar to MacFarlane's test. $\forall$ and $\exists$ are logical in both theories.

I now show how to adapt the invariance test of logicality to our semantic system. Given a universe U, consider the interpretations based on U and a bijection π between subsets of U. We define the transformation π^+ induced by π on the type-constructions from U as follows, where X^n is the n-ary Cartesian product of X and n^X is the n-ary power-set of X:

$$\pi^+(x) = \pi(x) \text{ if } x \in U$$
$$\pi^+(\langle x_1, .., x_n \rangle) = \langle \pi^+(x_1), .., \pi^+(x_n) \rangle$$
$$\forall n \forall m \text{ if } x \in m^{U^n} \text{ then } \pi^+(x) = \{y : \exists z(z \in x \land y = \pi^+(z))\}$$

We define invariance similarly to MacFarlane (2000, 204):

Definition (Invariance)**.** *An expression x of L is bijection-invariant in U exactly when, for all π between subsets of U, if $\pi^+(c_I) = c_H$, then $\pi^+([\![x]\!]_I) = [\![x]\!]_H$*

One can readily check that $\exists$ and $\forall$ satisfy the test. In fact, if we apply the test to characters, we have the stronger invariance result that $\pi^{+}([\![\exists]\!]_U) = [\![\exists]\!]_U$, for any π. If we wanted, we could strengthen the test by looking at bijections across different universes, like Woods does. This would make the result less contingent upon the initial choice of U. Similar results would be obtained in the strengthened theory.

5.3 Classical recapture

I believe that free logic is the most natural choice in view of an interpretational account of logical validity and the context-sensitivity of quantifiers. However, in this section I claim that one can recapture classical logic from within my system, contrary to what the literature argues about IR.

The fact that IR is unable to validate classical logic is one of Etchemendy's strongest complaints against it, and the fact that RE naturally fits classical logic is for him one of its great advantages over IR. Etchemendy argues that in (classical) model-theory we usually restrict the interpretation of the expressions inside the range of the quantifiers in each model. This holistic fitting procedure makes sense in representational terms: if a model is like a picture of a possibility, then all the expressions of the language ought to be 'squeezed' into the picture (1990, 66). To achieve the same result in interpretational terms, however, one would need to ban all spurious interpretations where the value of an expression outstrips the range of quantifiers. Yet, these 'cross-term restrictions' are hard to make sense of, since the interpretations banned are perfectly legitimate if not for making some rules or axioms invalid, which puts the cart before the horse.

No cross-term restrictions should be adopted in my proposal because, under a context where everything relevant is in on the table, we are clearly not reinterpreting the whole language so that everything fits on the table. 'Socrates' clearly still refers to Socrates, regardless. Yet, we can make sense of the classical 'holistic fitting' without resorting to cross-term restrictions, by taking a particular stance on how contexts affect the truth-value of sentences without quantifiers. We can insist that contexts act even on sentences that do not contain quantifiers, by proposing a pervasive context-sensitivity of truth, which is not triggered only by the presence of quantification. If Socrates is not relevant at a context, 'Socrates is human' is not true in any interpretation that specifies that context. A common option is to make the formula not false, either: a multi-valued logic comes natural.

Gauker (1997) develops a system of validity with these characteristics, under the name 'Kaplanesque three-valued free logic'. In such a logic there are no tautologies, since any object is irrelevant at some context, so any formula about that object is neither true nor false, when interpreted under that context.[15] As explained by Gauker, the rule of universal instantiation is invalid because the fact that everything relevant at a context is ϕ does not imply that Socrates ϕs, as Socrates might not be relevant at that context. On the other hand, if Socrates does satisfy some condition under a context, then he must be relevant at that context, so something relevant at it satisfies

[15] This can be proven by induction on the complexity of formulae. Some atomic formulae might not be about any object in particular, like 'It's cold'. These can be formalised as sentence letters, and they are true in any context if true, at all. This fact does not change the results of the theory because any sentence letter can arguably be reinterpreted as any other atomic formula.

that condition: the rule of existential generalisation is valid (1997, 196). Contraposition fails, *modulo* the equivalence between $\exists x\phi$ and $\neg\forall x\neg\phi$. Since there are no tautologies, the deduction theorem also fails. Similarly, proofs by contradictions are invalid: $\phi \wedge \neg\phi$ implies any formula and its negation, since it is not true under any interpretation; yet, $\neg(\phi \wedge \neg\phi)$ is not logically true.

What should we make of such a massive departure from classical logic? Gauker suggests that we should redefine logical validity by replacing truth and falsity in a context with assertibility and deniability in a context, respectively. If Socrates is not something relevant at the context at hand, then nothing can be asserted about him, nor can it be denied, for to deny something is to assert its negation. This is quite a conceptual shift, which Gauker acknowledges as such. However, we do not need to follow Gauker, as the semantics does not force us into Kleene validity. Instead, we can use Gauker's semantics to recapture classical logic within IR. Even if we insist that any sentence about Socrates is neither true nor false in a context where he is not relevant, we still have different choices on how to define validity. Sure, bivalence fails, but failure of bivalence does not by itself lead to Kleene logic, nor in general to non-classical logic (Rumfitt 2015). One way to recapture classical logic is to distinguish between a strong and a weak counterexample to the validity of a formula. A strong counterexample is given by an interpretation where the formula is false, a weak counterexample is given by an interpretation where the formula is not true. Absence of any counterexample results in Kleene validity, absence of only strong counterexamples results in classical validity. As I show in the appendix, classical validity is recaptured as follows: ϕ (classically) logically follows from Γ exactly when, under all interpretations and assignments based on U, if each member of Γ is true, ϕ is not false. The result works only if we further assume that the set of things relevant at a context is never empty, something which we quite left open so far. One could argue for it by insisting that in any context we must be talking about something, so something must always be relevant.

6 Conclusion

I argued that 'There are n things' is not logically true because, while from 'There are n things' we can logically infer 'There are $n-1$ things', from 'There are $n-1$ things' we cannot logically infer 'There are n things'; yet, we should be able to infer 'There are n things' from any premise if it was a logical truth. An interpretational theory of validity can account for this 'informational asymmetry' by making quantifiers context sensitive. I argued that my proposal is superior to the ones present in the literature: it does not require us to abandon the simplicity of the interpretational theory, nor does it rely on a shift in the choice of logical notions, nor does it force us to depart from a traditional notion of logical validity as formally and analytically truth-preserving. I showed quantifiers are still logical notions even if context-sensitive. Finally, I showed how, while an inclusive free logic is a natural choice for the context-sensitive version of IR, we can also recapture classical logic within IR by adopting a Kleene semantics and a non-standard notion of validity.

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Appendix A Appendix: Kleene semantics and classical validity

We can adapt the formal system in 4.3 to Kleene semantics as follows. First, we rule out any interpretation I such that c_I is empty. We leave the definition of $[\![]\!]$ unaltered. For every n-ary predicate P, the anti-extension of P relative to I is $c_I^n \backslash [\![P]\!]_I$: the set of n-tuples from elements in c_I that are not in $[\![P]\!]_I$. We denote it by $[\![^- P]\!]_I$. A valuation is the unique function from the set of formulae of L to $\{1, 0, \#\}$, defined as follows (when the value is not specified, it is #):

$$\text{(K1)} \qquad V(P(\alpha_1, .., \alpha_n), I, \mathfrak{a}) = \begin{cases} 1 & \text{iff } \langle [\![\alpha_1]\!]_{I,\mathfrak{a}}, .., [\![\alpha_n]\!]_{I,\mathfrak{a}} \rangle \in [\![P]\!]_I \cap c_I^n \\ 0 & \text{iff} \langle [\![\alpha_1]\!]_{I,\mathfrak{a}}, .., [\![\alpha_n]\!]_{I,\mathfrak{a}} \rangle \in [\![^- P]\!]_I \end{cases}$$

$$\text{(K2)} \qquad V(\alpha_n = \alpha_m, I, \mathfrak{a}) = \begin{cases} 1 & \text{iff } [\![\alpha_n]\!]_{I,\mathfrak{a}} = [\![\alpha_m]\!]_{I,\mathfrak{a}} \text{ and } \{[\![\alpha_n]\!]_{I,\mathfrak{a}}, [\![\alpha_m]\!]_{I,\mathfrak{a}}\} \subseteq c_I \\ 0 & \text{iff } [\![\alpha_n]\!]_{I,\mathfrak{a}} \neq [\![\alpha_m]\!]_{I,\mathfrak{a}} \text{ and } \{[\![\alpha_n]\!]_{I,\mathfrak{a}}, [\![\alpha_m]\!]_{I,\mathfrak{a}}\} \subseteq c_I \end{cases}$$

$$\text{(K3)} \qquad V(\neg\phi, I, \mathfrak{a}) = \begin{cases} 1 & \text{iff } V(\phi, I, \mathfrak{a}) = 0 \\ 0 & \text{iff } V(\phi, I, \mathfrak{a}) = 1 \end{cases}$$

$$\text{(K4)} \qquad V(\phi \wedge \psi, I, \mathfrak{a}) = \begin{cases} 1 & \text{iff } V(\phi, I, \mathfrak{a}) = V(\psi, I, \mathfrak{a}) = 1 \\ 0 & \text{iff } V(\phi, I, \mathfrak{a}) = 0 \text{ or } V(\psi, I, \mathfrak{a}) = 0 \end{cases}$$

$$\text{(K5)} \qquad V(\exists v\phi, I, \mathfrak{a}) = \begin{cases} 1 & \text{iff for some } x \in c_I \; V(\phi, I, \mathfrak{a}_v^x) = 1 \\ 0 & \text{iff for all } x \in c_I \; V(\phi, I, \mathfrak{a}_v^x) = 0 \end{cases}$$

As before, we can show how to extract the truth-value of a quantified formula directly from the character and content of the quantifiers. The content of $\forall$ in a universe U is $[\![\forall]\!]_U(c_I) = \{x \in \wp U : c_I \subseteq x\}$. K5 is equivalent to the following:

$$\text{(K5*)} \qquad V(\exists v\phi, I, \mathfrak{a}) = \begin{cases} 1 & \text{iff } \{x : V(\phi, I, \mathfrak{a}_v^x) = 1\} \in [\![\exists]\!]_U(c_I) \\ 0 & \text{iff } \{x : V(\phi, I, \mathfrak{a}_v^x) = 0\} \in [\![\forall]\!]_U(c_I) \end{cases}$$

We now prove that we can recapture classical validity (labelled $\vDash_{cl}$), in this semantic system. Define $\vDash_2$ as follows, where I and $\mathfrak{a}$ are based on U:

$$(\vDash_2) \qquad \Gamma \vDash_2 \phi \leftrightarrow_{df} \forall I \forall \mathfrak{a} (\forall \psi (\psi \in \Gamma \rightarrow V(\psi, I, \mathfrak{a}) = 1) \rightarrow V(\phi, I, \mathfrak{a}) \neq 0)$$

$\vDash_2$ considers only strong counterexamples. We show that it is equivalent to $\vDash_{cl}$.
Lemma 1 (Recapture). *For any Γ and ϕ of L, $\Gamma \vDash_2 \phi$ iff $\Gamma \vDash_{cl} \phi$.*

A classical model $\mathfrak{M}$ is a tuple $\langle D_{\mathfrak{M}}, [\![]\!]_{\mathfrak{M}} \rangle$ where $D_{\mathfrak{M}}$ is any set and, for any constant a and predicate P of arity n of L, $[\![a]\!]_{\mathfrak{M}} \in D_{\mathfrak{M}}$ and $[\![P]\!]_{\mathfrak{M}} \subseteq D_{\mathfrak{M}}^n$. We extend as before $[\![]\!]_{\mathfrak{M}}$ to $[\![]\!]_{\mathfrak{M},\mathfrak{a}}$. Truth in a classical model is defined in the usual way. By $V_{cl}(\phi, \mathfrak{M}, \mathfrak{a}) = 1$ we mean that, in the usual classical semantics, ϕ is true in $\mathfrak{M}$ and $\mathfrak{a}$ (where $\mathfrak{a}$ is an assignment based on $D_{\mathfrak{M}}$). As a consequence of the Löwenheim-Skolem theorem, classical validity (in first-order logic) is equivalent to preservation of truth in all countable classical models. So, to prove the lemma, we choose the universe U to be at least countably infinite, and we focus on classical models whose domain is a subset of U.[16]

To prove the lemma we need two sublemmata. By 'Kleene interpretation' I mean an interpretation I in a system where the valuation function is defined via K1–K5.

Lemma 1.1. *For any Kleene interpretation K and assignment $\mathfrak{b}$ from U there is a countable classical model $\mathfrak{M}$ and assignment $\mathfrak{a}$ from $D_{\mathfrak{M}}$ such that, for any ϕ of L, if $V(\phi, K, \mathfrak{b}) = 1$ then $V_{cl}(\phi, \mathfrak{M}, \mathfrak{a}) = 1$ and if $V(\phi, K, \mathfrak{b}) = 0$ then $V_{cl}(\phi, \mathfrak{M}, \mathfrak{a}) = 0$.*

Lemma 1.2. *For any classical model $\mathfrak{M}$ such that $D_{\mathfrak{M}} \subseteq U$ and assignment $\mathfrak{a}$ from $D_{\mathfrak{M}}$, there is a Kleene interpretation K and assignment $\mathfrak{b}$ from U such that, for any ϕ of L, if $V_{cl}(\phi, \mathfrak{M}, \mathfrak{a}) = 1$ then $V(\phi, K, \mathfrak{b}) = 1$ and if $V_{cl}(\phi, \mathfrak{M}, \mathfrak{a}) = 0$ then $V(\phi, K, \mathfrak{b}) = 0$.*

Proof. To prove sublemma 1.1, the relevant $\mathfrak{M}$ and $\mathfrak{a}$ are defined as follows: $D_{\mathfrak{M}} = c_K$, for any P of arity n $[\![P]\!]_{\mathfrak{M}} = [\![P]\!]_K \cap c_k^n$, and $[\![\alpha]\!]_{\mathfrak{M},\mathfrak{a}} = [\![\alpha]\!]_{K,\mathfrak{b}}$ if $[\![\alpha]\!]_{K,\mathfrak{b}} \in c_K$; $[\![\alpha]\!]_{\mathfrak{M},\mathfrak{a}}$ is some fixed e in c_K otherwise.[17] We can prove the sublemma by a straightforward induction on the complexity of ϕ.

We prove the base of the induction for $P(\alpha_1, .., \alpha_n)$:

$$\begin{aligned} V(P(\mathfrak{a}_1, .., \mathfrak{a}_n), K, \mathfrak{b}) = 1 &\rightarrow \langle [\![\alpha_1]\!]_{K,\mathfrak{b}}, .., [\![\alpha_n]\!]_{K,\mathfrak{b}} \rangle \in [\![P]\!]_K \cap c_K^n \\ &\rightarrow \langle [\![\alpha_1]\!]_{\mathfrak{M},\mathfrak{a}}, .., [\![\alpha_n]\!]_{\mathfrak{M},\mathfrak{a}} \rangle \in [\![P]\!]_{\mathfrak{M}} \\ &\rightarrow V_{cl}(P(\mathfrak{a}_1, .., \mathfrak{a}_n), \mathfrak{M}, \mathfrak{a}) = 1 \end{aligned}$$

The second step follows because $[\![P]\!]_{\mathfrak{M}} = [\![P]\!]_K \cap c_K^n$ and because, by definition of V, $V(P(\mathfrak{a}_1, .., \mathfrak{a}_n), K, \mathfrak{b})$ is 1 or 0 only if each of $[\![\alpha_1]\!]_{K,\mathfrak{b}}$, ..,$[\![\alpha_n]\!]_{K,\mathfrak{b}}$ are in c_K, and because by construction $c_K = D_{\mathfrak{M}}$.

Similarly:

$$\begin{aligned} V(P(\mathfrak{a}_1, .., \mathfrak{a}_n), K, \mathfrak{b}) = 0 &\rightarrow \langle [\![\alpha_1]\!]_{K,\mathfrak{b}}, .., [\![\alpha_n]\!]_{K,\mathfrak{b}} \rangle \in c_K^n \backslash [\![P]\!]_K \\ &\rightarrow \langle [\![\alpha_1]\!]_{\mathfrak{M},\mathfrak{a}}, .., [\![\alpha_n]\!]_{\mathfrak{M},\mathfrak{a}} \rangle \notin [\![P]\!]_{\mathfrak{M}} \\ &\rightarrow V_{cl}(P(\mathfrak{a}_1, .., \mathfrak{a}_n), \mathfrak{M}, \mathfrak{a}) = 0 \end{aligned}$$

The first line follows since $[\![^{-}P]\!]_I = c_K^n \backslash [\![P]\!]_K$ by definition. The second line follows because each $[\![\alpha]\!]_{K,\mathfrak{b}}$ must be in $D_{\mathfrak{M}}$ and because, if $x \in c_K^n \backslash [\![P]\!]_K$, then $x \notin [\![P]\!]_K \cap c_K^n$, so $x \notin [\![P]\!]_{\mathfrak{M}}$. The base of the lemma for $\alpha_n = \alpha_m$ can be proven using the fact that whenever $V(\alpha_n = \alpha_m, K, \mathfrak{b})$ is 1 or 0, both values of α_n and α_m must be in c_K, which is $D_{\mathfrak{M}}$. The base follows by construction of $\mathfrak{a}$ and $\mathfrak{M}$, respectively.

We assume the following induction hypothesis: sublemma 1.1 holds for all ϕ of complexity n, when the relevant $\mathfrak{M}$ and $\mathfrak{a}$ are constructed in the way explained above. The induction step is straightforward for $\neg\phi$ and $\phi \wedge \psi$. For $\exists v\phi$, by K5 and since $c_K = D_{\mathfrak{M}}$, $V(\exists v\phi, K, \mathfrak{b}) = 1$ iff for some $x \in D_{\mathfrak{M}}$, $V(\phi, K, \mathfrak{b}_v^x) = 1$. By IH, there is an $x \in D_{\mathfrak{M}}$ such that $V_{cl}(\phi, \mathfrak{M}', \mathfrak{a}') = 1$, where $\mathfrak{M}'$ and $\mathfrak{a}'$ are built from K and $\mathfrak{b}_v^x$ as

[16] If we don't want to rely on this result, we can choose proper-class theory as meta-theory, and pick U to be the proper-class of all sets.

[17] The resulting classical model is consistent. By definition of a classical model, the values in $\mathfrak{M}$ and $\mathfrak{a}$ of variables and constants can be any value in $D_{\mathfrak{M}}$, also a unique one. Logically, it follows that the values in $\mathfrak{M}$ and $\mathfrak{a}$ of variables and constants whose values in K and $\mathfrak{b}$ are in c_K can also be a unique value e in $D_{\mathfrak{M}}$.

follows: $D_{\mathfrak{M}'} = c_K$, $[\![P]\!]_{\mathfrak{M}'} = [\![P]\!]_K \cap c_k^n$ where P has arity n, and $[\![\alpha]\!]_{\mathfrak{M}',\mathfrak{a}'} = [\![\alpha]\!]_{K,\mathfrak{b}_v^x}$ if $[\![\alpha]\!]_{K,\mathfrak{b}_v^x} \in c_K$; $[\![\alpha]\!]_{\mathfrak{M}',\mathfrak{a}'} = e$ otherwise. Clearly $\mathfrak{M}' = \mathfrak{M}$. Consider any variable w such that $w \neq v$. Either $\mathfrak{b}_v^x(w) \in c_K$ or not. If the former then $\mathfrak{a}'(w) = \mathfrak{b}_v^x(w) = \mathfrak{b}(w) = \mathfrak{a}(w)$ by construction of $\mathfrak{a}'$, by definition of $\mathfrak{b}_v^x$, and by construction of $\mathfrak{a}$, respectively. If the latter then $\mathfrak{a}'(w) = e = \mathfrak{a}(w)$. Since $x \in c_K$, $\mathfrak{b}_v^x(v) = \mathfrak{a}'(v) = x$ by construction of $\mathfrak{a}'$, so $\mathfrak{a}'$ is $\mathfrak{a}_v^x$. So, there is an $x \in D_{\mathfrak{M}}$ such that $V_{cl}(\phi, \mathfrak{M}, \mathfrak{a}_v^x) = 1$, which implies $V_{cl}(\exists v\phi, \mathfrak{M}, \mathfrak{a}) = 1$. A parallel reasoning can be employed when $V(\exists v\phi, K, \mathfrak{b}) = 0$.

Sublemma 1.2 is more straightforward. The relevant K and $\mathfrak{b}$ are the following: $\mathfrak{b} = \mathfrak{a}$ and $K = \langle U, D_{\mathfrak{M}}, [\![]\!]_{\mathfrak{M}}\rangle$. The sublemma follows by induction on the complexity of ϕ, with a reasoning parallel to the one above.

We now prove the lemma. Assume $\Gamma \nvDash_2 \phi$. By definition of $\vDash_2$ there is a Kleene-interpretation K and $\mathfrak{b}$ from U such that, for all $\psi \in \Gamma$ $V(\psi, K, \mathfrak{b}) = 1$ but $V(\phi, K, \mathfrak{b}) = 0$. By sublemma 1.1 there is a classical model $\mathfrak{M}$ and $\mathfrak{a}$ from $D_{\mathfrak{M}}$ where, for any $\psi \in \Gamma$ $V_{cl}(\psi, \mathfrak{M}, \mathfrak{a}) = 1$ but $V_{cl}(\phi, \mathfrak{M}, \mathfrak{a}) = 0$, so $\Gamma \nvDash_{cl} \phi$. Assume $\Gamma \nvDash_{cl} \phi$. Then there is a classical model $\mathfrak{M}$ from U and an assignment whose range is $D_{\mathfrak{M}}$ such that for any $\psi \in \Gamma$ $V_{cl}(\psi, \mathfrak{M}, \mathfrak{a}) = 1$ but $V_{cl}(\phi, \mathfrak{M}, \mathfrak{a}) = 0$. By lemma 1.2, there is a Kleene-interpretation K and an $\mathfrak{b}$ based on U such that for all $\psi \in \Gamma$ $V(\psi, K, \mathfrak{b}) = 1$ but $V(\phi, K, \mathfrak{b}) = 0$, so $\Gamma \nvDash_2 \phi$. We conclude that $\Gamma \vDash_2 \phi$ exactly when $\Gamma \vDash_{cl} \phi$. □

References

Barwise J, Cooper R (1981) Generalized quantifiers and natural language. In: Philosophy, language, and artificial intelligence. Springer, p 241–301

Beany M (ed) (1997) The Frege Reader. Blackwell, Oxford

Bonnay D (2014) Logical constants, or how to use invariance in order to complete the explication of logical consequence. Philosophy Compass 9(1):54–65. https://doi.org/10.1111/phc3.12095

Cappelen H, Lepore E (2008) Insensitive Semantics: A defense of semantic minimalism and speech act pluralism. John Wiley & Sons

Etchemendy J (1990) The Concept of Logical Consequence. CSLI Publications

Frege G (1884) Die Grundlagen der Arithmetik. Eine logisch-mathematische Untersuchung über den Begriff der Zahl. Beaney, Michael, translated as: 'The Foundations of Arithmetic' in Beany (1997, 84-129)

Gauker C (1997) Universal instantiation: A study of the role of context in logic. Erkenntnis 46(2):185–214

Georgi G (2015) Logic for languages containing referentially promiscuous expressions. Journal of Philosophical Logic 44(4):429–451. https://doi.org/10.1007/s10992-014-9335-5

Gómez-Torrente M, et al (1996) Tarski on logical consequence. Notre Dame Journal of Formal Logic 37(1):125–151

Hanson WH (1997) The concept of logical consequence. The Philosophical Review 106(3):365–409

Hilbert D, Bernays P (1939) Grundlagen der Mathematik, Volume II. Springer, Berlin

Kaplan D (1989) Demonstratives: An essay on the semantics, logic, metaphysics and epistemology of demonstratives and other indexicals. In: Almog J, Perry J, Wettstein H (eds) Themes From Kaplan. Oxford University Press, p 481–563

Lewis D (1996) Elusive knowledge. Australasian Journal of Philosophy 74(4):549–567

MacFarlane JG (2000) What does it mean to say that logic is formal? PhD thesis, University of Pittsburgh

McGee V (1996) Logical operations. Journal of Philosophical Logic 25(6):567–580

Quine WV (1986) Philosophy of Logic. Harvard University Press

Ray G (1996) Logical consequence: A defense of Tarski. Journal of Philosophical Logic 25(6):617–677

Rumfitt I (2015) The Boundary Stones of Thought: An essay in the philosophy of logic. Oxford University Press

Sagi G (2018) Logicality and meaning. The Review of Symbolic Logic 11(1):133–159

Shapiro S (1998) Logical consequence: Models and modality. In: Schirn M (ed) The Philosophy of Mathematics Today. Oxford: Oxford University Press, p 131–156

Sher G (1991) The Bounds of Logic: A Generalized Viewpoint. MIT Press

Sher G (1996) Did Tarski commit 'Tarski's fallacy'? The Journal of Symbolic Logic 61(2):653–686

Sher G (2001) The formal-structural view of logical consequence. The Philosophical Review 110(2):241–261

Stalnaker RC (1978) Assertion. Syntax and Semantics 9:315–332

Stanley J, Gendler Szabó Z (2000) On quantifier domain restriction. Mind & Language 15(2-3):219–261

Tarski A (1936a) O pojęciu wynikania logicznego. Przegląd Filozoficzny 39:58–68. Translated as: 'On the concept of logical consequence' in Tarski 1983, 409-20

Tarski A (1936b) Über den begriff der logischen folgerung. Actes du Congrès International de Philosophie Scientifique, fasc 7 ((Actualités Scientifiques et Industrielles) 394:1–11. German version of Tarski 1936a; translated as: 'On the concept of logical consequence' in Tarski 1983, 409-20

Tarski A (1983) Logic, semantics, metamathematics: papers from 1923 to 1938. Oxford Clarendon

Williamson T (1999) Logic and existence. Aristotelian Society Supplementary Volume 173 (1):181–203

Williamson T (2003) Everything. Philosophical Perspectives 17.1:415–465

Williamson T (2017) Semantic paradoxes and abductive methodology. In: Armour-Garb B (ed) Reflections on the Liar. Oxford University Press, p 325–346

Woods J (2017) Characterizing invariance. Ergo 3:778–807