Scrutiny of Einstein's Geodesic *and* Field Equations

Mohamed E. Hassani
Institute for Fundamental Research
BP.197, CTR, Ghardaïa 47800, Algeria
E-mail: hassani641@gmail.com

Abstract: Since its final version and publication in 1916, it is widely reported in several specialized textbooks and research articles that general relativity theory may be reduced to the Newton's gravity theory in the limit of a weak gravitational field and slow motion of the material bodies. In the present paper, the so-called reducibility of Einstein's geodesic *and* field equations to Newton's equation of motion *and* Poisson's gravitational potential equation, respectively, is scrutinized and proven to be mathematically, physically and dimensionally wrong and also the geometrization of gravity is not really necessary.

Keywords: General relativity theory, Newton's gravity theory, physico-logical argumentation, dimensional analysis

1. Introduction

It is widely alleged and blindly believed that Newton's gravity theory is manifestly a limiting case of general relativity theory (GRT) for so-called weak gravitational fields and slow motions in spite of the fact that, strictly speaking, Newton's gravity theory is not physically and/or even mathematically contained in GRT. Hence, the main aim of the present paper is to show more conclusively that the Einstein's geodesic *and* field equations cannot be reduced to Newton's equation of motion and Poisson's gravitational potential equation, respectively, and the geometrization of gravity is not really necessary or even essential to physics. To clarify the problematic, let us firstly inspect the following *physico-logical viewpoints* which are in fact a tangible evidence .

–*First physico-logical viewpoint*: It was well shown in [1] that the exact equations of motion of the matter and Einstein's field equations lead to the existence of only vanishing integrals of the motion, so that at each stage of approximate calculations, including the Newtonian approximation, these integrals of the motion of GRT must vanish. Hence, it follows, in particular, that GRT does not have really and correctly a classical Newtonian limit, since the integrals of the motion of the two theories are not equal.

–*Second physico-logical viewpoint*: Since gravity is a universal attractive force in Newton's gravity theory and in GRT is not a force but rather is space-time, warped by the presence of mass and energy, however, in such a situation, *i.e.*, phenomenologically how can Newton's gravity theory become a limiting case of GRT even for so-called weak gravitational fields and slow motions?

–*Third physico-logical viewpoint*: It is frequently said that the stress-energy tensor serves a role in GRT very similar to that of mass distribution in Newton's gravity theory, more precisely, it tells space-time how to deform, creating what –we observe as gravity–.Therefore, space-time is not an inert entity. It acts on matter and can be acted upon. Consequently, curved space-time itself behaves like *a sort of matter*, not merely a geometrical seat in which arise the physical phenomena without specific dynamical properties.

–*Fourth physico-logical viewpoint*: Phenomenologically speaking, the geometrization of gravity implies the materialization of the curved space-time itself, and as a direct result the usual principle of causality is violated because the causal source of such *materialization* is absolutely without existence.

–*Fifth physico-logical viewpoint*: The presence of light speed in the Einstein's field equations is not a pure physical consequence, but a geometrical one, historically speaking, is due to the Minkowski space-time geometry. Without forgetting that so even the Schwarzschild's solution has been performed in the context of Minkowski space-time geometry because at infinity the Riemann's geometry reduces to space-time with the Minkowski metric. Therefore, the said presence of light speed in the Einstein's field equations does not necessary mean that gravitational fields propagate at light speed.

2. GRT's Irreducibility

2.1. Geometrization of gravity is Unnecessary

According to Lehmkuhl [2], it seems even Einstein did not seriously accept the geometrization of gravity as an indisputable physical reality. In 1926, Einstein's attitude was more clear about the fact that GRT should not be understood as a pure reduction of physics to Geometry. This attitude was motivated by a letter from the philosopher of science, Hans Reichenbach, who was at the time in connection with Weyl's and Eddington's theories, and wrote Einstein that he thought that seeing electricity as geometrical in Weyl's theory is not more than an illusion, one that, he argued, is equally possible (and equally trivial) in GRT. Einstein approved enthusiastically, and wrote [2]:

"*You are completely right. It is wrong to thing that 'geometrization' is something essential. It is only a kind of crutch for the finding of numerical laws. Whether one links 'geometrical' intuitions with a theory is a ... private matter.*"

Once again, from Einstein's response, we can affirm that the geometrization of gravity *was* and *is* in principle unnecessary. Furthermore, the skeptics and opponents of the geometrization of gravity are most notably represented by the two Nobel laureates Feynman and Weinberg. At the time of his lectures delivered at Caltech , Feynman was struggling to quantize gravity –that is, to forge a synthesis of general relativity and the fundamental principles of quantum mechanics. Feynman's whole approach to general relativity is shaped by his desire to arrive at a quantum theory of gravitation as straightforwardly as possible. For this purpose, geometrical subtleties seem a distraction; in particular, the conventional geometrical approach to gravitation obscures the telling analogy between gravitation and electrodynamics. Feynman preserves the force and motion concepts with their well-known classical meanings, and comments[3]:

" *It is one of the peculiar aspects of the theory of gravitation, that it has both a field interpretation and a geometrical interpretation ... the fact is that a spin-two field has this just marvelous. The geometrical interpretation is not really necessary or essential to physics.*"

And as reinforcement to the above comments, Weinberg wrote [4]:

" *In learning general relativity, and then in teaching it to classes at Berkeley and M.I.T. ,I became dissatisfied with what seemed to be the usual approach to the subject. I found that in most textbooks geometric ideas were given a starring role, so that a student who asked why the gravitational field is represented by a metric tensor, or why freely falling particles move on geodesics, or why the field equations are generally covariant would come away with an impression that this had something to do with the fact that space-time is a Riemannian manifold.*

Of course, this was Einstein's point of view, and his preeminent genius necessarily shapes our understanding of the theory he created.[...] Einstein did hope, that matter would eventually be understood in geometrical terms[...]. [I believe that]too great an emphasis on geometry can only obscure the deep connections between gravitation and the rest of physics."

As we have seen, even the preeminent scholars were explicitly in total disagreement about the idea of geometrizing the physics in general and gravity in particular.

2.2. Geodesic equation is irreducible to Newton's equation of motion: physico-logical argument

In order to show, without calculations, that the geodesic equation is irreducible to Newton's equation of motion because of the fact that Newtonian gravity theory is not contained in GRT, we begin by recalling the profound difference between mathematics and physics. Such a recall is indispensable for the reason that in the framework of GRT, there is no clear and explicit distinction between a physical equation (mathematical equation written in pure physical context) and mathematical equation (any equation written in pure mathematical context).

First, mathematics is not physics and physics is not mathematics. The inhabitants of the mathematical world are pure abstract objects characterized by an absolute freedom. However, the inhabitants of the physical world are pure concrete objects –in theoretical sense and/or in experimental/observational sense– and are characterized by very relative and so restricted freedom.

When applied outside its original context, mathematics should play the role of an accurate language and useful tool, and gradually should lose its abstraction. Let us illustrate these considerations by writing the following ODE in pure mathematical context:

$$a\,y' + b\,y = 0\,, \tag{1}$$

where $y \equiv y(x)$; $x, a, b \in \boldsymbol{R}$ and $(a,b) \neq (0,0)$ are constant that may be infinitely small or infinitely large. Remark, by virtue of the abstraction and freedom that characterize Eq.(1), we can rewrite it, without any major or minor worry, as follows

$$y' + \tfrac{b}{a}\,y = 0\,. \tag{2}$$

Also, we can rewrite Eq.(1), after performing a second derivation with respect to *x*:

$$a\,y'' + b\,y' = 0\,. \tag{3}$$

All what we have done is pure *abstraction* reinforced by high degree of *freedom*, that is, without a direct contact with the tangible reality. However, when we are dealing with physical equations, abstraction and freedom together lose their absolutism and become very relative and so restricted because each parameter containing in the physical equation has well-defined role fixed by its own physical dimensions as we will see. To this end, let us write the following physical equation

$$\frac{d^2\mathbf{r}}{dt^2} = -\nabla U\,, \tag{4}$$

in pure context of classical gravitational physics, with $\|\mathbf{r}\| \equiv r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ and U is the gravitational potential defined by

$$U \equiv U(r) = \begin{cases} \frac{2}{3}\pi G\rho\left(r^2 - 3R^2\right), & r < R \\ -\dfrac{GM}{r}, & r > R \end{cases}\,. \tag{5}$$

Where G, ρ, M, R and r are, respectively, the gravitational constant, mass density, mass of gravitational source which is supposed to be spherically symmetric, radius of source and the relative distance between the center of source and the center of test-body of mass m $(m < M)$. In classical gravitational physics, we can interpret the two above equations as follows: (*i*) Eq.(4) illustrates us the motion of a test-body of mass *m* under the action of gravitational field represented by $-\nabla U$; (*ii*) according to (5), the test-body is inside the gravitational source if $r < R$ and outside it if $r > R$; (*iii*) since we can write $d^2\mathbf{r}/dt^2 = d\mathbf{v}/dt$ this means that the test-body may be in relative motion with respect to the center of source and consequently Eq.(4) may be rewritten as

$$\frac{d\mathbf{v}}{dt} = -\nabla U\,, \tag{6}$$

that is to say, we have also an equation of motion; (i*v*) the quantity $-\nabla U$ has the physical dimensions of gravitational acceleration, therefore, we can rewrite Eq.(4) by using the usual notation, *viz*, $\mathbf{g} = d^2\mathbf{r}/dt^2$ and we get the well-known classical equation of gravitational field

$$\mathbf{g} = -\nabla U\,. \tag{7}$$

Finally, if we multiply the two sides of (7) by the scalar m ,*i.e.*, the mass of test-body, we obtain the expression of the gravitational force acting on the test-body

$$\mathbf{F} = m\mathbf{g} = -m\nabla U\,. \tag{8}$$

As we can remark it, phenomenologically, Eq.(7) is completely different from Eq.(8), this is not the case with Eqs.(1) and (2). In passing , it is worthwhile to note that the physical equations are permanently subject to the dimensional analysis (DA). The principal role of DA is to check and verify the correctness and the coherence of the physical equations in their proper context. But unfortunately, many physics students and professional physicists ignore or neglect the veritable goal and useful of DA. For example, the ignorance or negligence of DA is well reflected by the fact that we can find in many specialized textbooks and research articles some fundamental physical equations written without c^2 and G which are the light speed squared and the gravitational constant, respectively. Because are, by common ill-convention, supposed to be $c = 1$ and $G = 1$, which, perhaps, is an acceptable trick to facilitate mathematical calculations. But *physically*, it represents a loss of information and can lead to confusion and such equations cannot be checked by DA.

Let us turn now to the geodesic equation and apply the above ideas to Einstein's approach. At the beginning of the fourth Princeton lecture (1921), Einstein commences the discussion of the motion of material points, and stating that in GRT the law of inertia has to be generalized by generalizing the old concept of a straight line [2]:

"*According to the principle of inertia, the motion of a material point in the absence of forces is straight and uniform. In the four dimensional continuum of special relativity, this is a real straight line. The natural, i.e. the simplest, generalization of the straight line making sense in the conceptual scheme of the general (Riemannian) theory of invariants is the straightest (geodesic) line.*

Following the equivalence principle, we will have to assume that the motion of a material point subject only to inertia and gravity is described by the equation

$$\frac{d^2 x_\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 0\,. \tag{i}$$

Indeed, this equation becomes that of a straight line if the components $\Gamma^\nu_{\mu\sigma}$ of the gravitational field vanish.

Einstein then shows without any explicit physical support that in the Newtonian limit the geodesic equation becomes (with $dl^2 = dt^2 = ds^2$)

$$\frac{d^2 x_\mu}{dl^2} = \frac{\partial}{\partial x_\mu}\left(\frac{\gamma_{44}}{2}\right). \tag{ii}$$

where γ_{44} is defined by $g_{\mu\nu} = \delta_{\mu\nu} + \gamma_{\mu\nu}$.

Remark, it seems even Einstein has adopted the common ill-convention $c = 1$ and $G = 1$. Furthermore, there is a problematic behind Eq.(*ii*), since the quantity γ_{44} is not previously well-defined that is contrary to Eq.(4) in which the leading term U is well-defined in advance by (5). Therefore, we cannot apply DA to Eq.(*ii*) and consequently Eq.(*ii*) is a pure mathematical equation.

In order to dodge this inevitable impasse, Einstein simply and verbally points out a link between the geodesic equation and the Newtonian equation of motion for particles subject to gravitational fields:

This equation (ii) is identical with Newton's equation of motion of a point in a gravitational field if one identifies $-\gamma_{44}/2$ *with the gravitational potential. ... One look at Eqs.(i) and (ii) shows that the quantities* $\Gamma^{\nu}_{\mu\sigma}$ *play the role of the field strength of the gravitational field. These quantities are not tensorial.*

The above passage reflects a major problematic. Firstly, there is no a single physico-mathematical proof concerning the so-called reduction of geodesic equation to Newton's equation of motion. Instead of an explicit physico-mathematical proof, there is a sort of 'word-game' contained in the expression:

This equation (ii) is identical with Newton's equation of motion of a point in a gravitational field if one identifies $-\gamma_{44}/2$ *with the gravitational potential.*

Because if we substitute '*if one identifies*' with '*if one replaces*' we get a very equivalent expression:

This equation (ii) is identical with Newton's equation of motion of a point in a gravitational field if one <u>*replaces*</u> $-\gamma_{44}/2$ *with the gravitational potential.*

Hence, geodesic equation may be unphysically reduced to Newton's equation of motion, only *via* a replacement/substitution, and not *via* an explicit calculation done in pure physical context. Accordingly, and strictly speaking, we cannot consider the geodesic equation as physical equation.

Secondly, supposing that the gravitational potential U itself is completely unknown to Einstein and/or his successors so that he would unable to identify/replace $-\gamma_{44}/2$ with U .

2.3. Einstein's (gravitational) field equations are irreducible to Poisson's (gravitational) potential equation

As the title of the present subsection reveals it clearly, many specialized textbooks and research article incorrectly utilize the expression: Poisson's equation for/of gravitational field and also they consider the same equation as a generalization of Laplace's equation. However historically speaking, the French mathematician Siméon-Denis Poisson (1781-1840) who was one of the founders of mathematical physics discovered his equation in the framework of the potential theory and published it in 1813 –in Bulletin de la Société Philomatique, pp. 388-392–. And he never considered his equation as a generalization of Laplace's one. In modern language and notation, Poisson said in his paper that Laplace equation

$$\nabla^2 U = 0, \tag{9}$$

is applicable only if the material test-body is outside the gravitational source and when the test-body is inside the source we should take in consideration the mass density $\rho \equiv \rho(r)$ inside the radius $r < R$ and Laplace equation should be replaced with Poisson's equation

$$\nabla^2 U = 4\pi G\rho . \tag{10}$$

Thus, according to (5), the gravitational potential function $U \equiv U(r)$ is at the same time a fundamental solution for Eqs.(9) and (10) for the cases $r > R$ and $r < R$, respectively. Curiously, it seems until now, Poisson's equation is not correctly understood because lots of specialized textbooks and research articles claimed that Eq.(10) may be reduced to Eq.(9) when $\rho = 0$. However, the mass density ρ in Eq.(10) playing the role of gravitational source and in Eq.(9) this role is played by the mass M which is contained in the expression of U when $r > R$. Therefore, physically, Eq.(10) cannot reduce to Eq.(9) since the mass density itself gives rise to the gravitational field *via* the gravitational potential U and once again this implies that the mass density must play the role as a gravitational source and that's why ρ is explicitly present in the expression of U for the case $r < R$.

Therefore, according to (5), when $\rho = 0 \Rightarrow U = 0$ and Eq.(10) becomes an identity of the form 0 = 0 for this reason Poisson himself did not consider his equation as a generalization of Laplace's equation.

It is repetitively claimed that the Einstein's field equations (here without cosmological constant)

$$G_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{11}$$

reduce to the Poisson's equation (10) in the limit of a weak gravitational field and slow motion of the material bodies. In order to prove the irreducibility of Eqs.(11) to Eq.(10), we must apply the DA. So, by means of the DA, which is very robust tool, we can get the following dimensional expressions for Laplace's and Poisson's equations, respectively:

$$\mathrm{T}^{-2} - \mathrm{T}^{-2} = 0, \tag{12}$$

and

$$\mathrm{T}^{-2} = \mathrm{T}^{-2}. \tag{13}$$

Eqs.(12) and (13) are dimensionally identical. Moreover, if we multiply Eq.(12) or (13) by the dimensional quantity L (length) we get an acceleration and if we multiply the same equations by L^2 we obtain speed squared. Now, let apply the same DA to the Einstein's field Eqs.(11), and we find:

$$\mathrm{L}^{-2} = \mathrm{L}^{-2}. \tag{14}$$

As we can remark it more clearly, Eq.(14) is not dimensionally identical to Eqs.(12) and (13); hence, mathematically and physically, Einstein's field Eqs.(11) cannot reduce to Poisson's Eq.(10).

– Finally, in contrast to Einstein's approach, the expression of the constant κ cannot be deduced by comparison with Poisson's Eq.(10), therefore, we should have

$$\kappa \neq 8\pi G/c^4 . \tag{15}$$

3. Conclusion

In this article, the author proved that contrary to folklore, general relativity theory cannot be reduced to the Newton's gravity theory even in the so-called limit of a weak gravitational field and slow motion of the material bodies. With certain physico-logical argumentation, the geodesic equation is proven to be irreducible to the Newton's equation of motion, and finally by means of the dimensional analysis, the Einstein's field equations are shown to be irreducible to the Poisson's gravitational potential equation.

References

[1] V. I. Denisov and A. A. Logunov, Teor. Mat. Fiz. **43**, 187 (1980)
[2] D. Lehmkuhl, Studies in History and Philosophy of Modern Physics, **46**, 316 (2014)
[3] R. P. Feynman, Feynman Lectures on Gravitation, Addison-Wesley, New York, p.113 (1995)
[4] S. Weinberg, Gravitation and Cosmology, John Wiley & Sons, Inc, p.vii (1972)