The Cantorian Bubble

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Abstract: The purpose of this paper is to suggest that we are in the midst of a Cantorian bubble, just as, for example, there was a dot com bubble in the late 1990's.

"Humans herd."[1]

Some time ago, after hearing something along the lines of "the infinite differs from the finite" or "infinite numbers are different from finite numbers" for the umpteenth time, two things occurred to me. These are phrases reminiscent of the language used in bubbles. And infinite numbers need not differ from finite numbers. This paper develops those ideas.

Introduction

How do very intelligent people lose their minds and go off the rails? Now here we have a question that, when you run it by your thesis advisor, word comes back, "We have to narrow this down a bit." Let's focus on groups of people, over long timeframes (years, decades, centuries, etc.), who hold (what come to be seen as) obviously incorrect, odd ideas. How does this happen?

Though still broad, I suggest that in some cases *common sense wisdom* is tossed overboard as *countervailing ideas* are given *free reign*. With common sense wisdom gone, people (who are off the rails) do (at least) three things. 1t), they say odd things; uncharitably, they spew nonsense. 2t), to support the odd ideas that run counter to common sense wisdom, they say things are now different; "this time it's different" becomes a common refrain. And 3t), they disparage people who adhere to the old views, even if giants in the field.

A great deal has been written about financial bubbles, in which prices come unhinged from any sort of reasonable valuation or fundamentals.[2] The focus of the above analysis is on the underlying ideas, the underlying justification that drives the bubble.[3] The nice thing about financial bubbles is that they burst, shining a spotlight upon the wretched miscreants who failed to understand that you need to be a nimble trend follower to make money in a bubble. That is, there is a very clear sense in which your neighbor, who aims to turn 100k into 1m but instead manages to go from 100k to 200k to a final 7k, has failed. And so with financial bubbles we can look back and ask “what went wrong?” (whereas with bubbles without prices people can simply plow ahead for millennia). Looking at the dot com bubble that occurred at the end of the 1990’s, we can see the following that ties into our framework discussed above:

Common sense wisdom: money matters.[4]
Countervailing idea: the internet is radically transformative and important technology.
Countervailing idea given **free reign**: There will be a shortage of tech stocks, so buy at any price. Prices will only go up. Revenue and earnings of tech companies do not matter; eyeballs matter (if even that).

With common sense wisdom cast aside in order to give free reign to the countervailing idea, the three things listed above occur:

1t) People spew nonsense: Corporate revenue and earnings don’t matter.
2t) People constantly say “this time it is different.”
And, 3t) Warren Buffett, e.g., is attacked as not understanding investing anymore. He has, it is claimed, lost his touch.

A very important point is that the common sense wisdom is simply not recognized. Anyone who points to it is disparaged as out of touch (the countervailing idea has completely triumphed). Such people simply don’t understand the great progress that has been made, it is claimed. Once the bubble bursts, then the common sense wisdom is again sense as obvious and common.

We are in the midst of a Cantorian bubble that has been going on for 120 years.[5] We can see the following that ties into our framework discussed above:

Common sense wisdom: the number of terms in the series will be the last number of the series. (I will often write “the last number gives the number.”)
Countervailing idea: it’s worth investigating the collection of natural numbers and

subjecting them to numerical determination, if possible.

Countervailing idea given **free reign**: there is a number of natural numbers. The natural numbers form the basis of the extension of the concept of finite number into the infinite.

With common sense wisdom cast aside in order to give free reign to the countervailing idea, the three things listed above occur:

1t) People spew nonsense: You can have 500,000 balls in a pit, do nothing but add 1 ball to the pit, and wind up with 0 balls in the pit.[6]

2t) People constantly say “this time it is different” in this case in the form of: “the infinite differs from the finite.”

3t) Leibniz, e.g., is attacked as not understanding numbers and called naïve, on a point he states is “subtle.”

A very important point is that the common sense wisdom is simply not recognized. Anyone who points to it is disparaged as out of touch. Such people simply don’t understand the great progress that has been made, it is claimed. Since the time of Cantor, Cantorians simply do not understand the common sense wisdom that the last number in a series gives the number.

In what follows let’s look into these ideas in greater detail. Just as it is difficult, when in the midst of the dot com bubble, to see the truth of the common sense wisdom, so too will it be difficult (impossible) for Cantorians to see the truth of the common sense wisdom, and so all of what follows will be dismissed by Cantorians. It’s worth keeping in mind that there is a “dot com outlook,” that differs radically from the outlook outside of the dot com bubble. So too, I suggest, we are in the midst of a “Cantorian outlook.” The Cantorians have a narrative that hangs together. The view outside of the Cantorian outlook differs radically. At times I will use “Cantorian outlook” and “non-Cantorian outlook” to refer to these two completely different outlooks. At the end of this paper, I answer a series of question from each outlook. The questions, without answers, are in Appendix 1. The questions, answered, are at the end of this paper. Those who wonder what I mean by Cantorian—I mean those who answer the questions as

Cantorians do. Each section begins to consider these questions, that is, the next section introduces questions one and two.

1. and 2.

The purpose of this paper is to suggest that we are in the midst of a Cantorian bubble, just as, for example, there was a dot com bubble in the late 1990's. The are two large challenges with this undertaking. First, there are no prices in the world of Cantorian mathematics. So be it; we will proceed just the same. Even without prices, I believe a compelling argument can be made that there is a Cantorian bubble. My goal is to present a nuanced, balanced view of the Cantorian bubble. The second challenge is that the project is fundamentally one of futility and frustration (only an idiot writes that there is a dot com bubble in 1997, shorts the QQQ,[7] and waits for riches to flow in).[8] Cantorians will stay Cantorian, not understand the argument presented, and reject all the evidence presented.[9] Such is the nature of bubbles. Given the two challenges, I will be aim to be brief; there are places where the argument could be developed in greater detail and where more examples could be given. I can only hope that the seventeen people I expect to download the paper over the next decade, some of whom may even begin reading the paper, will find enough detail to follow the main thread of the argument. Since I am taking the dot com bubble of the late 1990's as my example of a financial bubble, I will choose examples (indicating a Cantorian bubble) from the same time period.[10]

There are many interesting lenses through which we can view the world. The conservative/progressive distinction is one of them. In this paper, I consciously adopt a conservative stance, which has nothing to do with standard political and social issues. Rather, I'm going to suggest that Cantorians have ignored genuine common sense wisdom of the past, which has launched them into strange and choppy waters. Of course, the Cantorian will reply that they have made genuine progress and everyone should join them in their paradise.[11]

Imagine that it is 500,000 B.C.[12] Two groups of people are out hunting. One group sees the other, and they count the other group: 1, 2, 3, 4, 5, 6, 7. Then, just to be sure, they count their own group: 1, 2, 3. Any fight will be 7 against 3, and so the 3 flee. Let's notice something here that is important, something that has been forgotten in the Cantorian bubble: a question of "how many?" is answered by counting, where the last number gives the number.[13]

Though we are in the Cantorian age, and though the importance of this fact has been lost to Cantorians, nothing has changed. You, brave reader, can run the experiment yourself. Throw those bananas you just bought onto the kitchen table. Count them: 1, 2, 3, 4, 5, 6. There are 6 bananas.[14] For hundreds of thousands of years, right through to the present, questions of how many, questions answered with a whole number, have been answered by counting and providing the last number. Let's zoom forward a bit from 500,000 B.C. People are beginning to wrestle with the collection of all whole (natural) numbers: 1, 2, 3… They want to know how many there are. And here a fundamental tension (one lost to the Cantorian mind) emerges. The last number gives the number.[15] But the collection of natural numbers has no last number.[16]

Pick your mathematician or philosopher after 1600 and before 1900, and he[17] likely has a quote recognizing this fundamental tension in some form *and he likely stated the common sense wisdom that the last number gives the number*. In no particular order: Descartes, Leibniz, Wallis, Euler, de L'Hopital, Gauss, Bolzano, Frege, Russell, etc. The Cantorian does not find, nor care about, these quotes because the Cantorian is blind to the common sense wisdom (and so when presented with such a quote, the Cantorian mind thinks along the lines of "oh this poor fool wasn't as smart as Cantor"[18]). To be very clear here and just to give a sampling: Leibniz stated that there is no number of natural numbers. Bolzano stated that there is no number of natural numbers. Russell stated that there is no number of natural numbers.[19] Frege stated that Cantor's infinite ordinals are not the infinite numbers. And on it goes. Why do people, between 1600 and 1900, keep saying that there is no number of natural numbers? Because they are still in tune with the common sense wisdom that the last number of a series gives the number in the series.

Of this list of names, I'll present quotes of Leibniz in the text below, Bolzano in Appendix 3 (Bolzano calls the tension described above the "first paradox of mathematics"), and Russell at the end of the paper. I've discussed several of the other figures in other papers. Here are some of the Leibniz quotes:

> "The number of finite numbers cannot be infinite…. If numbers can be assumed as continually exceeding each other by one, the number of such numbers cannot be infinite, since in that case the number of numbers is equal to the greatest number, which is supposed to be finite."[20]

> "…the number of terms of the series will be the last number.
> …of course the number of terms will be the last number of the series.
> But in fact there is no last number of the series…
> Thus if you say that in an unbounded series there exists no last finite number that can be written in, although there can exist an infinite one; I reply, not even this can exist, if there is no last number. The only other thing I would consider replying to this reasoning is that the number of terms is not always the last number of the series….
> This consideration is extremely subtle"[21]

And so let us begin our list of questions. These are questions that will be answered at the end of the paper from the Cantorian outlook and from the non-Cantorian outlook.

1) Should infinite numbers have a last number?
2) Is there a number of natural numbers?

3.

Are we in the midst of a Cantorian bubble? Ultimately the answer will hinge on whether Cantor was correct about certain underlying assumptions and concepts. I suggest that we are in a bubble. But let us pause, and note that progress can be made. Penicillin is better for some ailments than letches; let us not suggest that there is a medical bubble with regards to penicillin. The wisdom of those who came before is not infallible. But in some cases it can be a helpful guide, and in some cases it is genuine wisdom that should not be discarded. Against the chants of Cantorian progress, let us continue to consciously adopt a conservative attitude that seeks to at least understand the thinking of those who came before. Ultimately, we must wrestle with the question as to who is right, but for now let us again look to the past for the collective wisdom on the topic of the infinite.

Is the collection of natural numbers any sort of actual, determined infinite set? Here I note that prior to Cantor, the answer was: no. Aristotle held that infinity was a potential infinite.

So too Gauss. Anyone not trained in the Cantorian ways also tends to think that the infinite is merely potential, and not actual and determined. This gives rise to another question:

3) Is the set of natural numbers actual and determined?

I won't spend a great deal of time on this question. But this entire paper could be rewritten with the common sense wisdom being "the natural numbers are merely potentially infinite." The reason for choosing to focus on the last number is that this wisdom is, in some suitable sense, more basic. And it has been completely forgotten.

4.

In the late 1990's, there was a bubble in dot com stocks. Any stock thought to be connected to the internet soared to absurd heights. Only a few turned out to be worth their peak prices. QQQ, an etf launched in 1999, which held many of these companies, performed horribly when the bubble burst. Supporting the bubble were beliefs that were pushed too far. Those who noted that the internet would change the world, they were correct. Those who stated that internet stocks must be bought at any price, lest one find an extreme shortage of internet stocks to buy, those people were wrong. Supporting any bubble are beliefs that are pushed to far, together with a sort of collective delusion (money doesn't matter, this time is different, etc.). As there is no price action in the Cantorian realm, what evidence can be presented to suggest that we are in a Cantorian bubble? Let me further develop the three threads of thought discussed above. Recall: "I suggest that in some cases *common sense wisdom* is tossed overboard as *countervailing ideas* are given *free reign*. With common sense wisdom gone, people (who are off the rails) do (at least) three things. 1t), they say odd things; uncharitably, they spew nonsense. 2t), to support the odd ideas that run counter to common sense wisdom, they say things are now different; "this time it's different." And 3t), they disparage people who adhere to the old views, even if giants."

Let's take these points in order. In bubbles, smart people feel free to state silly things in all earnestness. In the dot com bubble, as noted, people felt that money no longer mattered. What sort of silly things do Cantorians state in all earnestness? Here hundreds of examples could be chosen. Let me whittle it down to one meant to appeal to the general reader: You can have a situation where there are 500,000 balls in a pit, and the only sort of action that ever occurs is the addition of one ball to the pit. Repeat the addition of one ball "infinitely many" times in the right manner, and no balls remain in the pit. The Cantorian will tell you this with a straight face.[22] "It's not measurable. That's external. The infinite differs from the finite. Everyone knows that already. Etc." The Cantorian is comfortable with anything short of an outright contradiction so long as the Cantorian has been taught the stock reply. Oh Cantorian, let us right your ship. 3.5) Can you have 500,000 balls in a pit, do nothing but add one ball infinitely many times, and wind up with 0 balls in the pit?[23]

And so in a bubble people state silly things because they discard long help wisdom and ignore the past. With the dot com bubble, people argued that revenue and earnings no longer mattered for companies. People said that "this time is different," even though clear historical parallels existed (including the fact that in the previous technology-advance-driven bubbles people also said "this time is different"). Some stated that eyeballs were what mattered, the idea being that so long as people looked at your website, the financials didn't matter. In some cases, companies without revenue had market caps into the billions of dollars. With the Cantorian bubble, the Cantorians have discarded long held wisdom. For hundreds of thousands of years, the last number gives the number. The Cantorian has discarded this wisdom. With odd results, the Cantorian must then often say "The infinite differs from the finite." As with financial bubbles, these statements occur so often as to not warrant any attempt at a full listing. Here is one example, chosen from the 1990's, that attempts to explain away Ross' paradox: "In brief this evaporation is simply an artefact of our subtraction of one infinite set from another. It is surprising but not contradictory. Such evaporation cannot happen with the subtraction of finite sets, where our intuitions are developed."[24]

Third, in a bubble, smart people feel comfortable criticizing giants, so long as the criticizer is working from assumptions operative to the bubble. And so in the dot com bubble, many people argued that Warren Buffett had lost his touch; he simply didn't understand how to invest in the brave new world. Of course, the bubble burst, and these yahoos are largely forgotten. Similarly, in the Cantorian bubble people feel free to criticize giants. Examples are many. Let's use one example touched on above and also taken from the 1990's. When Leibniz wrestles with the question of how many natural numbers there are and writes that the issue is subtle, is he stuck in a genuine tension? Or is he just not quite smart enough to make the brilliant Cantorian leap?

> 4) Is Leibniz wrestling with a genuine tension when he wrestles with the question – how many natural numbers are there? -- or is he just not as smart as Cantorians?

I'll return to this question. Here I simply note that from the Cantorian perspective, the answer is the latter. And so, for example, Samuel Levey feels free to imply that Leibniz is naïve.[25] The point isn't that smart people are always right about everything ("hey Descartes, can you give me some tips for starting up my doggie daycare business?"). Rather, the point is that Leibniz, co-inventor of calculus, is saying that a *mathematical* point is subtle. I suggest that you have to be in a bubble to feel comfortable implying that Leibniz is naïve with regard to a mathematical point that he claims is subtle.

5.

Let us continue to consider questions and think about the Cantorian response and the non-Cantorian response. Is there a set whose size is between the natural numbers and the continuum (and where size is cashed out in terms of bijections)? This is Cantor's continuum hypothesis (CH). The Cantorians will tell you they think the continuum is aleph-17, or there are many universes, or new axioms may settle the matter, and all manner of other things besides. It's as though you ask a lender for your mortgage rate, and one person replies that for a vial of unicorn blood they may be able to knock off 20 basis points. Then they pass you to the person who tells you they think you'll be fed funds plus 2%, but no one really knows. *These are not the clear, crisp answers you desired.*[26] You're perfectly justified in wondering what just happened. Similarly, the Cantorians are asked, what, on their view, should be a simple question, and they fumble about with an answer. Why?

> 5) Why doesn't CH have a neat, clean answer?

6.

Consider the following chart:

500,000 B.C.	Finite natural numbers
1880 A.D.	Cantor’s infinite ordinals and cardinals
1960 A.D.	Infinite natural numbers[27]

When we consider our sixth question -- 6) Which objects are the infinite natural numbers? Or, what is the correct extension of the concept of finite natural number into the infinite? – now the Cantorian does not seem to have a great reply. Things called the infinite natural numbers, that behave exactly like the finite natural numbers, well, these would seem to be the infinite natural numbers. They also have last elements, that can be thought of as giving the overall number. Leibniz and Bolzano and Russell (etc.) are justified and correct, if only Cantorians would recognize the importance of correctly identifying the infinite natural numbers. However, ignoring evidence that opposes one’s position is one factor that allows bubbles to grow and expand. The Cantorian will not be slowed.

7.

It grows wearisome, writing for a Cantorian audience. Let us turn to the list of questions, and answer them from the Cantorian outlook and from the non-Cantorian outlook. And then, enough. One must be a trend follower (Richard Cantillon, David Ricardo, Jesse Livermore, etc.) when bubbles are afoot. Mean reversion has no place. Though the Cantorians have pushed their assumptions well beyond any reasonable level, let us recall that “prices are never too high to begin buying” (so long as you know what you are doing). I capitulate. I will finish the paper, and then let us all be Cantorians.[28]

Questions and Answers, Cantorian Outlook:

1. Should infinite numbers have a last number?

    Not necessarily. There is an ordinal/cardinal distinction. Some infinite ordinals do have a last number, some do not.

2. Is there a number of natural numbers?

    Yes. Countably many. The order type is omega.

3. Is the set of natural numbers actual and determined, or is it merely a potential infinity?

    Actual and determined.

3.5 Can you have 500,000 balls in a pit, do nothing but add one ball infinitely many times, and wind up with 0 balls in the pit?

    Yes. And if you think there is a ball in the pit, go ahead and name it.

4) Is Leibniz wrestling with a genuine tension when he wrestles with the question – how many natural numbers are there? -- or is he just not as smart as Cantorians?

    Leibniz did not foresee the brilliant moves and advances that Cantor and his followers made. Any follower of Cantor is ahead of Leibniz on this point.

5. Why doesn't CH have a neat, clean answer?

    It's complicated. (Different Cantorians give different answers, at least to a degree.)

6. Which objects are the infinite numbers?

    Cantor's infinite ordinals and cardinals. Though if pressed, a Cantorian will sometimes reply that there are many types of infinite numbers (Cantorain, hyperreal, surreal, etc.).

7. How did Cantorians get to where they are?

    Cantor made bold and brilliant advances and intelligent people followed.

8. Did Cantor make bold clear progress or launch mathematics into a cul-de-sac?

    Bold and clear progress.

9. What is worth keeping of the Cantorian project and what is worth jettisoning?

    Everything is worth keeping.

10. When applied to the natural numbers, is the question – how many? – an easy question or a hard question? If hard, why?

Relatively easy.

11. Did Russell exactly state the paradox that lies at the heart of the Cantorian project?

No. And, what?

Questions and Answers, Non-Cantorian Outlook

1) Should infinite ***numbers*** have a last number?

Of course. For hundreds of thousands of years, people have recognized that the last number gives you the overall number in the case of finite number. People counting bananas realize this point. Around 1960, infinite numbers arrived, where these numbers have good claim to be the infinite numbers, and where meaning can be given to the idea that the last number gives you the overall number. Only those locked in a Cantorian bubble are unable to see the obvious nature of the "of course" reply to this question.

2) Is there a number of natural numbers?

No. There are too many (finite) natural numbers for them to be finite in number. And there are too few (again, finite) natural numbers for them to be infinite in number. As Cantorians can't even understand the answer to question 1, it's not worth going into a great deal of detail here, except to note that Russell, prior to being dragged into the Cantorian fold, stated this result perfectly.

3) Is the set of natural numbers actual and determined, or is it merely a potential infinite?

Merely potentially infinite. Aristotle, Gauss, and any non-brainwashed human being older than 4 can see this fact. The Cantorians have moved the literature on the actual/potential distinction in a painful direction. If you like pain, go read a paper on Zeno-spheres.

3.5) Can you have 500,000 balls in a pit, do nothing but add one ball infinitely many times, and wind up with 0 balls in the pit?

No. There is an important general point here, namely that when you are dealing with people who are willing to believe anything, it's time to realize that you are wasting your time trying to bring them around to the bright world of common sense.

4) Is Leibniz wrestling with a genuine tension when he wrestles with the question – how many natural numbers are there? -- or is he just not as smart as Cantorians?

If you assume that Cantor was correct about everything, then Leibniz is naïve. But if you engage in a bit of independent thought on questions such as (What is the importance of infinite natural numbers? Why isn't there a cardinal/ordinal distinction in the work of, and under the assumptions of, Mayberry (2001)? Which objects are the infinite numbers? Etc.) you may find that Leibniz suddenly does not seem so naïve. Leibniz was both smarter and more intellectually honest than Cantorians. Plenty of people will be forgotten, but vaguely remembered as people who criticized Leibniz (as, e.g., naïve). Others who are criticized unfairly are Bolzano, Frege, Russell, and any other thinker who genuinely wrestled with the question – How many? – when applied to the collection of natural numbers.

5. Why doesn't CH have a neat, clean answer?

   There is no number of natural numbers.

6. Which objects are the infinite numbers?

   Here we find the real power of herd behavior. Cantorians are completely untroubled by this question, happy to ignore it or wave hands about for a bit. Those capable of independent thought should wrestle with it for a bit. And find that the infinite numbers are the infinite numbers. There is a picture of an infinite number, M, in Appendix 2. Just as a finite whole number is a whole number on the real line, so too an infinite whole number is an infinite whole number on the hyperreal line.

7. How did Cantorians get to where they are?

   Now we are getting to interesting questions. How do very smart people go off the rails? Looking over history, and taking a slightly different tack from above, what you generally find are men following power and prestige, going off the rails like a bunch of lemming-sheep. (Then, if we take a wide enough scope of history, no matter how inane the idea, what you often see is people in power doing whatever they can to keep things going for awhile.[29]) In the specific case of Cantor, he did mathematical work which naturally lead to the infinite ordinals. So far so good. But then the trouble started when two things occurred. First, bijection was taken as the sole, correct way to judge the relative sizes of infinite sets. And two, Cantor argued that his infinite ordinals and cardinals are just as real as the finite natural numbers, and in fact, were the correct continuation of the finite natural numbers into the infinite. (This, to all but a Cantorian, is lunacy.) Hilbert came along and made CH his first problem, imparting a great deal of prestige and power to studying Cantorian issues. Gödel took a profoundly realistic/platonic philosophical outlook. So now you have a bunch of guys running around who think they are investigating the ultimate mysteries of the universe and uncovering the mind of God—generally a recipe for trouble. In reality, they are investigating something that doesn't exist (a standard model of the natural numbers) and investigating how that imaginary thing relates to such things as bijection, powerset etc. There's an element of laziness operative here. Why? Because the natural numbers are the *simplest, first-appearing* structure that is infinite. To then go base an entire theory of infinite number on the first and simplest thing you run across is laziness. If I'm an onion farmer, and I'm running around telling everyone that onions cure cancer, it's reasonable that they are skeptical. I should figure out what cures cancer, and forget the onions. Similarly,[30] if I'm Cantor and I need to work with the natural numbers, it's a bit much to base an entire theory of infinite number on the thing I happen to be working with. I should first figure out what the infinite numbers are.

8. Did Cantor make bold clear progress or launch mathematics into a cul-de-sac?

Cul-de-sac. The Cantorian will point out that mathematics has made progress. Two points here in reply. It is something of a miracle that no outright contradiction has been found in some portion of arithmetic. At the risk of unfairly beating up on men a bit more, men have spent a great deal of time investigating bigger and larger forms of the infinite. The proofs get longer and harder.[31] Not a great deal of work has been put into investigating axioms that reduce size, for example, some sort of inverse powerset operation that would be numerically akin to taking the log base 2. From a supertask perspective, you can start with a column of the natural numbers. Divide the one column into 2 columns, then 4, then 8…. When does the column become a flat row? Not at any finite number of steps. But countably many steps seems too many. This won't trouble the Cantorian, who will simply note that he knew all along that nothing raised to the second power is countable. But my suspicion is that the path to contradiction is to focus on reduction. If a standard model of the natural numbers exists, then such a focus may be pointless, and may be one of the reasons there hasn't been much research in this direction. Turning to the second point in reply to question 8., hundreds of billions of people-hours of work in any mathematical system is going to make progress. Math might wind up being be path dependent like the typewriter. It is also possible that math never recovers from the Cantorian pandemic/bubble.

9. What is worth keeping of the Cantorian project and what is worth jettisoning?

In the dot com bubble, some stocks emerged and went on to be great investments. The internet changed the world. The error was in taking sensible ideas and pushing them to far. Similarly, Cantor's infinite ordinals may very well be interesting and fruitful. The search for a better answer to CH is likely worth jettisoning—as the root of the problem is that there is no number of natural numbers. It's perfectly reasonable to adopt a formalist outlook and investigate axioms and the consequences of their adoption. But it should be recognized that the person is investigating "infinite bumber," not infinite number. Those who think that this would not be a radical change are missing a main thrust of this paper.

10. When applied to the natural numbers, is the question – how many? – an easy question or a hard question? If hard, why?

This is a hard question. It is hard because the question -- how many? -- is answered using the last number in a series, and the natural numbers have no last number.

11. Did Russell exactly state the paradox that lies at the heart of the Cantorian project?

Yes. Cantorians won't understand it, but here it is:

"The series of [natural] numbers is infinite. This is the common basis of all theories of infinity. But difficulties arise as soon as we examine this statement. For we can hardly say that there is an infinite number of finite numbers. For if there are n numbers, the last number must be n. If n is infinite, the last number is infinite, thus n is not the number of *finite* numbers. But if n is finite, there must be a finite number (n+1), and therefore again n is not the number of finite numbers."[32]

| | | | | | you've failed to count some finite numbers

| | | | | | | | | |… **how many natural numbers are there?**

| | | | | | | | | |… … …| | | | | |… … …| | | | | | | | you've counted some infinite numbers

Independent thought is not humanity's long-suit. The Cantorian bubble has a great deal of room to run yet. Go with it. The paper began with herds, let us end there as well. "Men, it has been well said, think in herds; it will be seen that they go mad in herds, while they only recover their senses slowly, and one by one."[33]

APPENDIX 1: QUESTIONS

1) Should infinite numbers have a last number?

2) Is there a number of natural numbers?

3) Is the set of natural numbers actual and determined, or is it merely a potential infinity?

3.5) Can you have 500,000 balls in a pit, do nothing but add 1 ball infinitely many times, and wind up with 0 balls in the pit?

4) Is Leibniz wrestling with a genuine tension when he wrestles with the question – how many natural numbers are there? – or is he just not as smart as Cantorians?

5) Why doesn't CH have a neat, clean answer?

6) Which objects are the infinite numbers?

7) How did Cantorians get to where they are?

8) Did Cantor make bold clear progress or launch mathematics into a cul-de-sac?

9) What is worth keeping of the Cantorian project and what is worth jettisoning?

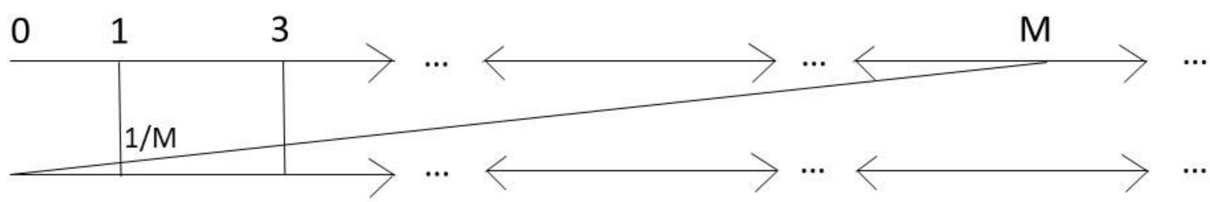
10) When applied to the natural numbers, is the question – how many? – an easy question or a hard question? If hard, why?

11) Did Russell exactly state the paradox that lies at the heart of the Cantorian project?

APPENDIX 2: For those who wish to escape the Cantorian bubble

Words are generally not adequate to jog one free from a bubble. Pictures and charts aren't either, but here they are.

1. A finite number is a number on the real line. An infinite number, such as M, is an infinite number on the hyperreal line.[34]



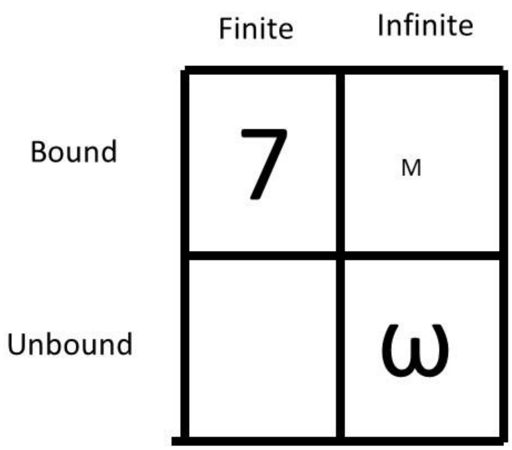
2. Dates again:

500,000 B.C.	Finite natural numbers
1880 A.D.	Cantor's infinite ordinals and cardinals
1960 A.D.	Infinite natural numbers

As David Isles writes (in making a different point), "Since the time of Skolem's work in the 1920's, mathematicians have been aware of non-standard models for arithmetic. Yet the "existence" of such non-standard integers has never really been taken seriously…"[35]

It's time to get to know the infinite natural numbers and give them their rightful place.

3. Low Hanging Fruit



The large font represents the low hanging fruit, that humanity has explored a bit. The suggestion of this paper is that numbers (such as 7 and M) are bound, and that it's time to dig into M a bit more. It's worth remembering that our axioms do what we want them to do. People have taken the outlook for so long of -- let's propose an axiom and see how it plays out -- that they're forgotten this point. If we want numbers with a last number, that is easy enough to accomplish.

4. Lived an infinite number of days

Joe was born, lived a finite number of days, and died. This story gives a certain structure to the days.

Joe was born, lived an infinite number of days, and died. This story gives a certain structure to the days. Thinking about this story can help begin to break one free from the Cantorian hold.[36]

Or again, Joe came to me and said, "Yup, I counted an infinite number of stars, 1, 2, 3 … M-3, M-2, M-1, M."

APPENDIX 3: Bolzano

"15. The existence of infinite sets, at least with non-actual members, is something which I now regard as sufficiently proved and defended; and also, that the set of all absolute truths is an infinite set. Arguments like those in 13 will win assent for the statement that the set of all numbers is infinite--that is, the set of all the so-called natural or whole numbers, as defined in 8. Yet this statement also sounds paradoxical, and we may regard it as the first paradox to appear in the realm of mathematics--for the one just considered belongs, properly speaking, to a science more general than that of quantity.

'If each number,' it might be protested, 'is by definition a merely finite set, how can the set of all numbers be infinite? If we contemplate the series of the natural numbers

1, 2, 3, 4, 5, 6…,

we become aware that the numbers of the series lying between the first (unity) and any particular one of them form a set which is enumerated by that particular one. For example, those from one to six form a set of six numbers. Consequently, the set of all numbers must be enumerated by the last number, and being therefore itself a number, not be infinite.'

The illusiveness of this argument disappears at once if we remember that in the natural series of natural numbers no term occupies the last place, and the notion of a last or highest number is an empty notion, being a self-contradictory one. In fact, the principle of construction of this series, as explained in 8, assigns to each of its terms a following term. This single remark must therefore be regarded as solving the present paradox.

16. ...The set of all [natural] numbers manifests itself immediately as an indisputable example of an infinitely great quantity. I say advisedly, as an example of a quantity; and certainly not as an example of an infinitely great number; for this infinitely great multitude cannot, as we remarked in the preceding paragraph, be given the name of number..."[37]

APPENDIX 4: Epistemology and Further Considerations

Everything so far is fairly obvious. The present state of thought on *infinite number* is deplorable. As Charlie Munger has said, give or take, “Think about it for a bit and you’ll see that I’m right, because you are smart, and I am right.” In this section, I’ll consider some less obvious considerations.

In a previous paper[38] I noted a coincidence, namely that the structure of an infinite (whole) number is the same structure that emerges when a large finite number is investigated under limitations. I suggested that this is not a mere coincidence. Here, I’d like to point out that epistemic limitations can *increase* the number of things one can talk about. Imagine a bar with 100 strokes on it. Two people who cannot count to 20 investigate it, beginning one at each end. They use the ellipses “…” to stand for more. Notice something. We know that there are 100 strokes on the bar. But the two people can get together and reason as follows. Call the number of strokes to roughly the middle from my end, P. The other person has Q strokes. The people also each have *a path* that they can talk about, a natural number type structure: 1, 2, 3…. So the two people can talk about 102 things. P + Q =100 and then the 2 paths makes 102.

Now shift gears and imagine a binary tree of infinite depth, M, where M is an infinite number. I am suggesting that the number of paths in the infinite binary tree is exactly and obviously equal to the number of nodes, but that the epistemic limitations are what is leading limited people to incorrectly see an explosion of paths relatively to nodes. In particular, people discussing the paths that are everything a finite distance from the root, they arrive at a “diagonal proof” and get to a whole new cardinality. But again, those less limited see that paths = nodes. These points are not worth explaining in greater detail at this time.

Shifting gears. The thrust of this paper is that those who wish to investigate the infinite are free to define the infinite however they like and do whatever they like. Those who wish to investigate infinite *number* are not free to decide what the infinite numbers are and are not free to do whatever they like. Those people are bound to focus on finite number, to at least use it as a guide. Then it rapidly becomes obvious what the infinite numbers are, and much follows, outlined above.

But what if we take all of the numbers, finite and infinite. Surely that then, some sort of super-ω, is actual and determined and subject to mathematical determination? No, it is not. The Cantorian fails to recognize how odd it is to say that, “Yes, I can begin tasks where I do things 1-by-1, I can finish all of the tasks, but there is no last task.” At the end of the day, we may need to wrestle with possibility that finite + 1 is, at some point and in some way, infinite.

If we meet the advanced aliens, let us ask them if they have had their Cantorian hiccup yet. This being the unsettling fact that the collection of natural numbers[39] is simple and appears first as a candidate for infinite number, while managing both to fail to be an infinite number, and to be too small to be counted by any infinite number.

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NOTES

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[1] William Berstein in conversation with Barry Ritholtz, April 18th 2019.

[2] Charles Kindleberger and Charles Mackay have written classic works on bubbles. More recently, Howard Marks has described bubbles as, I believe, situations where no reasonable future outcomes could possibly justify present prices. Didier Sornette takes a more mathematical approach in “Why Markets Crash.” The literature is, these days, vast.

[3] The underlying psychology (the drive/emotion/motivation), they often say, is that there’s nothing more painful than seeing your neighbor get rich. That is, much of the motivation comes from a fear of missing out. This feature of bubbles is almost certainly more important than the justifications or ideas involved. We’re all Humean, after all.

[4] The idea that money matters is just that, the notion that it is important to pay attention to money, because it matters (assuming we are in the financial realm and want, ultimately, to make money with an investment). And so money matters might mean-- pay attention to fundamentals, perhaps various valuation metrics. In real estate, at least look at how the property price relates to potential rental income, and understand how that relates to geography and historical averages. Stocks have P/E, P/S, P/B… ratios and the like. With things that do not cast off income or potential income, money matters admittedly becomes a bit hazier. Tulips, silver, bitcoin—what are these things worth? Should silver trade at $0.50, $5.00, $50, $500, or something else entirely in 1980? Obviously any detailed consideration of that question won’t happen here, but money matters at least means it’s worth being aware of how the item has traded historically. Unless you have some reason why you think the gold-silver ratio is low in 1980, it’s best to stay away. (Of course, if you do have an idea as to why the gold-silver ratio is low, and it’s ludicrous, it’s also best to stay away.) If paying attention to income streams is a way to avoid bubbles, an interesting question presents itself: Can you have a bubble with investments that are nothing but fixed income streams? That is, can you have a bond bubble? Again, there will be no detailed consideration here. I simply note that the answer to this question likely depends on your definition of bubble (moreso than with other assets). My sense is that many bondholders may well get drubbed at some point in the relatively near future. However, people who buy bonds are smarter than people who buy other things. (No bond buyer is thinking they will make 10x their money, only to subsequently lose 97%. All to say, the fixed income streams do provide some anchoring, even as rates may look crazy on a 5,000 year view.)

[5] The ideas were formed 140 years ago, and took 20 years, give or take, to triumph.

[6] Throw in two balls, and then throw out the lowest numbered ball. This is a version of what is often called “Ross’ Paradox.”

[7] Of course, the QQQ wasn’t created until March of 1999, but you get the idea.

[8] The point, as discussed above, is that one must be a trend follower in bubbles. Channel your inner Richard Cantillon, David Ricardo, Jesse Livermore, etc. Ignore mean reversion. To be a trend follower into the Cantorian bubble means that as people become more and more pro-Cantorian, so too do you. As Cantor’s stock rises, you buy more, so to speak.

[9] It's actually worse than that. The Cantorian will be so certain that Cantorian ideas are correct and the ideas presented in this paper are wrong, that the Cantorian will wonder if this whole paper is some sort of extended satire directed *against anti-Cantorians*.

[10] For fun, more than anything else.

[11] If we think of progressives as the engine and conservatives as the break, the implication is that more break is needed in the case of Cantorians.

[12] Spend a year or two reading all you can find of James Grant and odd thing occur. You wake up one morning to find that cash, which once seemed so solid and sure, looks like casino chips. You own gold that is stored in far flung countries. You can do little at dinner parties except blather uninterestingly and only semi-coherently about historically low interest rates, ten trillion dollars of negative yielding debt, and the doom that is sure to follow. You wonder if Grant pegs his subscription price to the price of an ounce of gold and you keep meaning to track down that data. And, when you write, you want to take the long view.

[13] There's no operationalist fallacy, nor need to ground numbers in human ability, going on here, when I note this fact of counting and the last number giving the number. The point is a structural one, which can be made in the absence of any discussion of human ability, counting, etc. Simply: numbers have last elements.

[14] Bolzano appears in appendix 3, where he discusses the number 6. Whether the motivation was the standard set of bananas is unclear. Lost to historians.

[15] The Cantorian mind has become so warped in the Cantorian bubble, and so inured to lunacy, that many Cantorians react by insisting that 6 must be {0,1,2,3,4,5}, as if this is an important point for a person counting bananas. Oh Cantorian, let us try to steer you back to the bright world of common sense.

[16] The Cantorian grows bored with such chatter. Surely everyone knows that Cantor made wonderous progress by developing his infinite ordinals and cardinals. Some of the ordinals have last members and some do not. The Cantorian has important things to do, and off he goes. But let us plodders proceed nonetheless.

[17] I'm going to stay "in character" as it were, in my consciously adopted conservative stance, and use every pronoun (I, you, he, we, they) except "she."

[18] Occasionally an "or me" slips in at the end of that sentence for the noble Cantorian.

[19] Purely from an interest in the history of science/mathematics, it's worth wondering why, or so I suggest.

[20] Leibniz, p. 51.

[21] Leibniz, pp. 99-101.

[22] In general, the Cantorian feels fine with any sort of outcome, so long as the Cantorian has some stock reply ready. Like a pouty 6 year-old who figures he is correct so long as he gets in the last word, so the Cantorian stumbles through the world.

[23] Imagine it is 1,000 years from now and the Cantorian bubble has burst. No-balls-in-the-pit is the sort of claim where, if money were involved, one can imagine a person in the distant future asking: Were people actually so silly as to believe that or was outright fraud occurring? Or again: Why wasn't anyone arrested?

[24] Earman and Norton, p. 240. Note the language: "[it] is simply" – the implication being that only a simpleton could fail to understand. Note also that any tethering to the finite realm is cast aside. We must abandon our finite intuitions. This paper goes so far as to note that the authors would be "disappointed" if anyone ever wrote about supertasks again in a manner not to their liking. It makes one think of the angry CEOs who yelled at analysts who asked about earnings in the dot com bubble, or beyond.

[25] Levey p. 83.

[26] J.P. Mayberry and Edward Nelson have good passages on, more or less, this point.

[27] Robinson.

[28] If you do some independent thought from time to time, you find that every now and again your ideas differ from the standard view. If you are in the financial realm, you often can then go and make a lot of money. If you are public enough with these trades, people will call you "contrarian." But you never set out to be contrarian. (Nothing is more irksome than someone who does.) You set out to think for yourself, and found (in a sense by definition, by the workings of markets) that the best money-making opportunities came in situations where your view differed from consensus (were contrary). The point here is that when your views differ from the standard, consensus view, if you have a way to financially benefit from this – that is a great thing. The "score keeping" nature of finance is wonderful to a certain sort of person. Matters are far more complex when you have a view that differs from consensus that may lead to your detriment. "Oh but truth will win out…" or some such is the naïve cry of a bright-eyed 20 year-

old. My advice: go into finance if you are capable of sensible, independent thought. On the other hand, if you can make people think you are articulate and thoughtful and your thoughts go with the herd, but your thoughts rarely succeed in worldly, objective sorts of ways, academia is the place for you. Most people will not fall neatly into either category, and so such advice is almost entirely worthless.

[29] On one view, the world's central bankers are following this pattern presently. That is, they are buying into inane economic ideas, that should have long ago been discredited (or at least in 2008-9), in order to keep a massive worldwide credit fueled bubble going, sometimes called "the everything bubble."

[30] Is it similar? Really?

[31] You get the idea.

[32] Russell, pp.121-2.

[33] Mackay.

[34] For an explanation of the picture see Gwiazda, Picturing the Infinte.

[35] Isles.

[36] William Lane Craig has a paper where he considers $\omega + \omega$ and bemoans the fact that a day exists with no yesterday, which, he writes, is absurd. Frege notes that in $\omega + \omega$, there is an element without a predecessor, which does not happen with finite numbers (ordinals). Frege is correctly pointing out that Cantor hasn't pinned down the infinite numbers. Craig is getting angry for no reason due to his being locked in a Cantorian framework. Put differently, if you don't like the structures Cantor talked about (the infinite ordinals), then don't use them. But for Pete's sake, don't use them and then get mad about it.

[37] Bolzano, p. 60-1.

[38] Gwiazda, 2012.

[39] Which of course, isn't an actual, determined thing.