

Frege's Contribution to Philosophy of Language

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1 Whence a Philosophy of Language?

Gottlob Frege's contributions to philosophy of language are so numerous and so fundamental that it is difficult to imagine the field without them. That this should be so was not, however, Frege's original intent. Frege was trained as a mathematician, and most of his non-foundational mathematical work lay at the intersection of geometry and complex analysis. That makes it at least somewhat surprising that he should have made any contribution to the study of language, let alone one so profound. But mathematics, in Frege's time, was itself in a state of upheaval, many of its most basic notions being subjected to a thorough re-examination. Among the central issues in Frege's intellectual environment was how we should understand the relation between geometry and arithmetic, the latter understood in a broad sense, as including the theory of all numbers, natural, real, and complex.¹ Though by no means universal, it was a common view that geometry was the more fundamental of the disciplines. In part, this view derived from Kant, who had famously argued that even our knowledge of basic number theory is founded upon pure *a priori* intuition of roughly the same sort that he claimed underlay our knowledge of the axioms of Euclidean geometry. The view had other sources, too. For example, it was as common then as it is now to think of the complex numbers as points (or perhaps vectors) in the Euclidean plane. Such a treatment of the complex numbers would suggest, again, that geometrical knowledge is more basic than arithmetical knowledge.

Frege, like many mathematicians of his time, rejected this Kantian view. His strategy for refuting Kant was to demonstrate that "...arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition" (Gg, §0) by identifying a small set of recognizably logical principles,

¹ Frege's relation to his mathematical environment has been the subject of much recent work (Tappenden, 1995a,b, 2000; Wilson, 1995).

defining the basic logical notions using strictly logical vocabulary, and then proving axioms for arithmetic² using only recognizably logical means of inference. This view is what has come to be known as *logicism*.

Frege quickly realized that he could not carry out this program unless he had some way to keep track of precisely which assumptions were being made in a given proof, and which means of inference were being used.

To prevent anything intuitive from penetrating here unnoticed, I had to bend every effort to keep the chain of inference free of gaps. In attempting to comply with this requirement in the strictest possible way, I found the inadequacy of language to be an obstacle; no matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. (Bg, pp. 5–6)

And so it was that Frege was led to invent his *begriffsschrift*,³ his “conceptual notation”. The invention consisted of two parts: a formal notation in which actual statements of mathematics could be written, and a detailed enumeration of a small number of modes of inference to be employed in formal arguments.⁴ What made the system work was that the two parts were properly balanced: The formal notation was sufficiently articulated that it could be determined which modes of inference could be applied in a given case simply by examining the *forms* of the representations. That is, the logical relationships between propositions were sufficiently exhibited in the notation that one could check whether a proof was correct—and, in particular, whether all necessary assumptions had been made explicit—purely mechanically. Of the many innovations that made this possible, perhaps most familiar today are Frege’s dual notions of scope and binding and how they are used to represent generality. Indeed, one would be hard pressed to challenge the claim that it is Frege’s conception of logical form that constitutes his most significant contribution to the study of language.

The reception of Frege’s work among his colleagues was generally lukewarm.⁵

² Frege never presents an explicit axiomatization of arithmetic, but there is a case to be made that he had one that was only slightly different from the now standard axiomatization due to Dedekind and Peano (Heck, 1995).

³ We will use the lowercase ‘*begriffsschrift*’ to refer both to Frege’s formal language and to the logical theory stated in that language. Context should disambiguate these uses. We will of course use upper case when referring to Frege’s book.

⁴ Arguably, Frege recognizes but one rule of inference, *modus ponens*, in *Begriffsschrift*. He does not clearly identify the other rules an adequate formalization would require—namely, universal generalization and substitution—and it is not clear that he regards them as rules of *inference*. See (BLC, p. 39), where he refers to *modus ponens* as “[t]he rule of inference”.

⁵ Of course, Russell and Wittgenstein are well-known exceptions. Among purely mathematical

This was especially so in the uncomprehending reaction by logicians of his day to *Begriffsschrift*, a volume we now rightly regard as firmly entrenched in the pantheon of thought about logic. Traditional logic—deriving from Aristotle but, just before Frege’s time, thoroughly updated by George Boole and his followers (Boole, 1847, 1854)—had been quite incapable of articulating enough of the logical structure of actual mathematical statements to make their logical relations plain. Nowhere was this more evident than with generality. Nonetheless, neither Ernst Schröder (the leading German member of the Boolean school) nor John Venn (he of the Venn diagram) could see any real value in Frege’s new notation.⁶ Frege penned several pieces in the years immediately following the publication of *Begriffsschrift* comparing his new logic to that of the Booleans (AppCN; AimCN; SJCN; BLC; BLF); not surprisingly, in response Frege touts the importance of his treatment of generality. More generally, Frege emphasizes that the *begriffsschrift*, inclusive of the treatment of generality, provides us with something more than just an alternative (if albeit superior) notational device. As Russell would remark somewhat later in a famous appendix to *The Principles of Mathematics* that is devoted to Frege’s thought:

Frege’s work abounds in subtle distinctions, and avoids all the usual fallacies which beset writers on Logic. His symbolism, though unfortunately so cumbrous as to be very difficult to employ in practice, is based upon an analysis of logical notions much more profound than Peano’s, and is philosophically very superior to its more convenient rival. (Russell, 1903, p. 501)

From the early 1880s on, explicating this “symbolism” became a central focus of Frege’s writings, and this required Frege to become more precise about the conceptual underpinnings of logic. Thus, while almost none of Frege’s most well-known doctrines are fully present in *Begriffsschrift* (though their kernels often are), they have all emerged in full force a decade and half later at the time of *Grundgesetze*. These doctrines—most notably, the articulations of content embedded in the concept–object and sense–reference distinctions—emerge as Frege struggled to carry the logicist program forward. In large part, these developments spawned from Frege’s understanding that, in order to derive the postulates of arithmetic from the laws of logic, he would have to operate in a setting in which proofs could be given rigorously and reliably. The logicist program simply could not be carried out otherwise. It was for this purpose that Frege had to clarify fundamental notions of

colleagues, the only one who seems to have had much time at all for Frege’s work was Peano, and his interest seems to have been short-lived.

⁶ Their reviews of *Begriffsschrift* are reprinted in (Frege, 1972).

logic and semantics: They were essential to articulating the relation between the language in which proofs are carried out, his *begriffsschrift*, and the mathematical claims he intended to be proving. Insofar as we speak of Frege's philosophy of language, then, it can only be understood properly if we keep clearly in mind that it was, first and foremost, a philosophy of logic that was an integral part of a larger scientific project, the project of logicism.

The treatment of generality is to a large extent the catalyst for the emergence of Frege's characteristic philosophical doctrines. This is so because Frege regarded his treatment of generality as demonstrating why logic is so important to the study of mathematics in general and to logicism in particular. Frege recognized that logical rigor depends, first and foremost, upon our being able to articulate the structure of sentences, and what Frege understood profoundly was that how sentences containing words of generality *compose* is fundamentally different from how sentences containing no such words *compose*. This difference Frege saw as a structural difference in conceptual content, and we must come to terms with this point if we are to begin to understand what Frege means by "conceptual contents" or, to use his later terminology, "thoughts": We must grasp that thoughts are, *qua* contents, compositionally complex. Nonetheless, compositionality for Frege is neither a metaphysical principle nor a psychological one. Rather, it is a semantic principle, integral to our understanding of how thoughts can be expressed by language. As such, compositionality ranks as the distinctively *linguistic* contribution Frege's philosophy of logic makes to philosophy of language, not just as a guiding maxim but also in particular aspects of Frege's more detailed proposals.⁷

Our plan for the remainder of the paper is as follows. In section 2, we discuss Frege's apparently metaphysical doctrine that concepts are 'unsaturated'. We argue that it is primarily a semantic thesis, an essential ingredient of Frege's conception of compositionality. In section 3, we discuss Frege's conception of truth. We argue that his seemingly puzzling doctrine that sentences denote objects, namely, truth-values, emerges from considerations about the logic of sentential connectives and the semantics of predicates and embodies an understanding of why, as Frege sees it, logic is so intimately concerned with the notion of truth. In section 4, we turn to Frege's notion of a thought and, more generally, to the distinction between sense and reference. Our first goal is to explain the philosophical pressures that lead Frege to draw this famous distinction. We then raise an interpretive question

⁷ There is much of interest in Frege's work that we shall not have space to discuss. We shall only touch very briefly upon Frege's treatment of intensional contexts (SM, opp. 36ff), barely mention his discussion of indexicality (Th, opp. 65ff), and, sadly, ignore his notion of a criterion of identity (? , §§62ff) and his discussion of presupposition (SM, opp. 39ff). That is not, of course, because we think there is nothing of value in these discussions. There is. But we have tried here to focus upon Frege's most general doctrines, which in turn shape his particular analyses.

that has not previously been clearly recognized, namely: How does distinguishing the sense of ‘the Morning Star’ from that of ‘the Evening Star’ allow Frege to explain why sentences containing these two names express different thoughts? Answering this question will require us to clarify yet further Frege’s conception of compositionality.

2 Concepts

In the logic developed by Boole, sentences were constructed from predicates using a small number of operators corresponding to traditional forms of judgment, such as universal affirmative judgments, which are of the form ‘All F s are G ’. Proper names, such as ‘Socrates’, were regarded as predicates, that is, as being of the same logical type as expressions like ‘is mortal’. Thus, the famous argument

All humans are mortal.
Socrates is a human.
Therefore, Socrates is mortal.

might have been represented as:⁸

All H are M .
All S are H .
All S are M .

The correctness of the argument then follows from the validity of the form of syllogism known as Barbara.

Frege’s way of representing generality required him to reject this traditional identification of names and predicates. As Frege saw the matter in *Begriffsschrift*, a sentence may be regarded as constructed from an *argument* and a *function*. In the case of ‘Socrates is mortal’, for example, we may take the argument to be ‘Socrates’ and the function to be “the part that remains invariant in the expression” when we replace ‘Socrates’ by other names, such as ‘Plato’ or ‘Thales’ (Bg, §9). Frege would then represent the sentence ‘Socrates is mortal’ in his logic, as: $M(s)$, and the generalization “Everything is mortal” as: $M(x)$, where the singular term ‘Socrates’ has been replaced by a variable. Note the absence of the quantifier: Frege’s view, at this time, was that generality is indicated by “letters”, that is, by variables (Bg, §1). The “concavity” in the more explicit representation

$$\ulcorner M(\alpha),$$

⁸ The second premise could, of course, also be taken to be: Some S are H .

serves, Frege says, only to “delimit[] the scope that the generality *indicated by the letter covers*” (Bg, §11, our emphasis).

Built into Frege’s logical notation, then, is an asymmetry between expressions that occur as functions and those that occur as arguments. It is obvious that this early distinction, between function and argument, has much in common with Frege’s later distinction between concept and object, but there are two important differences.

First, Frege insists in *Begriffsschrift* that the distinction between function and argument “has nothing to do with the conceptual content [of an expression]; it comes about only because we view the expression in a particular way” (Bg, §9). The distinction between function and argument is thus not absolute in the way Frege insists, in his later work, that the distinction between object and concept is: One may regard ‘Socrates’ as the argument and ‘is mortal’ as the function; but one may equally regard ‘is mortal’ as the argument and ‘Socrates’ as the function. Note that we have *not* said that one can regard ‘ ξ is mortal’ as the argument and the second-level concept ‘ $\Phi(\text{Socrates})$ ’ as the function, which is how Frege would have seen the matter in his later work: There is no notion of ‘second-level concept’ to be found in *Begriffsschrift*.⁹

Second, Frege speaks in *Begriffsschrift* as if functions and arguments are (parts of) expressions: When we replace ‘Socrates’ with other names in the expression ‘Socrates is mortal’, the function we discover is “the part that remains invariant in the *expression*” (Bg, §9, our emphasis). It would be uncharitable, we think, to saddle Frege with the view that functions are linguistic expressions: More reasonable is the supposition that he was, at that time, no clearer about the distinction between use and mention than were any of his contemporaries.

Frege’s view begins to change in both these respects fairly quickly. Consider the following remarks, taken from the unpublished paper “Boole’s Logical Calculus and the *Begriffsschrift*”, written about 1881:¹⁰

If...you imagine the 2 in the content of possible judgement $2^4 = 16$ to be replaceable by something else, by -2 or by 3 say, which may be

⁹ And similarly, there is no distinction between first- and second-order quantification. There is but one axiom of universal instantiation, proposition 58, and it is used indiscriminately to justify both what we would regard as first-order inferences and what we would regard as second-order inferences.

¹⁰ Frege speaks almost entirely of concepts in this paper, not of functions. But it is clear that this difference is only terminological: Boole speaks of concepts, and Frege is speaking as he does. Indeed, Frege’s examples of functions in *Begriffsschrift* are generally examples of concepts (or predicates).

Although this paper was not published during Frege’s lifetime, he submitted it for publication on three separate occasions, so we may nonetheless regard it as representing Frege’s considered view.

indicated by putting an x in place of the 2:

$$x^4 = 16,$$

the content of possible judgement is thus split into a constant and a variable part. The former, regarded in its own right but holding a place open for the latter, gives the concept ‘4th root of 16’. . . . And so instead of putting a judgement together out of an individual as subject and an already formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of possible judgement. (BLC, pp. 16–17)

Note how Frege speaks here not of replacing the *expression* ‘2’ by other *expressions* in the *sentence* ‘ $2^4 = 16$ ’ but rather of replacing the *object* 2 by other *objects* in the *content of possible judgement* $2^4 = 16$. The concept *fourth root of 16* is thus discovered in the *content* $2^4 = 16$: It is what remains invariant in the content when we imagine 2’s being replaced by other objects BLC, pp. 16–7. So already by 1881, Frege no longer regards concepts as parts of expressions that remain invariant when other parts are varied but rather as what such parts of expressions *denote*.¹¹ Note, moreover, how close Frege’s understanding of how concepts differ from objects is in 1881 to his mature view: The concept *fourth root of 16* is that part of the content $2^4 = 16$ that remains constant when one varies 2, where this part is “regarded in its own right but *holding a place open for*” a suitable argument (BLC, p. 16, our emphasis). To say that concepts have an ‘open place’ is very close to saying that they are ‘incomplete’ or ‘unsaturated’. Frege does not use that terminology until 1882, in a letter to Anton Marty (PMC, p. 101). But he comes close to using such terminology in a footnote attached to the end of the preceding quotation, in which he remarks that, in a concept, “one simply doesn’t have anything complete, but only the predicate of a judgement which as yet lacks a subject” (BLC, p. 17, note **). But, however unsettled Frege’s terminology may be, it is clear that he now regards concepts as fundamentally different from objects: “In the case of a concept it is always possible to ask whether something, and if so what, falls under it, questions which are senseless in the case of an individual” (BLC, p. 18).

Moreover, it is equally clear that Frege’s earlier view that the distinction between function and argument “has nothing to do with the conceptual content” (Bg, §9) has now been abandoned. If it is to make any sense at all to speak of replacing the object 2 in the content of possible judgement $2^4 = 16$ with other objects, then the object 2 must itself occur in that content—it must somehow be a part of it—as

¹¹ This change is noted explicitly by Philip Jourdain in his 1912 summary of Frege’s doctrines, on which Frege provided extensive commentary: “. . . [T]he traces of formalism in the *Begriffsschrift* vanished: a function ceased to be called a *name* or *expression*” (PMC, p. 204).

must what remains constant when it is varied. Frege himself makes a cognate point about expressions:

Of course, if the expression of the content of possible judgement is to be analysable in this way, it must already be itself articulated. We may infer from this that at least the properties and relations which are not further analysable must have their own simple designations. (BLC, p. 17)

(Here again, we see Frege distinguishing carefully between concepts, or properties, and expressions that designate them.) And Frege goes on to argue that these expressions, like their denotations, are incapable of occurring on their own:

... [I]t doesn't follow from this that the ideas of these properties and relations are formed apart from objects: on the contrary they arise simultaneously with the first judgement in which they are ascribed to things. Hence in the *Begriffsschrift*, their designations never occur on their own, but always in combinations which express contents of possible judgement. ... A sign for a property never appears without a thing to which it might belong being at least indicated, a designation of a relation never without indication of the things which might stand in it. (BLC, p. 17)

What is striking about these remarks is how they invert Frege's mature view of the matter. What Frege seems to be saying here is that predicates are unsaturated because what they designate is unsaturated.

These roles are reversed in Frege's mature work. Once he has made the distinction between sense and reference, Frege can no longer speak of replacing the number 2 in the content of the sentence ' $2^4 = 16$ '—that is, in the thought it expresses—for he denies that objects occur in thoughts.¹² And so, when Frege is attempting to explain his doctrine that functions are unsaturated in "What Is a Function?" he first explains his conception of a functional *expression* by asking us to consider sequences of expressions like 'sin0', 'sin 1', 'sin 2' (WF, opp. 663ff),¹³ much as he did in *Begriffsschrift*. The difference, of course, is that Frege no longer regards functions as expressions but as what functional expressions denote.¹⁴ That is to

¹² See the famous exchange about Mont Blanc and its snowfields in Frege's letter to Russell of 13 November 1904 (PMC, p. 163) and Russell's reply of 12 December 1904 (PMC, p. 169).

¹³ We shall cite Frege's published papers by the page number in the original publication.

¹⁴ There is no general agreement about how Frege's technical term '*Bedeutung*' and its cognates should be translated. We shall here generally translate it as 'denotation' but sometimes as 'reference', and we shall use the latter exclusively when contrasting *Bedeutung* with sense.

say: What Frege explains first is his view that functional *expressions* are unsaturated; he then explains the unsaturatedness of functions in terms of the unsaturatedness of predicates. Thus, Frege writes in “Comments on Sense and Meaning”:¹⁵

... [O]ne can always speak of the name of a function as having empty places, since what fills them does not, strictly speaking, belong to it. Accordingly I call the function itself unsaturated, or in need of supplementation, *because* its name has first to be completed with the sign of an argument if we are to obtain a meaning that is complete in itself. (CSM, p. 119, our emphasis)

One finds similar remarks in *Function and Concept* (FC, opp. 5ff), “On Concept and Object” (CO, opp. 194–5), and “What Is a Function?” (WF, opp. 665). So we take it that this aspect of Frege’s view stabilized by 1891: The unsaturatedness of predicates is what is basic; the unsaturatedness of functions and concepts is to be explained in terms of the unsaturatedness of predicates.

Predicates are not unsaturated in the very same way that concepts are: Frege does not, in his mature work, regard predicates as themselves being functions from, say, names to sentences.¹⁶ What, then, does it mean to say that a predicate is unsaturated? By insisting that predicates are unsaturated, Frege is expressing his deeper insistence on the fundamental syntactic distinction between names and predicates. It is clear enough that this claim has withstood the test of time, embedded as it is not only in contemporary logic but in syntactic theory, as well. The claim that the *denotations* of predicates are unsaturated, on the other hand, is often regarded as simply bizarre, and even Frege’s most sympathetic interpreters rarely seem to know what to make of it. It can easily seem as if Frege is projecting—and that is the perjorative use—his syntax onto the world. But this impression is the result of our mistaking for a metaphysical doctrine what is, in Frege’s hands, primarily a semantic one. After all, if the fact that predicates are unsaturated is to have any consequence whatsoever for the nature of what they denote—and, as we have said, in Frege’s mature work, the unsaturatedness of concepts is explained in terms of the unsaturatedness of predicates—then surely such consequences must issue from the nature of the *connection* between predicates and what they denote, that is, from something about the *semantics* of predicates.

¹⁵ The translation has the last word of the first sentence being “them”, as if it were anaphoric on ‘empty places’. It is clear, however, that what Frege means is, as he puts it in *Function and Concept*, that “the argument does not belong with a function” (FC, op. 6).

¹⁶ Of course, Frege did regard predicates as functions in *Begriffsschrift*, but that was for a different reason, namely, that he thought functions were expressions. As we shall see below, it is a delicate question to what the unsaturatedness of the *senses* of predicates amounts.

What it means for a predicate to be unsaturated is that it must always occur with an appropriate number of arguments: A sentence does not contain the one-place predicate ‘is mortal’ unless those words occur with an appropriate argument. If so, then the semantic clause for a predicate ought, one might well suppose, to reflect this fundamental syntactic fact about it. It is therefore only a slight exaggeration to say that, on Frege’s view, the question what ‘is mortal’ denotes need not be answered at all, since the predicate ‘is mortal’ can never occur on its own but only together with an appropriate argument. The semantic clause for ‘is mortal’ should, that is to say, begin not:

‘is mortal’ denotes. . . ,

but rather:

‘ Δ is mortal’ denotes. . . ,

where Δ is a syntactic variable ranging over expressions that might occur as arguments.¹⁷ This suggestion accords with Frege’s own practice: Witness his stipulations regarding the primitive expressions of the formal language of *Grundgesetze*.¹⁸ A Frege-inspired clause for ‘is mortal’ would thus take the form:¹⁹

(1) ‘ Δ is mortal’ denotes the True iff, for some x , denotes(Δ, x) and x is mortal.

But while clauses like (1) directly reflect the unsaturatedness of predicates, it is not clear what they imply about predicates’ denotations, since they do not explicitly assign denotations to predicates at all. The most obvious way of doing so would be:

(2) The predicate ‘is mortal’ denotes the concept *mortality*.

¹⁷ More formally, Δ ranges over what may be called ‘auxiliary names’: We suppose that the language can always be expanded by the addition of a new name, whose reference may then be any object one wishes. Formally, a truth-definition using such a device requires us to quantify over languages that expand the original one. See (Heck, 1999) for discussion, and the appendix of that paper for a sketch of a formal of a theory of this kind.

Frege uses some such device, and we have borrowed this use of Greek capitals from him. It is not clear, however, how Frege regarded these expressions, whose use he never explains. Sometimes, they seem to act like meta-linguistic variables ranging over objects; but then they also occur in quotation-marks, as in the semantic clause for identity in §7 of *Grundgesetze*, which suggests that they are substitutional variables. Auxiliary names let us have the best of both worlds.

¹⁸ We will quote one of these below, that for the horizontal. It is in no way exceptional. Regarding the other primitives, the clause for negation is in §6; identity, §7; the first-order universal quantifier, §8; the smooth breathing, §9; the definite article, §11; the conditional, §12; and the second-order universal quantifier, §24.

¹⁹ We will take up Frege’s view that sentences denote truth-values below.

But, of course, this will not do, for it leads directly to the infamous problem of the concept *horse*. Nevertheless, it is clear that Frege thought that a relation between a predicate and a concept was at least implied by a clause like (1). In section 5 of *Grundgesetze*, for example, Frege explains the meaning of the horizontal by the following postulate:

— Δ is the True if Δ is the True; on the other hand, it is the False if Δ is not the True,

and he takes this stipulation to be sufficient to assign a function as denotation of the horizontal, continuing: “Accordingly, — ξ is a function whose value is always a truth-value. . .”.

While Frege leaves matters at this pass, we can be more precise. Consider the relation that holds between a one-place predicate and its denotation. Since this is a relation of ‘mixed level’, taking as arguments an object—the predicate itself—and a concept, an expression denoting the relation between a predicate and its denotation must take as arguments a proper name denoting the predicate and a predicate denoting the concept.²⁰ This predicate, being unsaturated, must occur with an argument, which in this case will be a bound variable, there being nothing else for it to be. Thus, a ‘denotation clause’ for a predicate that is compatible with Frege’s commitments would have to have the following form:²¹

(3) $\text{denotes}_x(\text{‘}\xi \text{ is mortal’}, x \text{ is mortal})$.

Now, suppose we formulate our semantic theory using clauses of this form rather than clauses of form (1). To characterize the truth of atomic sentences, we will also need a principle governing the composition of simple sentences, say:²²

(4) $\lceil \Phi(\Delta) \rceil$ denotes the True if, and only if, for some ϕ and for some x , $\text{denotes}_x(\Phi(\xi), \phi x)$ and $\text{denotes}(\Delta, x)$ and ϕx .

We can now prove:²³

²⁰ The concept *horse* problem surely does make itself felt in these informal remarks. The point is that it need not make itself felt in the formal semantics.

²¹ To be fully faithful to Frege’s intentions, the clause would better be formulated so that it explicitly assigned a *function* to the predicate, but we’ll glide past that complication.

²² Here ‘ $\Phi(\xi)$ ’ is a syntactic variable ranging over predicates, with ξ indicating the argument-place.

It is here that it becomes important that Δ is an ‘auxiliary name’ and does not just range over such names as are available in the language itself. Note, however, that our being able to give this definition in no way depends upon our using the device of auxiliary names. The same trick could be pulled using satisfaction.

²³ For the proof, we also need a principle stating that every predicate denotes at most one concept: $\text{denotes}_x(\Phi\xi, \phi x) \wedge \text{denotes}_x(\Phi\xi, \psi x) \rightarrow \forall x(\phi x \equiv \psi x)$. But we need such a principle anyway, since

- (5) $\text{denotes}_x(\Phi(\xi), \phi x)$ iff, for every Δ , $\lceil \Phi(\Delta) \rceil$ denotes the True iff, for some x , $\text{denotes}(\Delta, x)$ and ϕx .

It follows that (1) is indeed sufficient to determine the denotation of ‘is mortal’, since (1) just is the right-hand side of the relevant instance of (5). It might therefore be thought that the question whether the semantics of predicates should be given by clauses like (1) or instead by clauses like (3) is of no real significance. We can take the latter as basic, in which case (3) and (4) obviously imply (1); or we can take (1) as basic, define denotation using (5), and then prove both (3) and (4). In that case, we could still regard (1) as assigning a denotation to ‘ ξ is mortal’ as directly as it is possible to assign one, since, as already noted, (1) is the right-hand side of an instance of (5).

From Frege’s perspective, however, the question whether (1) or (3) is more fundamental is critical. In a letter to Anton Marty, written in 1882, the language of unsaturatedness first appears explicitly when Frege writes:²⁴

A concept is unsaturated in that it requires something to fall under it; hence it cannot exist on its own. That an individual falls under it is a judgeable content, and here the concept appears as predicative and is always predicative. In this case, where the subject is an individual, the relation of subject to predicate is not a third thing added to the two, but it belongs to the content of the predicate, which is what makes the predicate unsatisfied. (PMC, p. 101)

What should we make of this claim that concepts are essentially “predicative”? that is, that predication itself somehow “belongs to the content of the predicate”? To understand what Frege is suggesting, we must understand what contrast he is trying to draw. The first question is therefore what the other case is, the case where the subject is *not* an individual. And the answer is that it is the case traditional logic takes as fundamental, the case in which the subject is itself a concept. Frege’s point is that “the linguistic schema of subject and predicate. . . contains what are logically quite different relations” (PMC, p. 101). The one present when the subject is an object he calls “falling under”: It is the relation between an object and a concept expressed in such sentences as ‘Socrates is mortal’. The one present when the subject is a concept he calls “subordination”: It is the relation between concepts expressed in such sentences as ‘Humans are mortal’. And in *this* case, Frege insists, the relation between subject and predicate *is* a “third thing added to the two”,

we’d otherwise not be able to prove, say, that ‘ $0 = 1$ ’ is false: For that argument, we need to know that ‘=’ denotes *only* the relation of identity. With this principle in place, we could then introduce an expression *true-of*(t, y), read ‘ t is true of y ’, as equivalent to: $\exists F(\text{denotes}_x(t, Fx) \wedge Fy)$.

²⁴ Frege’s use of traditional terminology here is, presumably, in deference to Marty.

namely, the relation Frege symbolizes thus:

$$\text{---} \overset{a}{\cup} \text{---} \left\{ \begin{array}{l} \Phi a \\ \Psi a \end{array} \right.$$

By thus reducing subordination to generality and the conditional, Frege is able to “reduce [Boole’s] primary propositions to his secondary ones” (BLC, p. 17).

Part of what Frege is claiming is thus that what we would call ‘atomic’ sentences are what are fundamental for logic. As he writes a decade or so later: “The fundamental logical relation is that of an object’s falling under a concept: all relations between concepts can be reduced to this” (CSM, p. 118). If atomic sentences are *truly* fundamental, however, then they cannot assert the existence of a *relation* between the subject and the predicate. The correct analysis of ‘Socrates is mortal’ is not: *falls-under*(M, s): That is, in effect, simply a version of the traditional view. The correct analysis is just: $M(s)$. It is in that sense that concepts must contain the relation of predication within themselves. But a theory that takes the semantics of predicates to be given by clauses like (3) does *not* treat “the relation of subject to predicate” as something that “belongs to the content of the predicate”; on the contrary, it is a “third thing”, recorded in (4), that must be “added to the two”. Frege’s doctrine that concepts are unsaturated is thus, as suggested above, primarily a semantic thesis, not a metaphysical one.

That the denotations of predicates—concepts—are unsaturated and so are fundamentally different from the denotations of proper names—objects—is Frege’s central point in “On Concept and Object” (CO). We can now see both why Frege held it to be almost incoherent to hold otherwise and why expressions like ‘the concept *horse*’ should not seduce us into thinking that concepts are objects after all. For Frege, effacing the concept–object distinction would beg the question of composition: If both predicates and proper names denoted objects, the question what bound them together would remain open. In striking contrast, taking the denotations of predicates to be functions from objects to truth-values provides a definite and precise answer to that question. Moreover, as Frege emphasizes, this answer can be generalized by extending the compositionality of concepts through a hierarchy of functions,²⁵ forming what we would characterize today as a type-hierarchy in which “[a]n object falls under a first-level concept [and] a [first-level] concept falls within a second-level concept” (CO, op. 201, emphasis removed). It is thus

²⁵ Concepts for Frege are those functions, at any level, that map their arguments to truth-values. Frege’s hierarchy is defined, however, in terms of the arguments of functions, not their values, so there are functions at higher levels of the hierarchy that do not have truth-values as their values. Most notable is the second-level concept that is the denotation of the ‘smooth-breathing’ operator, which forms names of extensions of concepts (or, more generally, of value-ranges of functions).

Frege's reduction of compositionality to function-application that lies at the heart of his conception of logical form.

Both these aspects of Frege's view remain familiar to us today. The former corresponds to taking predicates to denote characteristic functions, while the latter corresponds to taking generality words like 'every' or 'some' to denote generalized quantifiers, functions from characteristic functions (or the sets they determine) to truth-values.²⁶ There are, of course, alternatives. If, in "On Concept and Object", Frege was responding to the mistake of taking predicates to denote objects, he did not envisage denying that predicates denote at all. Donald Davidson, for one, is well-known for urging us not to take the step of assigning an entity to a predicate as its denotation (Davidson, 1984, p. 18). Opposition of this kind is sometimes motivated by ontological scruples—by a preference for desert landscapes, as Quine famously put it (Quine, 1953, p. 4)—but there is a more immediate concern, too, namely, whether Frege's treatment of predication is adequate to the empirical data. The issues between broadly Fregean and broadly non-Fregean accounts of predication are varied and subtle, and which we prefer will depend in part upon the context in which we are operating. In logic, the decision may well be a result of pragmatic choices in linguistic design. In linguistics, it will be rather empirical considerations about syntax, semantics, and their relation that are likely to come to the fore.²⁷ Adjudicating these matters goes far beyond what we can attempt here. But even if we abandon Frege's claim that predicates denote, what cannot be escaped is the demand that we "do justice at once to the distinction and to the similarity" between the modes of composition exhibited in 'Socrates is mortal' and in 'Everyone is mortal' (CO, op. 201). Frege's insight that such an account is semantically indispensable remains a watershed in the history of semantics.

3 Truth

For Frege, then, the concept denoted by the predicate 'is mortal' is a function. Its arguments are objects, such as Socrates. But what are its values? By the principle of compositionality, the denotation of 'Socrates is mortal' is the result of applying the function denoted by 'is mortal' to the object denoted by 'Socrates'. So the question what the values of concept-functions are is equivalent to the question what the denotations of sentences are. Frege's answer to that question, famously, is that

²⁶ The most well-known characterization in terms of characteristic functions is found in (Montague, 1974). See also (Lewis, 1970). For discussion of generalized quantifiers, see (Higginbotham and May, 1981), (Barwise and Cooper, 1981), and (Keenan and Stavi, 1986), among a considerable literature.

²⁷ For some of the consequences of this choice for the semantics of natural language, see (Chierchia and McConnell-Ginet, 1990), (Larson and Segal, 1995), and (Heim and Kratzer, 1998).

sentences denote truth-values. But what kinds of things are truth-values? Since the distinction between concepts and objects is both absolute and exhaustive, there are but two possibilities, and Frege famously opts for the latter: Sentences denote either the True or the False, and it is these objects that are the values of concept-functions. These claims may seem bizarre, but they are central to Frege's understanding of the special role truth plays in logic, a topic to which he devoted several essays (Log79; Log97; Tht). Truth-values are also central to Frege's philosophy of language, where their role in the emergence of the sense–reference distinction is critical.²⁸

Frege's reasons for his view of truth commence with considerations concerning the particular role sentences play in his formal system. As Frege describes his system in *Grundgesetze*, the horizontal, negation, and the conditional are all explicitly presented as truth-functional operators, in a strong sense: Each of them really does denote a *function* whose arguments and values are *truth-values*.²⁹ Negation, for example, is the function that maps the True to the False and the False to the True. It needs no emphasis how important and enduring Frege's conception of the sentential connectives has proven to be. But it was not the way he originally conceived of them. Although something very much like truth-tables appear in *Begriffsschrift*, the words 'true' and 'false' are not used in that connection. Frege speaks not of a content's being true or false but of its being "affirmed" or "denied", so that a conditional, for example, is said to "deny" the case in which the antecedent is "affirmed" and the consequent is "denied". Moreover, there is no indication in *Begriffsschrift* that Frege regarded the sentential operators as (being or denoting) functions.³⁰ So the expressions Frege would later regard as truth-functional connectives he regarded in *Begriffsschrift* neither as functional nor as specially concerned with truth-values.

Frege's discovery of the notion of a truth-function likely results, at least in part, from his reading Boole in the early 1880s. It was central to Boole's treatment of the sentential connectives that he regarded them as expressing functions, and one can well understand why that idea would have appealed to Frege. But if

²⁸ There is a famous argument, the so-called "slingshot", that derives the claim that sentences denote their truth-values from compositionality and a handful of other claims. See (Neale, 2001) for extensive discussion. The argument has sometimes been ascribed to Frege, but we know of no evidence he ever formulated it and so will leave it out of account.

²⁹ Since, for Frege, the truth-values are objects, these functions are defined for non-truth-values as well. But because negation and the conditional embed horizontals, they may, in effect, be regarded as defined only on the truth-values.

³⁰ Øystein Linnebo argues that Frege relies upon a quite traditional form-content distinction in *Begriffsschrift*, and the logical machinery belongs to the 'form' of a proposition (Linnebo, 2003). If so, then the notion of a function clearly belongs to the content side and so wouldn't have been applied to such logical notions as the conditional.

these operators are to be taken as expressing functions, the question arises³¹ what the arguments and values of these functions are. Boole's answer, contrary to what seems to be a popular misconception, was not "truth-values". In fact, the Booleans disagreed among themselves about what the correct answer was, and Boole's own view varies.³² But, whatever the arguments and values were, they had to be classes of some sort: That much was demanded by how Boole proposed to reduce the calculus of judgements (roughly, sentential logic) to the calculus of classes (roughly, monadic first-order logic). So while Frege may have borrowed from Boole the idea that the sentential connectives denote functions, the idea that they denote *truth*-functions is his own. Indeed, so far as we know, the first fully explicit appearance of the notion of a truth-function, as described above, is in Frege's 1891 lecture *Function and Concept* (FC, opp. 20–21, 22, 28), but we find Frege starting to use the terms 'true' and 'false' in connection with the arguments of the conditional as early as 1881 (BLC, p. 11).

Why does Frege so quickly settle upon this answer to the question what the arguments and values of his sentential connectives are?

Both in *Begriffsschrift* and in *Grundgesetze*, a form of Leibniz's Law applies to sentential variables: If the identity-statement connecting S and T is true, then S can be substituted for T . Under what circumstances, then, is an identity-statement connecting two sentences to be regarded as true? In *Begriffsschrift*, Frege's stated view was that ' $p \equiv q$ ' is true only if p and q have the same conceptual content. If so, however, substitution will be possible only rarely. But, we believe, Frege came to realize that much more extensive substitution ought to be possible in his formal language: We ought to be able to substitute any true sentence for any other true sentence, *salva veritate*.³³ Such extensive substitution would be permitted if we regarded sentences as referring to their truth-values and re-interpreted the "sign for identity of content": Then all true sentences name the same thing, ' $p = q$ ' may be taken to be true so long as p and q have the same truth-value, and substitution may

³¹ It will only arise for Frege once his views about what functions are have changed and he no longer regards them as expressions. As noted above, this change was well under way by 1881.

³² Boole's view in *The Mathematical Analysis of Logic* (Boole, 1847) is that the possible values of sentential variables are something like sets of circumstances: These are basically possible worlds as understood in the model theory of propositional modal logic. This was an important idea, to be sure, and some of the earliest investigations of modal logic were undertaken by members of the Boolean school, such as Hugh MacColl, around the turn of the twentieth century. (Thanks to Stephen Read for this information.)

Boole's later view, in *Laws of Thought* (Boole, 1854), is the one Schröder elaborates in his review of *Begriffsschrift* (Schröder, 1972, p. 224): The semantic value of a sentence-letter is taken to be the set of times when it is true. Frege has a lot of fun with that suggestion, but it too has a contemporary echo, in tense logic.

³³ Frege's reading of Boole probably played an important role here, too: Boole's logic is formulated as an algebraic system, and substitution of equals is one of its most basic principles.

proceed apace; the material biconditional has thus been reduced to identity. But identity here is *objectual* identity, so sentences must denote objects, namely, the True or the False.

If the True and the False are objects, the question then arises which objects they are. Frege's answer is that they can be any objects at all, so long as all true thoughts refer to the same object, all false thoughts refer to the same object, and true thoughts refer to a different object than do false thoughts. In section 10 of *Grundgesetze*, Frege argues that if any pair of distinct value-ranges satisfies these conditions, then every pair of distinct value-ranges does so.³⁴ Although Frege's particular implementation is vitiated by its being embedded in an inconsistent system—the proof that the third clause is satisfied in any given case will depend on the identity-criteria for value-ranges, and these, unfortunately, are given by the Basic Law V—the underlying idea is nevertheless clear and widely employed: Obviously, there is no significant difference between letting the truth-values be 0 and 1 and letting them be 27 and 34.

If, as we have seen, Frege argues for taking the truth-values to be objects by pointing to their role in logic, he is, of course, also aware that, in natural language, the word 'true' appears as an adjective: "Grammatically, the word 'true' looks like a word for a property" (Tht, op. 59). Frege argues, *via* the so-called regress argument, that such a property must be indefinable:

For in a definition certain characteristics would have to be specified. And in application to any particular case the question would always arise whether it were true that the characteristics were present. So we would go round in a circle. (Tht, op. 60)

This circularity would be particularly endemic, Frege observes, to truth defined as correspondence:

... [W]hat ought we to do so as to decide whether something is true? We should have to inquire whether it is *true* that an idea and a reality, say, correspond. ... And then we should be confronted by a question of the same kind, and the game could begin again. So the attempted explanation of truth as correspondence breaks down. (Tht, op. 60)

The breakdown would be in evidence even if we delimited our definition, as we properly should according to Frege, to the truth of sentences or, better, of thoughts. Having said this, however, Frege is quick to note that it does not follow that truth

³⁴ Value-ranges are the only option, since they are the only objects (other than truth-values, if truth-values are not value-ranges) in the domain of theory. See (Wehmeier and Schroeder-Heister, 2005) for a careful analysis of this argument, which turns out to contain a subtle flaw.

is not a property of thoughts. “With every property of a thing,” Frege says, “there is tied up a property of a thought, namely truth” (Tht, op. 61). On the other hand, ascribing this property to a thought does not result in a different thought with new content but simply gives one the same thought back, or so Frege claims.³⁵ Perhaps, then, truth is “something which cannot be called a property in the ordinary sense at all” (Tht, op. 61).

Some commentators have taken Frege to be arguing here that there is no property that all and only the true thoughts have.³⁶ But there is, we think, little evidence for this interpretation, and the regress argument simply does not establish this strong claim: If truth-values are the references of sentences, then “denotes the True” is a truth-predicate. Defenders of the interpretation just mentioned therefore take Frege to be arguing, more generally, that there can be no ‘semantic meta-perspective’ on logic: We cannot really speak of such a relation as that of denotation. But this suggestion seems to us desparate: The suggestion that Frege abjures all properly semantic discourse is simply at odds with too much of what he has to say about logic and, in particular, with the plain sense of Part I of *Grundgesetze*.³⁷ But there is nonetheless a puzzle here. Doesn’t the regress argument apply *mutatis mutandis* to the suggestion that truth, regarded as a property of thoughts, can be defined in terms of “denotes the True”? If not—if truth, regarded as a property of thoughts, *is* in the end definable as “denotes the True”—then what is Frege trying to establish with the regress argument?

The earliest presentation of the regress argument is in a draft of a textbook on logic, written in 1897. At the opening of the paper, Frege emphasizes that logic, as he understands it, is concerned with truth in a way no other science is:

Of course all the sciences have truth as their goal, but logic is concerned with ‘true’ in a quite special way, namely in a way analogous to that in which physics has to do with the predicates ‘heavy’ and ‘warm’ and chemistry with the predicates ‘acid’ and ‘alkaline’. . . . [L]ogic is the most general science of the laws of truth. (Log97, p. 128)

Frege then goes on to present the regress argument, concluding:

Truth is obviously something so primitive and so simple that it is not

³⁵ Even on Frege’s own view, this is incorrect or, at least, inexact. As Dummett notes, the equivalence of *A* and “It is true that *A*” cannot be maintained if one allows sentences not to have truth-values, as Frege does (Dummett, 1978b, pp. 4–5).

³⁶ This view mentioned is particularly associated with Thomas Ricketts (Ricketts, 1986, 1996). See also (van Heijenoort, 1967), (Dreben and van Heijenoort, 1986), (Weiner, 1990), and (Goldfarb, 2001). See (Stanley, 1996) and (Tappenden, 1997) for criticism and development of contrary views.

³⁷ See (Heck, 2007) for development of this claim. Some reason for it will also emerge below in section 4.

possible to reduce it to anything simpler. Consequently we have no alternative but to bring out the peculiarity of [this] predicate by comparing it with others. What, in the first place, distinguishes it from all other predicates is that predicating it is always included in predicating anything whatever. (Log97, p. 129)

What matters to Frege here is less the first claim he makes, that truth is irreducible, than the second one, that an attribution of truth is included in all predication or, better, in all judgement.³⁸ And the conclusion Frege means for his reader to draw is that it will be impossible to understand logic's special concern with truth so long as we regard truth as fundamentally a property of thoughts. There may, then, be a property that all and only the true thoughts have, and this property may even be definable in terms of "denotes the True". But the point is that no such property of thoughts could be what specially concerns logic.

Logic, Frege insists, "is not concerned with how thoughts, regardless of truth-value, follow from thoughts. . ." (CSM, p. 122). Rather, the premises and conclusion of an inference are always judgements, and judgement, as Frege understands it, is essentially directed at truth: To make a judgement is to acknowledge the truth of a thought (Tht, op. 62). Logic's special concern with truth is thus a consequence of its special concern with judgement and of judgement's intimate relation to truth. It is here that the regress argument becomes relevant: Isn't *acknowledging* something itself simply making a judgement? If so, then to acknowledge the truth of the thought that $2 + 3 = 5$ is simply to judge that this thought is true. But then that judgement too must be regarded as the acknowledgement of the truth of the thought judged—that is, of the truth of the thought it is true that $2 + 3 = 5$ —but to acknowledge the truth of *that* thought is in turn simply to judge that the thought that it is true that $2 + 3 = 5$ is true, and so on *ad infinitum*. What the regress argument is intended to establish is thus that the intimate relation between judgement and truth cannot be understood in terms of predication: To say that judgement is essentially directed at truth cannot be to say that, in every judgement, we *predicate* truth of a thought.³⁹

³⁸ It is now widely agreed that this claim is what drives the regress. See (Ricketts, 1986, p. 78) and (Kemp, 1995, p. XXX), for instance.

³⁹ There is, in fact, some evidence that Frege himself once so regarded judgement:

We can imagine a language in which the proposition "Archimedes perished at the capture of Syracuse" would be expressed thus: "The violent death of Archimedes at the capture of Syracuse is a fact". . . . *Such a language would have only a single predicate for all judgements, namely, "is a fact". . . . Our ideography is a language of this sort, and in it the sign \vdash is the common predicate for all judgements.* (Bg, §3)

The emphasis is Frege's. (Thanks to Ian Proops for this observation.) See (Geach, 1965, pp. 457–8) for some relevant remarks.

Frege has another argument for this same claim. This argument—which we might call the argument from content-redundancy—occurs already in “On Sense and Reference”. It purports to show that it is neither necessary nor sufficient for one to make a judgement that one should predicate truth of a thought.⁴⁰

One can say...: ‘The thought that 5 is a prime number is true’. But closer examination shows that nothing more has been said than in the simple sentence ‘5 is a prime number’. The truth claim arises in each case from the form of the assertoric sentence, and when the latter lacks its usual force, e.g., in the mouth of an actor upon the stage, even the sentence ‘The thought that 5 is a prime number is true’ contains only a thought, and indeed the same thought as the simple ‘5 is a prime number’. It follows that the relation of the thought to the True may not be compared with that of subject to predicate. (SM, op. 34)

Frege makes a similar claim shortly after he presents the regress argument in “Thoughts” and then remarks that what “explains why... nothing seems to be added to a thought by attributing to it the property of truth” is the fact that we do not need to use the word ‘true’ to express a judgement (Tht, op. 63). Rather, we may express our acknowledgement of the truth of a thought simply by asserting a sentence that expresses that thought (Tht, op. 62).

What, then, is judgement? What, in particular, distinguishes one’s *judging* that $2 + 3 = 5$ from one’s merely *entertaining the thought* that $2 + 3 = 5$? Frege often notes that, if one is merely entertaining a thought, then it does not matter whether its parts refer to anything: It is only if we are concerned to evaluate the thought—to decide whether we should assert it or deny it—that the references of the parts matter (SM, opp. 32–3): “... [I]n every judgement, no matter how trivial, the step from the level of thoughts to the level of reference [is] taken” (SM, op. 34).⁴¹ But, for Frege, reference is essentially compositional: If it matters whether the parts refer, and if so to what, then that must be because it matters whether the whole refers,

⁴⁰ We borrow the term ‘content-redundancy’ from Gary Kemp (1998). Kemp understands the argument differently from how we do, but we think he was right to highlight the connection between it and the regress argument.

See also the “Logic” of 1897, where Frege writes:

... [I]t is really by using the form of an assertoric sentence that we assert truth, and to do this we do not need the word ‘true’. Indeed we can say that even where we use the form of expression ‘it is true that...’ the essential thing is really the assertoric form of the sentence. (Log97, pp. 129)

Frege might also have mentioned that ‘it is true that p ’ can occur as antecedent of a conditional, in which case the thought that p need not be asserted.

⁴¹ Frege makes similar remarks in his other discussions of the regress argument and the argument from content-redundancy (Log97, p. 130), and (Tht, op. 63).

and if so to what. We may thus distinguish judgements from mere entertainings of thoughts by insisting that judgements are “advances from a thought to a truth-value” (SM, op. 35): When one judges that $2 + 3 = 5$, one is not just entertaining the thought that $2 + 3 = 5$; one is attempting to *refer* to something, namely, to the True. The relation between a thought and its truth-value, on Frege’s view, is thus not that of an object to a property but “that of sense to reference” (SM, op. 34).

This view is immune to Frege’s regress, even if we do take the next step and define ‘It is true that p ’ as: The thought that p denotes the True. The regress is driven by a ‘predicational’ conception of judgement, according to which making a judgement is ascribing truth to a thought. The view just outlined rejects that conception of judgement. The view is *not* that judging that p is ascribing the property *denoting the True* to the thought that p : That would indeed re-instate the regress. Rather, the view is that judging that p is attempting to refer, by thinking that p , to the True.

It is, of course, obvious that this conception of judgement—that is, of belief and, correlatively, of assertion—could use further development. Unfortunately, Frege says little more about it. But that, we suggest, is because his main concern was not with judgement itself but with logic. The point that matters to Frege is that truth-values are properly understood as the references, or semantic values, of sentences: That claim, as we saw earlier, is central to his treatment of the sentential connectives; since the semantic values of sentences are the values of concept-functions, it is central also to his conception of concepts and so to his understanding of composition. Here again, Frege’s basic point has survived. The role truth plays in logic is indeed more fundamental than is suggested by the familiar phrase ‘the truth-predicate’: The truth-values enter logic, in the first instance, as the semantic values of sentences, because the most basic notion of logic, that of a valid inference, cannot properly be explicated except in terms of the notion of truth.

4 Thoughts

Truth-values, on Frege’s view, are thus the references of sentences. But this doctrine gives rise to a problem, one that is most obvious if we translate Frege’s term ‘*Bedeutung*’ using its ordinary English equivalent, ‘meaning’. The problem is this: “If... the truth-value of a sentence is its meaning, then on the one hand all true sentences have the same meaning and so, on the other hand, do all false sentences” (SM, p. 35). Frege’s solution to this problem is, familiarly, to claim that the thought expressed by a sentence is distinct from its “meaning”. But, as we shall see, Frege was not content simply to draw this distinction. His commitment to the principle of compositionality, and his subtle appreciation of the demands it imposes, led him

to an account of what thoughts are and how they are expressed by sentences whose influence continues to be felt today.

The problem just mentioned is discussed in detail in *Function and Concept*:

... What ' $2^2 = 4$ ' means is the True just as, say, ' 2^2 ' means 4. And ' $2^2 = 1$ ' means the False. Accordingly, ' $2^2 = 4$ ', ' $2 > 1$ ', and ' $2^4 = 4^2$ ' all mean the same thing, viz. the True... The objection here suggests itself that ' $2^2 = 4$ ' and ' $2 > 1$ ' nevertheless tell us quite different things, express quite different thoughts. (FC, op. 13)

Sentences cannot “mean” their truth-values, the claim is, because otherwise all true sentences would have to “tell us” the same thing. Frege’s insight is that there is no incompatibility here at all. This objection poses no special problem for his view, he argues, because it stems from a much weaker assumption:

... [L]ikewise ' $2^4 = 4^2$ ' and ' $4 \times 4 = 4^2$ ' express different thoughts; and yet we can replace ' 2^4 ' by ' 4×4 ', since both signs have the same meaning. Consequently, ' $2^4 = 4^2$ ' and ' $4 \times 4 = 4^2$ ' likewise have the same meaning. We see from this that from sameness of meaning there does not follow sameness of the thought expressed. If we say ‘the Evening Star is a planet with a shorter period of revolution than the Earth’, the thought we express is other than in the sentence ‘the Morning Star is a planet with a shorter period of revolution than the Earth’; for somebody who does not know that the Morning Star is the Evening Star might regard the one as true and the other as false. And yet both sentences must have the same meaning; for it is just a matter of interchange of the words ‘the Evening Star’ and ‘the Morning Star’, which mean the same thing, i.e., are proper names of the same heavenly body. (FC, opp. 13–4)

What Frege is arguing here is that the principle that the “meaning” of a complex expression is determined by the “meanings” of its parts—that is, the principle of compositionality—together with the principle that the “meaning” of a proper name is its bearer, already implies that the thought expressed by a sentence is not determined by its “meaning”.⁴² If so, then it is no objection to Frege’s view that ' $2^4 = 4^2$ ' and ' $2 > 1$ ' have the same “meaning” that they express different thoughts.

There were a host of good reasons for Frege to endorse the premises behind this argument. (We revert now to speaking of denotation and reference rather than of “meaning”.) The principle of compositionality falls out of Frege’s treatment of concepts as functions; the thesis that sentences denote their truth-values we have

⁴² See (Hugly and Sayward, 2000) for a closely related interpretation.

just discussed; and Frege had held that the “content” of a proper name was its bearer even early on in *Begriffsschrift*. There is, of course, one other assumption that is needed, that “... ‘ $2^4 = 4^2$ ’ and ‘ $4 \times 4 = 4^2$ ’ express different thoughts”. This assumption has, of course, been much discussed in contemporary philosophy. At least until his correspondence with Russell, however, Frege does not seem even to have considered the possibility that it might be denied.⁴³

If we accept that premise, as Frege did, then the objection we have been considering may not pose a problem for his view that sentences denote their truth-values, but it does suggest a question, one Frege formulates in a letter to Peano as: “How can the substitution of one proper name for another designating the same object effect such changes?” (PMC, p. 169) Frege’s solution to this problem is, famously, to distinguish sense from reference. The distinction makes its first appearance in *Function and Concept*, immediately following the passage quoted above:

We must distinguish between sense and reference. ‘ 2^4 ’ and ‘ 4×4 ’ certainly have the same reference, i.e., are proper names of the same number; but they have not the same sense; consequently, ‘ $2^4 = 4^2$ ’ and ‘ $4 \times 4 = 4^2$ ’ refer to the same thing, but have not the same sense (i.e., in this case: they do not contain the same thought). (FC, op. 14)

Although the formulation in terms of sense and reference does not crystalize in Frege’s writings until the early 1890’s,⁴⁴ it is clear that Frege is aware of the distinction as early as *Die Grundlagen*,⁴⁵ and, as we shall see below, some of the machinery he deploys is already present in *Begriffsschrift*.

Though its application to identity-statements is extremely significant, it’s important to observe that the distinction between sense and reference does not emerge from any particular concern with identity-statements. At the time of *Begriffsschrift*, Frege treats mathematical equality as a notion distinct from ‘identity of content’, the latter being the notion governed by Leibniz’s Law. But Frege must quickly have realized that this view is incompatible with a central tenet of logicism, namely, that there are no arithmetical notions with irreducibly mathematical content. When, in “Boole’s Logical Calculus” (written, again, no more than two years later), Frege is demonstrating how actual mathematical arguments can be formalized in the *Begriffsschrift*, he takes the logical principles that, in *Begriffsschrift*, had governed ‘identity of content’ now to govern arithmetical equality (BLC, p. 29). In *Die Grundlagen*, Frege overtly takes the step of reducing arithmetical equality to to the

⁴³ See again the exchange about Mont Blanc and its snowfields, mentioned above in note 12.

⁴⁴ For some speculations on the dating, see (Sundholm, 2001).

⁴⁵ The notion that a single number may be given to us in different ways plays an important role in the central sections of *Die Grundlagen* (Gl, §§62–67).

general notion of objectual identity governed by Leibniz's Law (Gl, §65). Identity-statements then take pride of place within Frege's mathematical project, logicism: "... [I]dentities are, of all forms of proposition, the most typical of arithmetic" (Gl, §57), he tells us. But now there is a problem: Frege must explain why ' $2 + 2 = 4$ ' expresses something more than a mere triviality; he must solve what has become popularly known as 'Frege's Puzzle'.⁴⁶ But there is also something else Frege must show, which he intimates in the passage from *Function and Concept* displayed above, namely, why ' $2^4 = 4^2$ ' expresses a different true thought than does ' $4 \times 4 = 4^2$ '. The two puzzles are not the same, for in the latter case, unlike the former, substitution does not transform something of substance into a triviality: Both ' $2^4 = 4^2$ ' and ' $4 \times 4 = 4^2$ ' have non-trivial thought contents.⁴⁷

Although the sense-reference distinction is centrally implicated in accounting for both puzzles, a confluence that Frege must have thought highlighted the distinction's utility and importance, the reflections that led Frege to recognize its significance for these two puzzles are quite different, as befits the difference between them. Whereas the puzzle about identity-statements arises for Frege from deep concern about a foundational mathematical issue—how equalities *qua* identities can express substantive mathematical content—the more general version of the puzzle raises a more general semantic issue. How are we to understand how the parts of a sentence contribute to determining the thought it expresses if sentences expressing different thoughts can be composed in the same way from parts with the same references? The fact that Frege illustrates the problem with arithmetical examples shows that he views this matter as a semantic issue at the foundations of logic: It is an issue about the meaningfulness of expressions of *Begriffsschrift*.⁴⁸ But Frege did not take the issue to be limited to logic; rather he took it to be endemic, to be found in any language whose sentences express thoughts, including natural languages.

The crucial insight, for Frege, is that the distinction between sense and reference is not a distinction between content and something else but a distinction *within* content. In his initial remarks about content in *Begriffsschrift*, Frege talks of sentences' having "possible contents of judgement" as their "conceptual con-

⁴⁶ Frege's awareness of this issue is no doubt due to his colleague Johannes Thomae, who argued in (Thomae, 1880) that equalities would express trivialities if they were regarded as identities. Thomae took it as a distinct advantage of his formalist approach, on which arithmetic propositions are regarded as rules for the manipulation of formal symbols, that this problem does not arise. See (May, 2001) for elaboration.

⁴⁷ This could be doubted, on the ground that ' 4^2 ' is *defined* as ' 4×4 '. But there are obviously plenty of other examples.

⁴⁸ The issue is especially pressing for Frege since arithmetical equations, on his view, can be expressed by formulae of what we might call the 'pure' *Begriffsschrift*. But it would arise even if that were not so.

tent” (Bg, §8). But, in a famous passage from the preface to *Grundgesetze*, Frege emends this view, saying that he has “split” content into a complex of thought and truth-value (Gg, op. x).

A fair bit of effort has been expended trying to discover in what sense the earlier notion of conceptual content contained both the later notions within it. What difficulties have emerged may arise in large part from trying to see the break as structural, as if the notion of conceptual content were a kind of hybrid of the notions of thought and truth-value. But perhaps the break was functional: Frege had tried to use the one notion of conceptual content to do work for which he later decided two notions were needed. The rub came when conceptual content was asked to play a role more aptly enacted by truth-values. No doubt the peculiarities of taking conceptual contents as the denotations of sentences—so that negation and the conditional would be regarded as functions from conceptual contents to conceptual contents—would have been evident to Frege.⁴⁹ But the important point for present purposes is that there is no mystery about the way in which conceptual content was asked to play the role later played by thoughts: That is its most explicit theoretical purpose in *Begriffsschrift*, so much so that Frege would not, we think, have confused his readers had he simply continued using the term “conceptual content” instead of switching to the new term “thought”. In any event, the notion of a thought that Frege deploys in his discussion in *Function and Concept* is not one for which he would have had to search very far: the distinction between the sense and reference of *sentences*—that is, the distinction between thought and truth-value—was ready to hand. The problem the puzzle about substitution posed for Frege was therefore not that he could not see how to allow that two sentences can have the same truth-value but express different thoughts.⁵⁰ Rather, the question the puzzle posed was *why* the two sentences express the different thoughts they do.

Frege took himself to have shown that the thought expressed by a sentence is not determined by the references of its parts. If not, it must presumably be

⁴⁹ It is not clear whether Frege ever regarded conceptual contents as the values of the functions with which he identified concepts. Frege does not express a view on this matter in the papers on Boole. It is striking, however, that, during this period, he regards concepts as *intensional*, in the sense that there may be two concepts both true only of Venus (BLC, p. 18). It therefore seems that Frege was not by then taking truth-values to be the denotations of sentences. The matter is complicated, however, since Frege could have done so and still regarded concepts as intensional if he took functions generally to be intensional. Unfortunately, he does not express a view on that question in the papers on Boole.

⁵⁰ In (Thau and Caplan, 2001), Michael Thau and Ben Caplan make this sort of suggestion. While we disagree, and disagree more generally with their reading of Frege on identity—see (May, 2001) and (Heck, 2003)—it should be clear that we do think they were right to emphasize how important Frege’s view that truth-values are the referents of sentences is to the emergence of the distinction between sense and reference. The connection has been underappreciated.

determined by something else about the parts,⁵¹ which we might as well agree to call their *senses*: “The names, whether simple or themselves composite, of which the name of a truth-value [that is, a sentence] consists, contribute to the expression of the thought, and this contribution of the individual [component] is its *sense*” (Gg, §32). So far, then, the notion of a name’s sense is purely programmatic, but, as it happens, Frege also had the resources to make it somewhat less so ready to hand. In his discussion of identity-statements in *Begriffsschrift*, Frege remarks that:⁵²

... [T]he need for a sign for identity of content rests upon the following consideration: the same content can be completely determined in different ways; but that in a particular case *two ways of determining* it really yield the *same result* is the content of a *judgement*. Before this judgement can be made, two distinct names, corresponding to the two ways of determining the content, must be assigned to what these ways determine. (Bg, §8, Frege’s emphasis)

The idea that a proper name has both a “content”, taken to be the object it denotes, and an associated mode of presentation (its *Bestimmungsweise*, in the terminology of *Begriffsschrift*) is thus present early on in Frege’s thinking. The key insight that leads to the distinction between sense and reference, as we find it in Frege’s mature writings, was that the distinction between an object and how it is presented need not be regarded as a distinction between content and something else but can instead be regarded as a distinction within content itself.⁵³ From the perspective thus reached, Frege can therefore write in “On Sense and Reference”:

It is natural... to think of there being connected with a sign... , besides that which the sign designates, which may be called the reference of the sign, also what I should like to call the *sense* of the sign, wherein the mode of presentation is contained. (SM, op. 27)

This suggestion, that a name’s sense “contains” a mode of presentation of the object it designates, gives the otherwise programmatic notion of sense at least some substantial content: The senses of the parts of a sentence determine the thought it expresses because senses present the objects (and functions) the thought concerns, and these may be presented in different ways. Of course, how much illumination

⁵¹ An assumption is being made here to which we shall call attention later, namely, the assumption that the *sentence* expresses a thought.

⁵² Similarly, Frege writes in *Die Grundlagen*: “Why is it... that we are able to make use of identities with such significant results in such diverse fields? Surely it is... because we are able to recognize something as the same again although it is given in a different way” (Gl, §67).

⁵³ This was first observed by Ignazio Angelelli (1967, pp. 38–40).

is thus provided depends upon how much we take ourselves to know about ways in which objects (and functions) may be presented.

Infamously, Frege never says very much more about what modes of presentation are, nor about what the sense of a name is, than he does in the passages we have cited. In much of the secondary literature, especially the older secondary literature, it is assumed that a mode of presentation is a condition an object must satisfy if it is to be the denotation of an expression. If so, then, in some sense,⁵⁴ every proper name will be equivalent to a definite description ‘the ϕ ’, where ϕ abbreviates the mentioned condition. Part of what motivates this interpretation is the fact that the only examples Frege gives of modes of presentation are just such conditions.⁵⁵ But that does not show that modes of presentation *must* be descriptive conditions, only that they *may* be.

A deeper reason to endorse the descriptive interpretation of modes of presentation derives from Frege’s insistence that sense determines reference.⁵⁶ There are stronger and weaker interpretations of this doctrine. On the weaker interpretation, Frege means only that reference supervenes on sense, that is, that any two expressions that have the same sense must also have the same reference. On the stronger interpretation, what Frege is claiming is that a name has the reference it does *because* it has the sense it does. One can see why the stronger interpretation might lead one to suppose that the sense of a name had to be something like a description: If a name’s having the sense it does is to explain its having the reference it does, then the sense of the name must at least determine some condition that an object must satisfy if it is to be the name’s reference. Moreover, the condition must

⁵⁴ In *Naming and Necessity* (Kripke, 1980), Saul Kripke famously insisted that, for both Frege and Russell, descriptions ‘give the meaning’ of names in some sense strong enough to license substitution everywhere, in particular, in modal contexts. We know of no evidence that Frege held any such view. Even if the descriptive interpretation were correct, then, Frege need not have been vulnerable to the arguments in Lecture I of *Naming and Necessity*.

⁵⁵ See, for example, (SM, op. 27, fn). There is one passage that may be an exception:

... [I]f both Leo Peter and Rudolph Lingens understand by ‘Dr Gustav Lauben’ the doctor who is the only doctor living in a house known to both of them, then they understand the sentence ‘Dr Gustav Lauben was wounded’ in the same way; they associate the same thought with it. (Tht, op. 65)

It is not obvious that Frege is suggesting that Peter and Lingens both associate the description ‘the only doctor living in such-and-such a house’ with the name ‘Dr Gustav Lauben’. He could be suggesting, instead, that their grasp of the sense of the name is to be found in their acquaintance with him—not in Russell’s sense, of course, but in a sense close enough to Russell’s that some, such as Gareth Evans (1982), have thought it might do similar work.

⁵⁶ As expressed in such remarks as: “The regular connection between a sign, its sense, and its reference is of such a kind that to the sign there corresponds a definite sense and to that in turn a definite reference, while to a given reference (an object) there does not belong only a single sign” (SM, op. 27).

be to some extent independent of the name's referring to the object it does: Obviously, if the sense somehow incorporated the reference, the name's having that sense wouldn't explain its referring to the object it does.⁵⁷ But we know of no convincing evidence in favor of the stronger interpretation, and the weaker interpretation suggests no particular conception of what sense is.

The true reason the descriptive interpretation of the notion of sense was once so popular, however, is probably that it is simply not obvious what the alternatives are. How else are we to characterize modes of presentation? Nonetheless, we agree with most scholars of Frege's work that the descriptive interpretation is a misinterpretation.⁵⁸ We suggest, moreover, that if one seeks illumination about the notion of sense, one should not look for a direct answer to the question "What is the sense of a name?" There is no such answer to be found in Frege. What is to be found is a theory of logic and language in which the notion of sense has an important role to play. And it, like many other theoretical notions, inherits its content from the broader theoretical framework in which it makes its home.⁵⁹

Running through our discussion has been the claim, which Frege makes in numerous places, that the sense of a sentence is a thought. It is perhaps tempting to suppose that for Frege a thought just is the sense of a sentence, as if his remark in *Grundgesetze*, "The sense of a name of a truth-value I call a *thought*" (Gg, §2), were meant to be definitional. But it is not. For Frege, the notion of a thought is fundamentally a cognitive one. Like the earlier notion of conceptual content, it emerges from Frege's distinction between cognitive events, such as one's making a judgement or considering an hypothesis, and what it is that one judges or considers.⁶⁰ Frege insists, as against the 'psychologistic' logicians, that we must recognize in such episodes something objective that may be considered at one time and at another time judged, or affirmed by one person and denied by another. It is the objective element in such episodes that Frege calls a *thought*. Thoughts, that is, remain for Frege 'possible contents of judgement', to use the terminology of

⁵⁷ It may be that some such line of thought is at the bottom of Dummett's conception of sense. See (Dummett, 1978a).

⁵⁸ Dummett was perhaps the first explicitly to reject this interpretation (Dummett, 1981a, ch 5, appendix). See also (McDowell, 1977) and (Evans, 1985).

⁵⁹ Space limits how much we can say about this broader framework here. See (Dummett, 1981a, chs 5–6) for one classic discussion, as well as (Merrick, 2004), for a more recent one.

Given Frege's ontology of concepts and objects, if asked what senses are, then they are clearly objects. But this can be construed as a theoretical claim, rather than a metaphysical one: Senses are objects just because this is where the slot into the theory, not because there is an independent argument for their existence.

⁶⁰ Frege's most complete discussion of such issues is in (Tht), but similar themes surface in many other places, for example, in his various efforts to write a piece on "logic" (Log79; Log97; IntLog) and in the introduction to *Grundgesetze* (Gg).

Begriffsschrift, or, to use more modern terminology, possible contents of propositional attitudes.

The distinction between thought and judgement is especially evident in natural languages, for these have speakers who have cognitive attitudes towards thoughts.⁶¹ Speakers may judge thoughts, as well as know and believe them. Thus, in identifying thoughts as the senses of sentences, Frege is establishing a connection between language and cognition: he is claiming that with each sentence there is associated as its sense a possible content of a cognitive attitude. That naturally suggests that each sentence has a particular belief associated with it as the belief a literal utterance of that sentence would express. One might wonder, however, what justifies Frege's claim that thoughts are thus associated with the *sentences* of a language as opposed to the weaker claim that each speaker associates a thought with each sentence she understands, different speakers possibly associating different thoughts with the same sentence.⁶² Only if the stronger claim can be defended, one might argue, can the notion of sense be regarded as a *linguistic* notion as opposed to a merely psychological one.

How Frege arrived at his point of view about this matter seems clear enough. It was important to Frege that thoughts should be objective. He insisted, as against those who would confuse thoughts with "ideas", that it must be possible for you to believe the very same thing I do, or again to believe its negation.⁶³ But it was equally important to Frege—in large part because he was so impressed by the use of language as a tool of communication, in particular, as a tool of joint scientific inquiry—that *we express* such agreement and disagreement in our use of language. If Smith says "Aristotle was Greek" and Jones says "Aristotle was not Greek", Jones appears to contradict Smith. But that would not be so if the thought Jones associated with the sentence he uttered was not the negation of the thought Smith associated with the sentence he uttered. Someone who knew that Smith and Jones

⁶¹ It is less evident in logic, where the basic laws (axioms) are taken to be self-evident, and hence judged true universally. As Frege notes, however, the distinction is needed nonetheless, since thoughts can occur embedded within other thoughts, for example, as the antecedents of conditionals, where they need not be judged. (This is one reason to be unhappy with Frege's early use of the terms 'affirmed' and 'denied' rather than 'true' and 'false' in explaining the conditional.)

⁶² The issue we are raising here does not concern context-dependence. It arises as much for arithmetical statements as for empirical ones. But there are, of course, issues about context-dependence that do arise in this area, which Frege famously discusses in "Thoughts" (Tht, opp. 64–6). See the next footnote for some references to the extensive secondary literature on that discussion.

⁶³ There is a long-standing dispute concerning how extensive Frege's commitment to the shareability of thoughts is. The orthodox view, for a long time, was that it is exceptionless, so that the very idea of a thought only one person could entertain would be incoherent: See, for example, (Perry, 1993). That view was challenged, however, in the 1980s by Gareth Evans (Evans, 1985), and Evans's view has become the new orthodoxy. But the issue remains open. For some recent reflections, see (May, 2006a).

associated different thoughts with the sentence “Aristotle was Greek” might rationally regard both of them as speaking truly. For example, if Jones took the sentence to express the thought that the teacher of Alexander the Great was Greek whereas Smith took it to express the thought that the greatest student of Plato was Greek, their apparent disagreement might be merely verbal in the sense that, if they were apprised of this difference between them, they would no longer regard themselves as disagreeing.

That said, however, Frege was aware that to require speakers always to assign the same senses to their words is to set a high standard, one that is not always met in everyday communication. We might take this to indicate that the notion of sense incorporates an idealization, so that the sense of a name is, say, what speakers would, perhaps after reflection and consultation with other speakers, acknowledge as a common standard to which they were willing subject their own usage of the name. Sense would be constant from speaker to speaker by convention.⁶⁴ Frege’s view is different. His view is that we should simply recognize that, strictly speaking, those who associate different senses with a given proper name speak different languages and that communicability hews to weaker criteria than speaking the same language. Linguistic variation is a normal occurrence, and, as Frege himself notes, it is often unimportant that speakers should all associate the same sense with a given proper name. Whether an object falls under a given concept does not depend upon how the object is presented. It follows that communication can succeed between speakers who technically do not speak the same language so long as they can determine that they are speaking about the very same things: “So long as the reference remains the same,” Frege writes, “variations of sense may be tolerated” (SM, op. 27, fn).⁶⁵

But only to a point. It is not *always* unimportant whether the sense is the same for different speakers. The limits of tolerance are to be found where the thought itself matters. A contemporary philosopher might suspect that such cases are to be found most prominently where we attribute propositional attitudes. And, indeed, Frege famously regards sentences of the form ‘*N* believes that *S*’ as asserting a relation between *N* and the thought that *S*, which he takes to be the denotation of the phrase ‘that *S*’.⁶⁶ Such cases are undoubtedly of interest, and they have of course been much discussed. But for Frege, the crucial case is always that of *logic*.

⁶⁴ This sort of view is most strongly associated with Dummett, who writes: “The notion of sense is. . . of importance, not so much in giving an account of our linguistic practice, but as a means of systematizing it” (Dummett, 1981a, p. 105).

⁶⁵ See also (Tht, op. 65), which contains a somewhat more extensive discussion of this issue.

⁶⁶ We are simplifying substantially. Frege’s actual view is that, in certain contexts, expressions do not have their usual references but instead denote what are usually their senses, which Frege calls their “indirect” references (SM, opp. 47–8).

And so it is no surprise that Frege follows his remark that “such variations of sense may be tolerated” with the warning that “they are to be avoided in the theoretical structure of a demonstrative science and ought not to occur in a perfect language” (SM, op. 27, fn).

Why not? The following passage contains a hint:

As if it were permissible to have different propositions with the same wording! This contradicts the rule of unambiguousness, the most important rule that logic must impose on written or spoken language. If propositions having the same wording differ, they can do so only in their thought-content. Just how could there be a single proof of different thoughts? (Geo2, op. 385, fn)

On Frege’s view, logic is the enterprise of showing how true thoughts follow from other true thoughts. But showing this must meet the highest standards, those of proof. The criterion of sense-invariance is part of how we insure the reliability of proofs. If a given sentence, standing as the conclusion of a proof, could be associated with more than one thought, how could we be certain just which thought had been proven? That is why Frege is at pains to insist, in *Grundgesetze*, that his stipulations concerning the significance of his primitive expressions completely determine which thoughts are expressed by the sentences of his formal language. Referring to a much-discussed argument in the preceding sections, Frege writes in section 32 of *Grundgesetze*:⁶⁷

In this way it is shown that our eight primitive names have denotation, and thereby that the same holds good for all names correctly compounded out of them. However, not only a denotation, but also a sense, attaches to all names correctly formed from our signs. Every such name of a truth-value [that is, every well-formed sentence] *expresses* a sense, a thought. Namely, by our stipulations it is determined under what conditions the name denotes the True. The sense of this name—the *thought*—is the thought that these conditions are fulfilled. Now a proposition of *begriffsschrift* consists of the judgement-stroke and of a name. . . of a truth-value. . . It is now asserted by such a proposition that this name denotes the True. Since at the same time it expresses a thought, we have in every correctly-formed proposition of *begriffsschrift* a judgment that a thought is true; and here a thought certainly cannot be lacking. (Gg, §32)

⁶⁷ The elided material concerns sentences containing free variables—what Frege calls “Roman marks”—and is orthogonal to our concerns.

Frege is thus claiming that the theorems proven in *Grundgesetze* have been guaranteed to express completely determinate thoughts.⁶⁸

To the extent that Frege was concerned with language as a tool of communication between inquiring minds, it was to codify those aspects of language that allow it to operate as such in a rigorous, reliable, and sound fashion. Frege's goal, more precisely, was to isolate those aspects of language that are required for reasoning. All communication involves the communication of thoughts, Frege would insist, but if we are to ascertain whether communication abides by the laws of thought—that is, by the laws of logic—then it must be insisted that there be no variation of sense, so that we are dealing with just one language throughout. Otherwise the following sort of exchange might occur: Jones might prove a certain thought and then communicate it to Smith who, as it happens, actually associates a different thought with that same sentence; Smith then correctly derives some other thought from that one and then communicates it back to Jones, who in turn associates a different thought with *that* sentence, one that does not actually follow from the thought that he associated with the original sentence. Consequently, Frege insists, "... we must really stipulate that for every proper name that there shall be just one associated manner of presentation of the object so designated" (Tht, op. 65), although this condition may be relaxed (with the concomitant variation of language within a 'speech community') when not so much is on the line in our communicative interactions.⁶⁹

That languages are defined by the relation between certain linguistic forms (namely, sentences) and the thoughts they express is a central Fregean doctrine. Thus, it is not just that speakers associate thoughts with sentences; languages, including the sense relevant to logic, are individuated in part by what senses their expressions have. In this regard, it is a linguistic fact that

(M) The Morning Star is a planet

expresses a different thought than does

(E) The Evening Star is a planet.

And with this observation, we are close to an appreciation of Frege's solution to his famous puzzle. But we still lack one piece.

⁶⁸ Frege also takes himself to have shown, earlier in *Grundgesetze*, that the axioms of his theory are true and that his rules of inference preserve truth. It follows that, if there is at least one false sentence, not every sentence is a theorem, whence the theory is consistent. Frege apparently understood that the argument he is summarizing had this consequence, for in his response to Russell's letter informing him of the contradiction he writes: "It seems accordingly... that my law V... is false, and that my explanations in sect.~31 do not suffice to secure a meaning for my combinations of signs in all cases" (PMC, p. 132).

⁶⁹ For further elaboration, see (May, 2006b).

Grant that ‘the Morning Star’ and ‘the Evening Star’ are associated with different modes of presentation, so that they have different senses. How does that fact explain why (M) and (E) express different thoughts? That Frege intends such an explanation is clear from his language in *Function and Concept*:

‘ 2^4 ’ and ‘ 4×4 ’ certainly have the same reference, i.e., are proper names of the same number; but they have not the same sense; *consequently*, ‘ $2^4 = 4^2$ ’ and ‘ $4 \times 4 = 4^2$ ’ refer to the same thing, but have not the same sense (i.e., in this case: they do not contain the same thought). (FC, op. 14, our emphasis)

It is clear enough that Frege supposes that the sense of a sentence is determined by the senses of its parts, and this assumption will deliver the conclusion that ‘the Morning Star’ and ‘the Evening Star’ have different senses: If ‘the Morning Star’ and ‘the Evening Star’ had the same senses, then (M) and (E) would have to have the same senses, which they do not. But this assumption will not deliver the explanation Frege wants; it will not, that is to say, license his use of the term ‘consequently’, emphasized in the passage just quoted. The reference of the whole is also determined by the references of the parts, but the fact that the reference of ‘Venus’ differs from that of ‘Mars’ plainly does not require that ‘Venus is a planet’ and ‘Mars is a planet’ must have different truth-values; it merely makes it possible that they should. Similarly, then, if the sense of a sentence is determined by the senses of its parts, then the fact that (M) and (E) contain parts with different senses makes it *possible* that they should have different senses, but it in no way requires that they must. But what Frege seems to be saying here is precisely that (M) and (E) express different thoughts because they contain parts that have different senses.

Frege makes similar remarks in a letter to Peano:

... [T]he sense of the name ‘morning star’ is indeed different from that of the word ‘evening star’. *And so it happens* that the thought of [‘The evening star is the same as the evening star’] is different from that of [‘The evening star is the same as the morning star’]...; for the thought we express in a proposition is the sense of the proposition. (PMC, pp. 127–8, letter xxxiv/11, my emphasis)

Here again, we see evidence of an ambition to explain why (M) and (E), and similar pairs, express different thoughts. This explanatory ambition was, in fact, essential to Frege’s understanding of the notion of sense. The argument concerning (M) and (E) certainly does show—given Frege’s other assumptions—that the thought expressed by a sentence is not determined by the references of its parts. And so, as we said earlier, what thought the sentence expresses must presumably be determined by something else about the parts, which we may call their *senses*. This

way of characterizing the notion of sense is misleading, however, in so far as it suggests that the senses of the parts need only *determine* the sense of the whole. But if that were all that were required, we could simply take the sense of a primitive⁷⁰ expression to be, say, the complete history of its use. Since no two distinct primitives can have been used in precisely the same way—minimally, not both can be uttered at the same time by a given speaker—the thought expressed by a sentence will trivially be determined by the ‘senses’ of the parts. Such a theory would be utterly unsatisfying, however, precisely because the mere fact that ‘the Morning Star’ and ‘the Evening Star’ have different histories of use does not plausibly explain why (M) and (E) express different thoughts but merely makes it possible that they should. Frege’s own account of what sense is, on the other hand, is not vulnerable to this objection. At least, it is intended not to be vulnerable to it: That ‘the Morning Star’ and ‘the Evening Star’ are associated with different modes of presentation is supposed to explain why (M) and (E) express different thoughts.

The claim that the sense of a sentence is *determined by* the senses of its parts is thus much weaker than the claim that the sense of a sentence is *composed from* the senses of its parts. The former claim allows that the sense of the sentence might be something over and above the composed senses, so that in principle two different compositions of senses could converge on the same thought, just as two different compositions of references may converge on the same truth-value. On the latter conception, however, the composition of those senses into a whole determines a thought as the sense of the sentence by *being* that thought: If the senses of the parts differ, then the senses of the wholes they compose will, of necessity, also differ. What we have just argued, then, is that Frege’s explanatory ambitions *require* him to adopt the stronger interpretation of the principle of compositionality—the one that is arguably more deserving of the name—and so to regard thoughts as being composed of senses.

This view—which is explicitly stated, as we noted before, in *Grundgesetze*—is most vividly expressed in the opening remarks of his Frege’s published essay:

It is astonishing what language can do. With a few syllables, it can express an incalculable number of thoughts, so that even if a thought has been grasped by an inhabitant of the Earth for the very first time, a form of words can be found in which it will be understood by someone else to whom it is entirely new. This would not be possible if we could not distinguish parts in the thought corresponding to the parts of a sentence, so that the structure of the sentence can serve as a picture of

⁷⁰ We are assuming here that primitive expressions, unlike complex ones, cannot be meaningful though utterly unused. Whatever the status of that assumption, however, some such trivial account as the one we are considering ought to be possible.

the structure of the thought. (CT, op. 36)

Unfortunately, it is not at all obvious how to apply mereological notions to thoughts: Frege himself, following up the remark just cited, warns that “. . . we really talk figuratively when we transfer the relation of whole and part to thoughts. . . .” (CT, op. 36). The problem is no secret. It is all well and good to say that the sense of ‘The Morning Star is a planet’ contains the sense of ‘the Morning Star’ and the sense of ‘is a planet’ as parts (SM, op. 27, fn). But this is not enough to answer the question how the thought that the Morning Star is a planet is composed of these parts without an answer to the question how the parts are bound together, that is, without an account of how senses cohere to form thoughts. The sense of ‘John loves Mary’ cannot be a mere agglomeration of the senses of the parts, lest ‘Mary loves John’ have the same sense—which, sadly, it does not.

Frege was well aware of this need, writing in “On Concept and Object”:⁷¹

. . . [N]ot all parts of a thought can be complete; at least one must be ‘unsaturated’ or predicative; otherwise, they would not hold together. For example, the sense of the phrase ‘the number 2’ does not hold together with that of the expression ‘the concept *prime number*’ without a link. (CO, op. 205).

Frege insists that his terms ‘complete’ and ‘unsaturated’ are but “figures of speech”, meant in the context of the essay “to give hints” to the reader (CO, op. 205). But one natural way to understand Frege’s suggestion is to take the senses of predicates to be unsaturated in the very same way that their references are: the sense of a predicate is a function from the senses of names to thoughts. Thus, not only does ‘is a planet’ *refer* to a function, it also *expresses* one, namely, the one that maps the sense of ‘Venus’ to the thought that Venus is a planet; the sense of ‘Sirius’ to the thought that Sirius is a planet; and so forth. Compositionality for senses would then reduce to function-application, again in tandem with references. But this will not do. For one thing, it is not clear that it is coherent to regard senses *qua* functions as parts of a thought, however far we stretch that notion. Such functions would be strangely self-referential: The sense of ‘ ξ is a planet’ would be a function from the senses of names to complexes of which that very function was itself a part.⁷²

⁷¹ Similar remarks can be found in the late essay “Negation” (Neg, op. 155).

It is worth noting that, in remarks following the passage we quote, Frege effectively rejects the suggestion that thoughts are structured *if* the structure is conceived as some kind of framework into which senses slot, since we would then have to explain in what respect the structure itself is ‘unsaturated’, and no progress has been made.

⁷² For further discussion of the functional interpretation of the senses of predicates, see (Dummett, 1981b, Ch 13) and (Dummett, 1991a, Ch 6).

Perhaps that idea can be made coherent. Moreover, this view would break down the parallelism with reference, for we do not take concepts to be parts of truth-values.⁷³ But the deeper point is that, if the senses of predicates are functions, then the senses of the parts merely *determine* the sense of the whole. Without additional extrinsic stipulation, a ‘sense-function’ could map distinct senses onto the same thought, and then there is no explanation of why (M) and (E) express distinct thoughts.

There is, however, a more promising way of understanding Frege’s intentions. What we are going to suggest is that the unsaturatedness of the senses of predicates is parasitic on the unsaturatedness of their references. More precisely, our suggestion will be that what binds the sense of a predicate and the sense of a name together into a thought is the interaction of two more fundamental forces: the determination of sense by reference and the composition of references.⁷⁴

Let us think first about straightforwardly functional expressions, say ‘ $\xi^2 - 1$ ’ and ‘ $(\xi + 1)(\xi - 1)$ ’. These have the same reference, for they have the same value for every argument. They have different senses, however, because the common value the functions have for a given argument is determined in different ways: In the one case, we multiply the argument by itself and then subtract one; in the other, we multiply the result of increasing the argument by one by the result of decreasing it by one. These two ways of describing a single mapping from arguments to values correspond to the senses of the two expressions: The sense of a functional expression is a particular way in which values are associated with arguments. It is thus tempting to think of the senses of functional expressions as what are sometimes called ‘functions in intension’, that is, as arithmetic functions individuated intensionally rather than (as is nowadays common) extensionally. We are not suggesting that we should succumb to this temptation. But there is an idea here that is worth preserving, namely, that, since the sense of a functional expression is *a way in which a function may be given to us*, any such sense will inherit the ‘unsaturatedness’ of the function that is its referent. We can use this fact to explain how senses compose.

If the sense of a functional expression is ‘unsaturated’, then it must, in some sense, ‘need completion’ by something else. We suggest that what ‘completes’ the sense of a functional expression is simply an *object*. It might seem as if that’s impossible, since we would then be unable to distinguish the sense of, say, ‘ 4^2 ’ from that of ‘ $(2 \times 2)^2$ ’. And, indeed, if the sense of ‘ ξ^2 ’ were a function from

⁷³ In “On Sense and Reference”, Frege suggests that the references of the parts are parts of the reference of the whole (SM, opp. 35–6), but he later takes the suggestion back, for the obvious reason that Sweden is not part of the capital of Sweden (Darm, p. 255).

⁷⁴ If so, then the composition of senses is mediated by their relation to their references, and it becomes extremely natural to regard sentences containing parts without reference as exhibiting a grave semantic defect. Compare (Evans, 1985, pp. 297ff).

objects to thoughts, then of course the objection would be conclusive. But we are *not* saying that the sense of ' ξ^2 ' is such a function nor, for that matter, that it is any kind of function at all. It certainly does not follow from the fact that the senses of predicates are unsaturated that they are functions: Predicates are unsaturated, too (WF, opp. 663–4), but Frege certainly does not regard them as functions. What we are suggesting is that the sense of ' ξ^2 ' needs to be completed by an object *because* ' ξ^2 ' *denotes a function* which itself needs to be completed by an object. But when an object is provided as argument, it must be given to us in some particular way. What distinguishes the sense of ' 4^2 ' from that of ' $(2 \times 2)^2$ ' is how the argument is given to us. Let's suppose that the sense of ' 4 ' presents 4 as the successor of 3. Then the sense expressed by ' 4^2 ' may be characterized as follows: to one who understands the expression ' 4^2 ', this expression presents an object as the number that results if the successor of 3 is multiplied by itself. The expression ' $(2 \times 2)^2$ ' presents that same object in a different way.

It now becomes extremely tempting to say that the sense of a functional expression is completed by the sense of a name. Such an interpretation, the Siren notes, can be defended using the same move we just made: Deny that the senses of functional expressions are functions that takes senses as arguments. The difficulty, however, is that it would then be impossible to explain how the sense of ' ξ^2 ' is completed in a sentence like: $\forall x(x^2 > x)$, the problem being that variables do not have senses. On our reconstruction—it is hard to call it an 'interpretation'—there need be no such problem: The sense of the expression will be completed in virtue of its *reference's* being completed through the semantic analogue of variable binding.⁷⁵ Nonetheless, there is something right about the proposal, for there is a clear sense in which, on our view, the sense of a functional expression is *indirectly* completed by the sense of a name. What completes the sense of a functional expression is, most fundamentally, an object. But when we think of an object as the value of ξ^2 for some particular argument, we must think of the argument itself in some particular way. We will then be thinking of an object as the result of applying a function that is given to us in a certain way to an object that is given to us in a certain other way.

It should now be clear enough what we want to say about the senses of predicates and how they are bound together with the senses of names to form thoughts. Consider our sentence (M), 'The Morning Star is a planet'. The sense of the predicate ' ξ is a planet' we identify with a way of presenting a function from objects to truth-values.⁷⁶ It needs to be completed by an object because the function it

⁷⁵ A similar answer is available to the functional interpretation: Since the sense of ' ξ^2 ' is, on that view, a function, its argument-place can be bound by a higher operator.

⁷⁶ Of course, a finer analysis would distinguish parts within the predicate, too, recognizing at least the tense as another component. So something like a mode of presentation of the present would also

denotes needs to be completed by an object. The sense of the name ‘the Morning Star’ determines an object, and so this name’s sense indirectly completes the sense of the predicate and its referent completes both the sense and referent of the predicate. The references of the expressions can then compose, via function-application, to determine a truth-value for the whole sentence (the True, in this case). To entertain the thought expressed by (M) is thus to think of an object given in a certain way as being mapped to the True by a function given in a certain way. The thought expressed by (M) is thus, to a first approximation, that the last celestial body visible in the morning is mapped to the True by the function that maps all and only planets to the True. Or, much more precisely, albeit much less informatively: that the Morning Star is a planet. To entertain the thought expressed by (E), on the other hand, is to think of an object given as the first celestial body visible in the evening as being mapped to the True by the function that maps all and only planets to the True. The thought expressed is thus: that the Evening Star is a planet. The sentences (M) and (E) therefore express different thoughts, and they do so for just the reason that Frege cites: They are made up of parts with different senses.

In the end, then, no ‘sense-glue’ is needed to bind the parts of a thought together. The sense of a name is bound to the sense of a predicate because the sense of the name determines an object and the sense of the predicate determines a function which, being unsaturated, may be completed by that object. It follows that, absent reference-failure, a thought will have one of the two truth-values, and the senses of the parts will, through how they determine the references of the parts, determine under what condition that value will be the True. That is to say, the familiar Fregean doctrines that thoughts are truth-evaluable and that they determine truth-conditions emerge, on our account, as consequences of deep features of Frege’s conception of how senses combine to form thoughts. The same cannot be said for the alternative interpretation that takes the senses of predicates to be functions. There is, of course, nothing in that view that *precludes* thoughts from being truth-evaluable or from determining a truth-condition. But since thoughts are values of the functions with which the senses of predicates are identified, it will be impossible to characterize those functions absent an antecedent conception of what thoughts are (Dummett, 1981b, Ch 13). One will then have no alternative but to take it as axiomatic that thoughts are truth-evaluable and that they determine truth-conditions, and so will be unable to explain these facts.

In section 2, we were at pains to emphasize that Frege’s doctrine that concepts are unsaturated is primarily, and perhaps even entirely, semantic rather than

enter. How such context-dependence is to be handled in a broadly Fregean framework is a much discussed and very difficult question: For discussion, see (Perry, 1993), (Kaplan, 1989), (McDowell, 1977), (Burge, 1979), (Evans, 1985), (Heck, 2002), and (May, 2006a), among many other papers.

metaphysical. What it is for an object to fall under a concept will be, for Frege, illuminated by an account of linguistic meaning. Something similar can now be said about Frege's conception of thoughts. We emphasized earlier that, for Frege, the notion of a thought is fundamentally a cognitive one. That is *not* to say that it is fundamentally a *psychological* one, and Frege would vehemently have denied that it was. It is, indeed, as much a manifestation of Frege's genius as anything we know how carefully he separates these two claims. Both the danger, and Frege's strategy for avoiding it, are visible in his account of how thoughts cohere. The danger is that we will reach for the easy solution that seems to be offered by appeal to some psychological analogue of predication, some notion of 'applying a concept to an object'. The strategy for avoiding this danger is to occupy and steadfastly refuse to abandon the semantic perspective that pervades Frege's mature writings. Such a psychological notion may be needed to explain how or why we are able to *entertain* the thoughts we do, but it is not, Frege insists, needed to explain how the parts of a thought cohere. That, too, is to be illuminated by an account of linguistic meaning.

5 Closing Remarks

From Frege, what endures? If we were to distill his contribution to the study of language down to its essence, it would be his recognition of the necessity of compositionality to an account of truth and meaning. Although Frege never elaborated a formal theory of truth for *begriffsschrift*, as Tarski did for the calculus of classes, he did develop an informal theory that is no less mathematical for being informal⁷⁷ and which remains the first demonstration of how a compositional theory of truth for a language of reasonable expressive power can be given. The central aspects of this semantic theory remain with us, if not always in their particulars, then at least in their guiding ideas. Thus, underlying Frege's conception of concepts as unsaturated is his insistence that the semantics of predicates must reveal the role they play in determining truth-values; underlying his conception of thoughts is his insistence that the semantics of sentences must reveal the role their constituent parts play in composing entities that have truth-values.

As uncontentious as these doctrines may seem to us today, Frege's notion of

⁷⁷ For a defense of this way of reading Part I of *Grundgesetze*, see (Heck, 2007). This kind of reading originates, of course, with Dummett (Dummett, 1981a).

In application to the full *begriffsschrift*, Frege's theory of truth is inevitably flawed, since there is no consistent (classical) theory of truth for that language. Indeed, the part of the theory concerned with the smooth breathing, from which names of value-ranges are formed, is exceedingly peculiar. The remainder of the theory, however, is clean and familiar, differences from Tarski being due to Frege's different treatment of quantification.

sense still raises the hackles of many. But the underlying idea has persisted here as well: In a slogan, “No reference without information”. Thus, though the terminology differs, it is quite commonly held that, to use a proper name to speak of an object, a speaker must be in possession of conceptual information about the name’s reference: The speaker must have a way of thinking of it, a dossier of information about it, a body of knowledge concerning it, a guise through which she thinks of it, or—suggestive of the Fregean heritage—a mode of presentation of it.⁷⁸ How close a given view is to Frege’s is largely a function of how much information about the reference a speaker is required to have. But, whatever the other differences between them, these contemporary views all deny that the information the speaker associates with the name determines its reference.⁷⁹ It is here that the fundamental difference between ‘new’ theories of reference and Frege’s is to be found.

Frege himself no doubt would have viewed these newer accounts of sense, with their appeal to inherent subjectivity and speaker to speaker variability, as heralding a new psychologism no less pernicious than the old.⁸⁰ Contemporary writers are liable to dismiss this concern, and their tendency to do so is, we believe, indicative of a more fundamental change from Frege’s views, one concerning how language itself is conceived. For Frege, languages are inherently interpreted systems, characterized by the association between senses and symbols. Today, on the other hand, we are used to thinking of languages without this tight bond between syntax and semantics. The resulting perspective has proved extraordinarily fruitful. Metaling, as we know it, would not be possible without the idea that a fixed language can have various interpretations,⁸¹ and the question how syntax and semantics are related is among the most difficult and, therefore, most fruitful posed by contemporary linguistic theory. This change in how language is conceived places the notion of sense itself in a very different perspective: If languages are conceived as Frege conceived them, then sense is semantic by definition; for languages conceived as they are today, sense not only need not be semantic, it need not even be linguistic.

In this regard, the contemporary debate about sense seems to us to be one piece of a more general discussion that continues to animate the philosophy of

⁷⁸ There are more versions of this view than we can reasonably cite here. For a few different versions, see (Grice, 1969), (Schiffer, 1978), (Evans, 1982), (Salmon, 1986), (Castañeda, 1989), and (Forbes, 1990). Some, notably Michael Devitt (1996) and Saul Kripke (1980) have proposed that a mere causal link to the object is sufficient, even in the absence of information about the bearer. That view represents a complete break with Frege, but it remains a minority position.

⁷⁹ The *locus classicus* of such arguments against Frege is of course the second lecture of *Naming and Necessity* (Kripke, 1980).

⁸⁰ And Frege’s premier modern exponent, Michael Dummett, would agree with him (Dummett, 1991b).

⁸¹ Frege’s famous dispute with Hilbert was, most fundamentally, over how languages are to be characterized (Antonelli and May, 2000).

language: how we are to fit into a very different conception of language the very real semantical insights that Frege bequeathed to us.⁸²

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