METAPHYSICAL RELATIVITY THEORY I: M-LOGIC

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9th of February 2016

ABSTRACT. The present text provides a logical theory which originated in the unification of a number of well-known philosophical logics as well as the introduction and study of new operators. Further M-logic contains an object theory. With both the logical part and the object part we achieve a formal calculus that is able to express many metaphysical dogmas.

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1. T-LOGIC

The aim of t-logic is the study of important unary operators in philosophy. The possible/necessary distinction has been studied for decades now as well as the temporal sometimes/always distinction for propositions. However we give a new analysis of these operators via linear logic. Its expressive power is astonishing and there is a strong metaphysical argument, delivered by Dummet[D], that intuitionistic logic is of some importance. Moreover we introduce the apriori/empirical distinction and the analytic/synthetic distincion into propositional logic.

1.1. LINEAR TEMPORAL-MODAL LOGIC

We start with a convention. Let $(a_k)_k$ be a finite or countably infinite family of propositions. Then **sequences** are defined as the conjunction

$$... \to a_{l-1} \to a_l \to a_{l+1} \to ... \quad \overset{\text{def}}{\equiv} \quad \bigwedge_l (a_l \to a_{l+1}).$$

Consider the definition of classical material implication in linear logic[Gi]

$$a \to b \;=\; !a \multimap ?b.$$

We obtain a formula by applying the exponential isomorphism which describes the linear structure of classical disjunction

$$a \vee b \;=\; !(\neg a) \multimap ?b \;=\; ?a \parr ?b \;=\; ?(a \oplus b).$$

To avoid confusion between the endcoding of classical schemes in linear logic we denote $\neg$ for the unique involution in linear logic. The modal operator $\Diamond$

guarants $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$. To give a consistent account to possibilities in linear logic the following must be true

$$\Diamond(?a \parr ?b) \;=\; ?(\Diamond a) \parr ?(\Diamond b) \;=\; ?(\Diamond a \oplus \Diamond b).$$

From this we conclude that $\Diamond(?a) = ?(\Diamond a)$. Moreover both multiplicative and additive disjunction statisfy the property of classical disjunction

$$\Diamond(a \parr b) \;=\; (\Diamond a) \parr (\Diamond b), \qquad \Diamond(a \oplus b) \;=\; (\Diamond a) \oplus (\Diamond b). \tag{1}$$

Similarly the encoding of classical conjunction in linear logic is given by

$$a \wedge b \;=\; \neg(a \rightarrow \neg b) \;=\neg(!a \multimap ?(\neg b)) \;=\; !a \otimes !b \;=\; !(a \,\&\, b).$$

Because $\Box(a \wedge b) = \Box a \wedge \Box b$ the linear formulation must be

$$\Box(!a \otimes !b) \;=\; !(\Box a) \otimes !(\Box b) \;=\; !(\Box a \,\&\, \Box b)$$

which yields $\Box(!a) = !(\Box a)$ and likewise

$$\Box(a \otimes b) \;=\; (\Box a) \otimes (\Box b), \qquad \Box(a \,\&\, b) \;=\; (\Box a) \,\&\, (\Box b) \tag{2}$$

Obviously these properties can also be deduced by using the interdefinability $\Box a = \neg\Diamond\neg a$. The reader may consider the enumerated formulas as axioms and the logical yoga above as some sort of argument. The linear formulation of the scheme $\Box(a \vee b) \rightarrow \Box a \vee \Box b$ yields the formulas

$$\Box(a \oplus b) \multimap (\Box a) \oplus (\Box b), \qquad \Box(a \parr b) \multimap (\Box a) \parr (\Box b). \tag{3}$$

In the same manner we generalise $\Diamond(a \wedge b) \rightarrow \Diamond a \wedge \Diamond b$ as

$$\Diamond(a \,\&\, b) \multimap (\Diamond a) \,\&\, (\Diamond b), \qquad \Diamond(a \otimes b) \multimap (\Diamond a) \otimes (\Diamond b). \tag{4}$$

We head to the temporal operators. There are a number of those in philosophical logic. We may point out the interesting ones (and those that will be used in this text).

Definition 1.1.1. *Let p be a proposition then $\mathcal{A}(p)$ denotes the expression 'It is now the case that p'. Let $t_0 \in \mathbb{R}$ then $\bigcirc_{t_0} p$ denotes the expression 'It is at t_0 the case that p'. Last $\blacksquare p$ denotes the proposition 'It is always the case that p'.*

Note that $\blacksquare(.)$ comes with its dual $\blacklozenge p =$ "It is at some time the case that p" and we have the interdefinability $\blacksquare p = \neg\blacklozenge\neg p$. Hence this temporal logic is a modal logic that obeys a Kripke semantics[BRV]. We restrict ourselves to the study of those operators for one simple reason. If we have two inertia systems $\mathcal{I}$ and $\mathcal{I}'$ then the truth value of $\mathcal{A}(p)$ and $\bigcirc_{t_0} p$ depend on the choice of the inertia system by SRT. Hence we may write $\mathcal{A}^{\mathcal{I}}(p)$ and $\bigcirc_{t_0}^{\mathcal{I}} p$ for the respective truth values in the system $\mathcal{I}$ and drop the reference to the inertia system if the choice is obvious or does not change. We mimic the above argument as follows. Consider the linear formulation of $\blacklozenge(a \vee b)$

$$\blacklozenge(?a \parr ?b) \;=\; ?(\blacklozenge a) \parr ?(\blacklozenge b) \;=\; ?(\blacklozenge a \oplus \blacklozenge b). \tag{5}$$

Hence $\blacklozenge(?a) = ?(\blacklozenge a)$ and

$$\blacklozenge(a \oplus b) \;=\; (\blacklozenge a) \oplus (\blacklozenge b), \qquad \blacklozenge(a \parr b) \;=\; (\blacklozenge a) \parr (\blacklozenge b). \tag{6}$$

Note that $\blacksquare(a \wedge b) = \blacksquare a \wedge \blacksquare b$ hence

$$\blacksquare(!a \otimes !b) \;=\; !(\blacksquare a) \otimes !(\blacksquare b) \;=\; !(\blacksquare a \,\&\, \blacksquare b). \tag{7}$$

such that $\blacksquare(!a) = !(\blacksquare a)$ and

$$\blacksquare(a \otimes b) \;=\; (\blacksquare a) \otimes (\blacksquare b), \qquad \blacksquare(a \,\&\, b) \;=\; (\blacksquare a) \,\&\, (\blacksquare b). \tag{8}$$

For the sake of completeness we just state the linear schemes of implication

$$(\blacksquare a) \oplus (\blacksquare b) \multimap \blacksquare(a \oplus b), \qquad (\blacksquare a) \parr (\blacksquare b) \multimap \blacksquare(a \parr b), \tag{9}$$
$$\blacklozenge(a \,\&\, b) \multimap (\blacklozenge a) \,\&\, (\blacklozenge b), \qquad \blacklozenge(a \otimes b) \multimap (\blacklozenge a) \otimes (\blacklozenge b). \tag{10}$$

Definition 1.1.2. *The claim $\forall a[\blacklozenge a \to \mathcal{A}(a) \to \blacksquare a]$ is called* ***temporal actualism*** *and the negation $\exists a[(\blacklozenge a \wedge \neg\mathcal{A}(a)) \vee (\mathcal{A}(a) \wedge \neg\blacklozenge a)]$ is called* ***temporal eternalism****.*

There is a sequence $\blacksquare a \to \mathcal{A}(a) \to \blacklozenge a$ hence temporal actualism states that those operators are essentially the same. These are linguistic statements and we may give an ontological variation later.
Temporal-modal logic is an attempt to unify the modal logics of time and possibilities. In principle four types of expressions have to be clearified semantically and syntactically. Fix the axiomatic systems $T_\Box$, $T_\blacksquare$ for modal and temporal logic. Let $\Box a$ then

$$\Box a \to \blacksquare\Box a \quad \overset{T_\blacksquare}{\Longrightarrow} \quad \Box a \;=\; \blacksquare\Box a.$$

To justify the obove scheme we may consider the example of apriori statements a. Although there are instances of apriori propositions that seem to hold merely contingent apriori statements hold necessary to some extend. Since platonic ideas or abstract objects are atemporal and amorph it is reasonable to postulate that those expressions describing abstract objects are true at every time. For empirical propositions we consider the example $a_0 =$ "Protons consist of two up quarks and one down quark" which is true necessary such that $\blacksquare\Box a_0$.
Applying the negation on both sides of the above formula yields ($a' = \neg a$)

$$\neg\Box a' = \neg\blacksquare\Box a' \quad \Longrightarrow \quad \Diamond a \;=\; \blacklozenge\Diamond a.$$

Likewise consider the implication

$$\blacksquare a \to \Box\blacksquare a \quad \overset{T_\Box}{\Longrightarrow} \quad \blacksquare a \;=\; \Box\blacksquare a.$$

Again let a be statement that is true apriori. Then $\blacksquare a$ and the time independence is true necessary. As there are few examples of empirical assertions that are true necessary there are few examples of assertions that are true at every time. But those that statisfy this property like a_0 statisfy $\Box\blacksquare a$.

However the dual scheme is also given by the negation

$$\neg\blacksquare a' = \neg\Box\blacksquare a' \quad \Longrightarrow \quad \blacklozenge a \;=\; \Diamond\blacklozenge a.$$

There are two more implication describing the nature of mixed application of modal and temporal operators

$$\blacklozenge\Box a \to \Box\blacklozenge a, \qquad\qquad \Diamond\blacksquare a \to \blacksquare\Diamond a.$$

Assume the mentioned axioms are true. They lead to the final conclusion

$$\Box a \;=\; \blacksquare\Box a \;=\; \Box\blacksquare a \;=\; \blacksquare a.$$

In this sense $\Box a = \blacksquare a$ and likewise $\Diamond a = \blacklozenge a$ which means that this universe (including its temporal states) is a model of the multiverse. To justify these statements we give a set-theoretic semantics.

Definition 1.1.3. *An ordered pair $(\mathcal{W}, \Vvdash, \mathcal{R}, (\mathcal{W}', \Vdash, \mathcal{L}))$ is called **tK-model** if $\mathcal{W}$ is a set of worlds w_t depending on a time variable $\mathcal{W} = \{w_t \mid \mathbb{R} \overset{w}{\to} \mathcal{W}'\}$, a relation $\Vvdash$ between worlds $w_t \in \mathcal{W}$ and temporal-modal formulas and a relation $\mathcal{R} \subseteq \mathcal{W} \times \mathcal{W}$. Furthermore $(\mathcal{W}', \Vdash, \mathcal{L})$ is a Kripke model consisting of a set of time states of each world $w_t \in \mathcal{W}$ such that $\mathcal{W}' = \bigcup_{w \in \mathcal{W}} w(\mathbb{R})$, a relation $\Vdash$ between time states of worlds and modal formulas and a relation $\mathcal{L} \subseteq \mathcal{W}' \times \mathcal{W}'$ such that for $w_t \in \mathcal{W}$ and $w_{t_0} \in \mathcal{W}'$*

$$\begin{array}{rl}
\text{(i)} & w_t \Vvdash \neg p \;\equiv\; \neg(w_t \Vvdash p)\\
\text{(ii)} & w_t \Vvdash p \;\equiv\; w_{t_0} \Vdash p\\
\text{(iii)} & w_t \Vvdash (p \to q) \;\equiv\; \neg(w_t \Vvdash p) \vee (w_t \Vvdash q)\\
\text{(iv)} & w_t \Vvdash \blacksquare p \;\equiv\; \forall t_0[w_{t_0} \Vdash p]\\
\text{(v)} & w_t \Vvdash \Box p \;\equiv\; \forall u_t \mathcal{R} w_t[u_t \Vvdash p].
\end{array}$$

The formula $\blacklozenge p = \exists t_0[w_{t_0} \Vdash p]$ is obvious since $\neg\blacksquare\neg p = \neg\forall t_0[\neg(w_{t_0} \Vdash p)]$. If we impose certain properties on the relations $\mathcal{R}$ and $\mathcal{L}$ we get the scheme discussed above.

Proposition 1.1.1. *Let $(\mathcal{W}, \Vvdash, \mathcal{R}, (\mathcal{W}', \Vdash, \mathcal{L}))$ be a tK-model with the following property $\forall u_t, w_t \forall t_0[u_{t_0} \mathcal{L} w_{t_0} = u_t \mathcal{R} w_t]$ then $\Diamond\blacksquare a \;\to\; \blacksquare\Diamond a$ and $\blacklozenge\Box a \to \Box\blacklozenge a$ are valid schemes.*

Proof. Using the tK-semantics we obtain with the assumption on our relations and permutation of quantors

$$\blacklozenge\Box a \;\to\; \forall u_{t_0} \mathcal{L} w_{t_0} \exists t_0[u_{t_0} \Vdash a] \;=\; \forall u_t \mathcal{R} w_t \exists t_0[u_{t_0} \Vdash a] \;=\; \Box\blacklozenge a.$$

Hence $\blacklozenge\Box a \to \Box\blacklozenge a$ and with contraposition $\Diamond\blacksquare a \;\to\; \blacksquare\Diamond a$.

Proposition 1.1.2. *Let $(\mathcal{W}, \Vvdash, \mathcal{R}, (\mathcal{W}', \Vdash, \mathcal{L}))$ be a tK-model such that $u_t \mathcal{R} w_t = \forall t_0[u_{t_0} \mathcal{L} w_{t0}]$ then $\blacksquare\Box a = \Box a$ and $\Box\blacksquare a = \Box a$.*

Proof. Consider the set-theoretic formulation and the assumption

$$\Box a = \forall u_t \mathcal{R} w_t [u_t \Vvdash a] = \forall t_0 \forall u_{t_0} \mathcal{L} w_{t_0} [u_{t_0} \Vdash a] = \blacksquare \Box a.$$

Hence $\blacksquare \Box a = \Box a$ and furthermore

$$\Box \blacksquare a = \forall u_t \mathcal{R} w_t \forall t_0 [u_{t_0} \Vdash a] = \forall t_0 \forall t_0 \forall u_{t_0} \mathcal{L} w_{t_0} [u_{t_0} \Vdash a] = \blacksquare \Box a = \Box a.$$

Generalisation and particularisation are generalised conjunction and disjunction in classical logic which means that linear logic gives rise to four quantors. Let $\mathfrak{U} \in \{\Box, \blacksquare\}$ be a unary operator. We have the **linear Barcan**[Ba] **formulas**

$$\mathfrak{U}(\bigotimes_X \alpha X) \quad = \quad \bigotimes_X \mathfrak{U}(\alpha X) \tag{11}$$

for a property α. Likewise for the dual operator $\mathfrak{U}^{\text{op}} \in \{\Diamond, \blacklozenge\}$

$$\mathfrak{U}^{\text{op}}(\bigoplus_X \alpha X) \quad = \quad \bigoplus_X \mathfrak{U}^{\text{op}}(\alpha X). \tag{12}$$

1.2. LINEAR TEMPORAL-MODAL a-LOGIC

As proposed we introduce the apriori/empirically distinction into logic. This calculus will be called **a-logic**.

Definition 1.2.1. *Let a be a propostion then $\underline{a}$ is the proposition 'It is apriori true that a' and $\overline{a}$ is the proposition 'It is empirically true that a'. Furthermore we define $\underline{a} = a \wedge \neg \overline{a}$.*

One easy conclusion of the interdefinability is the symmetric formula $\overline{a} = a \wedge \neg \underline{a}$. Moreover note that both operators are truth-preserving due to the weakings $\overline{a} \to a$ and $\underline{a} \to a$ and $\underline{a} \dot\vee \overline{a} = a$ is exclusive.
The question wether there are propositions that hold apriori and empirically at the same time is to some extend reasonable. We may call such assertions **ampholytes** (following chemical nomeclature, ampholytes are particles that are able to react basic and acid). There are two examples that came to the authors mind first 'An apple plus another apple makes two apples' and 'Gravity warps space-time'. The first sentence is true due to natural arithmetic and the second is an apriori deduction of empirical axioms. In order to deal with ampholytes we use the following convention. If a proposition is true apriori then we say it is not true empirically.

Axiom 1.2.1. *We define $\underline{a \wedge b} = \underline{a} \wedge \underline{b}$ and $\underline{a \vee b} = \underline{a} \vee \underline{b}$.*

Why are the above schemes reasonable? If a is a proposition then four possible values occur. The statement can be true apriori: w/a $=$ $\underline{a}$, it can be false apriori: f/a $=$ $\underline{\neg a}$, it can be true empirically: w/e $=$ $\overline{a}$ and it can be false empirically: f/e$=$ $\overline{\neg a}$. The following table show that the formulas are true.

a	b	$\underline{a\wedge b}$	$\underline{a}\wedge\underline{b}$	$\underline{a\vee b}$	$\underline{a}\vee\underline{b}$
w/a	w/a	w	w	w	w
w/a	w/e	f	f	w	w
w/a	f/a	f	f	w	w
w/a	f/e	f	f	w	w
w/e	w/e	f	f	f	f
w/e	f/a	f	f	f	f
w/e	f/e	f	f	f	f
f/a	f/a	f	f	f	f
f/a	f/e	f	f	f	f
f/e	f/e	f	f	f	f

Proposition 1.2.1. *Under the assumption* $\underline{a\wedge b}=\underline{a}\wedge\underline{b}$ *we have* $\overline{a\wedge b} = (\overline{a}\wedge b)\vee(a\wedge\overline{b})$ *and if* $\underline{a\vee b}=\underline{a}\vee\underline{b}$ *then* $\overline{a\vee b} = (\overline{a}\wedge\neg\underline{b})\vee(\neg\underline{a}\wedge\overline{b})$.

Proof. These formulas can be verified with the following easy calculations

$$\overline{a\wedge b} = (a\wedge b)\wedge\neg(\underline{a\wedge b}) = (a\wedge b)\wedge(\neg\underline{a}\vee\neg\underline{b}) =$$
$$((a\wedge b)\wedge\neg\underline{a})\vee((a\wedge b)\wedge\neg\underline{b}) = (\overline{a}\wedge b)\vee(a\wedge\overline{b})$$

using distributivity and our definition $\underline{a} = a\wedge\neg\overline{a}$. In the same manner we gain

$$\overline{a\vee b} = (a\vee b)\wedge\neg(\underline{a\vee b}) = (a\vee b)\wedge(\neg\underline{a}\wedge\neg\underline{b}) =$$
$$((\neg\underline{a}\wedge\neg\underline{b})\wedge a)\vee((\neg\underline{a}\wedge\neg\underline{b})\wedge b) = (\overline{a}\wedge\neg\underline{b})\vee(\neg\underline{a}\wedge\overline{b}).$$

Definition 1.2.2. *The claim* $\forall a[a=\overline{a}]$ *is called* ***empirism****,* $\exists a[\underline{a}]$ ***apriorism*** *and* $\forall a[a=\underline{a}]$ ***rationalism****.*

Note that empirism and apriorism are converse to each other since $\neg\forall a[a = \overline{a}] = \neg\forall[a\to\overline{a}\wedge\overline{a}\to a] = \neg\forall a[a\to\overline{a}] = \exists a[a\wedge\neg\underline{a}] = \exists a[\underline{a}]$. Rationalism is therefore a strong apriorism. Certain semantic problems are not solved yet. As examples consider the syntactically correct expressions $\underline{\underline{a}}$ or $\underline{\neg a}$. The H-model of a-logic provides a set-theoretic semantics such that generalisations or unifications with other models are avaiable. Like Kripkes[BRV] theory the H-model is based on metaphysical ideas (platonism) but will be understood as a logical formalism. According to Wittgenstein[W1] a world w is a set of true propositions. These worlds split into two parts, the abstract worlds $\underline{w} = (\underline{w},\emptyset)$ and the concrete worlds $\overline{w} = (\emptyset,\overline{w})$.

Definition 1.2.3. *The* ***H-model*** *is an ordered pair* $(\mathcal{W},\Vdash)$ *consisting of a collection of worlds* $\mathcal{W} = \mathcal{W}^{\mathcal{A}!}\times\mathcal{W}^{\mathcal{O}!}$ *and a relation* $\Vdash$ *between these worlds* *and a-logical formulas such that for each* $(u,v) \in \mathcal{W}$

$$\begin{array}{rl}
\text{(i)} & \exists(\emptyset,\emptyset) \in \mathcal{W}[\neg\exists p[(\emptyset,\emptyset) \Vvdash p]] \\
\text{(ii)} & (u,v) \in \mathcal{W} \Longrightarrow (u,\emptyset) \in \mathcal{W} \wedge (\emptyset,v) \in \mathcal{W} \\
\text{(iii)} & (u,v) \Vvdash \neg p \implies \neg((u,v) \Vvdash p) \\
\text{(iv)} & (u \neq \emptyset \wedge v \neq \emptyset) \Longrightarrow ((u,v) \Vvdash \neg p \equiv \neg((u,v) \Vvdash p)) \\
\text{(v)} & (u,v) \Vvdash (p \to q) \equiv \neg((u,v) \Vvdash p) \vee ((u,v) \Vvdash q) \\
\text{(vi)} & (u,v) \Vvdash \underline{p} \equiv (u,\emptyset) \Vvdash p.
\end{array}$$

It is not necessary to define wether a proposition is true empirically or not since this can be deduced with the obove axioms as follows

$$\begin{aligned}
(u,v) \Vvdash \overline{a} &= ((u,v) \Vvdash a) \wedge ((u,v) \Vvdash \neg \underline{a}) \\
&= [((u,\emptyset) \Vvdash a) \dot\vee ((\emptyset,v) \Vvdash a)] \wedge \neg((u,\emptyset) \Vvdash a) = (\emptyset,v) \Vvdash a.
\end{aligned}$$

Moreover because $p = \underline{p} \dot\vee \overline{p}$ the following holds

$$(u,v) \Vvdash p \equiv ((u,\emptyset) \Vvdash p) \dot\vee ((\emptyset,v) \Vvdash p).$$

We define the recursion $\underline{a}_{(n)} = \underline{\underline{a}_{(n-1)}}$ with the start value $\underline{a}_{(1)} = \underline{a}$. Likewise $\overline{a}^{(n)} = \overline{\overline{a}^{(n-1)}}$ and $\overline{a}^{(1)} = \overline{a}$.

Proposition 1.2.2. *With the above notation* $\underline{a}_{(n)} = \underline{a}_{(n-1)} = ... = \underline{\underline{a}} = \underline{a}$ *and* $\overline{a}^{(n)} = \overline{a}^{(n-1)} = ... = \overline{\overline{a}} = \overline{a}$ *for each* $n \in \mathbb{N}$.

Proof. Consider the set-theoretic formulation of the iterations with $(u,v) \in \mathcal{W}$

$$\begin{array}{lllllll}
(u,v) \Vvdash \underline{\underline{a}} & = & (u,\emptyset) \Vvdash \underline{a} & = & (u,\emptyset) \Vvdash a & = & (u,v) \Vvdash \underline{a}, \\
(u,v) \Vvdash \overline{\overline{a}} & = & (\emptyset,v) \Vvdash \overline{a} & = & (\emptyset,v) \Vvdash a & = & (u,v) \Vvdash \overline{a}
\end{array}$$

and use induction on n.

Axiom (i) guarants the existence of a world $(\emptyset,\emptyset)$ of meontological character. We refer to $\emptyset$ and $(\emptyset,\emptyset)$ as **empty-worlds**. These worlds are important if we consider the schemes $\underline{(\overline{a})}$ or $\overline{(\underline{a})}$ due to its universal property. We denote $\bot = p \wedge \neg p$ for an arbitrary proposition p.

Proposition 1.2.3. *Let* a *be a proposition then* $\underline{(\overline{a})} = \bot = \overline{(\underline{a})}$. *Furthermore* $\underline{\neg a} \to \neg\underline{a}$ *and* $\overline{\neg a} \to \neg\overline{a}$.

Proof. We have for $(u,v) \in \mathcal{W}$

$$\begin{array}{lllll}
(u,v) \Vvdash \underline{(\overline{a})} & = & (u,\emptyset) \Vvdash \overline{a} & = & (\emptyset,\emptyset) \Vvdash a = \bot \\
(u,v) \Vvdash \overline{(\underline{a})} & = & (\emptyset,v) \Vvdash \underline{a} & = & (\emptyset,\emptyset) \Vvdash a = \bot.
\end{array}$$

Consider the H-semantics of the scheme

$$(u,v) \Vvdash \underline{\neg a} = (u,\emptyset) \Vvdash \neg a \implies \neg((u,\emptyset) \Vvdash a) = \neg((u,v) \Vvdash \underline{a}).$$

In conclusion $\underline{\neg a} \to \neg \underline{a}$ and the analogue scheme is given by

$$(u,v) \Vvdash \overline{\neg a} \;=\; (\emptyset, v) \Vvdash \neg a \implies \neg((\emptyset, v) \Vvdash a) \;=\; \neg((u,v) \Vvdash \overline{a})$$

hence $\overline{\neg a} \to \neg \overline{a}$. Alternatively we use the truth-preserving properties hence $\underline{\neg a} \to \neg a$ and $\overline{\neg a} \to \neg a$ and with contraposition we obtain the diagram

$$\begin{tikzcd}
\underline{a} \arrow[dr] & & \overline{a} \arrow[dl] \\
& a \arrow[dl] \arrow[dr] & \\
\neg(\overline{\neg a}) & & \neg(\underline{\neg a})
\end{tikzcd}$$

Hence $\underline{a} \to \neg(\underline{\neg a})$ and $\overline{a} \to \neg(\overline{\neg a})$ and again with the use of contraposition we get the desired schemes.

An interpretation of $\underline{(\overline{a})} = \bot$ could be that one cannot have apriori knowledge of empirical facts and converse. Next we give an analysis of quantified a-logic.

Axiom 1.2.2. *Let α be a property and X a variable then $\underline{\forall X[\alpha X]} = \forall X[\underline{\alpha X}]$ and $\underline{\exists X[\alpha X]} = \exists X[\underline{\alpha X}]$.*

Again why is this reasonable? If we think of quantifying as conjunction or disjunction over index systems of infinite cardinality ($\geq \aleph_0$) then the above formulas are just the abstraction of $\underline{a \wedge b} = \underline{a} \wedge \underline{b}$ and $\underline{a \vee b} = \underline{a} \vee \underline{b}$.

Proposition 1.2.4. *Let X be a variable and α a property. Under the assumption of the above axioms $\overline{\forall X[\alpha X]} = \exists Y[\overline{\alpha Y} \wedge \forall X[\alpha X]]$ and furthermore $\overline{\exists X[\alpha X]} = \exists Y[\overline{\alpha Y} \wedge \forall X[\neg \underline{\alpha X}]]$.*

Proof. These formulas can be obtained with some calculation as follows

$$\overline{\forall X[\alpha X]} = \forall X[\alpha X] \wedge \neg \underline{\forall X[\alpha X]} = \forall X[\alpha X] \wedge \exists X[\neg \underline{\alpha X}] = \forall X[\alpha X] \wedge \exists Y[\overline{\alpha Y}].$$

Because X is not bounded by Y we can quantify over Y first and get the first formula. Likewise

$$\overline{\exists X[\alpha X]} = \exists X[\alpha X] \wedge \neg \underline{\exists X[\alpha X]} = \exists Y[\alpha Y] \wedge \forall X[\neg \underline{\alpha X}].$$

In order to determine the interacting properties between the linear-logical and a-logical operators Lafonts[La] resource interpretation will be used. The corresponding formulas for the apriori-operator can be deduced formally. Let

$$\underline{!(a \,\&\, b)} = !(\underline{a} \,\&\, \underline{b}), \qquad \underline{!a \otimes !b} = !\underline{a} \otimes !\underline{b} \tag{13}$$

$$\underline{?(a \oplus b)} = ?(\underline{a} \oplus \underline{b}), \qquad \underline{?a \parr ?b} = ?\underline{a} \parr ?\underline{b} \tag{14}$$

be the linear schemes of $\underline{a \wedge b} = \underline{a} \wedge \underline{b}$ and $\underline{a \vee b} = \underline{a} \vee \underline{b}$. In conclusion $\underline{!a} = !\underline{a}$, $\underline{?a} = ?\underline{a}$ and

$$\underline{a \,\&\, b} = \underline{a} \,\&\, \underline{b}, \qquad \underline{a \otimes b} = \underline{a} \otimes \underline{b} \tag{15}$$

In the same manner the disjunction-formulas are given by

$$\underline{a \oplus b} \;=\; \underline{a} \oplus \underline{b}, \qquad \underline{a \parr b} \;=\; \underline{a} \parr \underline{b}. \tag{16}$$

Consider the following scheme together with its resource interpretation

$$\overline{a \,\&\, b} \;=\; (\overline{a} \,\&\, b) \oplus (a \,\&\, \overline{b}). \tag{17}$$

Assume we know empirically that we choose one resource. One can conclude that we have to choose one of the resources empirically otherwise $\overline{a \,\&\, b}$ must be false. Furthermore the choice between $\overline{a}$ and $\overline{b}$ was not made by us. Hence the additive disjunction has to be used. Conversely if $(\overline{a} \& b) \oplus (a \& \overline{b})$ is true obviously $\overline{a \,\&\, b}$ does also hold. To state symmetric identities we obtain

$$\overline{a \otimes b} \;=\; (\overline{a} \otimes b) \parr (a \otimes \overline{b}). \tag{18}$$

Analouge we consider the formula $\overline{a \vee b} \;=\; (\overline{a} \wedge \neg \underline{b}) \vee (\neg \underline{a} \wedge \overline{b})$. An additve and multiplicative formulation of this scheme yields

$$\overline{a \oplus b} \;=\; (\overline{a} \,\&\, \neg \underline{b}) \oplus (\neg \underline{a} \,\&\, \overline{b}), \qquad \overline{a \parr b} \;=\; (\overline{a} \otimes \neg \underline{b}) \parr (\neg \underline{a} \otimes \overline{b}). \tag{19}$$

Since $\underline{(.)}$ does commute with the modals of linear logic it is reasonable to take $\overline{!a} = !\overline{a}$ and $\overline{?a} = ?\overline{a}$ for granted.
We give a brief discussion on the interacting properties between the operators $\underline{(.)}$, $\overline{(.)}$ and $\Box(.)$, $\Diamond(.)$. A combinatorical argument shows that eight possible expressions occur. Each expression yields a formula in modal a-logic that is part of the calculus. Consider the syntactically correct expression $\Box\underline{a}$. Obviously $\Box\underline{a} \to \underline{a}$ and if we assume

$$\underline{a} \;\to\; \Box\underline{a} \quad\Longrightarrow\quad \Box\underline{a} \;=\; \underline{a}.$$

The scheme is correct if we suppose that those statements that are true apriori are true apriori in every possible world. Likewise it is not possible that a statement is true apriori in one possible world and not true apriori in another possible world. These considerations give rise to the implication

$$\Diamond\underline{a} \;\to\; \Box\underline{a} \quad\Longrightarrow\quad \Diamond\underline{a} \;=\; \underline{a}$$

if $\Box a \to \Diamond a$ and $\Box\underline{a} = \underline{a}$. Consider the famous example[Kr] of an assertion that is true empirically and necessary; "Water is H_2O.". Those examples statisfy the property of being empirically in every possible world as the following calculation shows. Since $\Diamond\underline{a} = \underline{a}$ and $\Box a = \Box a \wedge a$ in the modal system (T)

$$\Box\overline{a} \;=\; \Box(a \wedge \neg\underline{a}) \;=\; \Box a \wedge \neg\Diamond\underline{a} \;=\; \Box a \wedge \neg\underline{a} \;=\; \Box a \wedge \overline{a}. \tag{20}$$

A weaker formula holds for $\Diamond\overline{a}$ in this axiomatic system. Let

$$\Diamond\overline{a} \;=\; \Diamond(a \wedge \neg\underline{a}) \;\to\; \Diamond a \wedge \neg\underline{a}. \tag{21}$$

because $\Box\underline{a} \;=\; \underline{a}$. The stronger splitting property is false in this context. As an examples consider the assertion p_0 = Matter consists of positrons and antiprotons.". In conclusion $\neg\overline{p_0}$ since matter is made of electrons and protons but it is reasonable to argue that there is a possible world in which matter is antimatter from our point of view. Hence $\Diamond\overline{p}_0$. The implication

$$\Box a \to \underline{\Box a} \quad \Longrightarrow \quad \Box a = \underline{\Box a}$$

is true because it is impossible to perceive facts of other possible worlds. Under this assumption possibilities can be decided apriori if and only if they are not true empirically in the actual world and are possible

$$\underline{\Diamond a} = \neg\overline{a} \wedge \Diamond a.$$

Since knowledge of necessary statements is always apriori it is impossible to obtain knowledge of necessities empirically such that

$$\overline{\Box a} = \Box a \wedge \neg\underline{\Box a} = \Box a \wedge \neg\Box a = \bot. \tag{22}$$

Likewise we deduce under the assumption $a \to \Diamond a$

$$\overline{\Diamond a} = \Diamond a \wedge \neg\underline{\Diamond a} = \Diamond a \wedge \neg(\Diamond a \wedge \neg\overline{a}) = \overline{a} \wedge \Diamond a = \overline{a}. \tag{23}$$

Definition 1.2.3. *The* ***KH-model*** *is Kripke-model* $(\mathcal{W}, \Vdash\!\vdash, \mathcal{R})$ *such that* $(\mathcal{W}, \Vdash\!\vdash) = (\mathcal{W}^{\mathcal{A}!} \times \mathcal{W}^{\mathcal{O}!}, \Vdash\!\vdash)$ *is a H-model and both* $(\mathcal{W}^{\mathcal{A}!}, \models_{\mathcal{A}}, \mathcal{R}_{\mathcal{A}})$ *and* $(\mathcal{W}^{\mathcal{O}!}, \models_{\mathcal{O}}, \mathcal{R}_{\mathcal{O}})$ *are Kripke-models such that*

$$\begin{array}{rl} \text{(i)} & \forall(u,v)\forall(x,y)[(u,v)\mathcal{R}(x,y) = u\mathcal{R}_{\mathcal{A}}x \wedge v\mathcal{R}_{\mathcal{O}}y] \\ \text{(ii)} & \forall u[(u,\emptyset) \Vdash\!\vdash a \equiv u \models_{\mathcal{A}} a] \\ \text{(iii)} & \forall v[(\emptyset,v) \Vdash\!\vdash a \equiv v \models_{\mathcal{O}} a]. \end{array}$$

Proposition 1.2.5. *Let* $(\mathcal{W}^{\mathcal{A}!} \times \mathcal{W}^{\mathcal{O}!}, \Vdash\!\vdash, \mathcal{R})$ *be a KH-model with reflexive* $\mathcal{R}$ *such that* $\mathcal{W}^{\mathcal{A}!} = \{\emptyset, \mathfrak{I}\}$ *and* $\mathcal{R}_{\mathcal{A}} = \mathcal{W}^{\mathcal{A}!} \times \mathcal{W}^{\mathcal{A}!}$ *then* $\Box\underline{a} = \Diamond\underline{a} = \underline{a}$.

Proof. Let $(u,v) \in \mathcal{W}$ such that

$$(u,v) \Vdash\!\vdash \Box\underline{a} \equiv \forall(x,y)\mathcal{R}(u,v)[(x,\emptyset) \Vdash\!\vdash a].$$

By definition of the KH-model $(x,y)\mathcal{R}(u,v) = x\mathcal{R}_{\mathcal{A}}u \wedge y\mathcal{R}_{\mathcal{O}}v$ for some $(x,y) \in \mathcal{W}$. We assume wlog $x = \mathfrak{I}$ because otherwise $(x,\emptyset) \Vdash\!\vdash a = \bot$ in conclusion

$$(u,v) \Vdash\!\vdash \Box\underline{a} \equiv \forall(\mathfrak{I},y)\mathcal{R}(u,v)[(\mathfrak{I},\emptyset) \Vdash\!\vdash a] \equiv (\mathfrak{I},\emptyset) \Vdash\!\vdash a \equiv (\mathfrak{I},v) \Vdash\!\vdash \underline{a}$$

with $y\mathcal{R}_{\mathcal{O}}v$. Now $(u,v) \Vdash\!\vdash \underline{a} \to (\mathfrak{I},v) \Vdash\!\vdash \underline{a}$ since $(u,v) \Vdash\!\vdash \underline{a}$ is false if $u = \emptyset$ hence $\underline{a} \to \Box\underline{a}$ and with the scheme $\Box\underline{a} \to \underline{a}$ we gain equivalence. Likewise

$$(u,v) \Vdash\!\vdash \Diamond\underline{a} \equiv \exists(\mathfrak{I},y)\mathcal{R}(u,v)[(\mathfrak{I},\emptyset) \Vdash\!\vdash a] \equiv (\mathfrak{I},\emptyset) \Vdash\!\vdash a \equiv (\mathfrak{I},v) \Vdash\!\vdash \underline{a}$$

hence $\Diamond\underline{a} = \underline{a}$. All together $\Box\underline{a} = \Diamond\underline{a} = \underline{a}$.

Definition 1.2.4. We refer to $\mathfrak{I}$ as Platons **world of ideas**[Pl].

Corollary 1.2.1. *Let* $(\mathcal{W}^{\mathcal{A}!} \times \mathcal{W}^{\mathcal{O}!}, \Vdash\!\vdash, \mathcal{R})$ *be a KH-model with reflexive* $\mathcal{R}$ *such that* $\mathcal{W}^{\mathcal{A}!} = \{\emptyset, \mathfrak{I}\}$ *and* $\mathcal{R}_{\mathcal{A}} = \mathcal{W}^{\mathcal{A}!} \times \mathcal{W}^{\mathcal{A}!}$ *then* $\Box\overline{a} = \Box a \wedge \overline{a}$ *and* $\Diamond\overline{a} \to \Diamond a \wedge \neg\underline{a}$.

Proof. If $\mathcal{R}$ is reflexive then $\Box a \to a$ is valid hence $\Box a = \Box a \wedge a$. The rest follows from (20) and (21).

Proposition 1.2.6. *Let* $(\mathcal{W}^{\mathcal{A}!} \times \mathcal{W}^{\mathcal{O}!}, \Vdash, \mathcal{R})$ *be a KH-model such that* $(x,y)\mathcal{R}(u,\emptyset) = (x,y)\mathcal{R}(u,v)$ *for all* $(u,v),(x,y) \in \mathcal{W}$ *then* $\underline{\Box a} = \Box a$ *and* $\overline{\Box a} = \perp$.

Proof. Let $(u,v) \in \mathcal{W}$ then the under the assumption the follwing is obvious

$$\begin{aligned}(u,v) \Vdash \underline{\Box a} &\equiv \forall (x,y)\mathcal{R}(u,\emptyset)[(x,y) \Vdash a] \\ &\equiv \forall (x,y)\mathcal{R}(u,v)[(x,y) \Vdash a] \equiv (u,v) \Vdash \Box a.\end{aligned}$$

The formula $\overline{\Box a} = \perp$ follows from (22).

Proposition 1.2.7. *Let* $(\mathcal{W}^{\mathcal{A}!} \times \mathcal{W}^{\mathcal{O}!}, \Vdash, \mathcal{R})$ *be a KH-model such that* $\mathcal{R}$ *is an equivalence relation then* $\overline{\Diamond a} = \overline{a}$ *and* $\underline{\Diamond a} = \neg\overline{a} \wedge \Diamond a$.

Proof. Let $(u,v) \in \mathcal{W}$ such that $(u,v) \Vdash \overline{\Diamond a} \equiv \exists (x,y)\mathcal{R}(\emptyset,v)[(x,y) \Vdash a]$. We conclude by our assumption $x = \emptyset$ and $y = v$ hence

$$(u,v) \Vdash \overline{\Diamond a} \;\equiv\; (\emptyset,v) \Vdash a \;\equiv\; (u,v) \Vdash \overline{a}.$$

In conlcusion $\underline{\Diamond a} = \Diamond a \wedge \neg\overline{\Diamond a} = \Diamond a \wedge \neg\overline{a}$.

The interacting properties between temporal and a-logical operators are also determined by eight formulas describing the nature of mixed application. These schemes are similar to the formulas occuring in modal a-logic. As stated in the second chapter apriori assertions are atemporal which means that they do not depend on time. In conclusion they are true at every time and hence

$$\underline{a} \to \blacksquare\underline{a} \quad\Longrightarrow\quad \blacksquare\underline{a} = \underline{a}$$

If an assertion is true apriori at a certain time then it is true at every time therefore

$$\blacklozenge\underline{a} \to \blacksquare\underline{a} \quad\Longrightarrow\quad \blacklozenge\underline{a} = \underline{a}$$

since $\blacksquare a \to \blacklozenge a$. Because $\overline{a} = a \wedge \neg\underline{a}$ and $\blacksquare a = \blacksquare a \wedge a$ the scheme

$$\blacksquare\overline{a} \;=\; \blacksquare(a \wedge \neg\underline{a}) \;=\; \blacksquare a \wedge \neg\blacklozenge\underline{a} \;=\; \blacksquare a \wedge \neg\underline{a} \;=\; \blacksquare a \wedge \overline{a} \tag{24}$$

holds since $\blacklozenge\underline{a} = \underline{a}$. A weaker formula holds in the case of $\blacklozenge$. Let

$$\blacklozenge\overline{a} \;=\; \blacklozenge(a \wedge \neg\underline{a}) \;\to\; \blacklozenge a \wedge \neg\underline{a}. \tag{25}$$

As an example consider the assertion $p_0 =$ Platon is alive.". Obviously $\blacklozenge\overline{p_0}$ but $\neg\overline{p_0}$. We conclude that $\blacklozenge\overline{a} \neq \blacklozenge a \wedge \overline{a}$. Let $\blacksquare a$ then our knowledge of the fact that a is always true is apriori since it is impossible to perceive every possible state w_{t_0} of the world hence

$$\blacksquare a \to \underline{\blacksquare a} \quad\Longrightarrow\quad \blacksquare a = \underline{\blacksquare a}.$$

Likewise it is impossible to know wether a proposition is true at any time except we can decide the proposition at the actual time state empirically hence

$$\underline{\blacklozenge a} \;=\; \blacklozenge a \wedge \mathcal{A}(\neg\overline{a}) \;=\; \blacklozenge a \wedge \neg\mathcal{A}(\overline{a}).$$

If we use the convention $p = \mathcal{A}(p)$ we get the same formulas as in the case of $\Diamond$ and $\Box$. We conclude $\overline{\blacksquare a} \;=\; \blacksquare a \wedge \neg\underline{\blacksquare a} \;=\; \blacksquare a \wedge \neg\blacksquare a \;=\; \bot$ and furthermore

$$\overline{\blacklozenge a} = \blacklozenge a \wedge \neg\underline{\blacklozenge a} = \blacklozenge a \wedge \neg(\blacklozenge a \wedge \mathcal{A}(\neg\overline{a})) = \mathcal{A}(\overline{a}).$$

Definition 1.2.5. *A **tKH-model** is an ordered pair $(\mathcal{W}, \Vvdash, \mathcal{R}, (\mathcal{W}', \Vdash, \mathcal{L}))$ which in a tK-model such that $(\mathcal{W}, \Vvdash, \mathcal{R})$ and $(\mathcal{W}', \Vdash, \mathcal{L})$ are KH-models.*

Proposition 1.2.8. *Let $(\mathcal{W}, \Vvdash, \mathcal{R}, (\mathcal{W}', \Vdash, \mathcal{L}))$ be a tKH-model such that $\forall u_t \in \mathcal{W}^{\mathcal{A}!}[u_t = \text{const.}]$ then $\blacksquare\underline{a} = \blacklozenge\underline{a} = \underline{a}$.*

Proof. Let $(u_t, v_t) \in \mathcal{W}$ and apply the assumption such that $\forall t[u_t = u]$ then

$$(u_t, v_t) \Vvdash \blacksquare\underline{a} \;\equiv\; \forall t[(u, \emptyset) \Vdash a] \;\equiv\; \exists t[(u, \emptyset) \Vdash a] \;\equiv\; (u_t, v_t) \Vvdash \blacklozenge\underline{a}.$$

Likewise $(u_t, v_t) \Vvdash \blacksquare\underline{a} \equiv (u_{t_0}, v_{t_0}) \Vvdash \underline{a}$ all together $\blacksquare\underline{a} = \blacklozenge\underline{a} = \underline{a}$.

Corollary 1.2.2. *Under the above assumption $\blacksquare\overline{a} = \blacksquare a \wedge \overline{a}$ and $\blacklozenge\overline{a} \to \blacklozenge a \wedge \neg\overline{a}$.*

Proof. This was shown in (24) and (25).

Proposition 1.2.9. *Let $(\mathcal{W}, \Vvdash, \mathcal{R}, (\mathcal{W}', \Vdash, \mathcal{L}))$ be a tKH-model such that $\bigcap_{t\in\mathbb{R}} v_t \;=\; \emptyset$ for all $v_t \in \mathcal{W}^{\mathcal{O}!}$ then $\overline{\blacksquare a} = \bot$ and $\underline{\blacksquare a} = \blacksquare a$.*

Proof. Suppose $(u_t, v_t) \Vvdash \overline{\blacksquare a}$ we conclude

$$(\emptyset, v_t) \Vvdash \blacksquare a \;\equiv\; \forall t \in \mathbb{R}[(\emptyset, v_t) \Vdash a] \;\equiv\; a \in \bigcap_{t\in\mathbb{R}} v_t$$

which is a contradiction moreover $\underline{\blacksquare a} = \blacksquare a \wedge \neg\overline{\blacksquare a} = \blacksquare a \wedge \top = \blacksquare a$.

Observe that we fixed the **actual time** $t_0 \in \mathbb{R}_+$ in the definition of the tK-model. Therefore something like $w_{t_0} \Vvdash a \equiv w_t \Vvdash \mathcal{A}(a)$ is true. In the tKH-model every possible world v_t has its own actual time. But via transformation $\mathbb{R}^\kappa \xrightarrow{\mathcal{T}} \mathbb{R}^\kappa$ we can assume wlog that the actual time is t_0 in every possible world by letting $v_t = v_{\mathcal{T}t}$. Note that with this interpretation time is a vector $\mathbf{t} \in \mathbb{R}^\kappa$ and $\kappa = |\mathcal{W}|$.

Proposition 1.2.10. *Let $(\mathcal{W}, \mathcal{R}, \Vvdash, (\mathcal{W}', \mathcal{L}, \Vdash))$ be a tKH-model with $\bigcup_{t'} v_{t'} = v_{t_0}$ then $\overline{\blacklozenge a} = \overline{a}$ and $\underline{\blacklozenge a} = \neg\overline{a} \wedge \blacklozenge a$.*

Proof. The set-theoretic assumption $\bigcup_{t'} v_{t'} = v_{t_0}$ yields for $(u_t, v_t) \in \mathcal{W}$

$$(u_t, v_t) \Vvdash \overline{\blacklozenge a} \;\equiv\; \exists t'[(\emptyset, v_{t'}) \Vdash a] \;\equiv\; (\emptyset, v_{t_0}) \Vvdash a \;\equiv\; (u_t, v_t) \Vvdash \overline{a}.$$

In conclusion $\underline{\blacklozenge a} = \blacklozenge a \wedge \neg \overline{\blacklozenge a} = \blacklozenge a \wedge \neg \bar{a}$.

1.3. LINEAR TEMPORAL-MODAL A-LOGIC

A-logic is the formal study of the analytic/synthetic distinction[K]. We will see that pure A-logic has a high degree of similarity to a-logic. However the degree of similarity will depend strongly on the philosophical theory that is applied and we point out the crucial ones.

Definition 1.3.1. *Let a be a propostion then $a^\flat$ is the proposition 'It is analytically true that a' and $a^\sharp$ is the proposition 'It is synthetically true that a'. Furthermore we define $a^\flat = a \wedge \neg a^\sharp$.*

A conclusion of the interdefinability is the symmetric expression $a^\sharp = a \wedge \neg a^\flat$. Further note that both operators are truth-preserving $a^\sharp \to a$ and $a^\flat \to a$ and $a^\flat \dot{\vee} a^\sharp$.

Axiom 1.3.1. *We define $(a \wedge b)^\flat = a^\flat \wedge b^\flat$ and $(a \vee b)^\flat = a^\flat \vee b^\flat$.*

If a is a proposition then four possible values occur. The statement can be true analytically: w/a $= a^\flat$, it can be false analytically: f/a $= (\neg a)^\flat$, it can be true synthetically: w/s $= a^\flat$ and it can be false synthetically: f/s$= (\neg a)^\flat$. The following table show that the formulas are true.

a	b	$(a \wedge b)^\flat$	$a^\flat \wedge b^\flat$	$(a \vee b)^\flat$	$a^\flat \vee b^\flat$
w/a	w/a	w	w	w	w
w/a	w/s	f	f	w	w
w/a	f/a	f	f	w	w
w/a	f/s	f	f	w	w
w/s	w/s	f	f	f	f
w/s	f/a	f	f	f	f
w/s	f/s	f	f	f	f
f/a	f/a	f	f	f	f
f/a	f/s	f	f	f	f
f/s	f/s	f	f	f	f

Proposition 1.3.1. *Under the assumption $(a \wedge b)^\flat = a^\flat \wedge b^\flat$ we have $(a \wedge b)^\sharp = (a^\sharp \wedge b) \vee (a \wedge b^\sharp)$ and if $(a \vee b)^\flat = a^\flat \vee b^\flat$ then $(a \vee b)^\sharp = (a^\sharp \wedge \neg b^\flat) \vee (\neg a^\flat \wedge b^\sharp)$.*

Proof. These formulas can be verified with the following calculations

$$(a \wedge b)^\sharp = (a \wedge b) \wedge \neg (a \wedge b)^\flat = (a \wedge b) \wedge (\neg a^\flat \vee \neg b^\flat) =$$
$$((a \wedge b) \wedge \neg a^\flat) \vee ((a \wedge b) \wedge \neg b^\flat) = (a^\sharp \wedge b) \vee (a \wedge b^\sharp)$$

using distributivity and our definition $a^\flat = a \wedge \neg a^\sharp$. Furthermore we have

$$(a \vee b)^\sharp = (a \vee b) \wedge \neg(a \vee b)^\flat = (a \vee b) \wedge (\neg a^\flat \wedge \neg b^\flat) =$$
$$((\neg a^\flat \wedge \neg b^\flat) \wedge a) \vee ((\neg a^\flat \wedge \neg b^\flat) \wedge b) = (a^\sharp \wedge \neg b^\flat) \vee (\neg a^\flat \wedge b^\sharp).$$

Definition 1.3.2.[Q1] *The axiom $\forall a[a = a^\sharp]$ is called **confirmational holism** and $\exists a[a^\flat]$ **analycityism**.*

We want to give a set-theoretic semantics for A-logic. For a-logic we made use of products and now we make use of coproducts. Let w be a world i.e. a set of true propositions in this world. Then write $w = w_\flat \dot\sqcup w_\sharp$ whereas $w_\flat$ is the **linguistic part** of w and $w_\sharp$ is the **extra-linguistic part** of w.

Definition 1.3.3. *The $\mathcal{H}$**-model** is an ordered pair $(w = w_\flat \dot\sqcup w_\sharp, \Vdash)$ together with a decomposition $w = w_\flat \dot\sqcup w_\sharp$ such that $w_\flat \cap w_\sharp = \emptyset$ and*

$$\begin{array}{rl}
\text{(i)} & w \Vvdash \neg a \;\equiv\; \neg(w \Vvdash a) \;\equiv\; a \notin w \\
\text{(ii)} & w \Vvdash (a \to b) \;\equiv\; (a \notin w \vee b \in w) \\
\text{(iii)} & w \Vvdash a^\flat \;\equiv\; a \in w_\flat .
\end{array}$$

It is easy to conclude $w \Vvdash a_\sharp \equiv a \in w_\sharp$ if we consider the calculation

$$w \Vvdash a_\sharp \;\equiv\; w \Vvdash (a \wedge \neg a^\flat) \;\equiv\; (a \in w) \wedge (a \notin w_\flat) \;\equiv\; a \in w_\sharp .$$

Write $a^{\flat(1)} = a^\flat$ and defne the recursion $a^{\flat(n+1)} = (a^{\flat(n)})^\flat$ and likewise $a^{\sharp(1)} = a^\sharp$ with $a^{\sharp(n+1)} = (a^{\sharp(n)})^\sharp$ for some $n \in \mathbb{N}$. Note that $w_\flat \subseteq w$ is a submodel of w and therefore we have the equivalence $w_\flat \Vvdash a \equiv a \in w_\flat$.

Proposition 1.3.2. *With the above notation $a^{\flat(n)} = a^{\flat(n-1)} = \ldots = a^{\flat(1)} = a^\flat$ and $a^{\sharp(n)} = a^{\sharp(n-1)} = \ldots = a^{\sharp(1)} = a^\sharp$ for each $n \in \mathbb{N}$.*

Proof. Let $(w = w_\flat \dot\sqcup w_\sharp, \Vvdash)$ be a $\mathcal{H}$-model then for each $n \in \mathbb{N}$

$$w \Vvdash a^{\flat(n)} \equiv a^{\flat(n-1)} \in w_\flat \equiv a^{\flat(n-2)} \in w_\flat \equiv \ldots \equiv a \in w^\flat \equiv w \Vvdash a^\flat .$$

using the above remark. Likewise we obtain the chain $a^{\sharp(n)} = \ldots = a^{\sharp(1)} = a^\sharp$.

Corollary 1.3.1. *Let a be a proposition then $(a^\sharp)^\flat = \bot = (a^\flat)^\sharp$.*

Proof. This is deduced as follows using the above proposition

$$(a^\sharp)^\flat = a^\sharp \wedge \neg(a^\sharp)^\sharp = a^\sharp \wedge \neg a^\sharp = \bot = a^\flat \wedge \neg a^\flat = a^\flat \wedge \neg(a^\flat)^\flat = (a^\flat)^\sharp .$$

Proposition 1.3.3. *Let $(w_\flat \dot\sqcup w_\sharp, \Vvdash)$ be a $\mathcal{H}$-model then $(\neg a)^\flat \to \neg a^\flat$ and $(\neg a)^\sharp \to \neg a^\sharp$.*

Proof. We make use the truth-preserving properties hence $(\neg a)^\flat \to \neg a$ and $(\neg a)^\sharp \to \neg a$ and with the use of contraposition we obtain the diagram

$$\begin{tikzcd}
a^\flat \arrow[dr] & & a^\sharp \arrow[dl] \\
& a \arrow[dl] \arrow[dr] & \\
\neg(\neg a)^\sharp & & \neg(\neg a)^\flat
\end{tikzcd}$$

Hence $a^\flat \to \neg(\neg a)^\flat$ and $a^\sharp \to \neg(\neg a)^\sharp$ and again with the use of contraposition we get the desired schemes.

The converse $\neg a^\flat \to (\neg a)^\flat$ does not hold. Consider the example $p =$'*Water is* H_2O'. Then $p^\sharp$ and with weakening $\neg p^\flat$ but the negation $\neg p =$'*Water is not* H_2O' is not true analytically. In order to see that $\neg a^\sharp \to (\neg a)^\sharp$ is also not true consider the example $p =$'*Bachelors are unmarried*' then $\neg p^\sharp$ but $\neg(\neg p)^\sharp$. We give a short analysis of quantified A-logic.

Axiom 1.3.2. *Let X be a variable and α a first-order predicate then we set* $(\forall X[\alpha X])^\flat = \forall X[(\alpha X)^\flat]$ *and* $(\exists X[\alpha X])^\flat = \exists X[(\alpha X)^\flat]$.

Proposition 1.3.4. *Under the above assumption* $(\forall X[\alpha X])^\sharp = \exists Y[(\alpha Y)^\sharp \wedge \forall X[\alpha X]]$ *and* $(\exists X[\alpha X])^\sharp = \exists Y[(\alpha Y)^\sharp \wedge \forall X[\neg(\alpha X)^\flat]]$.

Proof. We deduce these formulas with an easy calculation

$$(\forall X[\alpha X])^\sharp = \forall X[\alpha X] \wedge \neg(\forall X[\alpha X])^\flat = \forall X[\alpha X] \wedge \exists Y[\neg(\alpha Y)^\flat]$$

and because αY we have $(\forall X[\alpha X])^\sharp = \forall X[\alpha X] \wedge \exists Y[(\alpha Y)^\sharp]$ and if we quantify over Y first we get the stated formula. Moreover

$$(\exists X[\alpha X])^\sharp = \exists X[\alpha X] \wedge \neg(\exists X[\alpha X])^\flat = \exists Y[\alpha Y] \wedge \forall X[\neg(\alpha X)^\flat]$$

and because $\forall X[\neg(\alpha X)^\flat]$ we have $(\alpha Y)^\sharp$ and therefore if we quantify over Y we get the desired formula.

Consider some linear A-logic. In order to mimic our axioms and conlcusion of A-logic we accept the following multiplicative schemes

$$(a \otimes b)^\flat = a^\flat \otimes b^\flat, \qquad (a \parr b)^\flat = a^\flat \parr b^\flat \tag{26}$$

and $(!a)^\flat =!a^\flat$ as well as $(?a)^\flat =?a^\flat$. Likewise we accept the additive schemes

$$(a \,\&\, b)^\flat = a^\flat \,\&\, b^\flat, \qquad (a \oplus b)^\flat = a^\flat \oplus b^\flat \tag{27}$$

and the induced formulas for the synthetic operator. Until now the formulas that were presented are uncontroversial. Whereas $\underline{a} \to \Box\underline{a}$ is a reasonable assumption it is not clear why $a^\flat \to \Box a^\flat$ should also hold. The question wether a proposition a is analytic in a certain world w is always relative to the language L_w that is spoken in this world. Also there are multiple languages spoken in the **actual world @**. Choose two languages $L_@$ and $L'_@$ of the actual world such that the syntactical expression a has two different meanings. To avoid this situation we think of a as the expression $\partial(a)$ which is the **meaning** of a in a **superlanguage**.

Definition 1.3.4. *The* ***superlanguage*** *is given by* $S = \bigcup_w \bigcup_{p \in w} \partial(p)$.

Being analytically true does not depend on the choice of the language but rather on the meaning of the expression which would justify schemes like $a^\flat = \Box a^\flat$ or $a^\flat = \blacksquare a^\flat$. As usual we develop a set-theoretic semantics to formalize talk about valid schemes of modal A-logic and temporal A-logic.

Definition 1.3.5. *The* ***K$\mathcal{H}$-model*** *is a Kripke model* $(\mathcal{W}, \mathcal{R}, \Vdash)$ *such that each* $w \in \mathcal{W}$ *is a* $\mathcal{H}$*-model* $(w_\flat \dot{\sqcup} w_\sharp, \vDash_w)$ *and for all* w

$$\begin{array}{ll} \text{(i)} & w \Vdash a \;\equiv\; w \vDash_w a \\ \text{(ii)} & w \in \mathcal{W} \;\Rightarrow\; w_\flat \in \mathcal{W} \wedge w_\sharp \in \mathcal{W}. \end{array}$$

With axiom (i) and (ii) we have $w \Vdash a^\flat \;\equiv\; w_\flat \Vdash a$ and $w \Vdash a^\sharp \;\equiv\; w_\sharp \Vdash a$.

Proposition 1.3.5. *Let* $(\mathcal{W}, \mathcal{R}, \Vdash)$ *be a K$\mathcal{H}$-model with reflexive* $\mathcal{R}$ *such that* $\forall w \forall u \mathcal{R} w [u_\flat = w_\flat]$ *then* $\Box a^\flat = \Diamond a^\flat = a^\flat$ *and therefore* $\Box a^\sharp = \Box a \wedge a^\sharp$ *and* $\Diamond a^\sharp \to \Diamond a \wedge \neg a^\flat$.

Proof. Consider $w \in \mathcal{W}$ then using our assumption

$$w \Vdash \Box a^\flat \equiv \forall u \mathcal{R} w [u_\flat \Vdash a] \equiv \forall u \mathcal{R} w [w_\flat \Vdash a] \equiv \exists u \mathcal{R} w [w_\flat \Vdash a] \equiv w_\flat \Vdash a$$

and hence $\Box a^\flat = \Diamond a^\flat = a^\flat$. Moreover because $\mathcal{R}$ is reflexive $\Box a^\sharp = \Box(a \wedge \neg a^\flat) = \Box a \wedge \neg \Diamond a^\flat = \Box a \wedge a \wedge \neg a^\flat = \Box a \wedge a^\sharp$. Further $\Diamond a^\sharp \to \Diamond a \wedge \Diamond(\neg a^\flat) = \Diamond a \wedge \neg a^\flat$.

Proposition 1.3.6. *Let* $(\mathcal{W}, \mathcal{R}, \Vdash)$ *be a K$\mathcal{H}$-model such that* $\forall w \forall u [u \mathcal{R} w \equiv u \mathcal{R} w_\flat]$ *then* $(\Box a)^\flat = \Box a$ *and* $(\Box a)^\sharp = \bot$.

Proof. Let $(\mathcal{W}, \mathcal{R}, \Vdash)$ be a K$\mathcal{H}$-model and $w \in \mathcal{W}$ then

$$w \Vdash (\Box a)^\flat \equiv \forall u \mathcal{R} w_\flat [u \Vdash a] \equiv \forall u \mathcal{R} w [u \Vdash a] \equiv w \Vdash \Box a.$$

We conlcude $(\Box a)^\sharp = \Box a \wedge \neg(\Box a)^\flat = \Box a \wedge \neg \Box a = \bot$.

Proposition 1.3.7. *Let* $(\mathcal{W}, \mathcal{R}, \Vdash)$ *be a K$\mathcal{H}$-model with* $\forall w \forall u [u \mathcal{R} w_\sharp \equiv (u = w_\sharp)]$ *then* $(\Diamond a)^\sharp = a^\sharp$ *and* $(\Diamond a)^\flat = \neg a^\sharp \wedge \Diamond a$.

Proof. Let $w \in \mathcal{W}$ where $(\mathcal{W}, \mathcal{R}, \Vdash)$ is a K$\mathcal{H}$-model then

$$w \Vdash (\Diamond a)^\sharp \;\equiv\; \exists u \mathcal{R} w_\sharp [u \Vdash a] \;\equiv\; w_\sharp \Vdash a \;\equiv\; w \Vdash a^\sharp.$$

Definition 1.3.6. *The* ***tK$\mathcal{H}$-model*** *is a tK-model* $(\mathcal{W}, \mathcal{R}, \Vdash, (\mathcal{W}', \mathcal{L}, \Vdash))$ *such that* $(\mathcal{W}, \mathcal{R}, \Vdash)$ *and* $(\mathcal{W}', \mathcal{L}, \Vdash)$ *are K$\mathcal{H}$-models.*

Proposition 1.3.8. *Let* $(\mathcal{W}, \mathcal{R}, \Vdash, (\mathcal{W}', \mathcal{L}, \Vdash))$ *be a tK$\mathcal{H}$-model such that the linguistic worlds are constant i.e.* $\forall w_t \forall t_0 [(w_{t_0})_\flat = w_0]$ *then* $\blacksquare a^\flat = \blacklozenge a^\flat = a^\flat$.

Proof. Let $w_t \in \mathcal{W}$ then

$$w_t \Vdash\!\vdash \blacksquare a^\flat \;\equiv\; \forall t_0[(w_{t_0})_\flat \Vdash a] \;\equiv\; \forall t_0[w_0 \Vdash a] \;\equiv\; w_0 \Vdash a \;\equiv\; w_t \Vdash\!\vdash a^\flat.$$

Corollary 1.3.2. *Let $(\mathcal{W}, \mathcal{R}, \Vdash\!\vdash, (\mathcal{W}', \mathcal{L}, \Vdash))$ be a tK$\mathcal{H}$-model such that $\blacksquare a^\flat = \blacklozenge a^\flat = a^\flat$ then $\blacksquare a^\sharp = \blacksquare a \wedge a^\sharp$ and $\blacklozenge a^\sharp \to \blacklozenge a \wedge \neg a^\flat$.*

Proof. We have $\blacksquare a^\sharp = \blacksquare(a \wedge \neg a^\flat) = \blacksquare a \wedge \neg\blacklozenge a^\flat = \blacksquare a \wedge a \wedge \neg a^\flat = \blacksquare a \wedge a^\sharp$ and $\blacklozenge a^\sharp = \blacklozenge(a \wedge \neg a^\flat) \to \blacklozenge a \wedge \neg\blacksquare a^\flat = \blacklozenge a \wedge \neg a^\flat$.

Proposition 1.3.9. *Let $(\mathcal{W}, \mathcal{R}, \Vdash\!\vdash, (\mathcal{W}', \mathcal{L}, \Vdash))$ be a tK$\mathcal{H}$-model such that*

$$\bigcap_{t_0} (u_t)_\flat|_{t=t_0} \;=\; \bigcap_{t_0} (u_t)|_{t=t_0}$$

for all $u_t \in \mathcal{W}$ then $(\blacksquare a)^\flat = \blacksquare a$ and $(\blacksquare a)^\sharp = \perp$ are valid schemes.

Proof. Let $u_t \in \mathcal{W}$ then under the set-theoretic assumption

$$u_t \Vdash\!\vdash (\blacksquare a)^\flat \equiv \forall t_0[(u_t)_\flat|_{t=t_0} \Vdash a] \equiv \forall t_0[(u_t)|_{t=t_0} \Vdash a] \equiv u_t \Vdash\!\vdash \blacksquare a.$$

In conclusion $(\blacksquare a)^\flat = \blacksquare a$ and therefore $(\blacksquare a)^\sharp = \blacksquare a \wedge \neg(\blacksquare a)^\flat = \perp$.

Proposition 1.3.10. *Let $(\mathcal{W}, \mathcal{R}, \Vdash\!\vdash, (\mathcal{W}', \mathcal{L}, \Vdash))$ be a tK$\mathcal{H}$-model with actual time $t_0 \in \mathbb{R}$ such that*

$$\bigcup_{t'} (u_t)_\sharp|_{t=t'} \;=\; (u_t)_\sharp|_{t=t_0}$$

for all $u_t \in \mathcal{W}$ then $(\blacklozenge a)^\sharp = a^\sharp$ and $(\blacklozenge a)^\flat = \blacklozenge a \wedge \neg a^\sharp$ are valid schemes.

Proof. Let $u_t \in \mathcal{W}$ then under the set-theoretic assumption

$$u_t \Vdash\!\vdash (\blacklozenge a)^\sharp \equiv \exists t'[(u_t)_\sharp|_{t=t'} \Vdash a] \equiv (u_t)_\sharp|_{t=t_0} \Vdash\!\vdash a \equiv u_t \Vdash\!\vdash a^\sharp.$$

An easy conclusion is $(\blacklozenge a)^\flat = \blacklozenge a \wedge \neg(\blacklozenge a)^\sharp = \blacklozenge a \wedge \neg a^\sharp$. Observe that modalities de re and de dicto can be formalized in terms of t-logical operators. We say that a is **necessary de dicto** iff $\Box a \wedge a^\flat$ and denote this with $\Box^\flat a$. A proposition a is **necessary de re** iff $\Box a \wedge a^\sharp$ in this case we write $\Box^\sharp a$. It is easy to check that

$$\Box a \;=\; \Box^\flat a \,\dot{\vee}\, \Box^\sharp a.$$

In the same manner we define **possibilities de re** and **de dicto** via $\Diamond^{\mathfrak{A}} a = \Diamond a \wedge \mathfrak{A} a$ for $\mathfrak{A} \in \{(.)^\flat, (.)^\sharp\}$. Moreover $\Diamond a = \Diamond^\flat a \,\dot{\vee}\, \Diamond^\sharp a$. Another treatment of de re and de dicto modalities was given by Zalta[Z1].

1.4. aA-LOGIC

aA-logic is concerned with the interaction of the operators of a-logic and A-logic. The following logical discussion is in parts inspired by classical analytic philosophy. The scheme $\underline{a} \to a^\flat$ is probably the only uncontroversial formula in this section and was emphasised by Carnap[C]. The set of axioms that the reader may accept depends on the philosophical position that he is willing to accept. However we try to shed light on all common positions.

Definition 1.4.1. *The claim $\forall a[\underline{a} = a^\flat]$ is called **neopositivism** and $\exists a[\underline{a} \wedge a^\sharp]$ is called **neorationalism**.*

Under the assumption $a^\flat \to \underline{a}$ it is easy to see that neorationalism is the negation of neopositivism. Note that neorationalism gives a positive answer to Kants main question in his critique of pure reason.

Definition 1.4.2. *The **$\mathbf{H}^2$-model** is a H-model $(\mathcal{W}, \Vdash\!\vdash)$ such that each $(u,v) \in \mathcal{W}$ is a $\mathcal{H}$-model $((u,v)_\flat \dot\sqcup (u,v)_\sharp, \Vdash_{(u,v)})$ and furthermore*

$$\begin{array}{ll} \text{(i)} & (u,v) \Vdash\!\vdash a^\flat \;\equiv\; (u,v) \Vdash_{(u,v)} a^\flat \\ \text{(ii)} & (u,v) \in \mathcal{W} \;\Rightarrow\; (u,v)_\flat \in \mathcal{W} \wedge (u,v)_\sharp \in \mathcal{W}. \end{array}$$

Proposition 1.4.1. *Let $(\mathcal{W}, \Vdash\!\vdash)$ be a H^2-model such that $\forall (u,v)[(u,v)_\flat \subseteq (u,\emptyset)]$ then $a^\flat \to \underline{a}$ and also $\overline{a} \to a^\sharp$.*

Proof. Let $(u,v) \in \mathcal{W}$ then obviously

$$(u,v) \Vdash\!\vdash a^\flat \;\equiv\; (u,v)_\flat \Vdash\!\vdash a \;\Rightarrow\; (u,\emptyset) \Vdash\!\vdash a \;\equiv\; (u,v) \Vdash\!\vdash \underline{a}.$$

We conclude $a^\flat \to \underline{a}$ hence $\neg\underline{a} \to \neg a^\flat$ and therefore $a \wedge \neg\underline{a} \to a \wedge \neg a^\flat$.

Proposition 1.4.2. *Let $(\mathcal{W}, \Vdash\!\vdash)$ be a H^2-model such that $\forall (u,v)[(u,v)_\flat = (u,\emptyset)]$ then $\underline{a^\flat} = a^\flat$ and $(\underline{a})^\flat = \underline{a}$.*

Proof. We start with a H^2-model $(\mathcal{W}, \Vdash\!\vdash)$ and $(u,v) \in \mathcal{W}$ then

$$(u,v) \Vdash\!\vdash (\underline{a})^\flat \;\equiv\; (u,v)_\flat \Vdash\!\vdash \underline{a} \;\equiv\; (u,\emptyset) \Vdash\!\vdash \underline{a} \;\equiv\; (u,v) \Vdash\!\vdash \underline{a}.$$

Note that for all worlds $(w_\flat)_\flat = w_\flat$ i.e. $(.)_\flat$ is idempotent. Therefore

$$(u,v) \Vdash\!\vdash \underline{a^\flat} \;\equiv\; (u,\emptyset)_\flat \Vdash\!\vdash a \;\equiv\; (u,v)_\flat \Vdash\!\vdash a \;\equiv\; (u,v) \Vdash\!\vdash a^\flat.$$

Corollary 1.4.1. *Let $(\mathcal{W}, \Vdash\!\vdash)$ be a H^2-model such that $\underline{a^\flat} = a^\flat$ and $(\underline{a})^\flat = \underline{a}$ then $\overline{a^\flat} = \bot = (\underline{a})^\sharp$.*

Proof. Consider the easy inference $\overline{a^\flat} = a^\flat \wedge \neg\underline{a^\flat} = a^\flat \wedge \neg a^\flat = \bot$. Likewise $(\underline{a})^\sharp = \underline{a} \wedge \neg(\underline{a})^\flat = \underline{a} \wedge \neg\underline{a} = \bot$.

Proposition 1.4.3. *Let $(\mathcal{W}, \Vdash\!\vdash)$ be a H^2-model such that $\forall (u,v)[(u,v)_\flat = (u,\emptyset)]$ then $(\overline{a})^\flat = \bot = \underline{a^\sharp}$.*

Proof. Let $(u,v) \in \mathcal{W}$ and assume $(\overline{a})^\flat$ is a true scheme then

$$(u,v) \Vdash\!\vdash (\overline{a})^\flat \;\equiv\; (u,v)_\flat \Vdash\!\vdash \overline{a} \;\equiv\; (u,\emptyset) \Vdash\!\vdash \overline{a} \;\equiv\; (\emptyset,\emptyset) \Vdash\!\vdash a$$

which is a contradiction since $(\emptyset, \emptyset) \Vvdash a$ is alway false. Because $(a^\flat)^\sharp = \bot$ we have a similar situation and with our assumption $(u, \emptyset)_\sharp = (u, v)_{\flat\sharp} = (\emptyset, \emptyset)$. Assume $\underline{a^\sharp}$ is true then we obtain a contradiction via

$$(u, v) \Vvdash \underline{a^\sharp} \;\equiv\; (u, \emptyset) \Vvdash a^\sharp \;\equiv\; (u, \emptyset)_\sharp \Vvdash a \;\equiv\; (\emptyset, \emptyset) \Vvdash a.$$

Corollary 1.4.2. *Let $(\mathcal{W}, \Vvdash)$ be a H^2-model such that $\forall(u, v)[(u, v)_\flat = (u, \emptyset)]$ then $\overline{a^\sharp} = a^\sharp$ and $(\overline{a})^\sharp = \overline{a}$.*

Proof. Consider the inference $\overline{a^\sharp} = a^\sharp \wedge \neg \underline{a^\sharp} = a^\sharp \wedge \neg\bot = a^\sharp$ and likewise $(\overline{a})^\sharp = \overline{a} \wedge \neg(\overline{a})^\flat = \overline{a} \wedge \neg\bot = \overline{a}$.

Definition 1.4.3. *The **language of propositional t-logic** L^t consists of the logical functors $\{\wedge, \neg, \rightarrow, =, \vee\}$, the unary operators $\{\Box, \Diamond, \blacksquare, \blacklozenge, \underline{(.)}, \overline{(.)}, (.)^\flat . (.)^\sharp\}$ and propositional variables denoted with latin letters.*
*The **t-standard axiomatic system** (tStd) of t-logic consists of the axioms for classical logic, the inference $a^\flat \rightarrow \underline{a}$, modal systems $\mathrm{S5}_\Box, \mathrm{S5}_\blacksquare$ with respect to the modal operators $\{\Box, \Diamond\}$, $\{\blacksquare, \blacklozenge\}$ and the equivalences of operators from the table*

$\equiv$	$\Box$	$\Diamond$	$\blacksquare$	$\blacklozenge$	$\underline{(.)}$	$\overline{(.)}$	$(.)^\flat$	$(.)^\sharp$
$\Box$	$\Box$	$\Diamond$	$\blacksquare$	$\blacklozenge\Box$	$\underline{(.)}$	$\Box \wedge \overline{(.)}$	$(.)^\flat$	$\Box \wedge (.)^\sharp$
$\Diamond$		$\Diamond$	$\blacksquare\Diamond$	$\blacklozenge$	$\underline{(.)}$		$(.)^\flat$	
$\blacksquare$	$\Box$	$\Diamond\blacksquare$	$\blacksquare$	$\blacklozenge$	$\underline{(.)}$	$\blacksquare \wedge \overline{(.)}$	$(.)^\flat$	$\blacksquare \wedge (.)^\sharp$
$\blacklozenge$	$\Box\blacklozenge$	$\Diamond$		$\blacklozenge$	$\underline{(.)}$		$(.)^\flat$	
$\underline{(.)}$	$\Box$	$\neg\overline{(.)} \wedge \Diamond$	$\blacksquare$	$\neg\overline{(.)} \wedge \blacklozenge$	$\underline{(.)}$	$\bot$	$(.)^\flat$	$\bot$
$\overline{(.)}$	$\bot$	$\overline{(.)}$	$\bot$	$\overline{(.)}$	$\bot$	$\overline{(.)}$	$\bot$	$(.)^\sharp$
$(.)^\flat$	$\Box$	$\neg(.)^\sharp \wedge \Diamond$	$\blacksquare$	$\neg(.)^\sharp \wedge \blacklozenge$	$\underline{(.)}$	$\bot$	$(.)^\flat$	$\bot$
$(.)^\sharp$	$\bot$	$(.)^\sharp$	$\bot$	$(.)^\sharp$	$\bot$	$\overline{(.)}$	$\bot$	$(.)^\sharp$

*The **language of first-order t-logic** extends the classical first-order logic by the unary operators $\{\Box, \Diamond, \blacksquare, \blacklozenge, \underline{(.)}, \overline{(.)}, (.)^\flat, (.)^\sharp\}$.*
*The **t-standard axiomatic system of first-order t-logic** $(\mathrm{tStd})^1$ extends (tStd) by the Barcan-style rules for $\mathfrak{U} \in \{\Box, \blacksquare, \underline{(.)}, (.)^\flat\}, \mathfrak{A} \in \{\underline{(.)}, (.)^\flat\}$*

$$\begin{array}{rrcl} \text{(i)} & \mathfrak{U}\forall X[\alpha X] & = & \forall X[\mathfrak{U}\alpha X] \\ \text{(ii)} & \mathfrak{A}\forall X[\alpha X] & = & \forall X[\mathfrak{A}\alpha X] \\ \text{(iii)} & \exists X[\mathfrak{U}\alpha X] & \rightarrow & \mathfrak{U}\exists X[\alpha X]. \end{array}$$

The above table reads from left to right i.e. the operator from the vertical row is applied to the operator in the horizontal row. Consider the set of t-logical operators together with a bijection $\{1, ..., 8\} \xrightarrow{\sigma} \{\Box, \Diamond, \blacksquare, \blacklozenge, \underline{(.)}, \overline{(.)}, (.)^\flat, (.)^\sharp\}$. Consider a vector $\pi \in \{1, ..., 8\}^n$ of operators for a natural number n. Then π is itself an unary operator

$$\pi a \;\stackrel{\text{def}}{=}\; (\pi_1, ..., \pi_n)a \;\stackrel{\text{def}}{=}\; \sigma(\pi_1)\sigma(\pi_2)...\sigma(\pi_n)a.$$

This leads to stochastic considerations because some pairs of operators kill each other for example $(a^\sharp)^\flat = \bot$ and most of them absorb each other like

$\Box\underline{a} = \underline{a}$. In order to prove the following theorem we use the convention

$$\Box\bot = \Diamond\bot = \blacksquare\bot = \blacklozenge\bot = \underline{\bot} = \overline{\bot} = (\bot)^\flat = (\bot)^\sharp = \bot.$$

Theorem 1.4.1. *Let $\pi_n \in \{1, ..., 8\}^n$ be an equally distributed random permutation of t-logical operators of lenght $n \in \mathbb{N}$ then in the axiomatic system* (tStd)

$$\lim_{n\to\infty} \mathbb{P}(\forall a[\pi_n a \equiv \bot]) = 1.$$

Proof. Fix a bijection $\{1, ..., 8\} \xrightarrow{\sigma} \{\Box, \Diamond, \blacksquare, \blacklozenge, \underline{(.)}, \overline{(.)}, (.)^\flat, (.)^\sharp\}$ and let $\pi_n \in \{1, ..., 8\}^n$ be a random permutation of t-logical operators. It is enough to show that $\forall a[\pi_n a \to \bot]$ because from this we conclude $\forall a[\pi_n a \equiv \bot]$. The calculation of the probability of the event $X_n = \forall a[\pi_n \equiv \bot]$ is quite complicated. This is why we give a lower bound that converges to 1. It is enough to consider the operator $\underline{(.)}$ which is indicated with $\sigma(\underline{(.)}) = 5$ and

$$\Box\underline{(.)} = \blacksquare\underline{(.)} = \underline{\underline{(.)}} = (\underline{(.)})^\flat = \underline{(.)}$$

i.e. the apriori-operator absorbs the above operators and moreover $\Diamond\underline{(.)} \to \Diamond$ as well as $\blacklozenge\underline{(.)} \to \blacklozenge$. Note that equivalence of unary operators U and U' is understood as $\forall p[Up \equiv U'p]$. Moreover $\overline{\underline{(.)}} = \bot$ and $(\underline{(.)})^\sharp = \bot$ with $\bot p = \bot$ for all p. Assume that the following event A takes place

$$\exists N \leq n[(\pi_n)_N = 5] \wedge \exists n' < N[(\pi_n)_{n'} = 6].$$

Then the whole unary operator becomes $\bot$ since starting at the apriori-operator at place N every operator that applies to this position vanishes until we have the expression $\overline{\underline{(.)}} = \bot$. We make use of the estimate

$$\begin{aligned} \mathbb{P}(6 \in (\pi_n)_{i\leq\lfloor\frac{n}{2}\rfloor})\mathbb{P}(5 \in (\pi_n)_{i\geq\lfloor\frac{n}{2}\rfloor+1}) &\leq \mathbb{P}(A) \leq \mathbb{P}(X_n) \leq 1 \\ (\frac{8^{\lfloor\frac{n}{2}\rfloor} - 7^{\lfloor\frac{n}{2}\rfloor}}{8^{\lfloor\frac{n}{2}\rfloor}})(\frac{8^{\lfloor\frac{n}{2}\rfloor-1} - 7^{\lfloor\frac{n}{2}\rfloor-1}}{8^{\lfloor\frac{n}{2}\rfloor-1}}) &\leq \mathbb{P}(X_n) \leq 1 \\ (1 - (\frac{7}{8})^{\lfloor\frac{n}{2}\rfloor})(1 - (\frac{7}{8})^{\lfloor\frac{n}{2}\rfloor-1}) &\leq \mathbb{P}(X_n) \leq 1. \end{aligned}$$

In conclusion $\lim_n \mathbb{P}(X_n) = 1$ because the left-side of the above estimate converges to 1.

Let $\mathfrak{A} \in \{\underline{(.)}, (.)^\flat\}$ then linear quantifiers behave as follows

$$\mathfrak{A}\bigotimes_X \alpha X = \bigotimes_X \mathfrak{A}(\alpha X), \qquad \mathfrak{A}\bigoplus_X \alpha X = \bigoplus_X \mathfrak{A}(\alpha X) \tag{28}$$

where α is a first-order predicate. For $\mathfrak{A}^{\mathrm{op}} \in \{\overline{(.)}, (.)^\sharp\}$ we have

$$\mathfrak{A}^{\mathrm{op}}\bigoplus_X \alpha X \;=\; \bigoplus_X \alpha X \wedge \mathop{\&}_X[\neg\mathfrak{A}^{\mathrm{op}}(\alpha X)] \tag{29}$$

where $\&_X$ is the additive generalisation. Likewise using multiplicative particularisation

$$\mathfrak{A}^{\mathrm{op}} \bigotimes_{X} \alpha X \;=\; \bigotimes_{X} \alpha X \wedge \mathop{\mathfrak{Y}}_{X}[\neg \mathfrak{A}^{\mathrm{op}}(\alpha X)]. \tag{30}$$

Oberserve that the scheme $a^{\flat} \to \underline{a}$ can be extended to the sequence

$$\bot \;\to\; a^{\flat} \;\to\; \underline{a} \;\to\; \Box a \;\to\; \blacksquare a \;\to\; a \;\to\; \top. \tag{31}$$

Contraposition at each arrow of the above sequence yields

$$\bot \;\to\; \neg a \;\to\; \blacklozenge \neg a \;\to\; \lozenge \neg a \;\to\; \neg \underline{a} \;\to\; \neg a^{\flat} \;\to\; \top. \tag{32}$$

Conjunction $(.) \wedge a$ at each object of the sequence gives

$$\bot \;\to\; \blacklozenge \neg a \wedge a \;\to\; \lozenge \neg a \wedge a \;\to\; \overline{a} \;\to\; a^{\sharp} \;\to\; a \;\to\; \top. \tag{33}$$

Note that in (33) **actual contingency** $a \wedge \lozenge \neg a$ cannot be analyzed further. This is the reason why there is no long sequence containing (31) and (33). We may point out that the scheme $\underline{a} \to \Box a$ is controversial. Again we refer to Kripkes example[Kr]. Let $w \in (\mathcal{W}, \mathcal{R}, \Vdash)$ be a element of some Kripke frame. Denote $\mathbf{U}_w$ for the w-prototype metre in w and $1\mathrm{m}_{@}$ for the @-metre defined as the length of $\mathbf{U}_{@}$. Consider the proposition $\nu^w \equiv (|\mathbf{U}_w| = 1\mathrm{m}_{@})$. Then ν^w is certainly apriori true in the actual world which we denote with $\mathcal{A}^{\Box}(\underline{\nu^w})$ and but there exists worlds such that ν^w is wrong i.e. $\exists w \in \mathcal{W}[\neg \nu^w]$ if we assume $\mathcal{W}$ to be big enough.
But if one assumes that $(*)$ $\forall w[\partial(w \Vdash p) = \partial(@ \Vdash p)]$ for some Kripke frame $(\mathcal{W}, \Vdash, \mathcal{R})$ then we may argue for $\underline{p} \to \Box p$. Condition $(*)$ may be called **world stability of** p. This notion can be of great help when rejecting scepticism. Denote $\mathfrak{h}$ for the sceptical hypothesis i.e. that every empirical proposition in a certain world is wrong. Assume that the **sceptical hypothesis** is world stable and $\mathfrak{h} \to \underline{\mathfrak{h}}, \mathrm{cont}(\mathfrak{h})$ since $\mathfrak{h}$ is not verifiable empirically and coningent, letting $\mathrm{cont}(a) = \lozenge a \wedge \lozenge \neg a$. Thus the diagram shows the contradiction

$$\begin{tikzcd}
 & \mathfrak{h} \arrow[dl] \arrow[dr] & \\
\underline{\mathfrak{h}} \arrow[d] & & \mathrm{cont}(\mathfrak{h}) \arrow[d] \\
\Box \mathfrak{h} & & \lozenge(\neg \mathfrak{h}).
\end{tikzcd}$$

2. T-LOGICAL FORMAL ONTOLOGY

Think of objects X as variables in some formal language L. The main topic of the next chapters is to develop an object calculus in order to formalize the philosophical discussion about the ontic status of metaphysical entities.

2.1. DESCRIPTIONS AND MEREOLOGY

Descriptions are one of our main logical tools. If ϕ is a set of first-order predicates the expression $\iota X\{\phi\}$ reads '*The* X *such that* $\forall\varphi \in \phi[\varphi X]$'. Usually the set of predicates is given arbitrary and hence there is probably a set of X statisfying ϕ. We indicate this using the **bundle desciption**[Kr]

$$\iota\iota X\{\phi\} \;=\; \{X \mid \forall\varphi \in \phi[\varphi X]\}.$$

Furthermore if $\mathcal{F}$ is a choice function given by the axiomatic system (ZFC) then the simple description is given by $\iota X\{\phi\} = \mathcal{F}(\iota\iota X\{\phi\})$.
Let $\mathcal{P} = (p_\alpha)_{\alpha\in A}$ be a set of propositions and write $\mathrm{cl}(\mathcal{P})$ for the **logical closure** i.e. the set $\{q \mid \mathcal{P} \Vdash q\}$. Consider the following set of propositions

$$\{p\}^{\rightarrow} \;=\; \{q \mid \exists n \exists q_1, ..., q_n[p = q_1 \rightarrow ... \rightarrow q_n = q]\}.$$

If $\mathcal{P} = (p_\alpha)_{\alpha\in A}$ is a set of propositions then it is easy to see that

$$\mathrm{cl}(\mathcal{P}) \;=\; \{\bigwedge_{\alpha\in A} p_\alpha\}^{\rightarrow}.$$

Definition 2.1.1. *Let X be an object then $\mathrm{Ext}(X) = \{\varphi \mid \varphi X\}$ is the* ***extension*** *of X. If φ is a first-order predicate then $\mathrm{Tor}(\varphi) = \{X \mid \varphi X\}$ is the* ***torsion*** *of φ.*

Both $\mathrm{Ext}(X)$ and $\mathrm{Tor}(\varphi)$ are considered to be sets (this is a convention). If $\Phi \subseteq \mathrm{Ext}(X)$ and $\mathfrak{U}$ is an unary operator we write $\mathfrak{U}\Phi = \{\mathfrak{U}\varphi \mid \varphi \in \Phi\}$ and $\Phi^{\mathfrak{U}} = \{\varphi \in \Phi \mid \mathfrak{U}\varphi X\}$. With this notation we define the **intension** of X as $\mathrm{Int}(X) = \mathrm{Ext}(X)^{\Box} \subseteq \mathrm{Ext}(X)$. Using the torsion we can give another formula for the bundle description

$$\iota\iota X\{\phi\} \;=\; \bigcap_{\varphi\in\phi} \mathrm{Tor}(\varphi).$$

Proposition 2.1.1. *The set of objects $\mathrm{Ob}(\mathbf{Ext}(X)) = \mathrm{Ext}(X)$ and the set*

$$\mathrm{Fl}(\mathbf{Ext}(X)) = \{\mathrm{Mor}(\alpha,\delta) = \begin{cases} \{*\}, & \forall Y[\alpha Y \rightarrow \delta Y] \\ \emptyset, & \exists Y[\alpha Y \wedge \neg\delta Y] \end{cases} \mid \alpha, \delta \in \mathrm{Ext}(X)\}$$

of arrows constitute the ***extension category*** *of X with products and coproducts. The category is denoted by $\mathbf{Ext}(X)$.*

Proof. For $\alpha \in \mathrm{Ext}(X)$ we have $\forall X[\alpha X \rightarrow \alpha X]$ hence $\alpha \rightarrow \alpha$. Let $\alpha, \delta, \gamma \in \mathrm{Ext}(X)$ with $\alpha \rightarrow \delta$ and $\delta \rightarrow \gamma$. Note that $\alpha \rightarrow \delta$ is equivalent to $\mathrm{Tor}(\alpha) \subseteq \mathrm{Tor}(\delta)$ because $\delta \rightarrow \gamma$ we conclude

$$\mathrm{Tor}(\alpha) \;\subseteq\; \mathrm{Tor}(\delta) \;\subseteq\; \mathrm{Tor}(\gamma) \;\Longrightarrow\; \mathrm{Tor}(\alpha) \subseteq\; \mathrm{Tor}(\gamma)$$

and therefore $\alpha \rightarrow \gamma$. We have $\mathrm{Tor}(\alpha \wedge \delta) = \mathrm{Tor}(\alpha) \cap \mathrm{Tor}(\delta)$ which yields injections $\mathrm{Tor}(\alpha\wedge\delta) \subseteq \mathrm{Tor}(\alpha)$ and $\mathrm{Tor}(\alpha\wedge\delta) \subseteq \mathrm{Tor}(\delta)$. If $\mathrm{Tor}(\gamma) \subseteq \mathrm{Tor}(\alpha)$ and $\mathrm{Tor}(\gamma) \subseteq \mathrm{Tor}(\delta)$ then $\mathrm{Tor}(\gamma) \subseteq \mathrm{Tor}(\alpha)\cap\mathrm{Tor}(\delta)$ which means that there exists a unique $\gamma \rightarrow \alpha \wedge \delta$ therefore $\wedge$ is the product. In the same manner one can show that $\vee$ is the coproduct and for arbitrary families $\{\alpha_i\}_{i\in I}$

$$\prod_{i\in I} \alpha_i(.) \;=\; \forall i \in I[\alpha_i(.)], \qquad \coprod_{i\in I} \alpha_i \;=\; \exists i \in I[\alpha_i(.)].$$

The **verum**-predicate is given by $\forall X[\top X = \top]$ and the **characteristic**-predicate of X is defined as $\forall Y[\chi_X Y \equiv (X = Y)]$. Obviously $\mathrm{Tor}(\chi_X) = \{X\}$ furthermore $\mathrm{Tor}(.)$ is a covariant functor

$$\mathbf{Ext}(X) \xrightarrow{\mathrm{Tor}} \mathrm{Ouv}(\mathfrak{V})$$

if we assume that $\mathrm{Tor}(\alpha)$ is a set for each $\alpha \in \mathrm{Ext}(X)$ and $\mathfrak{V}$ is the Von Neumann universe equipped with the finest topology $\mathfrak{P}(\mathfrak{V})$.
Let $\Phi = (\varphi_\alpha)_{\alpha \in A}$ be a set of predicates. Note that we can extend the logical operators to propositions i.e. for $\varphi, \varphi' \in \Phi$ we define $\varphi \wedge \varphi' = \forall X[\varphi X \wedge \varphi' X]$ further we write $\varphi \to \varphi'$ with the above definition then the logical closure of Φ is again denoted by $\mathrm{cl}(\Phi)$ and we have $\mathrm{cl}(\Phi) = \{\bigwedge_{\alpha \in A} \varphi_\alpha\}^{\to}$.

Definition 2.1.2. *The **subsistence-predicate** is given by $\mathbb{s}[X] = \top$ and the **existence-predicate** is written as $\mathbb{e}[X]$.*

For good reasons we let $\mathbb{e}$ undefined. Obviously different ontological theories give rise to different existence-predicates because their ontology is given by $\mathrm{Tor}(\mathbb{e})$. Later we will see that each world in a Kripke frame has its own existence predicate. Let us point out three additional things. First we have $\forall X[\mathbb{s}[X] \to \mathbb{e}[X]]$ and second an appealing definition of the existence-predicate might be the **predicate approach** $\mathbb{e}[X] = \exists \varphi[\varphi X]$. Third recall that the **standard approach** would suggest something along the lines of $\mathbb{e}[X] = \exists Y[X = Y]$.

Proposition 2.1.2. *The characteristic predicate χ_X is initial in $\mathbf{Ext}(X)$ and the verum-predicate $\top$ is terminal.*

Proof. Let $\varphi \in \mathrm{Ext}(X)$ then $X \in \mathrm{Tor}(\varphi)$ henceforth $\mathrm{Tor}(\chi_X) = \{X\} \subseteq \mathrm{Tor}(\varphi)$ in conclusion $\chi_X \to \varphi$ which shows that χ_X is initial. Likewise $\mathrm{Tor}(\top)$ is the set of all object therefore $\mathrm{Tor}(\varphi) \subseteq \mathrm{Tor}(\top)$ i.e. $\varphi \to \top$ which proves that $\top$ is terminal.

Definition 2.1.3. *The axiom $\forall X[\mathbb{e}[X] \to (\mathrm{Int}(X) \neq \emptyset)]$ is called **essentialism** and its negation $\exists X[\mathbb{e}[X] \wedge (\mathrm{Int}(X) = \emptyset)]$ **accidentialism**.*

A setless description of essentialism is $\forall X[\mathbb{e}[X] \to \exists \varphi[\Box \varphi X]]$ and a setless formulation of accidentialism is given by $\exists X[\mathbb{e}[X] \wedge \forall \varphi[\neg \Box \varphi X]]$. Recall Gödels[G] definition of **essential** properties φ of an object X

$$\varphi \,\mathrm{ess}\, X \;=\; \varphi X \wedge \forall \psi[\psi X \to \Box \forall Y[\varphi Y \to \psi Y]]. \tag{34}$$

Similarly we say that φ is an **accidential** property of X if

$$\varphi \,\mathrm{acc}\, X \;=\; \varphi X \wedge \exists \psi[\psi X \wedge \Diamond \exists Y[\varphi Y \wedge \neg \psi X]]. \tag{35}$$

With this information we define the **essence** of X as $\mathrm{Ess}(X) = \{\varphi \,|\, \varphi \,\mathrm{ess}\, X\}$ and the **accidence** $\mathrm{Acc}(X) = \{\varphi \mid \varphi \,\mathrm{acc}\, X\}$. From the above definition we conclude $\varphi X = \varphi \,\mathrm{ess}\, X \,\dot\vee\, \varphi \,\mathrm{acc}\,\mathrm{X}$ and therefore $\mathrm{Ext}(X) = \mathrm{Ess}(X) \,\dot\sqcup\, \mathrm{Acc}(X)$.

Proposition 2.1.3. *Let $\alpha, \delta \in \mathrm{Ext}(X)$ then $\alpha \,\mathrm{ess}\, X \wedge \delta \,\mathrm{ess}\, X \implies (\alpha \equiv$*

δ) hence the essence of X consists of one or zero elements up to logical equivalence $|\mathrm{Ess}(X)/_{\equiv}| \in \{0,1\}$.

Proof. We want to show $\forall Y[\alpha Y = \delta Y]$. Let $\alpha, \delta \in \mathrm{Ext}(X)$ with $\alpha \,\mathrm{ess}\, X \wedge \delta \,\mathrm{ess}\, X$ we infer under the assumption δX

$$\begin{aligned} \alpha \,\mathrm{ess}\, X &\longrightarrow (\alpha X \wedge \forall \psi[\psi X \rightarrow \Box \forall Y[\alpha Y \rightarrow \psi Y]]) \\ &\longrightarrow (\delta X \rightarrow \Box \forall Y[\alpha Y \rightarrow \delta Y]) \\ &\longrightarrow \Box \forall Y[\alpha Y \rightarrow \delta Y] \\ &\longrightarrow \forall Y[\alpha Y \rightarrow \delta Y]. \end{aligned}$$

Using symmetry we obtain $\forall Y[\alpha Y = \delta Y]$ therefore $\alpha \equiv \delta$. If there is no φ with $\varphi \,\mathrm{ess}\, X$ then $\mathrm{Ess}(X) = \emptyset$.

Proposition 2.1.4. *Write* $\mathrm{Ess}(X)/_{\equiv} = \{\varphi_0\}$ *then* φ_0 *is initial in* $\mathbf{Ext}(X)$.

Proof. Let $\mathrm{Ess}(X)/_{\equiv} = \{\varphi_0\}$ then

$$\begin{aligned} \varphi_0 \,\mathrm{ess}\, X &\longrightarrow (\varphi_0 X \wedge \forall \psi[\psi X \rightarrow \Box \forall Y[\varphi_0 Y \rightarrow \psi Y]]) \\ &\longrightarrow \forall \psi[\psi X \rightarrow \Box \forall Y[\varphi_0 Y \rightarrow \psi Y]] \\ &\longrightarrow \forall \psi[\psi X \rightarrow \forall Y[\varphi_0 Y \rightarrow \psi Y]] \\ &\longrightarrow \forall \psi \in \mathrm{Ext}(X) \forall Y[\varphi_0 Y \rightarrow \psi Y] \end{aligned}$$

therefore $\mathrm{Tor}(\varphi_0) \subseteq \mathrm{Tor}(\psi)$ i.e. $\varphi_0 \rightarrow \psi$ and because initial elements are equivalent up to isomorphisms $\varphi_0 = \chi_X$.

Consider a description $X = \iota Y\{\phi_X\}$ with $\phi_X \subseteq \mathrm{Ext}(X)$. Define for $\varphi \in \mathrm{Ext}(X)$ the predicate-minus $X \setminus \varphi = \iota Y\{\mathrm{Ext}(X) \setminus \varphi\}$. Using this notation we are able to define Parsons[Pa] nuclear/extranuclear distinction for properties and say that φ is **nuclear** if

$$\varphi \,\mathrm{nuc}\, X \;=\; \varphi X \wedge \neg \mathbb{e}[X \setminus \varphi]$$

and φ is **extranuclear** if $\varphi \,\mathrm{exn}\, X = \varphi X \wedge \mathbb{e}[X \setminus \varphi]$. An easy deduction is $\varphi X = \varphi \,\mathrm{nuc}\, X \,\dot{\vee}\, \varphi \,\mathrm{exn}\, X$. Obviously $\mathbb{e} \,\mathrm{nuc}\, X$ for an arbitrary object X.

Definition 2.1.4. *We denote the **extension graph** of X as* $\mathrm{G}[\mathbf{Ext}(X)]$ *where the set of vertices is* $\mathrm{V}[\mathrm{G}[\mathbf{Ext}(X)]] = (\mathrm{Ext}(X) \setminus \{\chi_X\})/_{\equiv}$ *i.e. the set of predicates mod logical equivalence and the set of edges is given by* $\mathrm{E}[\mathrm{G}[\mathbf{Ext}(X)]] = \{(\alpha, \delta) \in (\mathrm{Ext}(X) \setminus \{\chi_X\})/^2_{\equiv} \mid \alpha \rightarrow \delta\}$.

Theorem 2.1.1. *Let X be an object. Then the extension graph* $\mathrm{G}[\mathbf{Ext}(X)]$ *is an acyclic digraph.*

Proof. It is enough to show that $\mathrm{G} = \mathrm{G}[\mathbf{Ext}(X)]$ contains no cycle. The rest follows from general theorems in graph theory. Assume there is a cycle

$$\mathfrak{C} \;=\; \mathrm{G}[\{\alpha_1, ..., \alpha_\eta\}]$$

of length $\eta > 1$. By assumption we know $\alpha_\eta \rightarrow \alpha_1$ but also by going through the whole cycle $\alpha_1 \to \alpha_\eta$ therefore $\alpha_1 \equiv \alpha_\eta$ and because the set of vertices is given up to logical equivalence $\alpha_1 = \alpha_\eta$ hence $\eta = 1$ which is a contradiction.

Definition 2.1.5. *Let φ be a predicate then $\mathfrak{p}_\varphi = \iota Y\{\{\varphi\}\}$ is called $\varphi-$**part**.*

Definition 2.1.6. *Let $\{X_\alpha\}_{\alpha\in A}$ be a family of objects. We define their **mereological sum** and their **mereological product** as*

$$\underset{\alpha\in A}{\boxplus} X_\alpha \;=\; \iota Y\{\bigcup_{\alpha\in A} \mathrm{Ext}(X_\alpha)\}, \qquad \underset{\alpha\in A}{\boxtimes} X_\alpha \;=\; \iota Y\{\bigcap_{\alpha\in A} \mathrm{Ext}(X_\alpha)\}.$$

If we restrict ourselves to finite merological sums and products it is easy to see that they are commutative and associative. Moreover distributivity holds and if $\emptyset = \iota Y\{\emptyset\}$ is the **zero object** and $\top = \iota Y\{\emptyset^c\}$ is the **all object** we have neutral elements $X \boxplus \emptyset = X$ and $X \boxtimes \top = X$. Also $X \boxplus X^c = \top$ and $X \boxtimes X^c = \emptyset$ where $X^c = \iota Y\{\mathrm{Ext}(X)^c\}$ is the complement. Further it is possible to consider linear mereological sums. In order to generalise the above definition consider the equivalent formulation

$$\underset{\alpha\in A}{\boxtimes} X_\alpha \;=\; \iota Y\{\{\bigwedge_{\substack{\alpha\\ \Cap\\ A}} \; \bigwedge_{\substack{\varphi\\ \Cap\\ \mathrm{Ext}(X_\alpha)}} \varphi\}\}$$

Exchanging classical conjunction with linear conjunction yields additive and muplitplicative mereological products

$$\overset{\&}{\underset{\alpha\in A}{\boxtimes}} X_\alpha = \iota Y\{\{\underset{\substack{\alpha\\ \Cap\\ A}}{\&} \; \underset{\substack{\varphi\\ \Cap\\ \mathrm{Ext}(X_\alpha)}}{\&} \varphi\}\}, \qquad \overset{\otimes}{\underset{\alpha\in A}{\boxtimes}} X_\alpha = \iota Y\{\{\bigotimes_{\substack{\alpha\\ \Cap\\ A}} \; \bigotimes_{\substack{\varphi\\ \Cap\\ \mathrm{Ext}(X_\alpha)}} \varphi\}\}.$$

Equally additive and multiplicative mereological sums are constructed by replacing disjunction with the linear equivalents.

Theorem 2.1.2. *Let X be an object with extension* $\mathrm{Ext}(X)$ *then there is a decomposition into φ-parts which is called **part decomposition** such that*

$$X \;=\; \underset{\varphi\in\mathrm{Ext}(X)}{\boxplus} \mathfrak{p}_\varphi.$$

Proof. It is enough to show that the extensions are the same on both sides. Write $\Phi = \mathrm{Ext}(X)$ then $\mathrm{cl}(\mathrm{Ext}(X)) = \mathrm{Ext}(X)$ and

$$\mathrm{Ext}(\underset{\varphi\in\Phi}{\boxplus} \mathfrak{p}_\varphi) \;=\; \mathrm{cl}(\bigcup_{\varphi\in\Phi} \mathrm{Ext}(\mathfrak{p}_\varphi)) \;=\; \mathrm{cl}(\bigcup_{\varphi\in\Phi} \{\varphi\}\,) \;=\; \mathrm{Ext}(X).$$

The disticntion between objects and predicates was emphasised in the above considerations. Of course there are also higher predicates for example the positive-predicate $\mathcal{P}(.)$ from Gödels[G] ontological argument. Consider the following predicate defined with the recursion for $n \in \mathbb{N}$

$$\mathrm{pl}^{n+1}(X) \quad = \quad \exists Y[\mathrm{pl}^n(Y) \wedge X(Y)]$$

where $\mathrm{pl}^0(X)$ iff X is an object (or a variable in a given language). We say that X is a predicate of order n if $\mathrm{pl}^n(X)$. Obviously $\mathrm{pl}^{n+1}(\mathrm{pl}^n)$ for each $n \in \mathbb{N}_0$. Moreover the recursion can be resolves as follows

$$\mathrm{pl}^n(X) \quad = \quad \exists \alpha_n, ...\alpha_1[X(\alpha_n) \wedge \bigwedge_{j=1}^{n-1} \alpha_{j+1}(\alpha_j) \wedge \mathrm{pl}^0(\alpha_1)].$$

The definition of Ext(.) can be extended to predicates in the obvious way. The **higher Extension** of order $n \in \mathbb{N}_0$ is given by $\mathrm{Ext}^n = \mathrm{Ext}(\mathrm{pl}^n)$.

Theorem 2.1.3. *There exists maps $(\partial_n)_{n\in\mathbb{N}_0}$ and a sequence*

$$... \xrightarrow{\partial_{n-2}} \mathrm{Ext}^{n-1} \xrightarrow{\partial_{n-1}} \mathrm{Ext}^n \xrightarrow{\partial_n} \mathrm{Ext}^{n+1} \xrightarrow{\partial_{n+1}} ... \; .$$

Proof. Let $\mathcal{F}$ be a choice function. We write $\mathrm{Ext}(X)^{\mathcal{F}} = \mathcal{F}(\mathrm{Ext}(X))$. The map $\partial_n = \partial$ is given by $\alpha \mapsto \mathrm{Ext}(\alpha)^{\mathcal{F}}$. This yields a function $\mathrm{Ext}^n \xrightarrow{\partial_n} \mathrm{Ext}^{n+1}$ for each $n \in \mathbb{N}_0$. Moreover composing $m \in \mathbb{N}$ of these maps yields a map $\mathrm{Ext}^n \xrightarrow{\partial^m} \mathrm{Ext}^{n+m}$ given by

$$\prod_{j=1}^{m} \partial_n = \partial^m \;:\; \alpha \longmapsto \underbrace{\mathrm{Ext}(...\mathrm{Ext}(\mathrm{Ext}(\alpha)^{\mathcal{F}})^{\mathcal{F}}...)^{\mathcal{F}}}_{m-\text{times}}.$$

The above sequence induces a directed system $(\partial_{ij})_{ij} = (\partial^{j-i})_{ij}$ for $i < j$ on the system of objects $(\mathrm{Ext}^n)_{n\in\mathbb{N}_0}$. The maps are given by $\mathrm{Ext}^i \xrightarrow{\partial^{j-i}} \mathrm{Ext}^j$ and the identity $\mathrm{Ext}^i \xrightarrow{\mathrm{id}} \mathrm{Ext}^i$. Using this we define the **infinity extension** via the direct limit of this directed system

$$\mathrm{Ext}^\infty \quad = \quad \operatorname*{colim}_{n\in\mathbb{N}_0} \mathrm{Ext}^n.$$

Definition 2.1.7. *Consider a function* $\mathrm{Ext}(X) \xrightarrow{\rho} \mathrm{Ext}(Y)$ *for objects X and Y. Then ρ is called a* ***transformation*** *and this is denoted with $X \overset{\rho}{\rightsquigarrow} Y$.*

Example 2.1.1. Consider an inclusion $\mathrm{Ext}(X) \overset{i}{\hookrightarrow} \mathrm{Ext}(Y)$ then $X \overset{i}{\rightsquigarrow} Y$ is the usual mereological parthood. In this case we write $X \hookrightarrow Y$. Let X be an object and A a part of X i.e. $A \hookrightarrow X$. We can even define quotiens via $X/A = \iota Y\{\mathrm{Ext}(X)/\mathrm{Ext}(A)\}$. There exists a projection $\mathrm{Ext}(X) \overset{\pi}{\twoheadrightarrow} \mathrm{Ext}(Y)$ that gives rise to a transformation $X \overset{\pi}{\rightsquigarrow} X/A$. If $\rho = \mathrm{Id}$ then we have identity of objects.

Definition 2.1.8. *Let X be an object. Then $X^{\mathfrak{U}} = \iota\Sigma\{\mathrm{Ext}(X)^{\mathfrak{U}}\}$ for an unary operator $\mathfrak{U}$ is called* ***$\mathfrak{U}$-fication*** *of X.*

There is an obvious inclusion $X^{\mathfrak{U}} \hookrightarrow X$. Further if we assume that $p = \mathfrak{U}p \,\dot{\vee}\, \mathfrak{U}^{\mathrm{op}}p$ (as is the case of $\underline{(.)}$ and $(.)^\flat$) then the decomposition $X = X^{\mathfrak{U}} \,\dot{\boxplus}\, X^{\mathfrak{U}^{\mathrm{op}}}$

holds. One has the sequence of embeddings

$$\mathfrak{p}_{\chi X} \hookrightarrow X^{(.)^\flat} \hookrightarrow X^{\underline{(.)}} \hookrightarrow X^{\square} \hookrightarrow X^{\blacksquare} \hookrightarrow X.$$

if t-logical operators are used and under the assumption $(\chi_X X)^\flat$.

Theorem 2.1.4. *Let $\{X_\alpha\}_{\alpha\in A}$ be a family of objects and let $\mathfrak{U}$ be an unary operator of some propositional logic then*

$$\Big(\boxplus_{\alpha\in A} X_\alpha\Big)^{\mathfrak{U}} = \boxplus_{\alpha\in A} X_\alpha^{\mathfrak{U}}.$$

Proof. We write the respective sets of extensions as

$$\begin{aligned}
\Big(\bigcup_{\alpha\in A}\mathrm{Ext}(X_\alpha)\Big)^{\mathfrak{U}} &= \{\varphi\in\bigcup_{\alpha\in A}\mathrm{Ext}(X_\alpha)\mid \mathfrak{U}\varphi(\boxplus_{\alpha\in A} X_\alpha)\}\\
&= \{\varphi\in\bigcup_{\alpha\in A}\mathrm{Ext}(X_\alpha)\mid \exists\beta[\mathfrak{U}\varphi X_\beta]\}\\
&= \{\varphi\in\bigcup_{\alpha\in A}\mathrm{Ext}(X_\alpha)\mid \exists\beta[\varphi\in\mathrm{Ext}(X_\beta^{\mathfrak{U}})]\}\\
&= \{\varphi\in\bigcup_{\alpha\in A}\mathrm{Ext}(X_\alpha)\mid \varphi\in\bigcup_{\alpha\in A}\mathrm{Ext}(X_\alpha^{\mathfrak{U}})\}\\
&= \bigcup_{\alpha\in A}\mathrm{Ext}(X_\alpha^{\mathfrak{U}}).
\end{aligned}$$

And because we have equality of extensions we have equality of objects.

2.2. FORMAL WORLD THEORY

Wittgensteins[W1] conception of worlds was used throughout the whole inquiry. They appear as sets of all propositions that are true in the given world. However there are a number of interesting other formal models describing the nature of our universe in toto.

Definition 2.2.1. *A set of propositions w is called* ***Wittgenstein world****.*

This is exactly the conception that is used in the Kripke-semantics. Obviously this world theory does not rely on philosophical issues. We define the **w-existence predicate** for objects X as

$$\mathbb{e}_w[X] = (\mathbb{e}[X]\in w).$$

Let $(\mathcal{W},\mathcal{R},\Vdash)$ be a Kripke frame then the actual world is a unique element $@\in\mathcal{W}$. Hence we may write $(\mathcal{W},\mathcal{R},\Vdash,@)$ to denote this. Consider the **actual world** $@$ as a Wittgenstein world. As a Wittgenstein world it has infinitely many subworlds $w'\subseteq @$ which shows that this definition lacks some philosophical intuition because worlds are usually considered to be maximal in some sense.

Definition 2.2.2. *A Wittgenstein world w is called **empty-world** if $w = \emptyset$. Further w is called **impossible** if w is non-empty and $\exists p[(p \in w) \wedge (\neg p \in w)]$. In this case the world is denoted with $\mathcal{m}$. The **opposite world** of a Wittgeinstein world w is defined as $w^{\mathrm{op}} = \{\neg p \mid p \in w\}$.*

Obviously a world w is **possible** if $\forall p[(p \in w) \to (\neg p \notin w)]$. With the notion of impossible worlds we can give a first analysis of non-existent objects. Typical objects that are brought up in the discussion about non-existence are logically incoherent entities like ’the round square’ that we may denote with $\mathcal{RS}$. It is reasonable to argue that there exists some impossible world $\mathcal{m}$ such that $\mathbb{e}_{\mathcal{m}}[\mathcal{RS}]$.
Moreover non-existence in a given world can be interpreted as existence in the opposite world which means $\neg\mathbb{e}_w[X] = \mathbb{e}_{w^{\mathrm{op}}}[X]$. Descartes third step of his famous sceptizistic attack makes use of a thought experiment called genius malignus. This god-like entity implants for every true proposition p in @ the wrong proposition $\neg p$ into ones own mind except for $\mathbb{e}_@[🚹]$ whereas the symbol 🚹 refers to the reader of this symbol. Observe that the genius malignus can be identified with $\mathfrak{g}_m = @^{\mathrm{op}} \setminus \{\neg\mathbb{e}_@[🚹]\} \cup \{\mathbb{e}_@[🚹]\}$.

Definition 2.2.3. *Let w be a Wittgenstein world with w-existence predicate $\mathbb{e}_w$. The **set-world** is given by $w = \{X \mid \mathbb{e}_w[X]\} = \mathrm{Tor}(\mathbb{e}_w)$.*

We use the symbol w in these two cases in order to emphasise that we refer to the same entity. This is logically incoherent but expresses the philosophical view that the concept of a world is unique. The view that the extension of each world concept is equivalent may be called **world monism**.

Proposition 2.2.1.[Ga] *Let w be a set-world such that $\mathbb{e}_w[w]$ then w is not well-founded i.e. there exists an infinite two-sided chain*

$$\ldots \in w \in w \in w \in w \in w \in w \in w \in \ldots .$$

Proof. This is obvious because $\mathbb{e}_w[w] \Rightarrow (w \in w)$.

Let $(\mathcal{W}, \mathcal{R}, \Vdash)$ be a Kripke frame. This frame gives rise to a graph $\mathrm{G} = \mathrm{G}(\mathcal{W}, \mathcal{R}, \Vdash)$ with $\mathrm{V}[\mathrm{G}] = \mathcal{W}$ and $\mathrm{E}[\mathrm{G}] = \{(w, w') \in \mathcal{W}^2 \mid w\mathcal{R}w'\}$. For example Stalnakers[St] **modal actualism** can be represented with the loop

$$@ \circlearrowleft$$

that is the simplest non-trivial modal theory. Note that in this model we have the sequence $\Diamond a \to a \to \Box a$.
There is a famous trilemma popularized by Albert[A] that is called **Munchhausen trilemma**. To formalize this thought experiment we need counterfactual conditionals. We write $p \circlearrowright q$ for the proposition ’If it were the case that p, then it would be the case that q’. This logical connective has some interesting properties. It is well known that this relation is reflexive, not

transitive and not total. Moreover antisymmetry holds i.e. for all p, q

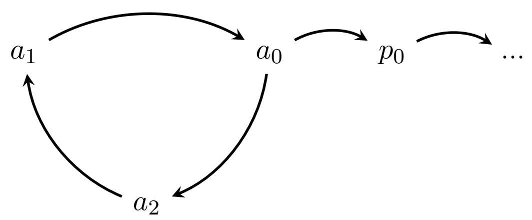
$$p \circlearrowright q \wedge q \circlearrowright p \implies p = q.$$

Definition 2.2.4. *Let w be a Wittgeinstein world. The graph $\mathcal{G}(w)$ given by the set of vertices $\mathrm{V}[\mathcal{G}(w)] = w$ and the set of edges $\mathrm{E}[\mathcal{G}(w)] = \{(p, q) \in w^2 \mid p \circlearrowright q\}$ is called **world graph** of w.*

Proposition 2.2.2. *Let $\mathcal{G}(w)$ be the world graph of a world w. There exists a binary relation $\mathcal{M} \subseteq w \times w$ on w that is transitive and reflexive.*

Proof. Let $p, q, r \in w$ then define $p\mathcal{M}q \equiv$ '$\exists \mathcal{P}_{p,q}$ path connecting p and q'. By convention each node $p \in w$ is connected to itself and further if $\mathcal{P}_{p,q}$ and $\mathcal{P}_{q,r}$ are the paths given by $p\mathcal{M}q$ and $q\mathcal{M}r$ then $\mathcal{P}_{p,r} = \mathcal{P}_{p,q} \oplus \mathcal{P}_{q,r}$ is the path connecting $p, r \in w$ therefore $\mathcal{M}$ is also transitive.

It is possible to give another definition of the relation $\mathcal{M}$. Fix the following notation $\{p\}^{\circlearrowright} = \{q \mid \exists n \exists q_0, ..., q_n[p = q_0 \circlearrowright ... \circlearrowright q_n = q]\}$. Let $p, q \in w$ then $p\mathcal{M}q \equiv q \in \{p\}^{\circlearrowright}$. A **primal ground** of a Wittgenstein world w is a unique proposition $\mathfrak{g}$ such that $\forall p \in w[p \in \{\mathfrak{g}\}^{\circlearrowright}]$. In search of the primal ground one starts with a node $p_0 \in w$ and walks back to an assertion a_0 such that $a_0 \circlearrowright p_0$. This procedure is continued as long as possible. The first possibility of the trilemma is the **cycle**. This example shows how the search of a ground for p_0 ends in the cycle $\{a_0, a_1, a_2\}$.



The relation $p\hat{\mathcal{M}}q \equiv p\mathcal{M}q \wedge q\mathcal{M}p$ is an equivalence relation on w that gives rise to the **reduced world graph** $\hat{\mathcal{G}}(w)$ that is given by the induced graph on the nodes $\mathrm{V}[\hat{\mathcal{G}}(w)] = w/_{\hat{\mathcal{M}}}$.

Theorem 2.2.1. *Let w be a Wittgenstein world. The reduced world graph $\hat{\mathcal{G}}(w)$ is an acyclic digraph with a partial order $\tau = \tilde{\mathcal{M}}$. There exists a decomposition into weakly connected acyclic digraphs $(\mathcal{AC}_l^w)_{l \preceq \kappa}$ for some ordinal κ*

$$\hat{\mathcal{G}}(w) = \coprod_{l \preceq \kappa} \mathcal{AC}_l^w.$$

Furthermore on each component $\mathcal{AC}_l^w$ there exists a well-ordering τ^l and a minimal element $r^l = \min_{\tau^l} \mathrm{V}[\mathcal{AC}_l^w]$ with respect to τ^l.

Proof. Assume there exists a cycle $\mathfrak{C} \subseteq \hat{\mathcal{G}}(w)$ that is given by the induced graph on the nodes $[q_0], ..., [q_n] \in w/_{\hat{\mathcal{M}}}$ for some $\eta \in \mathbb{N}_{>0} \dot{\sqcup} \{\aleph_\alpha\}_{\alpha \in \mathbb{N}_0}$

$$\mathfrak{C} \;=\; \mathrm{G}[[q_0], ..., [q_n]].$$

Then $q_0 \mathcal{M} q_n \wedge q_n \mathcal{M} q_0$ because by going through the cycle we get these pathes. Hence $q_0 \hat{\mathcal{M}} q_n$ and therefore $[q_0] = [q_n]$ and $\eta = 0$ which is a contradiction. This shows that $\hat{\mathcal{G}}(w)$ is acyclic. The relation $\tilde{\mathcal{M}}$ is given by the induced relation $[p]\tilde{\mathcal{M}}[q] = p\mathcal{M}q$ for $p, q \in w$. This is well-defined because all elements in the equivalence class are connected to each other. It is easy to see that $\tilde{\mathcal{M}}$ is still transitive and reflexive. Further if $[p]\tilde{\mathcal{M}}[q] \wedge [q]\tilde{\mathcal{M}}[p]$ we would have the cycle $\mathfrak{C}' = \mathrm{G}[[p], [q]]$ hence $[p] = [q]$ which shows that antisymmetry holds. Let $(\mathcal{AC}^w_l)_{l \preceq \kappa}$ be the family of weakly connected components of $\hat{\mathcal{G}}(w)$. If for some $l_0 \preceq \kappa$ the component $\mathcal{AC}^w_{l_0}$ would contain a cycle then $\hat{\mathcal{G}}(w)$ would also contain a cycle which is a contradiction.
By the Well-ordering theorem there exists a well-ordering τ^l on each components $\mathcal{AC}^w_l$ and a minimal element in this order.

Example 2.2.1. The second possibility in Munchhausens trilemma is the **infinite regress**. Starting with a node $[p_0] \in w/_{\hat{\mathcal{M}}}$ it could look like

$$\begin{tikzcd} \cdots \arrow[r, bend left] & {[a_1]} \arrow[r, bend left] & {[a_0]} \arrow[r, bend left] & {[p_0]} \arrow[r, bend left] & \cdots \end{tikzcd} .$$

The third case of the trilemma is the **termination** at a node that is not the primal ground of the world.

$$\begin{tikzcd} & {[a_2]} \arrow[r, bend left] & {[a_1]} \arrow[r, bend left] & {[a_0]} \arrow[r, bend left] & {[p_0]} \arrow[r, bend left] & \cdots \\ \cdots \arrow[r, bend left] & {[b_1]} \arrow[r, bend left] & {[b_0]} \arrow[ur, bend right] & & & \end{tikzcd}$$

In this case the algorithm terminates at the node $[a_2]$ which is not the primal ground. However there are worlds with a primal ground.

Proposition 2.2.3. *Let w be a Wittgenstein world. Then there exists a primal ground $\mathfrak{g}$ of w if and only if $\hat{\mathcal{G}}(w)$ has a minimal element $[\mathfrak{g}]$ with respect to $\mathcal{M}$ and $[\mathfrak{g}] = \{\mathfrak{g}\}$.*

Proof. Let $\mathfrak{g}$ be the primal ground of w then by definition

$$\forall p \in w[p \in \{\mathfrak{g}\}^{\circlearrowleft}] \;\equiv\; \forall p \in w[\mathfrak{g}\mathcal{M}p] \;\equiv\; \forall [p] \in w/_{\hat{\mathcal{M}}}[[\mathfrak{g}]\hat{\mathcal{M}}[p]].$$

Hence $[\mathfrak{g}]$ is minimal in $w/_{\hat{\mathcal{M}}} = \mathrm{V}[\hat{\mathcal{G}}(w)]$. Assume $[\mathfrak{g}] = \{[\alpha_1], ..., [\alpha_\eta]\}$ for some $\eta > 1$. Then each α_j for $j \leq \eta$ would be a primal ground too which contradicts the uniqueness of $\mathfrak{g}$. Therefore $[\mathfrak{g}] = \{\mathfrak{g}\}$. Conversely let $[\mathfrak{g}] \in w/_{\hat{\mathcal{M}}}$ be the minimal element. Then by the above equivalence chain $\forall p \in w[p \in \{\mathfrak{g}\}^{\circlearrowleft}]$ and since $[\mathfrak{g}] = \{\mathfrak{g}\}$ uniqueness holds. In conclusion $\mathfrak{g}$ is the primal ground of w.

Definition 2.2.5. *Let w be a Wittgenstein world with w-existence predicate $\mathbb{e}_w$. The* ***Lewis world*** *is given by the mereological sum*

$$w \;=\; \boxplus_{\mathbb{e}_w[X]} X.$$

This definition can be found in Lewis'[L1] philosophical papers. We define the expression $X^{\boxplus|A|} = \boxplus_{\alpha\in A} X$ for index systems A with the convention $X^{\boxplus|\emptyset|} = \emptyset$. Further the w-**torsion** is the torsion in a given world w i.e. $\mathrm{Tor}_w(\varphi) = \mathrm{Tor}(\varphi) \cap \mathrm{Tor}(\mathbb{e}_w)$. Let w be a Lewis world and write $X\mathrm{P}Y \equiv (X \hookrightarrow Y)$ for the parthood-relation of classical mereology. The non-well foundation of the set-world corresponds to the infinite chain of mereological parthood

$$...\ \mathrm{P}\ w\ \mathrm{P}\ w\ \mathrm{P}\ w\ \mathrm{P}\ w\ \mathrm{P}\ w\ \mathrm{P}\ w\ \mathrm{P}\ w\ \mathrm{P}\ ...\ .$$

under the assumption $\mathbb{e}_w[X]$ for our Lewis world w.

Theorem 2.2.2. *Let w be a Wittgenstein world with w-existence predicate. Then the Lewis world decomposes into parts*

$$w \;=\; \boxplus_{\varphi} \mathfrak{p}_\varphi^{\boxplus|\mathrm{Tor}_w(\varphi)|}.$$

Proof. Let w be a world with $\mathbb{e}_w$. We show that the following is true

$$w = \boxplus_{\mathbb{e}_w[X]} X \overset{*}{=} \boxplus_{\substack{X \\ \Cap \\ \mathrm{Tor}(\mathbb{e}_w)}} \boxplus_{\substack{\varphi \\ \Cap \\ \mathrm{Ext}(X)}} \mathfrak{p}_\varphi \overset{!}{=} \boxplus_{\substack{\varphi \\ \mathrm{Tor}_w(\varphi)\neq\emptyset}} \mathfrak{p}_\varphi \overset{**}{=} \boxplus_{\varphi} \mathfrak{p}_\varphi^{\boxplus|\mathrm{Tor}_w(\varphi)|}.$$

The equality $(*)$ is easily derived with the part decomposition of each existing X. In order to see that the second equality $(**)$ holds note that $X^{\boxplus|\Lambda|} = X$ if and only if $\Lambda \neq \emptyset$. And therefore $X^{\boxplus|\Lambda|} = \emptyset$ if and only if $\Lambda = \emptyset$. If we drop each zero object we obtain the second equality. The crux of this proof is to show that (!) holds. Let $\mathfrak{p}_{\varphi_0} \hookrightarrow$ (LHS) be a non-zero part of the left hand side. Then $\exists X[\mathbb{e}_w[X] \wedge \varphi_0 X]$ and therefore $\mathfrak{p}_{\varphi_0} \hookrightarrow$ (RHS) because $\mathrm{Tor}_w(\varphi_0) \neq \emptyset$. Conversely let $\mathfrak{p}_{\varphi_0} \hookrightarrow$ (RHS) be a non-zero part of the right hand side. Therefore $\mathrm{Tor}_w(\varphi_0) \neq \emptyset$ which means that there exists some X with $\mathbb{e}_w[X]$ and $\varphi_0 X$. Hence $\mathfrak{p}_{\varphi_0} \hookrightarrow$ (LHS) and because the parts on both sides are essentially the same we have equality.

Definition 2.2.6. *Let $\mathfrak{Z} : a = t_0 < ... < t_{2n+1} = b$ be a partition of length $\eta \in \mathbb{N}_0 \dot\sqcup \{\aleph_0, 2^{\aleph_0}\}$ of the interval $[a, b]$ for $a, b \in \mathbb{R} \dot\sqcup \{\pm\infty\}$. Let w be a Wittgenstein world and X an object then the* ***genesis-nemesis-cycle*** *of X in w is denoted with $\mathfrak{Z}_w^X$ and the following equivalence holds*

$$\mathfrak{Z}_w^X \;=\; \bigwedge_{j=0}^{\eta} (\bigcirc_{t_{2j}} \breve{\mathcal{G}}_w^X \wedge \bigcirc_{t_{2j+1}} \breve{\mathcal{N}}_w^X)$$

*where $\bigcirc_{t_{2j}} \breve{\mathcal{G}}_w^X = \forall\epsilon \in (t_{2j}, t_{2j+1})[\bigcirc_\epsilon \mathbb{e}_w[X]]$ is the w-**genesis** of X at t_{2j}*

and $\bigcirc_{t_{2j+1}}\breve{\mathcal{N}}_w^X = \forall\epsilon \in (t_{2j+1}, t_{2(j+1)})[\bigcirc_\epsilon \neg \mathbb{e}_w[X]]$ *is the* w-***nemesis*** *of* X *at* t_{2j+1}.

The nemesis-genesis cycle is always of even length $|\mathfrak{Z}_w^X| \in 2\mathbb{N}_0$ and $\blacksquare\neg\mathbb{e}_w[X] \equiv (|\mathfrak{Z}_w^X| = 0)$ by convention. It seems reasonable to argue that $\forall X[\mathbb{e}_@[X] \to (|\mathfrak{Z}_@^X| = 2)]$ is true. Abstract objects will never vanish in the actual world except the world itself vanishes hence $\mathcal{A}^!_@[X] \to (\bigcirc_t\breve{\mathcal{N}}_@^X \equiv \bigcirc_t\breve{\mathcal{N}}_@^@)$. Concrete objects may certainly vanish before the world vanishes. There is a famous though experiment called the infinite monkey theorem that illustrates the idea of the nemesis-genesis cylces for conrete objects. Consider the example of the mountain K2$^{\mathcal{A}}$ at the actual time. We have $\mathcal{O}^!_@[\mathrm{K2}^{\mathcal{A}}]$. There is a chance that there will emerge another K2' on a different planet such that $\mathrm{Ext}(\mathrm{K2}^{\mathcal{A}}) \cup P = \mathrm{Ext}(\mathrm{K2}') \cup P'$ with $|P|$ and $|P'|$ small. But equality will never hold under certain physicalistic assumptions like $\mathrm{d}S > 0$. **Infinite monkey worlds** w are worlds with

$$\exists X[\mathbb{e}_w[X] \wedge (|\mathfrak{Z}_w^X| \neq 0, 2)].$$

In these worlds there exists elements X that come into identical w-existence multiple times just like Shakespeare's Hamlet which is written by a monkey hitting keys at random on a keyboard.

Definition 2.2.7.[R] *Let* $(\mathcal{W}, \mathcal{R}, \Vdash\!\vdash, (\mathcal{W}', \mathcal{L}, \Vdash))$ *be a tK-model where the starting time* $t_0^w \in \mathbb{R}$ *for each world* w *is fixed. The standard claim* $\bigcirc_{t_0^w}\breve{\mathcal{G}}_w^w$ *is called* ***anti-lastthurstdayism****. The opposite assertion* $\exists t \neq t_0^w[\bigcirc_t\breve{\mathcal{G}}_w^w]$ *is called* ***lastthurstdayism****.*

Advocates of this position argue that it is not decidable wether the world came into existence last thurstday if all the properties of the world at this thurstday (including the memories of all humans) came into existence too.

2.3. OBJECT THEORY

Zalta[Z2] gave a logical account to abstract and concrete objects in his axiomatic metaphysics. This chapter extends this theory to other important metaphysical entities such as real objects or universals. Despite the clarification of these concepts we want to prove theorems on their interplay.

Definition 2.3.1. *An object* X *is called* ***individual*** *if* $\mathrm{pl}^0(X)$*. Individuals are denoted with* $\mathcal{I}^![X]$*. An object* X *is called* ***universal*** *if* $\mathrm{pl}^1(X)$ *and universals are denoted with* $\mathcal{U}^![X]$*. If* $\exists n > 1[\mathrm{pl}^n(X)]$ *the object* X *is called* ***ultra-universal*** *and this is indicated with* $\mathcal{U}^{!^n}[X]$.

General concepts like 'goodness' or 'beauty' are considered to be universals too. There exists a bijection between the set of general concepts and the set of predicates that give rise to these concepts. By convention we treat general concepts and the corresponding predicates as equivalent.
The following theorem describes a **individual-universal-link**.

Theorem 2.3.1. *Individuals and Universals are linked via the maps* $\Sigma \mapsto \mathrm{Ext}(\Sigma)^{\mathcal{F}}$ *and* $\Sigma \mapsto \mathrm{Tor}(\Sigma)^{\mathcal{F}}$ *i.e. there exists a diagram*

$$\begin{tikzcd} \mathrm{Tor}(\mathcal{I}^!) \arrow[r, bend left] & \mathrm{Tor}(\mathcal{U}^!) \arrow[l, bend left] \end{tikzcd}.$$

Proof. Obvious.

Corollary 2.3.1. *There is an extension of the individual-universal-link to ultra-universals.*

$$\begin{tikzcd} \mathrm{Tor}(\mathcal{I}^!) \arrow[r, bend left] & \mathrm{Tor}(\mathcal{U}^!) \arrow[l, bend left] \arrow[r, bend left] & \mathrm{Tor}(\mathcal{U}^{!!}) \arrow[l, bend left] \arrow[r, bend left] & \mathrm{Tor}(\mathcal{U}^{!!!}) \arrow[l, bend left] \arrow[r, bend left] & \mathrm{Tor}(\mathcal{U}^{!^n}) \arrow[l, bend left] \end{tikzcd}$$

Proof. Obvious.

Consider the definitions of abstracta and concreta that can be found in the Principia Metaphysica. Zalta[Z3] denotes $\mathcal{E}![X]$ for objects X with spatiotemporal existence. From this he defines abstract objects with $\mathcal{A}![X] = \Box\neg\mathcal{E}![X]$ and concrete objects via $\mathcal{O}![X] = \Diamond\mathcal{E}![X]$. We give our own definition using t-logic.

Definition 2.3.2. *Define* ***spatiotemporal existence*** $\mathcal{E}^![X] = \exists\varphi[\overline{\varphi X}]$ *of an object* X*. An object* X *is called* ***abstract*** *iff* $\mathcal{A}^![X] = \Box\neg\mathcal{E}^![X]$ *and* ***concrete*** *iff* $\mathcal{O}^![X] = \Diamond\mathcal{E}^![X]$*.*

It is important to mention that $\mathcal{E}^![X]$ is a second-order formula because we quantify over predicates (1-relations). An easy deduction is that for abstract objects $\mathrm{Ext}(X)^{\underline{(.)}} = \mathrm{Ext}(X)$. Concreta (concrete objects) may have apriori properties, for example identity-assertions like $\chi_X X$. We may assume $\forall X[\mathcal{E}^![X] \equiv \mathcal{E}![X]]$.

Proposition 2.3.1. *In the* (tStd)*-axiomatic system there exists a chain of logical equivalences for all* X

$$\Box\mathcal{A}^![X] = \blacksquare\mathcal{A}^![X] = \mathcal{A}^![X] = \underline{\mathcal{A}^![X]} = (\mathcal{A}^![X])^\flat.$$

Proof. Let q be any assertion of the form $q = \Box p$ for another proposition p. Then in the (tStd)-axiomatic system

$$\Box\Box p = \blacksquare\Box p = \Box p = \underline{\Box p} = (\Box p)^\flat.$$

If we set $p = \mathcal{E}^![X]$ we get the desired chain.

Proposition 2.3.2. *In the* (tStd)*-axiomatic system there exists a chain of logical equivalences for all* X

$$\Diamond\mathcal{O}^![X] = \blacklozenge\mathcal{O}^![X] = \mathcal{O}^![X] = \overline{\mathcal{O}^![X]} = (\mathcal{O}^![X])^\sharp.$$

Proof. Let q be any assertion of the form $q = \lozenge p$ for another proposition p. Then in the (tStd)-axiomatic system

$$\lozenge\lozenge p = \blacklozenge\lozenge p = \lozenge p = \overline{\lozenge p} = (\lozenge p)^\sharp.$$

If we set $p = \mathcal{E}^![X]$ we get the desired chain.

Axiom 2.3.1. *Set for all objects* $\mathcal{U}^![X] \to \mathcal{A}^![X]$.

A conclusin of this axiom is that every concrete object is also an individual $\mathcal{O}^![X] \to \mathcal{I}^![X]$. The converse does not hold. If we denote e for Eulers number then $\mathcal{A}^![e]$ but $\neg\mathcal{U}^![e]$. Obviously Zaltas predicates are universals too representing the concepts of abstracness, concreteness and spatiotemporal existence $\mathcal{U}^![\mathcal{A}^!] \wedge \mathcal{U}^![\mathcal{O}^!] \wedge \mathcal{U}^![\mathcal{E}^!]$. Using second order Barcan formulas one obtains the formulas for $\mathfrak{U}^{\mathrm{op}} \in \{\lozenge, \blacklozenge\}$ and for all X

$$\mathfrak{U}^{\mathrm{op}}\mathcal{E}^![X] = \mathfrak{U}^{\mathrm{op}}\exists\varphi[\overline{\varphi X}] = \exists\varphi[\mathfrak{U}^{\mathrm{op}}\overline{\varphi X}] = \exists\varphi[\overline{\varphi X}] = \mathcal{E}^![X].$$

In the same manner we get for $\mathfrak{U} \in \{\square, \blacksquare\}$ and for all X

$$\mathfrak{U}\mathcal{E}^![X] = \mathfrak{U}\exists\varphi[\overline{\varphi X}] \to \exists\varphi[\mathfrak{U}\overline{\varphi X}] = \exists\varphi[\mathfrak{U}\varphi X \wedge \overline{\varphi X}].$$

Definition 2.3.3. *An object* X *is called* ***real*** *iff*

$$\mathbb{r}[X] \quad = \quad \exists\alpha[(\alpha X)! \multimap \alpha X].$$

The predicate $\mathbb{r}[X] = \exists\alpha[\neg(\alpha X)! \invamp \alpha X]$ is called **real predicate**. The implication $X! \multimap Y$ is the intuitionistic implication in linear logic. This definition follows Dummets[D] considerations. He pointed out that classical logic commits to metaphysical realism. The anti-real predicate is given by

$$\mathbb{a}[X] = \neg\mathbb{r}[X] = \forall\alpha[(\alpha X)! \otimes \neg\alpha X].$$

Let w be a Wittgenstein world then $\mathbb{r}_w[X] = (\mathbb{r}[X] \in w)$ is the w**-real predicate** and $w^{\mathbb{R}} = \mathrm{Tor}(\mathbb{r}_w)$ is the w**-set-reality**. Again if the reality is real ($\mathbb{r}_w[w^{\mathbb{R}}]$) we get non-well foundation

$$\ldots \;\in\; w^{\mathbb{R}} \;\in\; w^{\mathbb{R}} \;\in\; w^{\mathbb{R}} \;\in\; w^{\mathbb{R}} \;\in\; w^{\mathbb{R}} \;\in\; w^{\mathbb{R}} \;\in\; \ldots \quad .$$

As an example of anti-real objects consider the world $\mathcal{X} = \mathrm{cl}((\mathrm{ZFC}))$. We consider the following objects for some $\eta \in \mathbb{N}_0$

$$\mathfrak{C}_\eta \;=\; \iota\Sigma[\aleph_\eta \prec |\Sigma| \prec 2^{\aleph_\eta}].$$

Because the existence of these sets is independent of (ZFC) we have $\forall\eta[\mathbb{a}_{\mathcal{X}}[\mathfrak{C}_\eta]]$. Let w be a set world and φ a first-order predicate such that $\forall X[\varphi X \dot\vee \neg\varphi X]$ then the set world decomposes into $w = \mathrm{Tor}(\varphi) \dot\sqcup \mathrm{Tor}(\neg\varphi)$. Thus

$$\mathrm{Tor}(\mathbb{r}_w) \dot\sqcup \mathrm{Tor}(\mathbb{a}_w) \;=\; w \;=\; \mathrm{Tor}(\mathcal{O}^!_w) \dot\sqcup \mathrm{Tor}(\mathcal{A}^!_w).$$

Let φ be a first-order predicate we define the following object

$$\overset{\varphi}{X} \;=\; \iota\Sigma\{\{\varphi\Sigma \wedge |\mathrm{Ext}(\Sigma \setminus X)| = \min!\}\}.$$

If $\varphi = \mathcal{A}^!$ the object $\overset{\mathcal{A}^!}{X}$ is called **abstraction of** X and if $\varphi = \mathcal{O}^!$ this construct ist called **concretization of** X. For example if we denote ♁ for the earth we would get an abstraction-transformation like

$$♁ \rightsquigarrow r_0 \mathbb{S}^2$$

where $r_0 = r_{♁}$ is the radius of the earth and $\mathbb{S}^2$ is the two-dimensional sphere. Obviously $\varphi(X^\varphi)$ and $\varphi X \to (X = X^\varphi)$. With abuse of nomenclature we refer to X^φ as the **φ-fication** of X.

Theorem 2.3.2. *There exists an **abstract-concrete-link** that is given by*

$$\begin{tikzcd} \mathrm{Tor}(\mathcal{E}^!) \arrow[r, bend left] & \mathrm{Tor}(\mathcal{O}^!) \arrow[l, bend left] \arrow[r, bend left] & \mathrm{Tor}(\mathcal{A}^!). \arrow[l, bend left] \end{tikzcd}$$

Proof. The respective maps are given by $\mathrm{Tor}(\varphi) \to \mathrm{Tor}(\psi)$, $X \mapsto X^\psi$ for $\psi, \phi \in \{\mathcal{E}^!, \mathcal{O}^!, \mathcal{A}^!\}$.

Definition 2.3.4. *An object X is called **fiction** iff $\neg\mathbb{e}[\overset{\mathcal{E}^!}{X}]$ in this case we write $\mathbb{f}[X]$. If $\neg\mathbb{f}[X]$ the object X is called **faction**. Let w be a Wittgenstein world then $\mathbb{f}_w[X] = (\mathbb{f}[X] \in w)$ is the w-**fiction**.*

Fictionalism is then understood as the claim $\exists X[\mathbb{e}[X] \wedge \mathbb{f}[X]]$ and the opposite claim **factionalism** is given by $\forall X[\mathbb{e}[X] \to \neg\mathbb{f}[X]]$. Baudrillards[B] **hyperrealism** can be identified with the assertion $\forall X[\mathbb{f}[X] = \mathbb{r}[X]]$.
As an example of the usefulness of real and fictional predicates we may state the absorbing though experiment **brain-in-a-vat**. Let $m \in \mathbb{Z}$ and define inductive the **simulated w-reality of level** m

$$w_m^{\mathbb{R}} \quad = \quad \mathfrak{p}_{\mathbb{f}_{(.)}[w_{m-1}^{\mathbb{R}}]}$$

that is a world in which $w_{m-1}^{\mathbb{R}}$ is fiction whereas $w_0^{\mathbb{R}} = w^{\mathbb{R}}$. In our example $@_1^{\mathbb{R}}$ could be a brain-in-the-vat that simulates the actual world. Putnam gave another analysis of this problem in [Pu]. The worlds at negative levels are implicitly given starting with $w_0^{\mathbb{R}} \quad = \quad \mathfrak{p}_{\mathbb{f}_{(.)}[w_{-1}^{\mathbb{R}}]}$. There exists an infinite two-sided chains of fictional parthood

$$\ldots \overset{\mathbb{f}}{\hookrightarrow} w_{-2}^{\mathbb{R}} \overset{\mathbb{f}}{\hookrightarrow} w_{-1}^{\mathbb{R}} \overset{\mathbb{f}}{\hookrightarrow} w_0^{\mathbb{R}} \overset{\mathbb{f}}{\hookrightarrow} w_1^{\mathbb{R}} \overset{\mathbb{f}}{\hookrightarrow} w_2^{\mathbb{R}} \overset{\mathbb{f}}{\hookrightarrow} \ldots \quad .$$

where $w \overset{\mathbb{f}}{\hookrightarrow} w' \equiv \mathbb{f}_{w'}[w]$ for Wittgenstein worlds w, w'. Let w be a Lewis world. A **place** p is a part of the Lewis world $p \subseteq w$. Recall that an object X is called **atomic** iff $\neg\exists y[y\mathrm{PP}X]$. In this case we write AX. Let p be an atomic place then ${}^p\mathcal{E}^!$ is spatiotemporal existence at p. If p is not atomic we set ${}^p\mathcal{E}^! = \perp$.

Definition 2.3.5. *Let w be a Wittgenstein world. An object T is called*

*w-**trope** iff $\mathbb{tr}_w[T]$ and w-**detrope** iff $\neg\mathbb{tr}_w[T]$ whereas*

$$\mathbb{tr}_w[T] \;=\; \mathcal{U}^![T] \wedge \exists!_{p\subseteq w}[{}^p\mathcal{E}^!_w\overset{\mathcal{E}^!}{[T]}].$$

Trope-realism is given by the claim $\exists X[\mathbb{e}[X]\wedge\mathbb{tr}[X]]$ and **trope-nominalism** is represented by $\forall X[\mathbb{e}[X] \rightarrow \neg\mathbb{tr}[X]]$. The relation between polylocal universals X and tropes (unilocal universals) is twofold. The **tropification at a place** $p \subseteq w$ of a universal X is denoted by $X|_p$ and is defined as

$$X|_p \;=\; \overset{{}^p\mathcal{E}^![\overset{\mathcal{E}^!}{\cdot}]\wedge\mathbb{tr}_w}{X}.$$

Tropification at a place $p \subseteq w$ induces a restriction map $\mathrm{Tor}(\mathcal{U}^!) \xrightarrow{r_p} \mathrm{Tor}(\mathbb{tr}_w)$, $X \mapsto X|_p$ which is a projection onto the image $r_p(\mathrm{Tor}(\mathbb{tr}_w))$ of tropes with spatiotemporal existence at p. Moreover since each trope is a universal we have the inclusion $\mathrm{Tor}(\mathbb{tr}_w) \hookrightarrow \mathrm{Tor}(\mathcal{U}^!)$. All together if we write $\mathbb{tr}^p_w[T]$ for universals T with ${}^p\mathcal{E}^!_w[T^{\mathcal{E}^!}]$ there exists a sequence of maps

$$\{*\} \;\rightarrow\; \mathrm{Tor}(\mathbb{tr}^p_w) \overset{i_p}{\hookrightarrow} \mathrm{Tor}(\mathcal{U}^!) \overset{r_p}{\twoheadrightarrow} \mathrm{Tor}(\mathbb{tr}^p_w) \;\rightarrow\; \{*\}$$

such that $i_p \circ \pi_p = \mathrm{id}$. Conversely if we were given a set of places $\Lambda_X = \mathrm{Tor}({}^{(\cdot)}\mathcal{E}^![X^{\mathcal{E}^!}])$ for some universal X there exists a decomposition into tropes

$$X \;=\; \underset{\lambda\in\Lambda_X}{\boxplus} X|_\lambda.$$

3. T-LOGICAL AXIOMATIC METAPHYSICS

The leading idea of this inquiry is the development of a formal language which is able to express metaphysics. What we have seen so far is that one starts with a logical system for example classical logic, intuitionistic logic or their unification linear logic and adds philosophical operators.
The majority of metaphysical theories can then be identified with first-order assertions using metaphysical 1-relations such as $\mathbb{e}, \mathbb{r}, \mathbb{f}, \mathbb{tr}, \mathcal{A}^!, \mathcal{O}^!, \mathcal{E}^!, \mathcal{U}^!, \mathcal{I}^!, \ldots$ and this section is devoted to these statements.

3.1 DEMONS AND DETERMINISM

Usually the Laplacian Demon is rejected as a consequence of the indeterminism of the actual world. We go the other way around.

Definition 3.1.1. *Let w be a Wittgenstein whereas each $p \in w$ can be formalized in n-order predicate logic for some $n \in \mathbb{N}$. A model $\mathfrak{T}$ of w is said to be an **ideal theory of** w iff $\forall p[p \in w \equiv \mathfrak{T} \vDash p]$.*

Let $\mathfrak{D}$ be an ideal theory of the empirical part of the actual world $\overline{@} = @^{\overline{(\cdot)}}$. Our theory $\mathfrak{D}$ is a prototype of the Laplacian demon. Fix the actual time

$t_@ \in \mathbb{R}$ then a fortiori $\mathfrak{D}$ is able to predict true propositions in the future

$$\forall_{t>t_@} \forall a \in \overline{@}_t[\mathfrak{D} \vDash a]. \tag{36}$$

However $\mathfrak{D}$ with the above property is not enough to give us a demon as it is possible with this definition to fill $\mathfrak{D}$ with indeterministic propositions over all times. We denote $\mathfrak{D}^{\bigcirc_t} = \{p \,|\, \mathfrak{D} \vDash \bigcirc_t p\}$. Observe that there exists a cover of $\mathfrak{D}$ where the second cover is given as a refinement

$$\mathfrak{D} \quad = \quad \bigcup_{t \in \mathbb{R}} \mathfrak{D}^{\bigcirc_t} \quad = \quad \bigcup_{\mathfrak{g} \leq t \leq \mathfrak{n}} \mathfrak{D}^{\bigcirc_t}$$

where the world $\overline{@}$ comes into existence at $\mathfrak{g} \in \mathbb{R} \,\dot\sqcup\, \{\infty\}$ and vanishes at $\mathfrak{n} \in \mathbb{R} \,\dot\sqcup\, \{\infty\}$. Laplacian demons fulfill a temporal consistency property.

Definition 3.1.2. *Let $\mathfrak{D}$ be an ideal theory of a Wittgenstein world w. Iff*

$$\exists_{\mathfrak{t}<\mathfrak{n}_w}[\mathfrak{D} = \bigcup_{\mathfrak{g}<t\leq\mathfrak{t}} \mathfrak{D}^{\bigcirc_t}]$$

*such that $\bigcirc_{\mathfrak{n}_w} \breve{\mathcal{N}}_w^w$ then $\mathfrak{D}$ is called a **forward demon**. Conversely let $\mathfrak{g}_w \in \mathbb{R}$ with $\bigcirc_{\mathfrak{g}_w} \breve{\mathcal{G}}_w^w$. An ideal theory $\mathfrak{D}$ of w is called **backward demon** iff*

$$\exists_{\mathfrak{t}>\mathfrak{g}_w}[\mathfrak{D} = \bigcup_{\mathfrak{t}\leq t<\mathfrak{n}} \mathfrak{D}^{\bigcirc_t}].$$

A slide variation of (36) yields for a forward demon $\mathfrak{D}$ of $\overline{@}$

$$\forall_{T>\mathfrak{t}} \forall a[\bigcirc_T \overline{a} \quad \rightarrow \bigcup_{\mathfrak{g}<t\leq\mathfrak{t}} \mathfrak{D}^{\bigcirc_t} \vDash a].$$

Now indeterministic assertions, like the EPR paradox[EPR], that take place after $\mathfrak{t}$ can be predicted by $\mathfrak{D}$. Note that (Hardin $\boxplus$ Taylor)[HT] showed that using the axiom of choice one is able to predict the future of any system S up to Lebesgue zero sets. A backward demon $\mathfrak{D}$ has the property

$$\forall_{T<\mathfrak{t}} \forall a[\bigcirc_T \overline{a} \quad \rightarrow \bigcup_{\mathfrak{t}\leq t<\mathfrak{n}} \mathfrak{D}^{\bigcirc_t} \vDash a].$$

A Laplacian demon would be both backward and forward. Extreme cases are of course $\mathfrak{D} = \mathfrak{D}^{\bigcirc_\mathfrak{g}}, \mathfrak{D}^{\bigcirc_{t_@}}$.

Definition 3.1.3. *A Wittgenstein world w is said to be **forward/backward deterministic** iff there exists a forward/backward demon $\mathfrak{D}_w$ of w. Conversely iff there exists no such forward/backward demon $\mathfrak{D}_w$ the world w is said to be **forward/backward indeterministic**.*

The notion of determinism between propositions of a world w can be recovered with the binary relation $p \circlearrowleft q$. All factors that determine q are given as the torsion $\mathrm{Tor}((.) \circlearrowleft q)$. Lewis'[L1] famous account to contrafactual conditionals is another way to clearify the notion of causality. Write $\det(q) = \exists p[q \circlearrowleft p]$ then the Wittgenstein world w is deterministic if and only if $\forall p \in w[\det(p)]$. If one interprets actions as maps $\mathfrak{a} : [a, b] \to w$ for some interval $[a,b] \subseteq \mathbb{R} \dot\sqcup \{\pm\infty\}$ one could define an analoug binary relation

$$\mathfrak{a} \circlearrowleft \mathfrak{b} \quad \equiv \quad \forall_{t \in \mathfrak{a}^{-1}(w) \cap \mathfrak{b}^{-1}(w)} [\mathfrak{a}_t \circlearrowleft \mathfrak{b}_t]. \tag{37}$$

For this formula to make sense we assume $\mathfrak{a}^{-1}(w) \cap \mathfrak{b}^{-1}(w) \neq \emptyset$. The space of all actions of w in $[a,b]$ is given as $w^{[a,b]} = \{\mathfrak{a}_i\}_{i \in I}$ and is directed via (37). The initial action of w is given as the inverse limit

$$\lim_{i \in I} \mathfrak{a}_i.$$

Note that there is an identification $w^{\{*\}} \cong w$.

3.2. LOGICAL DIALECTICS

We start with the basic theories of ontology and may drop the reference to the Wittgenstein world since the world-relative view can be expressed with different models.
The term **logical dialectics** is just an indication that for each metaphysical assertion $\mathfrak{m}$ which is topic of our study we are also interested in $\neg\mathfrak{m}$.

Definition 3.2.1. *The claim $\exists X[\mathbb{e}[X]]$ is called **weak quineanism** the opposite claim $\forall X[\neg\mathbb{e}[X]]$ is called **noneism**[Pr]. The assertion $\forall X[\mathbb{e}[X]]$ is called **quineanism**[Q2] and the converse $\exists X[\neg\mathbb{e}[X]]$ is called **meinongianism**[M]. If an object X is fixed we call the view represented by $\mathbb{e}[X]$ X-**quineanism**. Analogously the claim $\neg\mathbb{e}[X]$ will be called X-**meinongianism**.*

The existence predicate is somewhat fundamental to most other views in M-logic. Note that if $\mathrm{G}X = \forall\varphi[\varphi X = \mathcal{P}(\varphi)]$ is Gödels god-predicate and $\mathbb{g} = \mathfrak{p}_{\mathrm{G}}$ the position $\mathbb{g}$-quineanism is just **theism** and $\mathbb{g}$-meinongianism ist **atheism**.

Definition 3.2.2. *The claim $\exists X[\mathbb{e}[X] \wedge \mathbb{r}[X]]$ is called **weak realism** and the opposite proposition $\forall X[\mathbb{e}[X] \to \mathbb{a}[X]]$ is called **strong anti-realism**. The assertion $\forall X[\mathbb{e}[X] \to \mathbb{r}[X]]$ is called **realism** and the converse $\exists X[\mathbb{a}[X] \wedge \mathbb{e}[X]]$ is called **anti-realism**.*

Definition 3.2.3. *The claim $\exists X[\mathbb{e}[X] \wedge \mathcal{O}^![X]]$ is called **weak nominalism** and the opposite claim $\forall X[\mathbb{e}[X] \to \mathcal{A}^![X]]$ is called **strong platonism**. The assertion $\forall X[\mathbb{e}[X] \to \mathcal{O}^![X]]$ is called **nominalism** and the converse $\exists X[\mathbb{e}[X] \wedge \mathcal{A}^![X]]$ is called **platonism**.*

As we defined $\mathcal{O}^![X] = \Diamond\exists\alpha[\overline{\alpha X}]$ the view held by nominalist depends on the modal theory that is applied. Usually nominalists tend to argue for modal actualism but this is not included in our definition.

Definition 3.2.4. *The claim $\exists X[\mathbb{e}[X] \wedge \mathcal{I}^![X]]$ is called **weak universal-nominalism** and the opposite claim $\forall X[\mathbb{e}[X] \to \mathcal{U}^![X]]$ is called **strong universal-realism**. The assertion $\forall X[\mathbb{e}[X] \to \mathcal{I}^![X]]$ is called **universal-nominalism** and the converse $\exists X[\mathbb{e}[X] \wedge \mathcal{U}^![X]]$ is called **universal-realism**.*

Definition 3.2.5. *The claim* $\forall X[\text{e}[X] \to \text{e}_@[X]]$ *is called* ***ontological modal actualism*** *and the converse* $\exists X \exists w[(w \neq @) \wedge \text{e}_w[X]]$ *is called* ***ontological modal eternalism****. The claim* $\forall X[\text{e}[X] \to \mathcal{A}(\text{e}[X])]$ *is called* ***ontological temporal actualism*** *and the converse* $\exists X[\text{e}[X] \wedge \neg\mathcal{A}(\text{e}[X])]$ *is called* ***ontological temporal eternalism****.*

Proposition 3.2.1. *Noneism implies strong anti-realism, realism, strong platonism, nominalism, strong universal-realism, universal-nominalism, factionalism, essentialism, trope-nominalism, ontological temporal actualism and ontological modal actualism.*

Proof. This is obvious because the antecedens $\text{e}[X]$ is always false.

Proposition 3.2.2. *Strong universal-realism implies strong platonism. Weak nominalism implies weak universal-nominalism. Nominalism implies universal-nominalism and universal-realism implies platonism.*

Proof. This is an easy deduction of our axiom $\mathcal{U}^![X] \to \mathcal{A}^![X]$.

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