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# Redrawing Kant's philosophy of mathematics

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This essay offers a strategic reinterpretation of Kant's philosophy of mathematics in *Critique of Pure Reason* via a broad, empirically based reconception of Kant's conception of drawing. It begins with a general overview of Kant's philosophy of mathematics, observing how he differentiates mathematics in the *Critique* from both the dynamical and the philosophical. Second, it examines how a recent wave of critical analyses of Kant's constructivism takes up these issues, largely inspired by Hintikka's unorthodox conception of Kantian intuition. Third, it offers further analyses of three Kantian concepts vitally linked to that of drawing. It concludes with an etymologically based exploration of the seven clusters of meanings of the word drawing to gesture toward new possibilities for interpreting a Kantian philosophy of mathematics.

## Introduction

In this essay, I offer a strategic reinterpretation of Kant's philosophy of mathematics in *Critique of Pure Reason* via a broad, empirically based reconception of Kant's conception of *drawing*.[1] I will begin by sketching a general overview of Kant's philosophy of mathematics, observing how he differentiates mathematics in the *Critique* from both the dynamical and the philosophical, in the former case by mathematics' restriction to and legislation of the spatiotemporal world of appearances, and in the latter case by its method of synthetic a priori construction. Second, I will examine how a recent wave of critical analyses of Kant's constructivism takes up these issues, largely inspired by Jaako Hintikka's unorthodox conception of Kantian intuition. Third, I will offer further analyses of three Kantian concepts vitally linked to that of drawing, namely the 'manifold', 'schema' and 'imagination', ultimately characterising the manifold as a kind of homogeneous plurality mathematised by the imagination via the schemata. Fourth, I will conclude with an etymologically based exploration of the seven clusters of meanings of the word *drawing*—traction, attraction, extraction, protraction, construction, self-motion and its various adverb constructions (i.e. drawing up, drawing on, etc.)—to gesture toward new possibilities for interpreting Kant's critical mathematics.

In providing an overview of the role of mathematics and the mathematical, I will begin with mathematics in relation to physics and the dynamical. Kant first mentions mathematics in the first *Critique* in the preface to the A-edition, where he states that mathematics and physics are two examples of sciences 'whose grounds are well-laid' (A p. xi, n). This dyad quickly becomes a dichotomy that is repeated at three significant moments in the *Critique*. The first moment is in the Table of the Categories. The first two groups of categories Kant describes as 'mathematical', whereas the second two groups he terms 'dynamical' (i.e. relating to physics as the science of motion) (B p. 110). I will return to this distinction toward the end of my investigation, when I consider in greater detail the connections between mathematics and the imagination.

The second instance of this mathematics/physics dichotomy is in the table of the principles of the understanding. The first two principles are again described as 'mathematical' (in distinction to the latter two 'dynamical' ones), although only, according to Kant, because they justify 'applying

1 The German verb *ziehen* possesses an almost identical series of definitions to that of the English word *to draw*, which is itself of Old Germanic and Nordic extraction.

mathematics to appearances' (A p. 178, B p. 221). In other words, the mathematical principles explain how reality *is* such that mathematical analysis is objectively applicable to it, which is because the mathematical principles 'could be generated accordance with rules of a mathematical synthesis', and because they are 'constitutive' of possible experience (A p. 179, B p. 222). Along with the dynamical principles, the mathematical principles contain 'nothing but only the pure schema, as it were, for possible experience' (A p. 237, B p. 296).

Mathematics enjoys a particularly privileged role in the first principle of the understanding, which Kant calls the 'Axioms of Intuition'. This principle (as stated in the A-edition) is that 'All appearances are, as regards their intuition, **extensive magnitudes**.'[2] (A p. 162). In the B-edition it is restated as '**All intuitions are extensive magnitudes**' (B p. 202). Both mean for Kant that 'the representation of the parts makes possible the representation of the whole (and therefore necessarily precedes the latter' (A p. 162, B p. 203). (This stands in exact opposition to Kant's understanding of space, in which the concept of the whole is always prior to any part.)

Kant then offers the following example, significant because it contains the first use of the German word *ziehen*, 'to draw', in the sense of construction (according to the *Oxford-Duden German Dictionary* 2001).

> I cannot represent to myself a line, no matter how small it may be, without *drawing* it in thought, i.e., successively *generating* all its parts from one point, and thereby first *sketching* this intuition. (A p. 162, B p. 203, emphasis added)

Note that in this first appearance of the verb 'to draw', the word is both accompanied by another verb synonymous with drawing, and also expressed as synonymous with producing or creating. The latter point is also true with the following geometric example:

> If I say: "With three lines, two of which taken together are greater than the third, a triangle can be *drawn*," then I have here the mere function of the productive imagination, which *draws* the lines greater or smaller, thus allowing them to abut at any arbitrary angle. (A p. 164, B p. 205, emphasis added)

In Norman Kemp Smith's translation, the word 'drawn' above is instead rendered by 'described', which of course refers to *drawing* a circle, and which is thereby conjoined to the drawing of a straight line as a second exemplary activity of mathematical construction. The primary difference between the two is that describing a circle, at least for technical purposes, necessitates the use of an empirical device, namely, a protractor (literally: *drawer-out-er*). Perhaps this is the reason why the word *describing* is more rarely used in the sense of drawing or tracing out, and even then it is used almost exclusively in the case of circular shapes, as in 'the plane described a circle in the air', or 'the figure skater described a figure-eight on the ice'. On the other hand, if one considers the conventional use of 'describing' in academic prose, as a metaphorical drawing of a picture with words, as in 'Kant describes the concept in two ways', then description *qua* drawing appears everywhere. Also interesting in the above passage is that the drawing of the lines of the triangle is directly referred to as the 'function of the productive imagination'.

The last two instances of the mathematics/physics dichotomy are both to be found in the Transcendental Dialectic, in the Antinomies. The first concerns Kant's distinction between 'nature' and 'world', the latter of which is defined as 'the mathematical whole of all appearances and the totality of their synthesis' and the former of which consists of this mathematical 'world' 'insofar as it is considered as a dynamic whole' (A pp. 418–419, B pp. 446–447). Given that (1) mathematics is based in intuition, (2) human intuition is sensible, and (3) sensibility is merely our form of receptivity, then this definition would suggest that the entire 'world', inasmuch as it is 'mathematical', is limited to the self-constitutive world of appearances. Kant afterwards makes

2 All boldface in my essay follows Guyer and Woods' convention for indicating Kant's own indications of emphasis in his original text. My own emphases are indicated with italicisation. Thus, any word or phrase that is both boldfaced and italicised is being emphasised by both Kant and myself.

the interesting claim that ‘nowhere in mathematics do false assertions disguise themselves and make themselves invisible; for mathematical proofs always have to proceed along the lines of pure intuition, and indeed always through a self-evident synthesis.’ (A p. 424, B p. 452)

This means that the sensible basis of mathematics might more accurately be described as a sensible *process*, a visible synthesis as opposed instead of an inert, and thereby potentially deceptive, product. In other words, mathematical synthesis *qua* visible process, *draws attention* to itself in a disclosive, truth-ensuring way.

In the final instance of the dichotomy, repeating the structure applied to the categories and the principles, Kant describes the first two antinomies as ‘mathematical’ and the second two as ‘dynamic’. Toward the end of the section on the antinomies, Kant fleshes out these dichotomies at greater length, emphasising that the mathematical deals with ‘homogeneous’—that is, sensible—elements whereas the dynamical deals with ‘heterogeneous’—that is, *both* sensible *and* intelligible elements. Later, in the Ideal of Pure Reason, Kant offers the complementary observation that the mathematical antinomies, as opposed to the dynamical ones, are concerned ‘only with the combination of parts into a whole, or with the dissolution of a whole into its parts’, which is another way of saying it assembles and disassembles homogeneous spatiotemporal units. Put differently, mathematics constructs by either drawing [extracting] manifoldness from unity or drawing [reducing] a manifold into unity (A p. 560, B p. 588).

In distinguishing between the mathematical notion of *infinity* and the dynamical notion of *indefiniteness*, and valorising indefiniteness over infinity, Kant writes that ‘For although when it is said, “Draw a line” it obviously sounds more correct to add *in indefinitum* than if it were said *in infinitum*’, because we basically mean “‘Extend it as far as you want’” (A p. 511, B p. 539). The interesting thing here is that drawing a line, insofar as Kant intends to suggest geometric construction, falls under the realm of mathematics, yet it is physics’ *indefiniteness* that is described as preferable to mathematics’ *infinity* as the more accurate conception of open-ended drawing.

Could one perhaps say that drawing, which is both a dynamic, physical activity and also an activity for the sake of mathematics, finally bridges the gap between the mathematical and the dynamical? Perhaps this passage suggests that mathematics, though structurally a priori, is only fully realised when it is drawn [pulled] through drawing [construction] into the dynamical, physical world. This could be explored even further in terms of the freedom of the subject to draw or not draw, to engage in mathematical activity or not, to think or not; in this way, drawing as empirical activity would draw [extract] mathematics from its a priori *world* into a posteriori *nature*, complete with intelligibility’s freedom, but this is beyond the scope of this paper.

I now turn to Kant’s discussions on the distinctions between mathematics and philosophy. Lisa Shabel argues that the most significant historical fact against which to read Kant on mathematics is the ‘rationalist philosophy of mathematics’ of, in particular, ‘Leibniz, Wolff, and Mendelssohn’ (Shabel 2006: p. 98). Shabel identifies the ‘central tenet’ of this position as the following: ‘mathematical truth and metaphysical truth are equally certain due to their common method of reasoning, namely, conceptual analysis’ (Shabel 2006: p. 96).

For Kant, on the other hand, mathematical certainty requires a move beyond conceptual analysis to what he terms conceptual construction, an act ‘necessitated by the synthetic character of mathematical propositions’ (as opposed to the analytic character of philosophical propositions that limit it to conceptual analysis). For the former, one ‘must pass beyond [the concept] to properties which are not contained in this concept, but yet belong to it’.[3]

Kant’s most complete explication of mathematical construction is found in the following excerpt from the Discipline of Pure Reason:

> But to **construct** a concept means to exhibit *a priori* the intuition corresponding to it. For the construction of a concept, therefore, a **non-empirical** intuition is required, which

3 Shabel (2006: p. 580). This calls to mind Badiou’s terminology of *belonging* versus *inclusion* in the set-theory based ontology of *Being and Event*. Something belongs to a set if it presented there but not represented as being presented there; something is included in a set if it is represented. Mathematical construction, on this model, takes something belonging to (or ‘presented’ by) a concept and causes it to be included in (or ‘represented’ by) exhibition in intuition (Badiou 2007: pp. 81–84).

> consequently, as intuition, in an **individual** object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation the universal validity for all possible intuitions that belong under the same concept. Thus I construct a triangle by exhibiting an object corresponding to this concept, either through mere imagination, in pure intuition, or on paper, in empirical intuition, but in both cases completely *a priori*, without having had to borrow the pattern for it from any experience…for in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations…are entirely indifferent, which do not alter the concept of the triangle (A pp. 713–714, B pp. 741–742).

Put differently, the act of construction—which is the 'non-empirical intuition' *as act of intuiting as opposed to instance of intuiting*—gives necessity to the results of that construction, which is the initial first drawn object (and each subsequent 'intuition') as 'single object' of a mathematical concept. Note also that *both* productive imagination's metaphorical drawing *and* the literal drawing of an object of experience occur a priori according to Kant. This is true simply because it is the act of drawing, not the ontological status of what is drawn, that grants construction the status of necessity, and thereby apriority.[4]

Just below the above passage, Kant elucidates this notion of a 'non-empirical intuition'. Any 'non-empirical concept', he writes, 'either already contains a pure intuition in itself, in which case it can be constructed; or else it contains nothing but the synthesis of possible intuitions, which are not *a priori*' (A p. 719, B p. 747).

So the concept, and only the mathematical concept, actually contains an intuition in itself; and no intuition 'is given *a priori* except the mere form of appearances, space and time' (A p. 720, B p. 748).

I now proceed to a more in-depth discussion of what has emerged from the oppositions to both physics and philosophy as the most distinguishing characteristics of mathematics for Kant. At the beginning of the B-edition preface, Kant waxes poetic in his description of the origins of mathematics with 'that admirable people', the Greeks (B p. x). The first reason for Kant's admiration is what he considers to be the significantly greater difficulty mathematics had in becoming a sure science: 'Yet it must not be thought that it was as easy for [mathematics] as for logic – in which reason has to do only with itself – to find that royal path, or rather itself to open it up' (B pp. x–xi).

Norman Kemp Smith, by contrast, renders this last verb phrase as mathematics 'constructing' its own royal road: 'rather, I believe that mathematics was left groping about for a long time (chiefly among the Egyptians), and that its transformation is to be ascribed to a ***revolution***, brought about by the happy aspiration of a single man in an attempt from which the road to be taken onward could no longer be missed, and the secure course of a science was entered on and prescribed for all time and to an infinite extent' (B p. xi).

Note that though it was a happy 'aspiration' (for Smith, 'thought') that began the revolution that made mathematics scientific, the efficient cause of this revolution was an experiment, the advent of the empirical. 'For he found that what he had to do was not to trace what he saw in the figure, or even its mere concept, and read off, as it were, from the properties of the figure; but rather that he had to produce the latter from what he himself thought into the object and presented (through construction) according to *a priori* concepts' (B p. xiii).

Or, as Smith has it, he had to 'bring out what was necessarily implied in the concepts that he had himself formed *a priori*, and had put into the figure in the construction by which he presented it to himself'.

Mathematics does not begin with thinking about a mathematical concept. Nor does it begin with careful observation, visual inspection of a representation or symbol of a mathematical concept. Rather, in the act of constructing, literally drawing the mathematical object, in this case a triangle,

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4 It is worth noting here that according to one commentator, in an essay that we will consider in greater detail below, even the 'drawing' in the imagination is not conducted in some kind of theatre of the mind. '[O]ne's imaginative or mental shifting [of attention] is not a construction of or in private space. My attention, for example, traces out a figure in front of me.' In other words, even metaphorical imagination-drawing could be argued to be a imaginary projected tracing out of figures in the three-dimensional, phenomenal, 'outer sense' space (Melnick 1992: p. 248).

that triangle is infused with the mathematical conceptuality that the geometer himself put into it.

It is interesting that here, at the very beginning of the B-edition, Kant draws on resources for the conception of mathematical construction that did not appear until the very end of the A-edition, in the Discipline of Pure Reason. Insofar as it is well-known that Kant revised the B-edition specifically to combat the reception of his work as a naïve or conventional idealism along the lines of Berkeley, perhaps this is also the reason why Kant shifted his drawing-based conception of construction to the very beginning of the *Critique*.

Continuing on to the Introduction to the B-edition, one finds Kant's famous assertion that '**Mathematical judgments are all synthetic**' (B p. 14). Kant specifically mentions the entirety of arithmetic, as well as all but 'a few principles' of geometry, which 'are actually analytic' (B p. 16). One wonders what Kant means by pure mathematics, if even certain 'fundamental' principles of geometry do not qualify for the label. Just below this, one finds Kant's second famous assertion on this subject: 'properly mathematical propositions are always *a priori* judgments and are never empirical, because they carry necessity with them, which cannot be derived from experience (B p. 14).

However, Kant then concedes the possibility that this apriority only applies to '**pure mathematics**, the concept of which already implies that it does not contain empirical but merely pure *a priori* cognition' (B p. 15).

This seems to raise the question of pure mathematics again, because the geometric propositions above were allegedly disqualified because of their analyticity, on which this elaboration rests, not their syntheticity, on which their disqualification above was based. In brief, is it analyticity, aposteriority or both that causes a mathematical proposition to be empirical?

Perhaps the best clue to understanding Kant's conception of apriority is the following passage from the B-edition preface: for the geometer 'to know something securely *a priori* he had to ascribe to the thing nothing except what followed necessarily from what he himself had put in it in accordance with its concept' (B p. xiii).

The a priori is that which we contribute to experience and, if mathematics is a priori, then it too is something that we contribute to experience, not something that we are given ready-made. In the Transcendental Aesthetic, Kant criticises empiricist thinkers who hold exactly this 'readymade mathematics' sort of view, that mathematics consists merely of imaginative generalisations from experience (A p. 40, B p. 57).

In the Transcendental Deduction, Kant stipulates unequivocally that 'all intuition that is possible for us is sensible' (B p. 146). Without such intuition, the mathematical concepts of the pure intuitions of space-time 'are not by themselves cognitions, except insofar as one presupposes that there are things that can be presented to us only in accordance with the form of that pure sensible intuition (B p. 147).

It is due to this derivation from sensible intuition that mathematical principles are excluded from the principles of the understanding, which extend beyond mathematics' highly circumscribed domain of spatiotemporality (B p. 188).

Finally, toward the end of the Transcendental Analytic, Kant paraphrases his conclusions regarding the constructability of mathematics, noting that in it, 'the concept of magnitude seeks its standing and sense in number, but seeks this in turn in the fingers, in the beads of an abacus, or in strokes and points that pare placed before the eyes' (A p. 240, B p. 299). The concept of magnitude is schematised into the procedure of numeration, which actualises itself in such concrete practices as the drawing of strokes on a page. Contra the Platonists, 'mathematical cognitions have their object nowhere except in **possible** experience' (A p. 314, B p. 371n). We have thus observed in this section that mathematics described on its own terms is linked perhaps even more intimately to the empirical – but as constituting, not as derived from, the empirical.

I now turn to consider critical perspectives on Kant's philosophy of mathematics that focus on his constructivism. In the benchmark essay entitled 'Kant on the Mathematical Method', Hintikka first argues that a proper understanding of mathematical construction depends on an unconventional interpretation of Kant's concept of intuition, based more on the Doctrine of Method's presentation of intuition than on that of the Transcendental Aesthetic (Hintikka 1992: pp. 23–25). This proper understanding, for Hintikka, is that intuitions are merely particular ideas as opposed to general concepts. 'Everything, in other words, which in the human mind represents an individual is an intuition' and these individuals need not be empirical objects or sense-perceptions (Hintikka 1992: p. 23). The latter, according to Hintikka, are merely a subset of individuals in general (Hintikka 1992: p. 26).

In attempting to support this position, Hintikka examines in greater detail Kant's Euclidean inheritance, arguing that Kantian construction is modeled on the 'setting-out' and the 'auxiliary construction' parts of a Euclidean proposition. In the former part, Euclid, according to Hintikka, 'first applies the content of the enunciation [general statement of the problem] to a particular figure *which he assumes to be drawn*' (Hintikka 1992: p. 28, emphasis added). The latter part 'consisted in stating that the figure constructed in the setting-out was to be completed *by drawing certain additional lines, points, and circles*'. Note the centrality here of drawing.[5]

Hintikka links Euclid's constructions to his own interpretation of intuitions as individuals-added-during-construction, claiming that 'Kant's notion of a construction accommodates as a special case the usual geometric notion of construction' (Hintikka 1992: p. 30).

The upshot of Hintikka's argument is a challenge to Kant's assumption that all knowledge of particulars is through sense-perception; that is, Hintikka rejects the linkage of intuition to sensibility. The reason for this challenge is the implied passivity of Kant's view of human knowledge and perception. In its place, Hintikka suggests that we think of 'the processes by means of which we come to know the existence of individuals' as 'processes of searching for and finding than acts of perception' (Hintikka 1992: p. 40). Relating this back to construction, for Hintikka, it can be reduced to the modern symbolic logical rule of inference called 'existential instantiation' (Hintikka 1992: p. 35).

Another influential commentator on Kant's constructivism, Charles Parsons, argues that Hintikka's understanding neglects the centrality of immediacy (and thus of sensibility) for Kant's conception of intuition (Parsons 1992: p. 46). He also argues that being 'affected' by objects in Kant's account necessitates the very passivity on behalf of the subject that Hintikka wishes to replace with the searching-finding subject (Parsons 1992: p. 46). Lisa Shabel also understands, *contra* Hintikka, the intuitive as 'a singular and immediate mental representation' (Shabel 2006: p. 99).

While Parsons does agree with Hintikka that 'Euclid's postulates are what are in effect existence assumptions', he nevertheless refers these postulates to ostension. ('Ostension', meaning 'showing' in the sense of 'stretching toward', reverberates strongly with my preferred term of *drawing*; the former is arguably a Latinate equivalent of the Germanic term.) 'Kant emphasized that space was…understood in a way analogous to ostension, and the same ostensive understanding would be necessary for the particular primitives of Euclid' (Parsons 1992: p. 49).

However, seemingly *contra* both Hintikka and Parsons on this point, Kant writes in the Discipline of Pure Reason that mathematical problems are not 'about existence as such at all' (A p. 719, B p. 747). It seems that for Kant, while the notion of ostending or drawing is appropriate, the notion of existence is not, *at least not in Hintikka's sense*.

Shabel further disputes the position, allegedly held by Parsons among others, that Kant 'draw[s]' a rigid distinction between 'ostensive' (arithmetic and geometrical) versus 'symbolic' (algebraic) construction. Regarding ostensive construction, Shabel holds that the stroke used in arithmetic (as in making hash marks on a page) despite being ostensive, is nevertheless a more abstract mathematical tool than might appear from its sensible rendering (Shabel 2006: p. 100).

She argues that Kant uses '"symbolic construction" to designate that which symbolizes ostensive constructions' (Shabel 2006: p. 101). In other words, symbolic construction is merely ostensive

5 Hintikka (1992: p. 29, emphasis added). In fact, drawing plays a central role for Euclid, evidenced by the fact that his very first postulate concerns drawing: 'To draw a straight line from any point to any point'. More importantly, the final words of each Euclidean proposition are determined by whether or not synthetic construction (conceptual drawing) is involved therein. If the proposition contains construction, the proposition ends in the abbreviation Q.E.F., 'precisely what was required to be done'. If the proposition does *not* involve such construction, it ends in (the more familiar) Q.E.D., 'that which was to be demonstrated' or, perhaps more accurately, 'precisely what was required to be proved' (pp. xxiv–xxv). In other words, the doing or activity of construction, the drawing, is so important for Euclid that it alone determines how each proposition ends (Euclid 2002: p. 2).

construction once removed, the ostension of ostension; otherwise, according to Shabel, algebra fails to 'be the exhibition of an intuition in Kant's sense…' (Shabel 2006: p. 101).

Anticipating my reinterpretation of the manifold below, one could say that the ostension of ostension allegedly accomplished by algebra constitutes a kind of spatiotemporal reduction. It can constitute a spatial reduction in that fewer strokes, for example the 11 strokes necessary to write out "$x + y = 200$" (two strokes each for the $x$, the $y$, the plus sign, the equals sign, and one stroke each for the 2 and the two 0's) can function to represent a potentially greater number of strokes, such as the 200 strokes necessary (when $x = 150$ and $y = 50$) to carry out the sum with hash marks. This ostension of ostension can also constitute temporal reduction, in so far as each variable can stand for any combination of chronologically successive numbers, formulas, etc. For example, let $X = 2$, let $Y = X + 3$, and let $Z = X + Y + 4$. $Z$, in this case, is described in one mathematical operation (or temporal succession of strokes), even though it can also be construed as an abbreviation of two or three successive mathematical operations.

Shabel also reads Kant's synthetic construction as resting on the 'act of construction' that produces 'singular and concrete representations', referring the reader to the Schematism (Shabel 2006: pp. 110, 112). 'In the case of mathematical concepts', Shabel argues, 'schemata are strictly redundant: no "third thing" is needed to mediate between a mathematical concept and the objects that instantiate it since mathematical concepts come equipped with determinate conditions on and procedures for their construction' (Shabel 2006: p. 111).

Shabel then applies this conception to the arithmetical case of 'the mental act we perform in *exhibiting the strokes or points*' (Shabel 2006: p. 112, emphasis added). Here one encounters further support for both the centrality of drawing and the equivalence of mathematics and schematism. I will now consider how this is also true in an essay by J. Michael Young.

Arithmetic is Young's branch of choice for articulating his interpretation of Kant's philosophy of mathematics, reworking Parsons' arithmetic essay in order to suggest a distinct reinterpretation of Kantian imagination (Young 1992: p. 159). Young agrees with Parsons' criticisms of Hintikka's view of intuitions as simply singular representations, but criticises Parsons' focus on symbolic, as opposed to ostensive, construction of arithmetic concepts (Young 1992: p. 162). He thus falls in line with Shabel's focus on the primacy of the activity of ostension or drawing. Young suggests that Kant would understand numeral systems as providing 'procedures by which we can generate ostensive constructions of numerical concepts' (Young 1992: p. 162). He then elaborates this latter point through the arithmetical example in which we represent 'a collection of perceptible particulars that possesses the number in question', and 'represent that collection as nothing but a collection possessing that number'.

Young then constructs his own example of this procedure, namely by identifying correctly spelled versions of the English word 'synthetic' (Young 1992: p. 164). He notes that the word *synthetic* has nine letters, and that this is a clue to finding a correctly spelled token of the word-type 'synthetic'. For example, if we ask someone to write out the word *synthetic* and then count 10 letters on the page, then we already know that their word is *not* a correct token of the type *synthetic* (Young 1992: pp. 164–165). In short, 'The rule that determines how a word is to be spelled thus determines how many characters there must be in any of its (correctly spelled) tokens' (Young 1992: p. 167).

Young's argument hinges on his observation that, while a word 'is of course a symbol', it is nevertheless also the case that '[a] token of that word…but a perceptible instance of it' (Young 1992: p. 165). Reapplying this to arithmetic, Young asserts when we represent a collection of, say, 7 + 5 strokes, 'we represent it merely as something that conforms to these rules [specifying how such a collection is to be identified], ignoring the size, color, composition, etc., of the strokes. We even ignore the fact that they are strokes, representing them merely as units or 'ones', i.e., as arbitrary instances of an arbitrary concept. We therefore attribute to the collection of strokes only what is required by the general rules that mark it as a collection of 7 + 5' (Young 1992: p. 166).

In general, Young argues that for Kant, 'we have to construe discourse about number as having to do with the quantity of collections of sensible things – with the "numerosity" or "how-manyness" of such collections' (Young 1992: p. 167).

Just as with his ‘synthetic example’, ‘the rules that specify how to represent the number 7 + 5 are simply procedures for identifying perceptible collections of 7 + 5’, which is simply to ‘enumerate its members’ (Young 1992: p. 167). In summarising his view, Young asserts that ‘given the form of our intuition, the construction of arithmetical concepts requires that we command procedures for generating or identifying collections of *n* objects, and these procedures must be temporally successive’ (Young 1992: p. 169).

Put differently, every pattern must be patterned in every instance, because it is a procedure, and a procedure is enacted by ‘running through’ the procedure. In a sense, this reduces the static permanence of any mathematical concept to the activity of constructing it.

Furthermore, these procedures, according to Young (and Shabel), are nothing other than the ‘schemata’ of the imagination (Young 1992: p. 169). Young’s unconventional understanding of Kantian imagination is that the imagination is the faculty, not, as is usually thought, of ‘mental imaging’, but of ‘construing’. Young references ordinary usages of the word such as ‘a child imagin[ing] his stick is a gun’ and ‘[a] paranoid…imagin[ing] that a passing glance is threatening’ and explains that what is meant is that ‘someone construes, or views, or takes what is sensibly present as something other, or at least as something more, than what is immediately presents itself as being’ (Young 1992: p. 169).

One could say that, instead of thinking imagination as the power of making the (temporal) past present, Young thinks it as making (spatial) absence into presence.[6] Young argues that is only the imagination’s ‘construing’ that allows the schemata to present ‘sensible things as constituting collections of definite number’ by presenting ‘them as conforming to certain general rules or procedures’ (Young 1992: p. 171).

As a consequence of his reinterpretation of Kantian imagination, Young rejects the apparent or surface meaning of Kant’s claim that imagination is necessary for perception, which he claims ‘would be of little philosophical interest even if it were plausible’ (Young 1992: p. 170). This is arguably a major burden on Young’s theory, given Kant’s extensive insistence in the Transcendental Deduction that the transcendental synthesis of the imagination is an absolutely ineradicable element of experience, of constituting the human world. Modifying Young’s understanding of imagination as construing *more* in the thing that it grasps, I would like to suggest that the imagination qua mathematiser, through the schemata qua mathematisation procedures, instantiated by the activity of drawing, actually construes that which it grasps as *less* than it actually is, where what ‘it actually is’ should be understood as the empirical richness of the construed. Put differently, the imagination’s drawing is a process that takes its constructions, its objects, as merely opportunities for exercising its own activity, and thereby in a way reduces that which it draws. This view squares more successfully with Kant’s insistence on the necessity of imagination for perception, in that—as cognitive science has shown—human beings could never function without the ability to reduce the richness of their sensory input by such operations as concentration, habituation and attention.

Arthur Melnick, in his essay ‘The Geometry of a Form of Intuition’, argues that any ‘empirical intuition involves positing or setting objects outside oneself’, which one can understand as ‘pointing or circumscribing, or delineating, or tracing out or otherwise gesturing with one’s finger’ (Melnick 1992: p. 245). Put concisely, Melnick argues that the form of intuition for Kant to be ‘spatial behavior or activity…ostending’, or drawing (Melnick 1992: p. 246). Most importantly for the present investigation, Melnick seems to be the first interpreter of Kantian constructivism to take the concept of drawing seriously enough to consider how other meanings and usages of the word might helpfully explicate Kant’s thought. Regarding the concept of ‘ostension’, Melnick notes that ‘one may also regard this spatial behavior as directing or drawing out or focusing attention (either one’s own or others’ attention)’ (Melnick 1992: p. 246). He then offers an example.

> Suppose I draw a triangle. Then I am *drawing out your and my own attention* with the pencil as an extension of my finger (Melnick 1992: p. 248, emphasis added).

6 For an extensive discussion of Kantian imagination and the absence/presence duality, see Sallis (1980: pp. 164–166).

Though sympathetic to these claims, my interpretation of Kant differs from that of Melnick in two important ways. First, I do not share his final conclusion, which is to reject Kant's claim that geometry is a priori (Melnick 1992: p. 255). Second, I wish to *both* concretise *and* extend Melnick's notion of ostension back into the notion of drawing from which it was derived in order to apply it to all three branches of Kantian mathematics—arithmetic, algebra *and* geometry. To begin this process, I must first lay out the groundwork by offering analyses of three other concepts in Kant vitally connected to the notion of drawing—the manifold, schemata and imagination.

'Manifold', like 'drawing', is of Germanic origin. Like many German cognates in English, it means exactly what it sounds like – 'that which is constituted by many folds'. What is interesting is that, although by its grammatical/syntactical structure it is a singular noun, a discrete substantive, its semantic structure suggests a pure plurality. Manifold thus stands as a concrete linguistic example of the many and the one harmonised. Also, interestingly, in contemporary mathematics the word *manifold* refers to a 'topological space' that possesses the homogeneity of the form of 'the interior of a sphere in a Euclidean space', where Euclidean space means one in which parallel lines never meet, the perfect grid of Newtonian space.

One typical way to conceptualise the pure manifold of space-time, the field of the productive imagination in its a priori constituting of experience, would be as a blank sheet of paper, or as a pure Cartesian coordinate system on which points, lines, shapes, figures, etc., can be drawn. These constructions or drawings would then be available to structure actual empirical experience of objects in the world. Another, less conventional way to conceptualise the manifold would be as a kind of billowing white fabric that is given to sensibility, a sheet that moves with its own rhythms, creating natural risings, fallings, creases and folds before our eyes. On this model one could perhaps think of the spontaneity of the understanding, including the synthesis of the imagination, as producing or *pinching* folds in the fabric, and *gathering* or *stringing together* these folds. Every non-conceptual synthesis would then be a gathering of the fabric up into folds, and every concept would be a grasping together of those folds. In this way, each act of the mind on the manifold would be to separate it into levels of greater and lesser folding – smooth patches, gentle curves, scattered folds, folds of folds, large structures of multiple complex folding, etc.[7]

A still more unorthodox conceptualisation of the manifold, and one more in harmony with my privileging of the activity of drawing, can be achieved by using the analogy of the popular children's toy called a 'Magna-Doodle'. The Magna-Doodle is a drawing device that consists of a board with a translucent grey screen, behind which is a narrow box filled with tiny metal shavings, and across which one draws a special plastic 'pen' with a small magnet at its 'writing' tip. At every point and along every line at which one contacts the screen with the pen, the ferrous material rises, pinned to the other side of the screen, resulting in the appearance of a kind of blurry pencil drawing. In this analogy, the screen would be the pure manifold, the iron filings would be the given intuition of sensibility, and the activity of drawing with the pen would be the activity of the imagination as it synthesises—brings together, gathers the filings of—the manifold. This is also a reduction of the large, blank grey screen into a narrow, focused line. But how exactly should one use a Magna-Doodle, how does one draw with it, most effectively produce lines, shapes and figures, in space and time? For this, one needs a schema.

The etymology of *schema* is 'form, figure' and it is derived from the word 'scheme', which literally means 'a holding'. The word *schema* is also defined as a 'coding' or 'organisation', as a 'diagrammatic representation', and as a 'draft', the latter of which is itself defined repeatedly in terms of drawing. A schema is a form or design that exists because it holds itself in that form, perhaps as a procedure carried out. Moving from the 'folding' understanding of the manifold, one could say that a schema is a procedure for folding the manifold into certain shapes or patterns and holding the manifold in those shapes and patterns through its procedural design. Alternatively, thinking of the Magna-Doodle analogy, one could say that a schema is an instruction for how to use the 'pen' to effectively make marks on the screen.

7 This conception is of course indebted, at least in spirit, to the work of Gilles Deleuze (1993) in *The Fold: Leibniz and the Baroque*.

Put simply, the schema is, not the *what* of an image, but the *how* by which the imagination 'draws' the image. This *how* is simply in space and time, the lawful domain of mathematics. Kant distinguishes between an image and a schema by describing the image as the product of the empirical or reproductive imagination, and the schema as the product, or, 'as it were, monogram, of pure *a priori* imagination, through which, and in accordance with which, images themselves first become possible'.

Kantian schematisation is, ultimately, mathematisation, meaning that it consists in making the pure concepts of the understanding mathematical so that they can apply to the world of appearances, of which mathematics is the universal law. Kant attests to this in the Axioms of Intuition: 'The synthesis of spaces and times, as the essential form of all intuition, is that which at the same time makes possible the apprehension of the appearance, thus every outer experience, consequently all cognition of its objects, and what mathematics in its pure use proves about the former is also necessarily true of the latter' (A pp. 165–166, B pp. 206–207).

Put more briefly, in Smith's translation of this passage, 'Whatever pure mathematics establishes in regard to the synthesis of the form of appearances is also necessarily valid of the objects apprehended.'

Every appearance must conform to mathematics in order to even appear for our sensibility in the first place. Further, Kant speaks in the Discipline of Pure Reason of mathematics as enabling one to become 'master over nature' (A p. 725, B p. 753). In the antinomies, Kant notes that, though mathematics 'cannot give [one] any satisfaction in regard to the highest and most important ends of humanity', nevertheless mathematics 'guides reason's insight into nature', specifically by supplying the philosophy of nature 'with appropriate intuitions' (A pp. 463–464, B pp. 491–492). Knowledge of the empirical world as ordered and regular depends entirely on mathematics quite simply because mathematics is the governing law of that world.

If this is true, then it would mean that the imagination, via its operation of schematising (while under the legislative autonomy of the understanding for speculative thought) could be aptly understood as the mathematiser of reality. The word *imagination*, derived from the Latin middle-voice verb that means 'drawing a picture for oneself', also means 'scheming; plan, scheme'. Connecting this back up to the manifold and the schemata, one could think of the manifold as a blank sheet of paper covered with dots, in which case the (productive) imagination would be the function or power connecting the dots according to a rule-governed procedure or schema—or it would be the person pinching folds in the fabric, or it would be the pen attracting the filings into an ordered design—allowing for the construction of, for example, geometric figures, arithmetically countable strokes and algebraic symbols drawn to duplicate the results of those (chronologically or conceptually) previous strokes.

Kant utilises the concept of drawing to describe the functioning of the imagination in both the A- and B-editions of the Transcendental Deduction, though with a noticeable difference in characterisation or emphasis between the two accounts. In the A-edition, in a separate section with the word 'imagination' in the title—a section that refers to the imagination as a separate 'faculty'—Kant writes 'if I draw a line in thought…I must necessarily first grasp one of these manifold representations after another in my thoughts (A p. 102).

In the B-edition, with no separately titled section, and with the imagination reduced to being merely 'synthetic influence of the understanding on inner sense', the above passage from the A-edition is replaced by the following:

> We cannot think of a line without ***drawing*** it in thought, we cannot think of a circle without ***describing*** it… we cannot even represent time without, in ***drawing*** a straight line (which is to be the external figurative representation of time), attending merely to the action of the synthesis of the manifold… (B p. 154)

Several things are noteworthy here. First, what seems to be the literal sense of drawing, since it is linked to 'outer' representation, is made a necessary condition for representing time. Second, this activity of drawing, which I have located in practically every section of the *Critique* dealing with mathematics, is apparently intimately linked also to the imagination, which would seem to support my characterisation of the imagination as the mathematiser of reality.

Third, recollecting my above hypothesis that Kant moved his A-edition treatment of mathematical construction to the beginning of the B-edition in order to make his critical philosophy appear more empirical and less idealistic, perhaps a similar motive is at stake here. Perhaps Kant reduced the imagination from a being a 'faculty' distinct from the understanding to being merely a 'title' of the understanding, and also eliminated it from its own separately titled section, because he felt that his previous emphasis on the imagination as distinct facilitated an overly idealistic reading of the *Critique*. At any rate, one sees that, in moving from the A- to the B-edition, the imagination seems to recede in importance, whereas the (empirically founded) activity of drawing simultaneously rises in prominence, moving from the Transcendental Logic up to the Preface.

Lastly, on the imagination in its connection to mathematics and drawing, in the Ideal of Pure Reason, Kant discusses what he terms 'the creatures of the imagination' by returning to the above-mentioned metaphor of the monogram: 'they are, as it were, **monograms**, individual traits, though not determined by any assignable rule, constituting more a wavering sketch, as it were, which mediates between various appearances, than a determine image, such as what painters and physiognomists say they have in their heads, and is supposed to be an incommunicable silhouette of their products or even of their critical judgments' (A p. 570, B p. 598).

This remarkable passage deserves extended attention. I will begin with the central idea here of a monogram. One obsolete meaning of *monogram*, according to the *Oxford English Dictionary* (2013), is 'a picture drawn in lines without shading or colour; a sketch'. The monogram is thus pure form, pure figure without matter or content, reminiscent of Kant's statement that mathematics can construct the shape of a cone, but not its color, a priori. *Sketch* is a word for a drawing that connotes its unfinished or in-process state. The primary current meaning of *monogram* is that of 'A motif consisting of two or more letters, esp. the initials of a person's name, written together and usually interwoven…' Combining these two definitions, one finds that the (currently) more familiar meaning of 'initials signifying personal possession' becomes itself the less familiar 'picture or sketch' as the two or more letters are 'interwoven' or constructed as a geometric shape. Kant obviously has these manifold senses in mind when he writes that the imagination's monograms form, in Smith's translation, 'blurred sketches *drawn* [pulled] from diverse experiences'.

Another interesting aspect of this passage is the transition to the 'painters and physiognomists'. The imagination's monograms are like the aesthetic images 'in the heads' of visual artists and of the practitioners of what the dictionary refers to as 'the study of appearance', those who 'read faces or other physical features to discern character personality, etc.' One observes here Kant drawing one in with a narrative in which letters, words, the drawing thereof, images, pictures, appearances all blur together. Later, in the Discipline of Pure Reason, Kant speaks of the proper, or at least desirable, feature of the imagination as, in Smith's translation, 'not simply enthuse to be a *visionary*, but is to poetize [*dichten soll*] under the strict surveillance of Reason' (A p. 770, B p. 798). This seems an appropriate description of what Kant is doing here for the reader, recalling Kant's famous dictum that 'Mathematics is pure poetry', with which I transition to the concluding section of my investigation, with a polysemous proliferation of the meanings of the word *drawing*.[8]

As I mentioned at the beginning of my investigation, the *Oxford English Dictionary* (2013) identifies seven primary clusters of meaning for the word 'to draw': traction, attraction, extraction, protraction, construction, self-motion and its various adverb constructions (i.e. to draw out, to draw in, to draw from, etc.) Traction, or pulling, is the central root of the word, which can be thought as pulling the pencil across the page. Attracting (with Melnick) can be thought as drawing one's attention, which is what happens when one draws an image in one's mind or watches something being drawn on a page. Extraction could mean drawing out the intuitive content of the mathematical concepts. Protracting, the lengthening out of the line that is being drawn.

8 See Immanuel Kant, *Opum Postumum*, p. 139.

Construction, the model on which I am interpreting all the other senses of drawing. Self-motion, emphasising how the drawing is spontaneous activity. Finally, the last cluster of meanings is what happens when one constructs a phrase out of 'drawing' and any of various prepositions.

Playing with these various clusters of meanings, I would like to suggest thinking of drawing in Kant in several different ways. First, in the sense of protracting, or drawing out, the self, insofar as space and time are merely forms of our sensibility (and thus of all human intuition), and not separate, distinct, objective aspect of the world. Put differently, if space-time is the form of our sensibility, then space-time is a part or an aspect of us, which constitutes an extension of the Self from simply body and mind to include the pure a priori manifold. Given that each of us shares, in some sense, the a priori manifold with all other finite intelligences, this new, broader Self is also a kind of inter-subjective Self. In this way, mathematical construction is a drawing that is also a drawing out of selves into a common, pure manifold. The results of these constructions are, of course, applied to a posteriori, empirical experience. This would also suggest that mathematics is a kind of self-knowledge, both of the empirical, embodied aspect of the human being, and also of the spontaneous drawing activity of the imagination.

Second, drawing can be thought of as a kind of creative construction accomplished through a constricting of the manifold—turning a homogeneously dotted page into one or more designs (on the dotted page analogy)—folding it more tightly (on the fabric analogy)—gathering the ferrous material into points, lines and figures (on the Magna-Doodle analogy)—through the mathematising schemata.[9] After all, construction in the ordinary contemporary sense of constructing, say, a parking deck, involves pulling together materials from various places (lumber, glass, metal, etc.) and reducing the plurality, diversity, many-ness of these diverse elements into the synthetic unity of a new building. In a similar way, one draws the manifold into new shapes, thus constructing new concepts that can be realised in possible objects of experience.

Third, given this conception of construction as a kind of constriction of the manifold, one can observe that this construction of points, lines, figures, etc., also serves to constrict or focus the attention of both the constructor and any viewers of the construction. This is an important consequence of Melnick's argument in the above essay. Fourth, this mathematical synthesis of drawing (in all its senses) can be characterised as nothing other than the 'self-motion' or spontaneity of the understanding *qua* imagination. And fifth, the final cluster of meanings of 'drawing'—the cluster termed 'adverb constructions'—makes drawing even more open-ended, as drawing in, up, through, on…not infinitely, but indefinitely.

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9 One could, following Badiou, think of this as a kind of opening onto a subtractive or reductive mathematical ontology, insofar as, despite marked differences between this project and that of Badiou throughout *Being and Event*, they perhaps share the assertion that mathematics creates by reducing the manifold.

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