Picturing the Infinite

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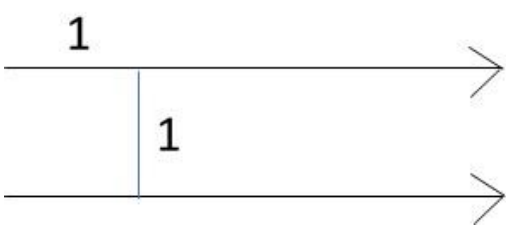
Abstract: The purpose of this note is to contrast a Cantorian outlook with a non-Cantorian one and to present a picture that provides support for the latter. In particular, I suggest that: i) infinite hyperreal numbers are the (actual, determined) infinite numbers, ii) ω is merely potentially infinite, and iii) infinitesimals should not be used in the di Finetti lottery. Though most Cantorians will likely maintain a Cantorian outlook, the picture is meant to motivate the obvious nature of the non-Cantorian outlook.

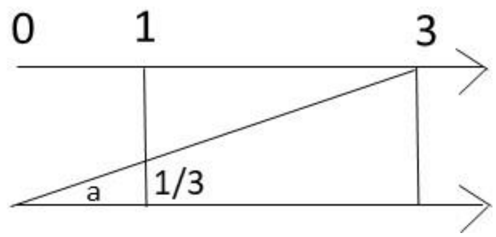
How many finite natural numbers, (1, 2, 3…), are there? On the Cantorian view, if we consider order, then we arrive at Cantor's smallest infinite ordinal, ω. Moving away from order, any set that can be mapped bijectively with the natural numbers is countable, Cantor's smallest infinite cardinal number. In general, Cantor's infinite ordinals and cardinals are the infinite numbers. Cantor argued that ω is actual and determined, and not merely a potential infinity. If we consider the natural numbers to be an actual and determined set, it is natural to think about random equiprobable selections from the natural numbers. Such a random selection is sometimes called the "di Finetti Lottery." What is the probability of selecting one specific natural number, e.g. 7, from all of them? 0 seems to be too small a probability[1] and any real number is too large a probability. A positive infinitesimal number is larger than 0 and smaller than any real number—which seems to be exactly what we are looking for. Indeed, people continue to suggest using infinitesimals for such a lottery.[2] In the chart below, let me summarize the Cantorian outlook outlined above and contrast the non-Cantorian outlook.

Cantorian Outlook	**Non-Cantorian Outlook**
Cantor's ordinals and cardinals are the infinite numbers.	Infinite hyperreal numbers are the infinite numbers.
The Cantorian infinite, for example ω, is actual and determined. These infinities are not merely potential.	ω, and also a ray, are merely potential. They are not actual and determined.
Infinitesimals should be used as the correct probability in the di Finetti Lottery.	Infinitesimals should not[3] be used as the correct probability in the di Finetti Lottery.

In what follows, pictures are used to motivate the non-Cantorian outlook. Consider two parallel real rays. Begin at 0, place them 1 unit apart, and draw a perpendicular line at 1 unit:

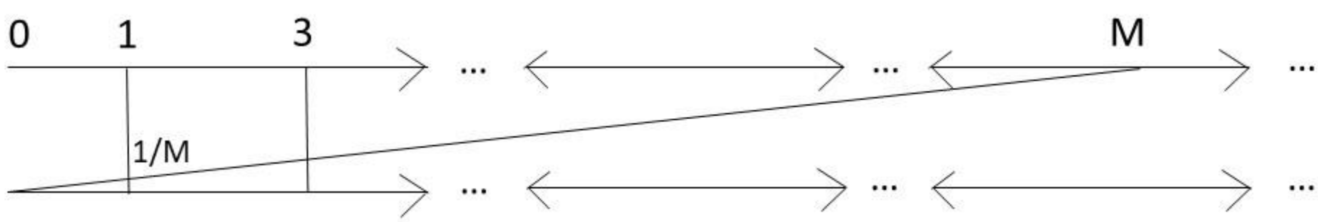


We can then pick any real number on the top ray, draw a line to 0 on the bottom ray, and consider where this line crosses our unit line. In the diagram below, we see that when we select 3 on the top line, the line to the origin of the bottom line crosses our unit line at 1/3. In general, a consideration of similar triangles (consider opposite over adjacent for angle a – the ratio is 1/3 for the larger and the smaller triangles) means that selecting x on the top ray results in a line to the origin that crosses the unit line at 1/x. We have a geometric interpretation of the reciprocal.



Let us carry these considerations into the hyperreal realm, by considering two hyperreal rays. These rays again begin at 0, but now contain infinitesimal and infinite numbers. The rays are non-Archimedean, meaning that any infinite number cannot be "reached" by adding a finite number a finite number of times. A positive infinitesimal is greater than 0 and less than any real number. The inverse of an infinitesimal is an infinite number (not necessarily a whole number). After the initial hyperreal ray extending from the origin, we have copies of hyperreals that are densely-linearly-ordered without endpoints.

We can again pick any number on the top ray. If we select an infinite number, M, the line to the origin crosses our unit line at 1/M.



If we make the straightforward assumption that the correct probability of selecting one number from a whole number n, in an equiprobable selection, is 1/n, then we run into a problem when we consider the di Finetti Lottery (an equiprobable, random selection from the natural numbers). Imagine the person who comes along and says: "I see how the picture and procedure above let me arrive at the probability of an equiprobable selection from 3 numbers or from M numbers. But how do I arrive at the probability of selecting a number from the natural numbers (the di Finetti Lottery)?” The picture is meant to motivate the view that there is no good answer to give the person. As the person noted, we can arrive at the correct probability of a selection from a finite number of outcomes. And we can arrive at the correct probability of selection from an infinite hyperreal number of outcomes. But there is no place to sink our teeth in, as it were, in order to arrive at the probability for the di Finetti Lottery.[4] What we want, in some sense, is to corral all of the (finite) natural numbers in the top line, find the spot where this is, and then draw the line to the origin of the bottom line. But there is no there, there; we simply have nowhere from which to begin drawing our line. There is an obvious sense in which any finite number is too far to the left; any infinite hyperreal number is too far to the right.[5] Such considerations suggest that the set of finite natural numbers is an example of the potentially infinite, whereas any specific infinite hyperreal number is an example of the actually infinite.[6]

We do not have to begin at the top line, nor do we have to work in a discrete setting (to this point we have focused on integers, whole numbers). Let us shift the story. Imagine that we allow a person to select a number on the unit line. We then draw a line from 0 on the bottom line, through the unit line’s selection, to the top line. (So for example, if the person chooses 1/3 on the unit line, the line from the origin of the bottom line, through 1/3 on the unit line, meets the top line at 3.) Perhaps the person receives a prize of a length of golden rope from 0 to the spot on the top line where the diagonal line crosses. For example, a person who selects $1/\pi$ on the unit line receives rope from 0 to π, that is, $[0, \pi]$. A person who selects the infinitesimal ε on the unit line (so $X = 1/\varepsilon$ is an infinite hyperreal) receives rope from 0 to X.

Now imagine that a person comes along and says “What I really want is the initial ray, that is, the length of rope that is all and only those parts of rope that are a finite distance from 0. How do I get that? What spot on the unit line do I select to arrive at that piece of rope?” Well, we seem to be in a bit of a conundrum. Any positive real number selected on the unit line (e.g., $1/3$, $1/\pi$, or $1/10^{10}$) will result in a finite length of rope. Any infinitesimal number, ε, will result in an infinite length of rope of length $X = 1/\varepsilon$, i.e., $[0, X]$, where X is infinite. There is no point on the unit line that we can select to arrive at the ray that the person desires. But the obvious reason seems to be because the person is neither asking for a finite length of rope nor for an infinite length of rope. Rather, the person is asking for a length of rope that is merely potential (not actual and determined).

The picture of the hyperreal lines (separated by a unit line) and the procedures described above are meant to motivate the position that each particular finite number is determined and actual, as is each particular infinite, hyperreal number. But the set of all finite natural numbers, and the ray, are not actual and determined infinities. (Each also awkwardly falls strictly between the finite and the actual infinite.) That is why there is no equiprobable selection from the natural numbers and why no number can be selected on the unit line to arrive at the ray. The picture and procedures also suggest that numbers should be taken to be numbers that occur on the hyperreal line and that infinitesimals are not the correct probability of selection for the di Finetti Lottery.[7]

References

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Notes

[1] If we must assign a probability using the reals, then it must be 0. Of course, a countable sum of 0's is 0 (and not 1, as we would like).

[2] For example see Gwiazda (2008) and Benci, Horsten, and Wenmackers (2018). Note that the latter builds on the authors' earlier work. Also note that I do not claim that Cantor himself endorsed the view of using infinitesimals for the di Finetti lottery (indeed Cantor did not like infinitesimals). Rather the claim is that many modern Cantorians consider the di Finetti lottery and suggest using infinitesimals, in part motivated, I suggest, by Cantor's claim that the natural numbers are a completed, actual, determined infinite set.

[3] Cantorians often argue (beginning from Cantorian assumptions) that infinitesimals can consistently be used in the di Finetti lottery. The consistency issue is more complicated than some believe, I suggest, though it must be admitted that no outright contradiction has been generated to date. Here I note that claims of consistency are different from the claim that infinitesimals *should* be used.

[4] Gwiazda (2012) and Pruss (2014) have argued, in the manner followed here, that infinitesimals are not the probability of selection from the di Finetti lottery. However, see Benci, Horsten, and Wenmackers (2018) for an opposing view.

[5] Put another way, any finite number is too small to count the natural numbers; any infinite number is too large to count the natural numbers.

[6] Note that Gwiazda (2013) presented a test to determine whether an infinity is potential or actual, namely: do random selections seem to get larger through sequential selections? If yes, the infinity is merely potential.

[7] Note that of course Cantorian developments were needed for the development of the hyperreals; that point does not rescue the Cantorian outlook. Put another way: from the point of the view of the non-Cantorian outlook, basing a system of infinite numbers on the collection of natural numbers is a historical accident. The collection of natural numbers was the first and easiest example of a collection that is clearly not finite. Only the utmost laziness could allow one to conclude from this that it follows that the collection of natural numbers is an example of an infinite (whole, natural) number. The Cantorians have chosen this path, choosing to ignore the basic question: what is the correct extension of the concept of finite, whole number into the infinite?