Quantifier variance without collapse

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1 Introduction

In the land of metaphysics, there is a debate raging about the thesis of "quantifier variance." As one metaphysician puts it, quantifier variance asserts that, "there are many meanings for the existential quantifier that are equally neutral and equally adequate for describing all the facts" (Wasserman, 2013). Now, this thesis isn't logically precise — and so we shouldn't expect it to be easily refutable by means of formal logic. And yet, several prominent philosophers have claimed just that, namely, that quantifier variance is formally incoherent. Their argument — called the "collapse argument" — runs roughly as follows: if you have two existential quantifiers, then by eliminating the first and introducing the second, it follows that they are really the same quantifier (Dorr, 2014; Hale and Wright, 2009; Harris, 1982; Rossberg, 2008).

The conclusion of the collapse argument states that in any system of first-order logic, there is only one existential quantifier. But I'm not sure then how anyone could have made this argument with a straight face, for its conclusion is known to be false: there *are* consistent systems of first-order logic with multiple existential quantifiers. In what follows, then, I'll remind the reader of the existence of many-sorted first-order logic, which permits multiple existential quantifiers, and which stands in no danger from the collapse argument. I then look at the most famous example of quantifier variance: Putnam's example of the the mereological nihilist and the universalist. I show that, in one precise sense, Putnam was absolutely correct in claiming that these two theories are equivalent, even though they disagree on the number of existing objects. The upshot of my investigation is a novel explication and defense of quantifier variance.

To date, none of the numerous discussions of quantifier variance have noted the relevance of many-sorted logic. This paper is intended to remedy this unfortunate omission.[1]

2 Carnap and the Polish logician

Following Putnam's famous example, let's suppose that Carnap is a mereological nihilist; i.e. he thinks that only simples exist. When Carnap says that something exists, he really means that some simple exists. In contrast, the Polish logician is a mereological universalist; i.e. he believes that any composite of simples also exists. Of course, the Polish logician can translate Carnap's existential quantifier $\exists^S$ into his unrestricted existential quantifier $\exists^U$ as follows:

$$\exists^S x\, \phi(x) \;\equiv\; \exists^U x\, (S(x) \wedge \phi(x)),$$

where $S(x)$ is the predicate "x is a simple."

The collapse argument is supposed to run as follows (using here something like Lemmon's system of natural deduction).

$$\begin{array}{llll} 1 & (1) & \exists^U x\, \phi(x) & \text{A} \\ 2 & (2) & \phi(c) & \text{A} \\ 2 & (3) & \exists^S x\, \phi(x) & 2\ \text{EI} \\ 1 & (4) & \exists^S x\, \phi(x) & 1,2,3\ \text{EE} \end{array}$$

For example, if $\phi(x)$ is the property of having a proper part, then we have shown that Carnap is committed to the existence of something with a proper part.

But the advocate of quantifier variance will say that this "proof" contains an error. If this proof is supposed to be carried out in the Polish logician's language, then the EI rule for (the translation of) the nihilist's quantifier $\exists^S$ will not be the same as the EI rule for the universalist's quantifier $\exists^U$. Rather, the nihilist's EI rule would look like this:

$$\frac{S(c) \wedge \phi(c)}{\exists^S x\, \phi(x)}$$

In other words, the nihilist shouldn't take $\phi(c)$ alone as sufficient for establishing $\exists^S x\, \phi(x)$. He should also require $S(c)$ to be established before he grants grants existence in his sense.

Let's make an important distinction between two questions:

[1] For more philosophically sophisticated responses to the collapse argument, see (Warren, 2015) and (Turner, 2012). My aim here is primarily pedagogical: to remind philosophers of sorted logic, and to explain how it resolves the quantifier variance debate.

1. Can a single language (or theory) have two distinct existential quantifiers?

2. Can two different languages (or theories) have, in some sense, different existential quantifiers?

We'll soon see that these questions are related. However, the collapse argument couldn't possibly provide a negative answer to the second question. For, different theories might be formulated in different languages, and it simply doesn't make sense to write a proof that begins in one language and ends in another. So, can the collapse argument provide a negative answer to the first question? No, it cannot: there are consistent systems of natural deduction with multiple distinct existential quantifiers.

There are many formal and philosophical advantages in adopting a framework that allows many different existential quantifiers. And there is a simple device for keeping track of different quantifiers and their distinct inference rules. That device is *sorts*. To introduce distinct quantifiers is to introduce distinct sort symbols. And if we're keeping track of sorts, we also need to divide our variables and constant symbols into sorts, and we need to specify the sorts of the arguments of our relation symbols.

To be precise, a many-sorted **signature** Σ is a set of sort symbols, predicate symbols, function symbols, and constant symbols. Σ must have at least one sort symbol. Each predicate symbol $p \in \Sigma$ has an **arity** $\sigma_1 \times \ldots \times \sigma_n$, where $\sigma_1, \ldots, \sigma_n \in \Sigma$ are (not necessarily distinct) sort symbols.[2] Likewise, each function symbol $f \in \Sigma$ has an **arity** $\sigma_1 \times \ldots \times \sigma_n \to \sigma$, where $\sigma_1, \ldots, \sigma_n, \sigma \in \Sigma$ are again (not necessarily distinct) sort symbols. Lastly, each constant symbol $c \in \Sigma$ is assigned a sort $\sigma \in \Sigma$. In addition to the elements of Σ we also have a stock of variables. We use the letters x, y, and z to denote these variables, adding subscripts when necessary. Each variable has a sort $\sigma \in \Sigma$.

A **Σ-term** can be thought of as a "naming expression" in the signature Σ. Each Σ-term has a sort $\sigma \in \Sigma$. The Σ-terms of sort σ are recursively defined as follows. Every variable of sort σ is a Σ-term of sort σ, and every constant symbol $c \in \Sigma$ of sort σ is also a Σ-term of sort σ. Furthermore, if $f \in \Sigma$ is a function symbol with arity $\sigma_1 \times \ldots \times \sigma_n \to \sigma$ and $t_1, \ldots, t_n$ are Σ-terms of sorts $\sigma_1, \ldots, \sigma_n$, then $f(t_1, \ldots, t_n)$ is a Σ-term of sort σ. We will use the notation $t(x_1, \ldots, x_n)$ to denote a Σ-term in which all of the

[2]The symbol $\sigma_1 \times \cdots \times \sigma_n$ has no intrinsic meaning. To say that, "p has arity $\sigma_1 \times \cdots \times \sigma_n$" is simply an abbreviated way of saying that p can be combined with n terms, whose sorts must respectively be $\sigma_1, \ldots, \sigma_n$.

variables that appear in t are in the sequence $x_1, \ldots, x_n$, but we leave open the possibility that some of the x_i do not appear in the term t.

A **Σ-atom** is an expression either of the form $s(x_1, \ldots, x_n) = t(x_1, \ldots, x_n)$, where s and t are Σ-terms of the same sort $\sigma \in \Sigma$, or of the form $p(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are Σ-terms of sorts $\sigma_1, \ldots, \sigma_n$ and $p \in \Sigma$ is a predicate of arity $\sigma_1 \times \ldots \times \sigma_n$. The **$\Sigma$-formulas** are then defined recursively as follows.

- Every Σ-atom is a Σ-formula.
- If ϕ is a Σ-formula, then $\neg\phi$ is a Σ-formula.
- If ϕ and ψ are Σ-formulas, then $\phi \to \psi$, $\phi \wedge \psi$, $\phi \vee \psi$ and $\phi \leftrightarrow \psi$ are Σ-formulas.
- If ϕ is a Σ-formula and x is a variable of sort $\sigma \in \Sigma$, then $\forall_\sigma x\phi$ and $\exists_\sigma x\phi$ are Σ-formulas.

A **Σ-sentence** is a Σ-formula that has no free variables.

If we abide by the syntactic laws of many-sorted logic, then the collapse argument doesn't even get off the ground. In particular, in order for $\exists^U x\, \phi(x)$ to be a wff, the sort of the variable x must match the sort of the quantifier $\exists^U$. i.e. you can't put any quantifier in front of any variable. What's more, if y is a variable of different sort than x, then $\phi(y)$ will not be wff. And an instance $\phi(c)$ of $\phi(x)$ is a wff only if the name c is of the same sort as the variable x. And if we then wish to apply existential generalization to $\phi(c)$, we must again replace c with a variable of the same sort. The grammar of many-sorted logic forbids replacing c with a variable of sort S. To repeat, the string of symbols $\exists^S x\, \phi(x)$ is not a wff, because the variable x is of type U, and the quantifier $\exists^S$ only pairs with variables of type S. In short, from $\phi(c)$, we are not licensed to infer $\exists^S x\, \phi(x)$, for the simple reason that $\exists^S x$ is not a proper quantifier. On the other hand, if y is a variable of sort S, then neither $\phi(y)$ nor $\exists^S y\, \phi(y)$ are wffs.[3]

Some might object, however, that many-sorted logic is beside the point, since it is reducible to unsorted logic (see Barrett and Halvorson, 2016b; Bell et al., 2001; Manzano, 1996). But beware the word "reducible," which hides a multitude of sins! Yes, for each many-sorted theory T there is an equivalent unsorted theory T^*. However, this fact doesn't entail that the choice between single- and many-sorted logic is philosophically otiose. For example, Hook (1985) shows that there are many-sorted theories T_1 and T_2 such that T_1 is

[3]Sider's (2007) rebuttal of the collapse argument is similar, but without explicitly adopting many-sorted logic.

interpretable into T_2, but T_1^* is *not* interpretable in T_2^*. Thus, for example, a user of many-sorted logic might grant that T_1 is reducible to T_2, but a user of single-sorted logic would say that it's not. Similarly, Barrett and Halvorson (2016a) give examples of many-sorted theories T_1 and T_2 such that T_1 and T_2 are equivalent, but T_1^* and T_2^* are inequivalent.[4] (We will give another such example below, namely the theories of the mereological nihilist and the universalist.) It seems to me, in fact, that the debate about quantifier variance is bound up tightly — if it's not synonymous with — the question of whether to adopt a single- or many-sorted logical framework. What's clear is that if the collapse argument presupposes a single-sorted framework, then it cannot be a good argument against a many-sorted framework.

3 Quantifier variance vindicated

Is Carnap's theory equivalent — in a many-sorted sense — to the Polish logician's theory? The answer depends on two things: the standard of equivalence that we adopt, and the precise formulation of these two theories. In this section, I'll advocate for a notion of equivalence in many-sorted logic called "Morita equivalence." And then I'll explain why the nihilist's and the universalist's theories are Morita equivalent.

Let Carnap's theory T_C be the single-sorted theory in signature $\Sigma = \{\sigma\}$ which says that there are exactly three things. (Here σ is a sort symbol, and Carnap's quantifier should be written $\exists_\sigma$ to indicate quantification over sort σ.) Suppose that we take the Polish logician's theory T_P to be a theory in signature $\Sigma_P = \{\sigma', P\}$, where P is a binary relation symbol, and where T_P says that:

1. P is a mereological relation of parthood (hence irreflexive, transitive, and asymmetric).

2. There are exactly three simples, exactly three composites of two simples, and exactly one composite of three simples.

Note, crucially, that Carnap and the Polish logician have different existential quantifiers $\exists_\sigma$ and $\exists_{\sigma'}$. There is no *a priori* reason to think that Carnap and the Polish logician mean the same thing by "there exists".

As everyone knows, Carnap's theory can be translated into the Polish logician's theory: in particular, Carnap's quantifier $\exists_\sigma$ is translated as the

[4]Caution: I'm using the word equivalent in two different senses here. For T_1 and T_2, I mean Morita equivalent. For T_1^* and T_2^*, I mean not definitionally equivalent.

restriction of the Polish logician's quantifier $\exists_{\sigma'}$ to the simples. Let's think of this translation as a map $F : T_C \to T_P$.

But is there a translation of the Polish logician's theory into Carnap's theory? Yes, but we have to be a bit more sophisticated. Can Carnap define more quantifiers out of his original existential quantifier? Yes, first he can define an existential quantifier for pairs: $\exists x \exists y$. Is $\exists x \exists y$ a legitimate quantifier? I cannot see why it wouldn't be. It has legitimate introduction and elimination rules. Second, Carnap can define an equivalence relation on those pairs of elements that are related by a permutation, and he can define a new quantifier $\exists'$ that ranges over the equivalence classes of these pairs. Is this $\exists'$ a legitimate quantifier? I contend that it is, but let me be very clear: I am *not* supposing that Carnap is willing to quantify over classes. No, what I'm saying is that Carnap is already implicitly committed to quantification over equivalence classes that are definable in his theory T_C. Given that this claim might be seen as tendentious, I'll explain in further detail.

There is a formula $\theta(x, y)$ in Carnap's language, and Carnap's theory T_C entails that $\theta(x, y)$ is an equivalence relation (reflexive, symmetric, transitive). And now Carnap can count the number of equivalence classes of $\theta(x, y)$ *without going beyond the resources of first-order logic.* For example, the claim that there is only one equivalence class is expressed by the formula $\forall x \forall y \theta(x, y)$. Similarly, the claim that there are at least two equivalence classes is expressed by the formula $\exists x \exists y \neg\theta(x, y)$. In fact, numerical sentences involving θ have exactly the same form as standard numerical claims involving the equality relation.

I contend then that introducing a quantifier over definable equivalence is merely making explicit something that is already implicit in T_C. Hence, introducing a quantifier over definable equivalence classes should *not* be taken as a genuine expansion of ontology. A genuine expansion of ontology is tantamount to adopting a *new* theory. But I'm not suggesting that Carnap adopt a new theory. I'm saying that his current theory is quantifying over definable equivalence classes.

I suspect that some people will be puzzled by what I just said. What exactly is Carnap's ontology? How many things exist according to T_C? Three, or more than three? What I am recommending is that there is no determinate answer to this question. I am recommending that the number of existing individuals is *not* invariant under theoretical equivalence — which means that there is not always a definite answer about how many entities a theory posits.

Back to the main line of argument, Carnap can unify the original domain of simples with the domain of composites of simples, giving him a single existential quantifier $\exists^U$ over everything. That is, Carnap can extend his original theory T_C by making appropriate definitions to a theory T_C^+ that is logically equivalent to the Polish logician's theory T_P. In the terminology of Barrett and Halvorson (2016a), T_C and T_P are Morita equivalent theories.

So, yes, on a plausible account of equivalence of theories, Carnap and the Polish logician have equivalent theories. To accept one of these theories is to accept the other. A similar result was proven by Warren (2015). However, Warren's result shows only mutual relative interpretability of T_P and T_C, which is known to be weaker than Morita equivalence. We have independently good reasons for taking Morita equivalence as the best explication of theoretical equivalence.

4 Conclusion

Before concluding, let me address an obvious objection to the things I've said above. I suspect the opponent of quantifier variance will say something like:

> (U) For any theory T, there is always one maximal, unrestricted existential quantifier $\exists$.

But even if we accept (U), the collapse argument doesn't go through. The reason, once again, is that even if a theory T has an unrestricted existential quantifier $\exists$, T might be equivalent to a theory T^+ whose existential quantifier $\exists^+$ is a proper extension of $\exists$. (That happens precisely when the elements in the domain of $\exists^+$ are definable from the elements in the domain of $\exists$.) In fact, that's exactly what's going on with the mereological nihilist and universalist. The nihilist's theory has an unrestricted existential quantifier. And his theory is equivalent to the universalist's theory, even though the universalist's existential quantifier $\exists^+$ is a proper extension of the nihilist's. But does this mean that the nihilist's quantifier wasn't really unrestricted in the first place? No, not unless you're willing to say the same about the universalist's quantifier, viz. that it's not really unrestricted. For the universalist's theory T_P also permits further definition of sorts, so that T_P is equivalent to a theory T_P^+ that says there are more than seven things.

Shouldn't we be looking, then, for the completely unrestricted theory, with its completely unrestricted quantifier? Isn't that theory really the true theory to which both the nihilist and the universalist are ultimately committed? But there are two problems with this suggestion. The first problem is that all these theories really are *equivalent*, which means that no particular formulation is to be preferred for ontological purposes. The second problem is that there may not be any such maximal, and completely unrestricted theory.

Postscript: my suggestion to use many-sorted logic is similar to Turner's (2012) suggestion to use free logic. The effect on the collapse argument is the same: $\phi(c)$ does not entail $\exists x\, \phi(x)$ unless c and x are of the same sort. The advantage of many-sorted logic is that it has proved independently useful in the sciences (see Makkai and Reyes, 1977; Meinke and Tucker, 1993), and it's expressively equivalent to classical first-order logic.

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