On Infinite Number and Distance

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> Context • The infinite has long been an area of philosophical and mathematical investigation. There are many puzzles and paradoxes that involve the infinite. **> Problem** • The goal of this paper is to answer the question: Which objects are the infinite numbers (when order is taken into account)? Though not currently considered a problem, I believe that it is of primary importance to identify properly the infinite numbers. **> Method** • The main method that I employ is conceptual analysis. In particular, I argue that the infinite numbers should be as much like the finite numbers as possible. **> Results** • Using finite numbers as our guide to the infinite numbers, it follows that infinite numbers are of the structure $\omega+(\omega^*+\omega)\alpha+\omega^*$. This same structure also arises when a large finite number is under investigation. **> Implications** • A first implication of the paper is that infinite numbers may be large finite numbers that have not been investigated fully. A second implication is that there is no number of finite numbers. Third, a number of paradoxes of the infinite are resolved. One change that should occur as a result of these findings is that "infinitely many" should refer to structures of the form $\omega+(\omega^*+\omega)\alpha+\omega^*$; in contrast, there are "indefinitely many" natural numbers. **> Constructivist content** • The constructivist perspective of the paper is a form of strict finitism.
> Key words • **Cantor, infinite number, infinity, ordinals, infinite distance.**

1. Introduction

Which objects are the infinite numbers? As order is taken into account, we limit our search to the order types of total orderings.[1] For the purposes of this paper, it will be sufficient to work with strokes and dots, and to think of such things as order types, and to take a subset of these order types to be the numbers. This is to say that 3 is

|||

Likewise 7 is

|||||||

These are two finite numbers. In stroke-dot notation, ω is[2]

||||||||||...

The question is, is this an infinite number? In general, which objects are the infinite numbers? Georg Cantor claims that the infinite ordinals, ω, ω+1, etc, are the infinite numbers. I claim that these are not the infinite numbers, but rather that different objects are the infinite numbers. How are we to know which objects are the infinite numbers? I suggest that we take the finite numbers to be our guide to the infinite numbers. The infinite numbers should be as much like the finite numbers as possible.[3] Cantor wrote:

> "All so-called proofs of the impossibility of actually infinite numbers are false in that they begin by attributing to the numbers in question all the properties of finite numbers, whereas the infinite numbers, if they are to be thinkable in any form, must constitute quite a new kind of number as opposed to the finite numbers, and the nature of this new kind of number is dependent on the nature of things and is an object of investigation, but not of our arbitrariness or prejudice." (Cantor 1955: 74)

Cantor's remark presupposes that the infinite numbers must differ from the finite numbers. And certainly this is trivially true in that the finite numbers are finite, whereas the infinite numbers are infinite. But what additional differences should there be? Let us attempt to construe the infinite numbers by taking as our guiding principle that the infinite numbers should be as much like the finite numbers as possible.

2. Infinite numbers: Informal transfer

Which objects are the infinite numbers? If the goal is to arrive at infinite numbers that differ as little as possible from finite (natural) numbers, what would such numbers look like? Let us arrive at the infinite numbers by moving (some) properties that all finite numbers share to the infinite. We can think of such numbers as being composed of ordered *elements*, or strokes, in base 1. Note that I will use "elements" to refer to the units (strokes) that comprise the number. Then any finite number has a last element, every element but the first has a predecessor, every element but the last has a successor, every number is odd or even, and subtracting 1 from a number yields a different and smaller number. Let us move *these properties*[4] to the infinite, where we begin by simply assuming that an infinite number, M, exists. Then when we subtract 1 from M, we arrive at a smaller infinite number. We can

1| 0, 1, 2 under < is an ordering. 2, 3, 7 under < is an ordering. These are two different orderings, but they have the same order type as they are order isomorphic. Order enters into stroke-dot notation, the notation that I use in this paper, via the assumption that any stroke to the left of another is less than that other.

2| Also note that ω* is: ... ||||||||||

3| One reason for taking the finite numbers to be our guide is that we do not seem to have any other access to the infinite numbers. That is, if the finite numbers are not our guide to the infinite numbers, then what is?

4| It might be asked at this point: How can we assume that these properties hold for infinite numbers? And why don't we include, for example, well-ordering? The answer: I am going to argue that infinite numbers are large finite numbers. If true, obviously these properties will hold of infinite numbers. Well-ordering does not; I suggest why below.

continue subtracting units from M, $M-1$, etc., and so the picture that emerges is:

$$1, 2, 3 \ldots M-2, M-1, M$$

which is of order-type $\omega+\omega^*$. Next, numbers are either odd or even. Let us choose an infinite M that is even. Then $M/2$ is an infinite number, and so it cannot be in the ω or the ω^* copy thus far,[5] and so the picture is:

$$1, 2, 3 \ldots M/2 \ldots M-2, M-1, M$$

We can subtract and add units to $M/2$, resulting in:

$$1, 2, 3 \ldots M/2-2, M/2-1, M/2, M/2+1, M/2+2 \ldots M-2, M-1, M$$

Our number now is of the structure $\omega+(\omega^*+\omega)+\omega^*$. And we can continue in this vein, assuming that M is divisible by 4, thus arriving at $M/4$ and $3M/4$. Each of $M/4$ and $3M/4$ gives rise to a copy of $(\omega^*+\omega)$, as again we can add and subtract units from each. The structure that emerges is: $\omega+(\omega^*+\omega)\,\alpha+\omega^*$. By an informal transfer of properties of finite numbers to the infinite, we have arrived at infinite numbers of structure $\omega+(\omega^*+\omega)\,\alpha+\omega^*$. Perhaps these are *the* infinite numbers. Let us review. Cantor points at

$$||||||||||\ldots$$

and says, "There is an infinite number." By contrast, I point at

$$||||||||||\ldots||||||||||$$

and say "There is an infinite number." The difference occurs at the level of form, that is, we both agree to take strokes and dots as order types, and to take order types as (potential) numbers. The debate, at this point, is not occurring at a deep metaphysical level. Cantor, to defend his claim, must explain why the objects he claims are the infinite numbers lack properties that hold of finite numbers. Cantor's quote above indicated that infinite numbers must differ from finite numbers in significant ways, but we have seen that this is not so. A good methodological principle is that infinite numbers should differ from finite numbers in as few ways as possible. Then objects of the form $\omega+(\omega^*+\omega)\,\alpha+\omega^*$ are the infinite numbers.

Further support for the position that order types of the form $\omega+(\omega^*+\omega)\,\alpha+\omega^*$ are the infinite numbers comes from people having called such objects the infinite numbers. For example, Abraham Robinson (1996: 51) discusses infinite numbers, writing, e.g., "Thus any finite natural number is less than any infinite natural number."[6] As "+" and "·" are in the language, the order type of Robinson's infinite natural numbers is $\omega+(\omega^*+\omega)\,\alpha+\omega^*$, as above, where α is a dense linear order without endpoints. This form is very similar to that arrived at above. Indeed, I believe that objects of the form $\omega+(\omega^*+\omega)\,\alpha+\omega^*$ are the infinite numbers. The problem is that these objects are not actually taken to be the infinite numbers. Such a structure is not a strange byproduct of nonstandard models; rather, objects of this form truly are the infinite numbers. In this paper, I argue that when one says, "There is an infinite number," one should be pointing at an object of the form $\omega+(\omega^*+\omega)\,\alpha+\omega^*$.

3. Infinite numbers and large finite numbers

"The point of view that there are no non-experienced truths… has found acceptance with regard to mathematics much later than with regard to practical life and to science." (Brouwer 1983: 90)

In this section, we arrive at the structure discussed above, but from a different perspective. Let us take seriously the idea that for a (natural, whole) number to be finite and determined, it must be constructed (or counted)[7] by a subject (or subjects) in base 1. Below, I present a thought experiment where some (of a very large number of) strokes that are on a steel bar are counted. Whether and to what extent this restriction to base 1 might be relaxed is beyond the scope of this paper, and so by working in base 1, I do not mean to imply that only numbers in this form are finite and determined. Some notations (e.g., Arabic numerals) and some operations (e.g., addition and multiplication) may be used to denote finite, determined numbers in some cases. And yet there are legitimate concerns as to whether or not some operations, such as exponentiation, preserve finitude. Rohit Parikh (1971: 507) writes: "Does the Bernays' number $67^{257^{729}}$ actually belong to every set which contains 0 and is closed under the successor function? The conventional answer is yes but we have seen that there is a very large element of phantasy in conventional mathematics which one may accept if one finds it pleasant, but which one could equally sensibly (perhaps more sensibly) reject." Edward Nelson (1986: 50) suggests that 80^{5000} may be infinite. Let us then link base 1 construction or counting to finitude to see what follows, while recognizing that the condition may be relaxed.

Imagine a steel bar that is 50 kilometers long, 5 centimeters tall, and 1 centimeter thick. It is not immediately apparent that there is anything on the bar. However, with the use of the most powerful microscope on earth, you are able to see 1 centimeter strokes wherever you look on one face of the bar. Thinking that the bar might contain a very important message of some sort or other, you investigate the bar. Now, of course, you have no idea what is on the bar, but I will tell you what is on the bar. The bar contains nothing except for 80^8 strokes. These strokes can only be distinguished by using the microscope. Also, imagine that the strokes are not evenly spaced, and so their density varies a great deal over the face of the bar. This condition prevents you from determining (or perhaps even estimating accurately) the number of strokes on the bar.

5| $M/2$ is infinite because if it were finite, then M would be finite, which it is not. We are assuming that finite numbers are closed under addition and multiplication. Exponentiation is not. Also, $M/2$ cannot be in the copies thus far, because then the difference between $M/2$ and M (or $M/2$ and 1) would be finite, which again, it is not.

6| The order type $\omega+(\omega^*+\omega)\,\alpha+\omega^*$ is arrived at by taking a non-standard, or infinite, N, and considering the order type of the elements less than N, which again displays a striking parallel with the finite numbers. That is, another way to run my argument is: take some N in a nonstandard model of arithmetic and consider the elements less than N. If N is finite then the order type is a finite number; if N is infinite then the order type is an infinite number.

7| If the number is being created, then the number is constructed; if some number of already existing objects is being determined, then the number is counted.

Your first step is to order more microscopes to be built, so that several teams can investigate the bar at the same time. The microscopes can be turned out at the rate of 1 every week. Given that you have 1 microscope, you decide to start at the beginning, or the leftmost end of the bar. You send a team with the microscope to start investigating the bar. They start at the leftmost stroke, and investigate. The microscopes are able to mark each stroke that has been investigated. Each night, the team stops work, first reporting what they have found to you. Now, given that more teams are soon to investigate the bar, you decide you need a code to keep track of what the teams have found. You decide upon the following:

- ω will indicate that a team has found a group of strokes, and that they have space (more bar) to investigate to the right of the group they have investigated, but no space to the left.
- ω^* will indicate that a team has found a group of strokes, and that they have space (more bar) to investigate to the left of the group they have investigated, but no space to the right.
- $(\omega^*+\omega)$ will indicate that a team has found a group of strokes, and that they have space (more bar) to investigate to the left and to the right of the group they have investigated.

You also use a number to indicate how many strokes they have found. Of course, as you have no idea what is on the bar, you certainly entertain the possibility that you may need more symbols if anything other than a stroke appears (which we know will not happen). Thus, at the end of the first day, perhaps the team starting at the left of the bar has seen 50 strokes (working with the microscope is slow going). Thus, you jot down in your notebook "50ω" (let us imagine that a team investigates 50 strokes per day). Let us further imagine that when the 2nd microscope is ready, you assign a new team to investigate the bar starting at the rightmost edge. And so of course they investigate right to left. At the end of Team 2's first day (which is the 8th day for Team 1), when the teams have reported in, you jot down in your notebook "$400\omega+50\omega^*$." The "+" is just a symbol used to keep the teams' reports separate. Again, the meaning of your notation "$400\omega+50\omega^*$" is that Team 1 has found 400 strokes, and has space on the bar to investigate to the right, but not to the left. Team 2 has found 50 strokes and has space on the bar to investigate to the left, but not to the right. Teams 1 and 2 plod on their weary way, when microscope 3 becomes ready. You send Team 3 roughly to the middle of the bar (25 kilometers from each end) and tell them to go to it. Team 3 will have space to investigate on both sides. Thus, at the end of day 15, your notebook would read "$750\omega+50(\omega^*+\omega)+400\omega^*$."

From this point on, the structure of the bar (your jotting each night, ignoring numbers) is $\omega+(\omega^*+\omega)\alpha+\omega^*$. In this case, α will be finite (but growing through time). A picture of this situation after 5 microscopes are in use is the following:

||||||	||||||	||||||	||||||	||||||
ω	$(\omega^*+\omega)$	$(\omega^*+\omega)$	$(\omega^*+\omega)$	ω^*

The structure $\omega+(\omega^*+\omega)\alpha+\omega^*$ results; which is the same structure that arose in Section 2, and the same structure of infinite integers in nonstandard models of arithmetic. One explanation for this "coincidence," and the explanation I believe, is that the structure of infinite numbers is $\omega+(\omega^*+\omega)\alpha+\omega^*$, and that infinite numbers are large finite numbers that have not been fully investigated.

4. Taking stock

Let us take stock. The overarching question that this paper addresses is: Which objects are the infinite numbers, when order is taken into account? In Sections 2 and 3, this question was addressed in very different ways. First, it is worthwhile noting that I believe that the argument in Section 2 succeeds, independent of Section 3. That is, I believe that there are better and worse ways for our concepts to carve up the world, and insofar as objects of the structure $\omega+(\omega^*+\omega)\alpha+\omega^*$ look and behave more like finite numbers than do Cantor's infinite ordinals, this provides evidence that objects of the structure $\omega+(\omega^*+\omega)\alpha+\omega^*$ are the infinite integers.

Section 3 suggests that the finite/infinite divide is relative to an observer and relative to time. What is finite to one observer may be infinite to another. What is infinite at one time to a group of observers may become finite at a later time. If a very powerful being can count 80^8 strokes at a glance, then this number is finite to this being. Yet for you and the teams investigating the steel bar, it contains an infinite number of strokes until they are all counted (at which time the bar contains a finite number, 80^8, of strokes). Perhaps some numbers simply cannot be counted, due to physical limitations. Such numbers may then be infinite in an absolute, or at least a very strong, sense. Physics and computer science may, someday, be seen as more fundamental than mathematics.

A person picks up a photo album that contains 100 pages. She flips through a few pages of pictures at the front of the album, then a few pages at the back, and finally through two clusters of middle pages. There is a sense in which this is an experience of the infinite, insofar as the photo album is experienced as the structure $\omega+(\omega^*+\omega)2+\omega^*$. Of course, there is some absurdity in suggesting that 100 is an infinite number. And as noted above, the restriction whereby the finite is only arrived at via base 1 counting (or constructing) is undoubtedly too strong. And yet there is *a sense* in which this experience is an experience of the infinite. Ultimately, I believe that there is no hard and fast distinction between the finite and the infinite; rather, the finite gradually bleeds into the infinite. The fashion in which this occurs may be messy. Not only is the finite/infinite divide relative to subjects and time, but it may be the case that some larger, neater numbers may be finite, whereas smaller, messier numbers may be infinite. Still, and though it may sound strange, 100 is more infinite than 3. 100 is less infinite than 80^8. 80^{5000} (or, perhaps numbers of this size that are not quite so "neat") may be infinite in a very strong sense.

5. Infinite distance

The considerations above are not limited to number, but also apply to distance. Any finite distance is bound by two points or things. Let us attempt to arrive at infinite distance while trying to keep infinite distance as much like finite distance as possible. As any finite distance is bound by two things, let us simply assume that two points, Point1 and Point2, are an infinite distance

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apart. Then traveling from Point1 towards Point2, any finite distance will not allow one to arrive at Point2 (on pain of contradicting the assumed infinite distance between the two). Similarly a finite distance from Point2 will not get one to Point1. Thus we arrive at the picture below, which should be compared with $\omega + \omega^*$:

Point1 ⟶   ⟵ Point2

Developing the picture a bit further, start in the middle of Point1 and Point2, and you cannot get to either Point1 or Point2 by traveling any finite distance (again given infinite distance between them). So the picture that emerges is (compare this picture with $\omega + (\omega^* + \omega) + \omega^*$):

Point1 ⟶   ⟵⟶   ⟵ Point2

I suggest that the two pictures above are examples of infinite distance. Furthermore, in the "white space" between Point1 and Point2 we can continue to find "⟷" copies. Thus if we call a ray opening to the right "r" and to the left "r^*," then the general form of infinite distance is $r + (r^* + r)\,\alpha + r^*$, the same general form as that of the infinite numbers. Just as the infinite numbers have not been correctly identified, neither has infinite distance.

The above considerations suggest a structure for infinite distance that mirrors Section 2 above, where informal transfer of properties from the finite to the infinite was used to arrive at infinite number (in section 2) and infinite distance (in the paragraph above). To mirror section 3, imagine a steel bar that is 80^8 kilometers long. Now imagine the attempt to paint such a bar with multiple painters; the painters here play the role of the (investigators with) microscopes above. The experience of attempting to paint this bar is $r + (r^* + r)\,\alpha + r^*$, where, for example, r means that there is no bar remaining to paint to the left, but there is more bar to paint to the right. Infinite distance is large finite distance that has not been fully investigated.

5.1 There is no length of a ray

I suggest that there is no length of a ray, as any finite distance is a proper part of a ray, whereas a ray is a proper part of any infinite distance. If we then make the second assumption that the part is smaller than the whole, it follows that there is no answer to the question: How long is a ray? A ray is longer than any finite distance, but shorter than any infinite distance. A picture, in increasing order of length, is:

A finite distance	⊢—⊣
A ray	⊢⟶
An infinite distance	⊢⟶ … ⟵⟶ … ⟵⊣

6. On the number of finite, natural numbers

Similarly, in the conception of infinite number presented above, there is no number of natural numbers because the natural numbers are too large to be any finite number but too small to be any infinite number. Let us consider an infinite number, M. Then the structure of this number is:

$$1, 2, 3 \ldots M/2-1, M/2, M/2+1, \ldots M-1, M$$

All the numbers in the leftmost ω-copy, namely 1, 2, 3…, are finite numbers. All other numbers are infinite numbers, e.g., $M-1$. Note then that this infinite number counts some infinite numbers, in addition to all of the finite ones, and so is not the number of finite numbers. If M is an infinite number, the last number is infinite, thus M is not the number of finite numbers. Any infinite number counts too many numbers to be the number of finite numbers. Bertrand Russell (1983: 121–122) saw this, when he wrote, "The series of [natural] numbers is infinite. This is the common basis of all theories of infinity. But difficulties arise as soon as we examine this statement. For we can hardly say that there is an infinite number of finite numbers. For if there are n numbers, the last number must be n. If n is infinite, the last number is infinite, thus n is not the number of *finite* numbers." To read this correctly, it is of course necessary to understand by "infinite number" a structure of the form $\omega + (\omega^* + \omega)\,\alpha + \omega^*$.

Of course, any finite number fails to count some finite numbers, and so also fails to count the number of finite (that is, natural) numbers. Any finite number counts too few numbers to be the number of finite numbers. When it comes to natural numbers, infinite numbers count too many numbers, finite numbers count too few numbers. There is no number of natural numbers. $m < \omega < M$ for any finite number m and any infinite number M, where M is of the structure $\omega + (\omega^* + \omega)\,\alpha + \omega^*$ and "$x < y$" means that x is a proper initial segment of y. A picture is as follows:

A finite number	|||||
A/the set of finite numbers	|||||||| …
An infinite number	|||||||| … |||| … ||||||

Here is another, slightly different, presentation of the argument that there is no number of natural numbers. The argument follows from two claims: (1) Infinite numbers are of the form $\omega+(\omega^*+\omega)\,\alpha+\omega^*$; and (2) the correct way to judge relative sizes of sets is Euclidean: the part is smaller than the whole (this holds due to the fact that infinite numbers are large finite numbers, and so, just as with finite numbers, adding one yields a larger number, subtracting one yields a smaller number). It immediately follows that any finite number is smaller than ω, as any finite number is a proper subset (proper initial segment) of ω. And ω is smaller than any infinite number, as ω is a proper subset (proper initial segment) of any infinite number. ω is too large to be finite and too small to be infinite. There is no number of natural numbers.[8] For a good discussion of this Euclidean method of judging relative size, see Mancosu (2009) and Parker (2009), both for historical discussion as well as a discussion of the consistency of the Euclidean position that the part is smaller than the whole.

7. Conclusion

In this section, I consider a potential objection, briefly discuss the importance of the above considerations, and conclude. One potential objection is that there is no single correct conception of the structure of an infinite natural number. The reply, I believe, is that there is a correct conception of the structure of infinite number. Cantor himself believed that his infinite ordinals properly extended the sequence of finite, whole numbers into the infinite. However, the correct conception of the structure of an infinite number is $\omega+(\omega^*+\omega)\,\alpha+\omega^*$. Note that the reason that well-ordering fails, or is not carried into the infinite, is epistemic. That is, it is not possible, with infinite numbers, to get from the last element to the first. To put the point in reverse, once an observer has gotten from the last element to the first, then the number is finite. Otherwise, infinite numbers are exactly the same as finite numbers, where "exactly the same" consists mainly of the first-order structural properties discussed above (e.g., having a last element, being odd or even, etc.).

Is there any importance to recognizing the correct conception of infinite number? I believe that there is. As discussed, we find that the (or perhaps, a) set of natural numbers is an indefinite size. There is no number of natural numbers. In terms of language, I suggest that we say that the natural numbers are of "indefinite" size. "Infinite number" should refer to the sorts of structures I described above. Recognizing the correct conception of infinite number also dissolves many paradoxes of the infinite. To give one example, imagine Thomson's Lamp, where a lamp button is pressed an infinite number of times. Any infinite number (properly construed) is either even or odd. Thus, if the button is pressed an even number of times, then the lamp is in its starting state. If the button is pressed an odd number of times, then the lamp is opposite its starting state. As another example, consider a version of Zeno's paradox. Let us divide up a run of 1 kilometer into an infinite number of tasks, that is, into an infinite number of runs of infinitesimal length. On the correct conception of infinite number, there is no paradox. There is a first task. There is a last task. If there are M tasks, then each must be completed in $1/M$ hours, if the run is to be completed in one hour. I suggest that recognizing the correct conception of infinite number dissolves many paradoxes of the infinite.

Which objects are the infinite numbers? I have argued that infinite numbers are of the form $\omega+(\omega^*+\omega)\,\alpha+\omega^*$ and are large finite numbers that have not yet been fully counted (or, if they are being created, not yet fully constructed). Because the part is smaller than the whole for large finite numbers, so too the part is smaller than the whole for infinite numbers. It follows that there is no number of natural numbers: as any finite number counts too few numbers, any infinite number counts too many. On the correct conception of infinite number, there is no number of finite natural numbers, just as on the correct conception of infinite distance, there is no length of a ray. I have also suggested that recognizing these facts explains a number of paradoxes of the infinite.[9]

References

Brouwer L. E. J. (1983) Consciousness, philosophy, and mathematics. In: Benacerraf B. & Putnam H. (eds.) Philosophy of mathematics: Selected readings. Second edition. Cambridge University Press, Cambridge: 90–96.

Cantor G. (1955) Contributions to the founding of the theory of transfinite numbers. Dover, New York.

Mancosu P. (2009) Measuring the size of infinite collections of natural numbers: Was Cantor's theory of infinite number inevitable? The Review of Symbolic Logic 2: 612–646.

Mayberry J. P. (2000) The foundations of mathematics in the theory of sets. Cambridge University Press: Cambridge.

Nelson E. (1986) Predicative arithmetic. Princeton University Press. Princeton NJ.

Parikh R. (1971) Existence and feasibility in arithmetic. The Journal of Symbolic Logic 36: 494–508.

Parker M. W. (2009) Philosophical method and Galileo's paradox of infinity. In: Van Kerkhove B. (ed.) New Perspectives on mathematical practices. World Scientific, New Jersey: 76–113. Preprint available at http://philsci-archive.pitt.edu/4276/

Robinson A. (1996) Non-standard analysis. Princeton University Press: Princeton NJ.

Russell B. (1983) The collected papers of Bertrand Russell. Volume 3. George Allen & Unwin: London.

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8| A potential objection is that the paper is only concerned with ordinal number and so Cantor's conception of cardinal number is unaffected. Indeed, throughout this paper, order is involved (taken into consideration). But in the Euclidean position I endorse, there is little difference between cardinal and ordinal number – ordinal number simply has a little more structure, as there is only one ordinal for each cardinal. See Mayberry (2000: 291) for a further discussion of this point.

9| Note, finally, that I have proceeded by considering epistemic limitations. It may be possible, and some may prefer, to replace these with set-theoretic limitations.