Frege's Principle*

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> In my *Grundlagen der Arithmetik*, I sought to make it plausible that arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition. In the present book this shall now be confirmed, by the derivation of the simplest laws of Numbers by logical means alone. (Frege, 1962, v. I, p. 1)

In his *Grundgesetze der Arithmetik*, Frege does indeed prove the "simplest laws of Numbers", the axioms of arithmetic being among these laws. However, as is well known, Frege does not do so "by logical means alone", since his proofs appeal to an axiom that is not only not a logical truth but is a logical falsehood. The axiom in question is Frege's Basic Law V, which governs terms of the form "$\grave{\epsilon}\Phi(\epsilon)$", terms that purport to refer to what Frege calls "value-ranges". For present purposes, Basic Law V may be written:[1]

$$(\grave{\epsilon}F\epsilon = \grave{\epsilon}G\epsilon) \equiv \forall x(Fx \equiv Gx)$$

The formal theory of *Grundgesetze*, like any (full)[2] second-order theory containing this sentence, is thus inconsistent, since Russell's Paradox is derivable from Basic Law V in (full) second-order logic.

In *Die Grundlagen*, Frege does not present a formal proof of the axioms of arithmetic. Instead, he merely sketches the proofs of a number of basic facts about numbers. The proofs of the corresponding results in *Grundgesetze* follow these sketches closely, for the most part.[3] In his proof-sketches, Frege does not refer to value-ranges, but to what he calls "extensions of concepts", of which value-ranges are a generalization; as we shall see below, value-ranges and Basic Law V are later introductions to his system.

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[1] Strictly speaking, Basic Law V is:

$$(\grave{\epsilon}F\epsilon = \grave{\epsilon}G\epsilon) = \forall x(Fx = Gx)$$

For Frege, truth-values are objects in the domain of the first-order variables, so the Law states that the truth-value of "$\forall x(Fx = Gx)$" is identical with that of "$\grave{\epsilon}F\epsilon = \grave{\epsilon}G\epsilon$". So formulated, it introduces, not just extensions of concepts, but the value-ranges of functions in general.

[2] This restriction is needed because Basic Law V is consistent both with simple and with ramified predicative second-order logic (Heck, 1996).

[3] As noted in the postscript to Chapter **??**, my views on this have changed over the years.

Presumably, however, any other formal principle by means of which Frege might have intended to formalize his informal use of extensions of concepts in *Die Grundlagen* would also have been inconsistent.

Frege's informal proofs, in *Die Grundlagen*, begin with his showing how the notion of one-one correspondence can be defined in logical terms: Frege explains, as is now standard, that the Fs can be correlated one-one with the Gs just in case there is a one-to-one relation R that relates each F to a G and which is such that, for each G, there is some F that R relates to it (Frege, 1980a, §§70–2). This done, Frege reminds the reader that, according to his usage, a concept F is equinumerous with a concept G just in case the Fs can be correlated one-one with the Gs. He then gives his explicit definition of names of numbers, which is:

> the number belonging to the concept F is the extension of the concept "equinumerous with the concept F". (Frege, 1980a, §72)

The number of Fs is thus, as it were, the class of concepts having the same cardinality as F. Frege then turns immediately to the derivation, from this definition, of HP:

> the number belonging to the concept F is identical with the number belonging to the concept G if [and only if] the concept F is equinumerous with the concept G. (Frege, 1980a, §73)

We will discuss the role HP plays in Frege's philosophical views below.

Once he has proven HP, Frege outlines proofs that each number has at most one predecessor, that each natural number has exactly one successor, and so forth. In these proofs, Frege makes no further appeal to his explicit definition: The proofs depend only upon HP itself. Frege's method suggests that we may divide his proofs of the axioms of arithmetic, in *Die Grundlagen*, into two parts: First, a proof of HP from the explicit definition; Second, a proof of the axioms of arithmetic from HP, a proof in which the explicit definition plays no role whatsoever and in which, perhaps, the extensions of concepts too play no role.[4] Indeed, not only do Frege's informal proofs of the axioms neither appeal to the explicit definition nor make any explicit use of the notion of an extension, proofs of the axioms from HP really can be given within second-order logic (Wright, 1983). It is this beautiful and surprising result that George Boolos has urged us to call *Frege's Theorem*.

It is natural to wonder at this point whether Frege's formal arguments in *Grundgesetze* have a similar structure, that is, whether Frege derives HP from his explicit definition (now given in terms of value-ranges) and then derives the axioms of arithmetic from HP alone, no further use being made either of the explicit definition or of value-ranges. The explicit definition does not, it is true, play any further role. But the answer to our question, so formulated, is "No", because Frege makes use of value-ranges

[4] The first to note that Frege's proofs can be so read was Peter Geach (1955), though Charles Parsons (1995a) is more explicit about the point.

throughout Part II of *Grundgesetze* (in which he proves the axioms). But the question ought really to be: Does Frege derive the axioms of arithmetic from HP without making any *essential* use of value-ranges? Does he use value-ranges, in those proofs, merely for convenience? The answer to this question is "Yes", as we saw in Chapter **??**. So the only essential appeal Frege makes to Basic Law V is in the proof of HP: All the proofs in Part II of *Grundgesetze*, except that of HP itself, are therefore carried out in a consistent sub-theory of the formal theory of *Grundgesetze* (excepting inessential uses of value-ranges), namely, in Frege Arithmetic—second-order logic plus HP. So Frege proved Frege's Theorem, which is as it should be.[5]

This fact has enormous importance for our understanding of Frege's philosophy, in particular, for our understanding of his philosophy of mathematics. What I wish to do here is to begin an investigation of its import by discussing certain historical questions that arise immediately. Before turning to that discussion, however, we must address a prior question, namely, whether Frege knew that the axioms of arithmetic could be proven in Frege Arithmetic. For, if Frege did not know that—if it were, so to speak, a happy accident that Part II of *Grundgesetze* can be read as a formal proof of Frege's Theorem—the fact that it can be so read would presumably have no significance for our understanding of Frege's thought.

There are many more questions about the role HP plays in Frege's thought than we shall even begin to answer here. My hope is that our discussion will demonstrate that a proper understanding of Frege's work requires an understanding of his attitude toward Basic Law V; of why it came to occupy so central a place in his philosophy; of why its refutation, in his own opinion, brought a large part of his life's work to ruin. It is commonly assumed that Frege abandoned his attempt to prove the axioms of arithmetic within (higher-order) logic because he believed that, with Basic Law V refuted, he could no longer do so. But the importance of Basic Law V does not lie in its formal role, for there was, in a clear sense, no *formal* obstacle to the logicist program, even after Russell's discovery of the contradiction, and Frege knew it. Indeed, we can only understand Frege's attitude toward Basic Law V once we appreciate the ridiculously meager formal role it plays.

1 Numbers as Extensions of Concepts

Crispin Wright has suggested that Frege's logicism may be reformulated, in the wake of Russell's Paradox, as follows:

> [I]t is possible, using the concepts of higher-order logic with identity, to explain a genuinely sortal notion of cardinal number; and hence to deduce appropriate statements of the fundamental

[5] Actually, Frege does make one use of value-ranges that it is not trivial to eliminate, namely, in his definition of ordered pairs. But Frege does not use ordered pairs in his proofs of the axioms of arithmetic, only in his proof of their categoricity. And, in any event, Frege arguably knew his use of ordered pairs to be eliminable, too (Heck, 1995).

> truths of number-theory...in an appropriate system of higher-order logic with identity to which a statement of that explanation has been added as an axiom. (Wright, 1983, p. 153)

Wright's suggestion is that Frege's heirs may, and Frege should, just abandon the explicit definition of number and install HP as the fundamental axiom of the theory of arithmetic. Since Basic Law V is used only in the formulation of the explicit definition and the derivation of HP from it, no appeal whatsoever to Basic Law V is then required, and Russell's Paradox ceases to be an obstacle.

The first question which ought to strike one, once one realizes that Frege knew he could derive the axioms of arithmetic from HP, is: Why did he not adopt this course himself?

To answer this question, we must first understand why Frege introduces extensions of concepts in the first place, namely, to resolve the so-called 'Caesar problem'. According to Frege, a proper explanation of names of numbers must yield an explanation of the senses of identity statements containing names of numbers (Frege, 1980a, §62). Frege first argues that "the sense of the proposition 'the number which belongs to the concept F is the same as that which belongs to the concept G'" is given by HP: "The number of Fs is the same as the number of Gs" is true just in case the Fs can be correlated one-one with the Gs (Frege, 1980a, §63). The question of the central sections of *Die Grundlagen* is whether HP, on its own, can be taken as a complete explanation of the senses of identity statements concerning numbers.

Famously, Frege considers three objections to the claim that it can, quickly rejecting the first two (Frege, 1980a, §§63–5). The third objection is that HP provides only for the resolution of questions of the form, "Is the number of Fs the number of Gs?" It does not determine the answer to such questions as, "Is the number of Fs Julius Caesar?" Hence, HP does not provide a general explanation of the senses of numerical identities: It explains only certain such statements, ones in which what are recognizably names of numbers—in the first instance, names of a quite particular form, viz., "the number of Fs"—occur on both sides of the identity-sign (Frege, 1980a, §§66–7). Frege then considers several attempts to resolve this problem but finds them lacking, eventually settling for an explicit definition of names of numbers, namely, that mentioned earlier: The number of Fs is the extension of the concept "concept which can be correlated one-one with the concept F" (Frege, 1980a, §68). The question whether Caesar is a number then reduces to the question whether he is an extension of a certain kind. Frege says that he "assume[s] that it is known what the extension of a concept is" (Frege, 1980a, §68, note), and it is natural to interpret him as assuming it known whether Caesar is an extension and, if so, which extension he is.

Now, it would be natural to suppose that the explicit definition makes HP otiose. However, this would merely be an understandable mistake. When Frege recapitulates the argument of *Die Grundlagen* at the end of the book, he writes:[6]

> The possibility of correlating one-to-one the objects falling under a concept F with the objects falling under a concept G, we recognized as the content of a recognition-statement concerning numbers. Accordingly, our definition had to lay it down that a statement of this possibility means the same as [*als gleichbedeutend mit*] a numerical identity. (Frege, 1980a, §106)

A proper definition of (names of) numbers is, in a sense to be explained, required to specify that the sense of an identity statement connecting names of numbers is given by HP.[7] It need not do so directly, and the Caesar problem obstructs any attempt to explain statements of numerical identity by means of HP alone: Nonetheless, the explanation of the senses of such statements must, in a sense to be explored below, yield HP. Frege's explicit definition yields HP in the strongest possible sense, since it provably implies it.

Frege's objection is thus not that HP fails to capture the senses of those identity statements which *do* have numerical terms (of a certain kind) on either side of the identity-sign. His objection is that it fails to explain the senses of other sorts of identity statements involving names of numbers. Most importantly, we cannot take ourselves thus to have explained the senses of (open) sentences of the form "x is the number of Fs", sentences in which a name of a number and a free variable flank the identity-sign, and this would appear to imply that we cannot, by means of HP alone, "obtain... any satisfactory concept of Number" (Frege, 1980a, §68). The concept "ξ is a number" can only be defined thus: For some F, ξ is the number of Fs (Frege, 1980a, §72). But we have just admitted that we have not explained what it is for "ξ is the number of Fs" to be true of an object, but only what it is for certain very special sentences containing this concept-expression to be true.

Now, as said earlier, extensions of concepts, and the explicit definition of numbers, are introduced by Frege to resolve the Caesar problem, which prima facie is a philosophical problem, not a formal one: That Frege appeals to the explicit definition only in the derivation of HP suggests, further, that extensions and the explicit definition are introduced only to resolve the Caesar problem. If Frege could not abandon Basic Law V, it is therefore because he could not abandon the explicit definition; and if that is so, he could not abandon Basic Law V because he could not solve the Caesar problem without it. Of course, it is antecedently possible that Frege thought that there was some formal problem, a formal reflection of the Caesar problem, that could only be overcome with the help of the explicit definition: In fact,

[6] I have altered Austin's translation here. The German "*erkannten wir*" is translated by Austin "we found", which does not imply quite so strongly as does "we recognized" that we *correctly* found.

[7] I am deeply indebted to Dummett's discussion of this issue (Dummett, 1991, pp. 178–9), as should be clear from what follows.

there is no such problem, at least within the theory of arithmetic, since axioms for second-order arithmetic can be derived from HP without appeal to the explicit definition. Since, as I shall shortly argue, Frege knew that the axioms could be derived from HP, he knew there was no such formal problem: no problem that arises within the formal theory for whose resolution the explicit definition was needed. Whether there is some meta-theoretic problem for whose resolution the explicit definition is required is another question, one to which we shall return.

2 The Importance of HP in Frege's Philosophy of Arithmetic

I have claimed that Frege's derivation of the axioms of arithmetic, in *Grundgesetze*, requires appeal only to HP, the only essential appeal to Basic Law V being in the proof of HP itself. Unless Frege himself knew as much, however, this fact is of little significance for the interpretation of his work. There is, however, good reason to think that Frege did know that neither the explicit definition nor Basic Law V was not needed for the proofs of the axioms. Moreover, there is good reason to think that he thought it significant that the explicit definition was needed only for the derivation of HP. Let me discuss the second point first.

In *Frege: Philosophy of Mathematics*, Michael Dummett argues that Frege's explicit definition of numerical terms is intended to serve just two purposes: To solve the Caesar problem, that is, to "fix the reference of each numerical term uniquely", and "to yield" HP (Dummett, 1991, ch. 14). The explicit definition is in certain respects arbitrary, since numbers may be identified with a variety of different extensions (or sets, or possibly objects of still other sorts): There is, e.g., no particular reason that the number six must be identified with the extension of the concept "is a concept under which six objects fall"; it could be identified with the extension of the concept "is a concept under which only the numbers zero through five fall" or that of "is a concept under which no more than six objects fall".[8] Now, one might wonder how a definition that is arbitrary in this way could possibly be a definition of names of numbers, as we ordinarily understand them. Yet it is essential that it be just such a definition, for Frege's goal is not to show that some formal theory bearing but a syntactic relationship to arithmetic can be developed within logic: His goal is to show that arithmetic as we ordinarily understand it is analytic, that our concept of number is logical in character, and that the truths of arithmetic we take ourselves to know are analytic of our concept of number.

This point can be made more vivid if we consider the relationship between Frege's views about geometry and those about analysis, i.e., the theory of the real numbers. Frege held not only that arithmetic, but analysis too, was analytic. Now, given a definition of ordered pairs (Frege, 1962, v. I,

[8] See the Postscript for some reservations about this claim.

§144), geometrical objects can be represented in the Cartesian plane (more generally, in Cartesian n-space). Not only that: It is easy to prove that the axioms of Euclidean geometry hold for such objects. Why, then, is Frege not committed to the view, which he repeatedly rejects, that geometry is analytic (or, more generally, that it has the same epistemological status as analysis)? Have we not explicitly defined what points are, what lines are, and so forth, and produced a proof of the axioms of Euclidean geometry?—I do not know exactly what Frege would have said about this question, but the most natural reply, and the one I expect he would have made, is as follows.[9] The fact that there is a definition of the fundamental geometrical concepts from which the axioms of Euclidean geometry are provable does not imply that there is a *correct* definition from which the axioms are provable. Or again: The representability of geometrical objects in real 3-space does not necessarily yield a proof of the axioms of Euclidean geometry, not if these axioms are supposed to have the same content as the axioms as we ordinarily understand them.

To reject the analyticity of geometry, Frege must reject the identification of geometrical objects with parts of Euclidean 3-space. At the same time, however, he must allow the identification of numbers with the extensions of certain concepts. Something must thus distinguish the explicit definition of (names of) numbers from this (alleged) explicit definition of (names of) points, lines, and so forth. The suggestion made by Dummett, and hereby endorsed by me, is that, for Frege, an acceptable explicit definition of names of Fs is one that immediately yields[10] an explanation of the senses of identity statements connecting names of Fs as we ordinarily understand them. There is no obvious reason the definition must yield any verdict on the truth-values of statements of the form "$t = x$", where t is our usual sort of name for an F and x apparently not a name of an F. Hence, to be an explicit definition of numbers is immediately to yield HP, since it is in terms of HP that the senses of identity statements connecting names of numerical terms are to be explained.[11]

What makes Frege's explicit definition a definition of numerical terms is thus that it has HP as an immediate consequence: It is this that constrains the explicit definition, that allows Frege to claim that, even though his identification of numbers with extensions of concepts is required by nothing in our ordinary understanding of numerical terms, the definition is nonetheless a definition of numerical terms. Indeed, according to Dummett, it is HP that, according to Frege, "embodies the received sense of" numerical terms. If so, then, an explicit definition that yields HP thereby answers to

[9] Frege's two series of papers entitled "On the Foundations of Geometry" contain several points very much along the lines of what I am about to say.

[10] Much more could be said about this notion of "immediately yielding" the explanation, but I shall not pursue such questions here.

[11] One might wonder, however, if we have quite so much freedom in choosing the objects with which we identify numbers. It seems likely, to me anyway, that our knowledge that Caesar is *not* a number plays an important role in Frege's understanding of the problem. This point is taken up in Section **??**.

everything in our ordinary understanding of names of numbers (Dummett, 1991, p. 179).

Now, Frege's formal theory of arithmetic, constructed along the lines of *Grundgesetze*, is to contain an explicit definition of names of numbers. As said, however, different explicit definitions could be given, and each of these will yield a different class of theorems: So, one might ask, on what basis is it claimed that the logical consequences of Frege's explicit definition are analytic of the concept of number? The answer to this question should now be clear: Not all the theorems of any particular such theory are claimed to be analytic of the concept of number, only those whose proofs do not depend upon arbitrary features of the explicit definition. Since a given explicit definition is non-arbitrary precisely in so far as it implies HP, the theorems that are analytic of the concept of number are just those that follow from HP. It is therefore essential to Frege's philosophical project that the explicit definition, arbitrary as it is, should not figure in the proofs of any of the axioms of arithmetic—except, of course, that of HP itself. It is for this reason that the explicit definition must "immediately yield" HP, for the explicit definition must fall entirely out of consideration *once* it has yielded HP.

3 The Role of Basic Law V in Frege's Derivation of Arithmetic

Frege's views about arithmetic thus require him to derive the axioms of arithmetic in two stages: First, to derive HP from his explicit definition, and then to derive the axioms of arithmetic from HP, making no further appeal to his definition. This does not yet show, however, that Frege knew that no further appeal to Basic Law V was needed. Further argument is needed conclusively to establish this point.

Circumstantial evidence for this claim comes from the Introduction to *Grundgesetze*, in which Frege writes that "internal changes in my Begriffsschrift. . . forced me to discard an almost completed manuscript"; one of the most important of these changes was the introduction of value-ranges and Basic Law V (Frege, 1962, v. I, pp. ix–x).[12] It seems likely that this early derivation of the axioms of arithmetic from HP did not make any mention at all of extensions of concepts. Without Basic Law V, or something very much like it, Frege would have had no means for effecting reference to the extensions of first-level concepts.

One may feel some resistance to this claim, because one wants to ask: How could Frege have derived HP from the explicit definition without re-

[12] It is unclear how much of the formal derivation Frege had when he wrote *Die Grundlagen*. That he had quite a bit of it is apparent from the letter from Stumpf to Frege, written in 1882 (Frege, 1980b, pp. 171–2). Moreover, almost all of the interesting logical results needed in the derivation of arithmetic from HP are already present in *Begriffsschrift*, with the exception of *Grundgesetze*, Theorem 141. On the other hand, as argued in Chapter **??**, there is a subtle confusion in Frege's proof in *Die Grundlagen* which he could not have made had he formally worked out the whole proof. I discuss the matter further elsewhere (Heck, 2019).

ferring to the extensions of concepts, without some axiom like Basic Law V? How could he even have formulated the explicit definition? The most plausible answer, prima facie, is that, while he did not have terms for value-ranges in his system, Frege did have terms for the extensions of second-level concepts; and, furthermore, that the formal system contained some axiom, similar to Basic Law V, governing them. However, the extensions of second-level concepts are of much less general utility than are those of first-level concepts: It is obvious how one might make quite general use of the extensions of first-level concepts, since these are essentially (naïve) sets, and Frege does make quite general use of value-ranges in *Grundgesetze* (though, as mentioned, these more general uses are inessential). But it is harder to imagine how one might make general use of second-level extensions. So, even if Frege did have terms for the extensions of second-level concepts in his formal theory, they probably were not used in the derivation of the axioms of arithmetic from HP, even if they were used in the formulation of the explicit definition and the derivation of HP from it.

Further evidence is provided by Frege's remark, toward the end of *Die Grundlagen*, that he is not committed to the claim that the Caesar problem can be solved only by means of his explicit definition, that he "attach[es] no decisive importance even to bringing in the extensions of concepts at all" (Frege, 1980a, §107). This would be a strange remark for him to make if his formal system contained an axiom governing extensions that was essential to the proofs of the axioms from HP.

The evidence thus far mentioned in not conclusive, however. Conclusive evidence is provided by a letter Frege wrote to Russell in 1902. Discussing whether it might be possible to do without value-ranges, or classes, in a logicist development of arithmetic, Frege writes:

> We can also try the following expedient, and I hinted at this in my *Foundations of Arithmetic*. If we have a relation $\Phi\xi\eta$ for which the following propositions hold: (1) from $\Phi(a, b)$, we can infer $\Phi(b, a)$, and (2) from $\Phi(a, b)$ and $\Phi(b, c)$, we can infer $\Phi(a, c)$; then this relation can be transformed into an equality (identity), and $\Phi(a, b)$ can be replaced by writing, e.g., "$\S a = \S b$"... But the difficulties here are [][13] the same as in transforming the generality of an identity into an identity of value-ranges. (Frege, 1980b, p. 141)

The suggestion is thus precisely that Basic Law V be abandoned and HP be taken as an axiom. The difficulty we must face if we choose this option is presumably the Caesar problem, which Frege has been discussing just prior to this passage. In any event, since it would be utterly pointless to replace the explicit definition with HP if value-ranges were going to be needed in the proof of the axioms of arithmetic from HP, Frege surely did know that

[13] At this point, the translation inexplicably contains the word "not", which is not found in Frege's original letter.

the axioms of arithmetic could be derived from HP without the help of Law V. That is, he knew that, in so far as reference to extensions was required in his formal theory, such reference was required only in order to formulate the explicit definition and to derive HP from it.

4 Frege's Derivations of HP

Basic Law V, value-ranges, and the extensions of concepts play only a very limited formal role in Frege's derivation of the basic laws of arithmetic: They are required only for the derivation of HP. How, then, could the refutation of an axiom of such little formal import have had, in Frege's own estimation, such a devastating effect upon his philosophy of arithmetic? It is not easy to take this question seriously. It is easy to say that Basic Law V was obviously needed if Frege was to show arithmetic to be a branch of logic. But the question is why *that* is: why Frege himself regarded the use of Law V as indispensible to the logicist project.

What I wish to discuss, henceforth, is not this question, but a slightly different one, namely: What reason there is to suppose that the Frege of *Die Grundlagen* intended his explicit definition to be given within his formal theory in the first place? What reason is there to suppose that extensions were to play *any role at all* in his derivation of the laws of arithmetic? By answering this question, we can hope to understand a little better why Frege could not just reject Basic Law V, why it was essential to his philosophy of arithmetic. But, more importantly, the investigation of Frege's attitude toward Law V must, I think, begin with a proper understanding of the options he thought himself to have. For example, one might wish to know what Frege's position was *circa* 1884, because that would tell us what his position was before the discovery of Basic Law V. If Frege thought, at that time, that extensions had some important role to play in his formal derivation of arithmetic, if at that time he thought it necessary to appeal to some analogue of Law V governing the extensions of second-order concepts, then the "retreat" to an earlier view that he suggests to Russell would not have helped very much: The obvious analogue of Law V governing the extensions of second-order concepts[14] can be show to be inconsistent. But, if his earlier view were that extensions had no formal role to play, such retreat might have been an option for him.

Above, I raised the question, "How, *circa* 1884, could Frege formally have derived HP from the explicit definition?" If we accept the presupposition of the question, there can be no answer other than that he had, in his formal system, terms for the extensions of second-level concepts and an axiom governing them. But it is important to recognize that the question presupposes that, *circa* 1884, Frege envisioned, or would have required, a formal derivation of HP from his explicit definition (even if he did not then have such a derivation). We ought not just to assume that he would have.

[14] See below for the analogue.

Our question is thus: Can it coherently be maintained that the Frege of *Die Grundlagen* did not intend HP to be derived, in the formal theory, from the explicit definition? that the explicit definition was not even to be stated in the formal theory? What makes this position seem so implausible is that, in *Die Grundlagen*, Frege sketches a *proof* of HP; during that discussion, he appeals directly to the explicit definition. It therefore cannot be denied that the explicit definition plays a role in this proof; the question is what role it plays.

Frege's sketch of the derivation of HP in *Die Grundlagen* begins as follows:

> Our next aim must be to show that the Number which belongs to the concept F is identical with the Number which belongs to the concept G if the concept F is equinumerous with the concept G....
>
> On our definition, what has to be shown is that the extension of the concept "equinumerous with the concept F" is the same as the extension of the concept "equinumerous with the concept G", if the concept F is equinumerous with the concept G.

By "our definition", Frege means his explicit definition. He continues:

> In other words: it is to be proved that, for F equinumerous with G, the following two propositions hold good universally:
>
> > if the concept H is equinumerous with the concept F, then it is also equinumerous with the concept G;
>
> and
>
> > if the concept H is equinumerous with the concept G, then it is also equinumerous with the concept F. (Frege, 1980a, §73)

Frege here transforms the statement of the identity of the extensions of the concepts "equinumerous with the concept F" and "equinumerous with the concept G" into the statement that the concepts falling under the one are just the concepts falling under the other: He transforms the statement that their extensions are identical into the statement that the concepts themselves are co-extensional.[15] Frege thus seems to be appealing,

[15] In Chapter **??**, Boolos and I note that Frege does mention extensions once in *Die Grundlagen* after he proves HP, in §83. Austin's translation of that passage is somewhat misleading. The relevant sentence is: "And for this, again, it is necessary to prove that this concept has an extension identical with that of the concept 'member of the series of natural number ending with *d*'". The German is: "Und dazu ist wieder zu beweisen, dass dieser Begriff gleichen Umfanges mit dem Begriffe 'der mit *d* endenden natürlichen Zahlenreihe angehörend' ist". What Frege says is that we must prove that one concept "is the same in extension" as another: One natural translation of "gleichen Umfanges mit... ist" would be is "is co-extensional with".

without mentioning that he is, to an axiom governing names of extensions of second-level concepts: This axiom, a natural analogue of Basic Law V, states that the extension of a second-level concept $\Phi_x(\phi x)$ is the same as that of $\Psi_x(\phi x)$—ϕ marks the argument-place—just in case, for every F, $\Phi_x(Fx)$ just in case $\Psi_x(Fx)$. Using this axiom, it is easy to derive HP from Frege's explicit definition.[16]

The proof of HP in *Grundgesetze* is in two parts. Frege proves in Part II, A(lpha), that, if F is equinumerous with G, then the number of Fs is the same as the number of Gs (Theorem 32). The central lemma in this proof is Theorem 32δ.[17] If we abbreviate "the concept Φ is equinumerous with the concept Ψ" as "$\mathrm{Eq}_x(\Phi x, \Psi x)$", then Theorem 32δ is:

$$\mathbf{Eq}_x(Fx, Gx) \to \forall H[\mathbf{Eq}_x(Fx, Hx) \equiv \mathbf{Eq}_x(Gx, Hx)]$$

That is: If F is equinumerous with G, then the concepts equinumerous with F are just the concepts equinumerous with G. The consequent is exactly what, in the passage from *Die Grundlagen* just cited, Frege says we must prove if we are to show that the number of Fs is the number of Gs.

Frege's proof of Theorem 32δ makes only inessential appeal to Law V and so may be reconstructed in second-order logic. The proof follows the sketch in *Die Grundlagen* closely. Theorem 32δ will follow easily once it has been shown that $\mathrm{Eq}_x(\Phi x, \Psi x)$ is an equivalence relation or, more precisely, that it is transitive and symmetric. If $\mathrm{Eq}_x(Fx, Gx)$, then $\mathrm{Eq}_x(Gx, Fx)$, by symmetry; hence, if $\mathrm{Eq}_x(Fx, Hx)$, then $\mathrm{Eq}_x(Gx, Hx)$, by transitivity. Similarly, if $\mathrm{Eq}_x(Gx, Hx)$, then $\mathrm{Eq}_x(Fx, Hx)$, again by transitivity. All the work in the proof thus goes into establishing that the relation of equinumerosity is transitive and symmetric.

Once he has established (32δ), Frege infers, via Basic Law V and the explicit definition, that the number of Fs is the number of Gs.[18] In fact, however, the full strength of Basic Law V is not required for this inference.[19] What is required is obviously just the following:

$$\forall H[\mathbf{Eq}_x(Fx, Hx) \equiv \mathbf{Eq}_x(Gx, Hx)] \to \mathbf{N}x : Fx = \mathbf{N}x : Gx$$

[16] For a proof of the inconsistency of this axiom, see footnote **??** of Chapter **??**, on p. **??**.

[17] By this I mean the intermediate result marked "δ" that occurs *during* the proof of (and so *prior* to the appearance of) Theorem 32.

[18] The second-level relation $\mathrm{Eq}_x(\Phi x, \Psi x)$ would be represented by its value-range, so, strictly speaking, we should thus have something like $\mathrm{Eq}(\grave{\epsilon}F\epsilon, \grave{\epsilon}G\epsilon)) \to \forall u[\mathrm{Eq}(\grave{\epsilon}F\epsilon, u) \equiv \mathrm{Eq}(\grave{\epsilon}G\epsilon, u)]$, where "$u$" ranges over value-ranges (as well as other objects). (Frege does not actually have a special term for our "$\mathrm{Eq}_x(\Phi x, \Psi x)$".) Basic Law V then yields: $\mathrm{Eq}(\grave{\epsilon}F\epsilon, \grave{\epsilon}G\epsilon) \to \grave{\alpha}\mathrm{Eq}(\grave{\epsilon}F\epsilon, \alpha) \equiv \grave{\alpha}\mathrm{Eq}(\grave{\epsilon}G\epsilon, \alpha)$. Since, by definition, $\mathcal{N}(x) = \grave{\alpha}\mathrm{Eq}(x, \alpha)$, we have: $\mathrm{Eq}(\grave{\epsilon}F\epsilon, \grave{\epsilon}G\epsilon) \to \mathcal{N}(\grave{\epsilon}F\epsilon) = \mathcal{N}(\grave{\epsilon}G\epsilon)$. The formal derivation, using extensions of second-level concepts, would proceed similarly.

[19] On a slightly different note, Robert May has observed that only Basic Law Va is needed for the proof of Theorem 32 from (32δ); it is used to infer (32ϵ). This is the 'safe' direction of Law V, which Frege calls Law Va:

$$\forall x(fx = gx) \to \grave{\epsilon}f(\epsilon) = \grave{\epsilon}g(\epsilon)$$

That we only need Va here is not surprising, since (32) is the 'safe' direction of HP, i.e., the one that has no consequences for the cardinality of the domain.

That is: If the concepts equinumerous with F are just the concepts equinumerous with G, then the number of Fs is the number of Gs.

The other direction of HP—if the number of Fs is the number of Gs, then the Fs are equinumerous with the Gs—is Theorem 49 of *Grundgesetze*, which is proven at the beginning of section B(eta). Frege's proof of Theorem 49 is extremely peculiar, and it makes essential use of the very specific way Frege defines the numbers.[20] I think he must have been overcome by the slickness of that proof, as it would have been just as easy, and ultimately more informative, to give a proof along the lines sketched in *Die Grundlagen*. That proof needs only the converse of the principle just mentioned, namely:

$$\mathrm{N}x : Fx = \mathrm{N}x : Gx \rightarrow \forall H[\mathbf{Eq}_x(Fx, Hx) \equiv \mathbf{Eq}_x(Gx, Hx)]$$

Given the reflexivity of $\mathrm{Eq}_x(\Phi x, \Psi x)$, the proof is then easy. Suppose that $\mathrm{N}x : Fx = \mathrm{N}x : Gx$. By *modus ponens* and instantiation of "H" with "G", $\mathrm{Eq}_x(Fx, Gx)$ iff $\mathrm{Eq}_x(Gx, Gx)$. Since $\mathrm{Eq}_x(\Phi x, \Psi x)$ is reflexive, $\mathrm{Eq}_x(Gx, Gx)$, so $\mathrm{Eq}_x(Fx, Gx)$.

The derivation of HP therefore requires only what we may call *Frege's Principle*:[21]

$$\mathrm{N}x : Fx = \mathrm{N}x : Gx \;\equiv\; \forall H[\mathbf{Eq}_x(Fx, Hx) \equiv \mathbf{Eq}_x(Gx, Hx)]$$

That is: The number of Fs is the number of Gs if, and only if, the concepts equinumerous with F are just the concepts equinumerous with G. Or: The number of Fs is the number of Gs just in case the (second-level) concept $\mathrm{Eq}_x(Fx, \Phi x)$ is co-extensional with the (second-level) concept $\mathrm{Eq}_x(Gx, \Phi x)$.

I conclude that Basic Law V and the explicit definition serve no essential purpose in Frege's proofs of the axioms of arithmetic in *Grundgesetze* other than to yield Frege's Principle. This is what I had in mind when I said earlier that Basic Law V plays a ludicrously meager formal role, and so this is what makes pressing the question why the truth, indeed the logical truth, of Basic Law V was so essential to Frege's philosophy of mathematics.

5 HP versus Frege's Principle

Our question is whether it can coherently be maintained that, *circa* 1884, Frege did not intend formally to derive HP from his explicit definition. As said, one of the chief reasons to think that it cannot is that Frege sketches a derivation of HP in *Die Grundlagen*: He presumably intended to derive it from something; hence, if we deny that he intended to derive HP from his

[20] Frege first derives Theorems 45 and 46α from the explicit definition. We can record these as: $\mathrm{Eq}(w, z) \rightarrow w \in \mathcal{N}(z)$ and $w \in \mathcal{N}(z) \rightarrow \mathrm{Eq}(w, z)$. Given that equinumerosity is reflexive (39, 42), Frege then uses (45) to get Theorem 48: $w \in \mathcal{N}(w)$. Identity then yields: $\mathcal{N}(w) = \mathcal{N}(z) \rightarrow w \in \mathcal{N}(z)$, and Theorem 49 follows immediately from this and (46α).

[21] Something very much like this principle is discussed by Boolos under the name "Numbers" (Boolos, 1998, p. 186).

explicit definition, we are left with the question from what he did intend to derive it. We have seen, however, that, in the first instance, Frege derives HP from Frege's Principle; and it is clear that, in *Grundgesetze*, anyway, he would have derived Frege's Principle from the explicit definition and Basic Law V. What dispute there might be concerns the status of Frege's Principle in *Die Grundlagen*. The standard view would be that Frege intended somehow to derive Frege's Principle from his explicit definition; an alternative view is that Frege's Principle itself was to be the fundamental axiom of the theory of arithmetic. Is this alternative view tenable?

The natural objection to it is that it would then be utterly unclear why Frege gives the explicit definition at all, what role it is supposed to play. More to the point, Frege introduces the explicit definition to resolve the Caesar problem: But Frege's Principle is no more immune to the objections he brings against HP than is HP itself; the one no more settles whether Caesar is a number than does the other. Hence, numbers must still be identified with the extensions of certain concepts. But, then, how it can be held both that Frege's Principle was to be the fundamental axiom of the theory of arithmetic and that numbers were to be identified with extensions? To put the point another way: What advantage, exactly, is Frege's Principle supposed to have over HP?

The natural suggestion is that the identification of numbers with extensions is not made in the formal theory but in the meta-theory. Consider again Frege's sketch of the proof of HP in *Die Grundlagen*. At the very beginning of that proof, he writes that the sentence "The concepts equinumerous with F are just the concepts equinumerous with G" states the identity of the number of Fs and the number of Gs, as he explicitly defines them, "in other words". Now, in *Grundgesetze*, the explicit definition of numbers as extensions is used in a formal proof of Frege's Principle; on the standard view, this is also what is envisaged in *Die Grundlagen*. On the alternative view, what we have here is (not a short formal proof but) an *informal justification* of Frege's Principle in terms of the explicit definition, which is itself being given in the meta-theory, not in the formal theory. Thus, Frege's Principle does have an advantage over HP, for it can straightforwardly be informally justified in terms of the explicit definition.

Why shouldn't the explicit definition be given in the meta-theory? The Caesar problem is not a formal problem, not, that is, a problem upon whose solution the successful execution of the formal part of the logicist program depends. The derivability of second-order arithmetic from HP shows that the Caesar problem is not a formal problem in this sense. Furthermore, the identification of numbers as value-ranges does not fully resolve the Caesar problem: A version of the Caesar problem arises again in *Grundgesetze*, the question, this time, being whether either of the two truth-values is a value-range and, if so, which value-ranges they are (Frege, 1962, v. I, §10). From our current perspective, what is interesting about Frege's solution to this problem is that *it is not incorporated into the formal theory*. As said, the Caesar problem is not a formal problem, so no solution to it needs to be incorporated into the formal theory: It is sufficient that it be resolved in the meta-theory, and that is precisely where Frege resolves it in *Grundgesetze*. According to the alternative view, Frege resolves it there in *Die Grundlagen*, also.

As yet, then, we have seen no reason to suppose that, *circa* 1884, Frege should not also have been satisfied to resolve the Caesar problem, for numerical terms, in the meta-theory: That is, we have discovered no objection to the claim that the explicit definition was to be given in the meta-theory, that Frege's Principle was to be justified informally in terms of it, and that Frege's Principle was to be the fundamental axiom of the formal theory of arithmetic.

6 Frege's Principle and the Explicit Definition

The philosophical value, for Frege, of formalizing arithmetic is that doing so affords insight into its epistemological status. The derivation of Frege's Principle from the explicit definition, and indeed the formulation of the explicit definition itself, could, however, be argued to involve an appeal to intuition: Perhaps an appeal to intuition is required to justify the transition from "the extension of F is the same as the extension of G" to "the Fs are exactly the Gs"; perhaps the notion of the extension of a concept is itself an intuitive one. One could attempt to answer such objections in a variety of ways. Presumably, objections similar to those Frege offers against the view that numbers are given in intuition could also be made against the view that the extensions of concepts are given in intuition: For example, the extensions of concepts, the objects falling under which are not given in intuition, are presumably also not given in intuition (Frege, 1980a, §§14, 24). Conclusively to answer such an objection, however, Frege would have had to show that the theory of the extensions of concepts is itself part of logic; he would, that is, have had to show that the explicit definition could be formulated in logical terms and that Frege's Principle could be derived from it in accord with general logical principles.

If Frege is going to derive Frege's Principle from his explicit definition, then he must do so formally: To show arithmetic to be a branch of logic, the derivation of Frege's Principle must be carried out formally, so that the epistemological basis of the principles employed in that derivation can be uncovered. On the alternative view we have been discussing, Frege's Principle is to be derived from the explicit definition within the meta-theory, rather than within the formal theory; but it is still to be derived from the explicit definition. What the objection just considered shows is that this claim cannot stand: One cannot interpret Frege as deriving Frege's Principle from the explicit definition only informally. Since, as was said earlier, Frege does appear to derive Frege's Principle from something, the alternative view surely fails as an interpretation of Frege.

In fact, however, this argument is less conclusive than it seems. What we need to know is not just that Frege proposed to derive Frege's principle from the explicit definition. There might be many different reasons one would want to do that. What we need to know is that Frege proposed to *justify* Frege's Principle in terms of the explicit definition. The argument just given assumes that he did. Did he?

7 The Caesar Problem Revisited

There is an important difference between Frege's resolution of the Caesar problem in *Die Grundlagen* and his resolution of the corresponding problem in *Grundgesetze*. As said earlier, the problem that arises in *Grundgesetze* is whether either of the truth-values is a value-range and, if so, which value-ranges they are. Frege's solution to the problem consists of two parts (Frege, 1962, v. I, §10). The first part is a proof that, compatibly with Basic Law V, one may identify each of the two truth-values with the value-range of any distinct functions one chooses; more precisely, if a bit anachronistically, given any model of the theory at all, there is, for each function in the domain of the model, a model in which, say, Truth is the value-range of that function.[22] The second part is a stipulation that Truth is to be its own unit class; Falsity, its own unit class. Of course, this does not solve the Caesar problem in any generality: It tells us neither whether Caesar is a value-range nor, if so, which one he is. But this appears not to trouble Frege: He says simply that the formal system, as it stands, contains only terms that refer either to value-ranges or to truth-values and that, if terms referring to other sorts of objects are introduced, additional stipulations will have to be made.[23]

In *Die Grundlagen*, the Caesar problem is resolved by giving an explicit definition from which HP follows; in *Grundgesetze*, the corresponding problem is resolved by making assignments to (stipulations regarding the references of) certain terms, which assignments are then proven to be consistent with Basic Law V. One can imagine a similar strategy being employed in *Die Grundlagen*: We make some stipulation about what the referents of numerical terms are to be, a stipulation that would resolve the Caesar problem compatibly with HP; these stipulations could be made by means of the explicit definition itself. To show that the assignment was consistent with HP, it would be sufficient, though not necessary, to derive the latter within some theory in which a statement of the former assignment is taken as an

[22] Of course, the full theory of *Grundgesetze* has no models, since it is inconsistent. But the first-order fragment of the theory does, and with respect to it this claim can be given a proper proof, though it needs some restriction: There are some very special pairs of concepts whose extensions cannot simultaneously be taken to be Truth and Falsity (Schroeder-Heister, 1987; Parsons, 1995b).

[23] Indeed, Frege does not even consider the possibility, in the main text, that additional primitive singular terms may be introduced into the language: He considers only the introduction of additional, primitive functions. It is natural to read the footnote, however, as concerning primitive singular terms.

axiom (which would then amount to a relative consistency proof). We probably would not find it natural then to say that our stipulations were made by means of a definition (except within the context of the relative consistency proof): Frege's calling it a definition[24] may be good reason to think that he did so intend it. Nevertheless, such a stipulation would resolve the Caesar problem, in the sense that it would fix the truth-values of identity statements of the form "the number of Fs is t", for every term t.[25]

If HP is to be justified in terms of the explicit definition, then the principles used in that derivation must be examined, if we are to be clear whether HP itself is thereby shown to be analytic, to be a truth of logic. On the other hand, suppose that HP is not to be justified in terms of the explicit definition, that the explicit definition is merely a tool for resolving the Caesar problem. Then the only requirement on the explicit definition is that it be consistent with HP. In that case, even if the definition made use, say, of geometrical notions, is it obvious that that would show HP not to be a logical truth? Some of our knowledge of numbers—of their identity with and distinctness from objects not given to us as numbers—would be infected with knowledge derived from intuition. But our knowledge of the basic laws of arithmetic would not be, since HP is not, on this view, to be to be thought of as justified by, or as known because it follows from, the explicit definition.[26]

Whether a stipulation of the sort under discussion would resolve the Caesar problem depends, of course, on what we take the Caesar problem to be.[27] The question we need to consider, at this point, is: Is there some reason that, in the case of numerical terms, the problem must be resolved not just by giving an explicit definition but by giving one in terms of which HP is supposed to be justified? It is important to remember that the Caesar problem is not a problem upon whose solution the proper development of the formal theory of arithmetic depends. As Frege presents the problem whether the truth-values are value-ranges, it is a formal problem, but explicitly a meta-theoretic one: The problem is to fix the truth-values of certain statements, so that every well-formed sentence of the theory will have a unique, well-determined truth-value (Frege, 1962, v. I, §§10, 30–1). The Caesar problem, as it arises in *Die Grundlagen*, on the other hand, is presented in nothing like this way: Rather, the problem is primarily philosophical in character. This difference is presumably of fundamental importance:

[24] It is worth mentioning here that Austin translates both "erklären" and "definieren" as "define", and similarly for cognate expressions. Frege may use them interchangeably, but this is not clear to me. I have not, however, studied this question.

[25] At least, it will do so if the corresponding explicit definition does so, so the explicit definition has no advantage on this score.

[26] And there is, in a way, nothing novel about this way of looking at things. As said earlier, even if the explicit definition is given in the formal theory and is given in purely logical terms, not all of its consequences are going to be analytic of the concept of number, anyway, since some of its features are arbitrary.

[27] Similarly, whether an appeal to non-logical notions in a solution of the Caesar problem would undermine the claim of HP to be a truth of logic will depend upon whether our knowledge of the truths of arithmetic is supposed somehow to depend upon how the Caesar problem is solved.

The problem of *Grundgesetze*, which Frege thinks he can solve by stipulation, is presumably not the same problem as the problem of *Die Grundlagen*, a problem he would not (at least by *Grundgesetze*) have been willing to resolve by stipulation. But that means that we have come, after much wandering, back to where we began, for what we need now to understand is what objection, exactly, Frege raises against HP in *Die Grundlagen*. The question, once again, is: What exactly is the Caesar problem?

8 Closing

When he wrote *Die Grundlagen*, Frege might well have been prepared, if confronted with Russell's Paradox, to renounce appeal to extensions and to install HP as an axiom. Extensions, as we have seen, are introduced only to resolve the Caesar problem: Abandoning the explicit definition, and so use of extensions of concepts, would require Frege to resolve the Caesar problem in some other way. At the time he wrote *Die Grundlagen*, however, Frege thought that, in principle, the Caesar problem could be resolved in some other way (Frege, 1980a, §107), though he may well have thought it probably could not be. Perhaps the Caesar problem could be resolved by giving a different explicit definition, which did not "bring[] in the extensions of concepts"; perhaps it could be resolved by making certain sorts of stipulations, as discussed above. Perhaps it could even be shown not really to be a problem, HP, together with more general sorts of considerations, actually settling the truth-values of 'mixed' identity statements, as Wright (1983, §§xiv–xv) would have it.

But by 1903, Frege's position had changed. As he wrote to Russell:[28]

> I myself was long reluctant to recognize value-ranges and hence classes; but I saw no other possibility of placing arithmetic on a logical foundation. But the question is, How do we apprehend logical objects? (Frege, 1980b, pp. 140–1)

This question is closely related to that raised in §62 of *Die Grundlagen*: "How, then, are numbers to be given to us, if we can have no ideas or intuitions of them?" Logical objects are certainly given neither in intuition nor in perception.[29] To answer this earlier question, Frege had once suggested, it would be sufficient to "fix the sense of a numerical identity": The difficulty, however, was that HP alone did not suffice to do this. Frege continues:

> And I have found no other answer to it than this, We apprehend them as extensions of concepts, or more generally, as value-ranges of functions. I have always been aware that there are dif-

[28] I have altered the translation slightly.
[29] The delicacy here is required by the fact that neither are directions given in intuition or perception, though they are not (purely?) logical objects.

> ficulties connected with this, and your discovery of the contradiction has added to them; but what other way is there? (Frege, 1980b, p. 141)

The reason that abandoning Basic Law V was not an option for Frege in 1902 was thus this: Only by identifying numbers with the extensions of certain concepts could he answer a certain epistemological question, first raised in *Die Grundlagen*, namely, how we apprehend numbers with the aid of neither intuition nor perception. Frege's logicist program thus did not fail, even by his own lights, because he could not derive the axioms of arithmetic from principles that have some claim to be logical in character. By his own lights, his program failed because he could not explain how we can apprehend the objects of arithmetic as logical objects; it failed because he was unable to resolve a particular philosophical problem, the Caesar problem, and it is the nature of this problem that we, as Frege's interpreters, must come to understand.

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