Consistent theories of truth for languages which conform to classical logic

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Abstract

Every language which conforms to classical logic is shown to have an extension for which a consistent definitional theory of truth is formulated. Also a consistent semantical theory of truth is formulated for that extension if every sentence of the object language is valuated by its meaning either as true or as false. These theories of truth contain both a truth predicate and a non-truth predicate. The case where they compose with predicates having several free variables is also included. Theories are equivalent when sentences of the object language are valuated by their meanings.

1 Introduction

Based on 'Chomsky Definition' (cf. [2]) we assume that a language is a countable set of sentences with finite length, and formed by a countable set of elements. A theory of syntax is also assumed to provide symbols and rules to construct well-formed sentences and possible formulas for that language.
A language is said to conform to classical logic if it has, or if it can be extended to have at least the following properties ('iff' means 'if and only if'):
(i) It contains logical symbols $\neg$ (not), $\vee$ (or), $\wedge$ (and), $\rightarrow$ (if...then), $\leftrightarrow$ (if and only if), $\forall$ (for all) and $\exists$ (exists), and the following sentences: If A and B are (denote) sentences, so are $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$ and $A \leftrightarrow B$. If $P(x)$ is a formula of the language, then P is called a predicate with domain D_P iff for every object of D_P there is a term b which names it, and $P(b)$ is a sentence of that language. Denote by N_P the set of those terms. $\forall x P(x)$ and $\exists x P(x)$ are then sentences of the language.
(ii) The sentences of that language are so valuated as true or as false that the following rules of classical logic are valid : If A and B denote sentences of the language, then A is true iff $\neg A$ is false, and A is false iff $\neg A$ is true; $A \vee B$ is true iff A or B is true, and false iff A and B are false; $A \wedge B$ is true iff A and B are true, and false iff A or B is false; $A \rightarrow B$ is true iff A is false or B is true, and false iff A is true and B is false; $A \leftrightarrow B$ is true iff A and B are both true or both false, and false iff A is true and B is false or A is false and B is true. If P is a predicate with domain D_P, then $\forall x P(x)$ is true iff $P(b)$

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is true for every $b \in N_P$, and false iff $P(b)$ is false for some $b \in N_P$; $\exists x P(x)$ is true iff $P(b)$ is true for some $b \in N_P$, and false iff $P(b)$ is false for every $b \in N_P$.
(iii) The language is bivalent, i.e., every sentence of it is either true or false.
Main results of this paper are:
For every language which conforms to classical logic an extension which has properties (i)–(iii) is constructed, and a consistent definitional theory of truth is formulated for it. Also a consistent semantical theory of truth is formulated for that extension if every sentence of the object language is valuated by its meaning either as true or as false. These theories of truth contain truth and non-truth predicates. In Section 6 these theories are extended so that both truth predicate and non-truth predicate compose with predicates having several free variables.

2 Extended languages

Assume that an object language L_0 conforms to classical logic, and is without a truth predicate. Before Section 6 'predicate' means a predicate with one free variable.
The first extension L_1 of L_0 is constructed by adding those sentences $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$, $A \leftrightarrow B$, $\forall x P(x)$ and $\exists x P(x)$ which are not in L_0 when A and B go through all sentences of L_0 and P its predicates. Valuate the sentences of L_1 so that properties (ii) and (iii) are valid. This can be done since L_0 conforms to classical logic.
Replacing L_0 by L_1 and so on, we obtain a sequence of languages L_n, $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, so valuated that the properties (ii) and (iii) are valid. The union L of languages L_n, $n \in \mathbb{N}_0$, called a basic extension of L_0, has properties (ii) and (iii).
If A and B denote sentences of L, there exist n_1 and n_2 such that A is in L_{n_1} and B is in L_{n_2}. Denoting $n = \max\{n_1, n_2\}$, then A and B are sentences of L_n. Thus the sentences $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$ and $A \leftrightarrow B$ are in L_{n+1}, and hence in L. If P is a predicate of L_0, then the sentences $\forall x P(x)$ and $\exists x P(x)$ are in L_1, so that they are in L. Thus L has also properties (i), and hence all properties (i) – (iii).
The language L_T is formed by adding to L extra formulas $T(x)$ and $\neg T(x)$, and sentences $T(\mathbf{n})$ and $\neg T(\mathbf{n})$, where $\mathbf{n}$ goes through all numerals which denote numbers $n \in \mathbb{N}_0$. Neither valuation nor meaning is yet attached to these sentences. Numerals are added, if necessary, to terms of L_T. Choose a Gödel numbering to sentences of L_T. The Gödel number of a sentence denoted by A is denoted by $\#A$, and the numeral of $\#A$ by $\lceil A \rceil$, which names the sentence A.
If P is a predicate of L_0 with domain D_P, then $P(b)$ is a sentence of L for each $b \in N_P$, and $\lceil P(b) \rceil$ is the numeral of its Gödel number. Thus $T(\lceil P(b) \rceil)$ and $\neg T(\lceil P(b) \rceil)$ are sentences of L_T for each $b \in N_P$, so that they are determined by predicates of L_T with domain D_P. Denote these predicates by $T(\lceil P(\dot{x}) \rceil)$ and $\neg T(\lceil P(\dot{x}) \rceil)$, where $\lceil P(\dot{x}) \rceil$ stands for the result of formally replacing the variable x of $P(x)$ by the term of N_P (cf. [4]).
Denote by $\mathcal{L}_0$ the language which contains L_T, and sentences $\forall x Q(x)$, $\exists x Q(x)$, $\forall x Q(\lceil P(\dot{x}) \rceil)$ and $\exists x Q(\lceil P(\dot{x}) \rceil)$, where Q is T or $\neg T$, and P is a predicate of L_0 or T or $\neg T$. When a language $\mathcal{L}_n$, $n \in \mathbb{N}_0$, is defined, let $\mathcal{L}_{n+1}$ be a language which is formed by adding to $\mathcal{L}_n$ those of the following sentences which are not in $\mathcal{L}_n$: $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$ and $A \leftrightarrow B$, where A and B are sentences of $\mathcal{L}_n$.
The language $\mathcal{L}$ is defined as the union of languages $\mathcal{L}_n$, $n \in \mathbb{N}_0$. Extend the Gödel numbering of the sentences of L_T to those of $\mathcal{L}$, and denote by $\mathcal{D}$ the set of Gödel numbers of sentences of $\mathcal{L}$.

Denote by $\mathcal{P}$ the set of predicates of L_0 (with one free variable). Divide $\mathcal{P}$ into three disjoint subsets as follows.

$$\begin{cases} \mathcal{P}_1 = \{P \in \mathcal{P} : P(b) \text{ is true for all } b \in N_P\},\ \mathcal{P}_2 = \{P \in \mathcal{P} : P(b) \text{ is false for all } b \in N_P\}, \\ \mathcal{P}_3 = \{P \in \mathcal{P} : P(b) \text{ is true for some but not for all } b \in N_P\}. \end{cases} \tag{2.1}$$

Define subsets $Z_1(U)$, $Z_2(U)$, $U \subset \mathcal{D}$, and Z_i, $i = 1 \ldots 5$, of $\mathcal{L}$ by

$$\begin{cases} Z_1(U) = \{T(\mathbf{n})\colon\ \mathbf{n} = \lceil A \rceil, \text{ where } A \text{ is a sentence of } \mathcal{L} \text{ and } \#A \text{ is in } U\}, \\ Z_2(U) = \{\neg T(\mathbf{n})\colon\ \mathbf{n} = \lceil A \rceil, \text{ where } A \text{ is a sentence of } \mathcal{L} \text{ and } \#[\neg A] \text{ is in } U\}, \\ Z_1 = \{\neg\forall x T(x), \exists x T(x), \neg\forall x \neg T(x), \exists x \neg T(x)\}, \\ Z_2 = \{\forall x T(\lceil P(\dot{x}) \rceil), \exists x T(\lceil P(\dot{x}) \rceil), \neg(\forall x \neg T(\lceil P(\dot{x}) \rceil)), \neg(\exists x \neg T(\lceil P(\dot{x}) \rceil)) : P \in \mathcal{P}_1\}, \\ Z_3 = \{\forall x \neg T(\lceil P(\dot{x}) \rceil), \exists x \neg T(\lceil P(\dot{x}) \rceil), \neg(\forall x T(\lceil P(\dot{x}) \rceil)), \neg(\exists x T(\lceil P(\dot{x}) \rceil)) : P \in \mathcal{P}_2\}, \\ Z_4 = \{\neg(\forall x T(\lceil P(\dot{x}) \rceil)), \exists x T(\lceil P(\dot{x}) \rceil) : P \in \mathcal{P}_3 \cup \{T, \neg T\}\}, \\ Z_5 = \{\neg(\forall x \neg T(\lceil P(\dot{x}) \rceil)), \exists x \neg T(\lceil P(\dot{x}) \rceil) : P \in \mathcal{P}_3 \cup \{T, \neg T\}\}. \end{cases} \tag{2.2}$$

Subsets $L_n(U)$, $n \in \mathbb{N}_0$, of $\mathcal{L}$ are defined recursively as follows.

$$L_0(U) = \begin{cases} Z = \{A : A \text{ is a true sentence of } L\} \text{ if } U = \emptyset \text{ (the empty set)}, \\ Z \cup Z_1(U) \cup Z_2(U) \cup Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \text{ if } \emptyset \subset U \subset \mathcal{D}. \end{cases} \tag{2.3}$$

When a subset $L_n(U)$ of $\mathcal{L}$ is defined for some $n \in \mathbb{N}_0$, and when A and B are sentences of $\mathcal{L}$, denote

$$\begin{cases} L_n^0(U) = \{\neg(\neg A) : A \text{ is in } L_n(U)\}, \\ L_n^1(U) = \{A \vee B : A \text{ or } B \text{ is in } L_n(U)\}, \\ L_n^2(U) = \{A \wedge B : A \text{ and } B \text{ are in } L_n(U)\}, \\ L_n^3(U) = \{A \to B : \neg A \text{ or } B \text{ is in } L_n(U)\}, \\ L_n^4(U) = \{A \leftrightarrow B : \text{both } A \text{ and } B \text{ or both } \neg A \text{ and } \neg B \text{ are in } L_n(U)\}, \\ L_n^5(U) = \{\neg(A \vee B) : \neg A \text{ and } \neg B \text{ are in } L_n(U)\}, \\ L_n^6(U) = \{\neg(A \wedge B) : \neg A \text{ or } \neg B \text{ is in } L_n(U)\}, \\ L_n^7(U) = \{\neg(A \to B) : A \text{ and } \neg B \text{ are in } L_n(U)\}, \\ L_n^8(U) = \{\neg(A \leftrightarrow B) : A \text{ and } \neg B, \text{ or } \neg A \text{ and } B \text{ are in } L_n(U)\}, \end{cases} \tag{2.4}$$

and define

$$L_{n+1}(U) = L_n(U) \cup \bigcup_{k=0}^{8} L_n^k(U). \tag{2.5}$$

The above constructions imply that $L_n^k(U) \subseteq L_{n+1}^k(U)$ and $L_n(U) \subset L_{n+1}(U) \subset \mathcal{L}$ for all $n \in \mathbb{N}_0$ and $k = 0, \ldots, 8$. Define a subset $L(U)$ of $\mathcal{L}$ by

$$L(U) = \bigcup_{n=0}^{\infty} L_n(U). \tag{2.6}$$

3 Properties of consistent subsets of $\mathcal{D}$

Recall that $\mathcal{D}$ denotes the set of Gödel numbers of the sentences of $\mathcal{L}$. When U is a subset of $\mathcal{D}$, denote by $G(U)$ the set of Gödel numbers of the sentences of $L(U)$ defined by (2.6):

$$G(U) = \{\#A : A \text{ is a sentence of } L(U)\}. \tag{3.1}$$

A subset U of $\mathcal{D}$ is said to be consistent if for no sentence A of $\mathcal{L}$ both $\#A$ and $\#[\neg A]$ are in U.

Lemma 3.1. *Let U be a consistent subset of $\mathcal{D}$. Then for no sentence A of $\mathcal{L}$ both A and $\neg A$ belong to $L(U)$, and $G(U)$ is consistent.*

Proof. We shall first show that there is no sentence A in $\mathcal{L}$ such that both A and $\neg A$ belong to $L_0(U)$. If $U = \emptyset$, then $L_0(U)$ is by (2.3) the set Z of true sentences of L. If A is in Z, then $\neg A$ is false, and hence not in $Z = L_0(U)$, since L has properties (i)–(iii).
Assume next that U is nonempty. As a consistent set U is a proper subset of $\mathcal{D}$.
Let $\mathbf{n}$ be a numeral. If $T(\mathbf{n})$ is in $L_0(U)$, it is in $Z_1(U)$, so that, by (2.2), $\mathbf{n} = \lceil A \rceil$, where $\#A$ is in U. Since U is consistent, then $\#[\neg A]$ is not in U. Thus, by (2.2), $\neg T(\mathbf{n})$ is not in $Z_2(U)$, and hence not in $L_0(U)$. This result implies also that $T(\mathbf{n})$ is not in $L_0(U)$ if $\neg T(\mathbf{n})$ is in $L_0(U)$.
(2.2) and (2.3) imply that sentences $\exists x T(x)$, $\neg\forall x T(x)$, $\neg\forall x \neg T(x)$ and $\exists x \neg T(x)$ are in Z_1, and hence in $L_0(U)$, but their negations are not in $L_0(U)$.
By the definitions (2.2) and (2.3) of Z_2, Z_3, Z_4, Z_5 and $L_0(U)$ neither both $\exists x Q(\lceil P(\dot{x}) \rceil)$ and $\neg(\exists x Q(\lceil P(\dot{x}) \rceil))$, nor both $\forall x Q(\lceil P(\dot{x}) \rceil)$ and $\neg(\forall x Q(\lceil P(\dot{x}) \rceil))$, are in $L_0(U)$ for any $Q \in \{T, \neg T\}$ and $P \in \mathcal{P} \cup \{T, \neg T\}$.
The above proof shows that for no sentence A of $\mathcal{L}$ both A and $\neg A$ belong to $L_0(U)$.
Make the induction hypothesis: There exists an $n \in \mathbb{N}_0$ such that

(h0) For no sentence A of $\mathcal{L}$ both A and $\neg A$ belong to $L_n(U)$.

If $\neg(\neg A)$ is in $L_{n+1}(U)$, it is in $L_n^0(U)$, so that A is in $L_n(U)$. Thus, by (h0), $\neg A$ is not in $L_n(U)$ so that $\neg(\neg(\neg A))$, is not in $L_n^0(U)$, and hence not in $L_{n+1}(U)$.
If $A \vee B$ is in $L_{n+1}(U)$, then it is in $L_n^1(U)$, whence A or B is in $L_n(U)$. $\neg(A \vee B)$ is in $L_{n+1}(U)$ iff it is in $L_n^5(U)$, in which case $\neg A$ and $\neg B$ are in $L_n(U)$. Thus $A \vee B$ and $\neg(A \vee B)$ are not both in $L_{n+1}(U)$, for otherwise both A and $\neg A$ or both B and $\neg B$ are in $L_n(U)$, contradicting with (h0).

$A \wedge B$ and $\neg(A \wedge B)$ cannot both be in $L_{n+1}(U)$, for otherwise $A \wedge B$ is in $L_n^2(U)$, i.e., both A and B are in $L_n(U)$, and $\neg(A \wedge B)$ is in $L_n^6(U)$, i.e., at least one of $\neg A$ and $\neg B$ is in $L_n(U)$. Thus both A and $\neg A$ or both B and $\neg B$ are in $L_n(U)$, contradicting with (h0).

If $A \to B$ is in $L_{n+1}(U)$, then it is in $L_n^3(U)$, so that $\neg A$ or B is in $L_n(U)$. $\neg(A \to B)$ is in $L_{n+1}(U)$ iff it is in $L_n^7(U)$, whence both A and $\neg B$ are in $L_n(U)$. Because of these results and (h0) the sentences $A \to B$ and $\neg(A \to B)$ are not both in $L_{n+1}(U)$.

$A \leftrightarrow B$ is $L_{n+1}(U)$ iff it is in $L_n^4(U)$, in which case both A and B or both $\neg A$ and $\neg B$ are in $L_n(U)$. If $\neg(A \leftrightarrow B)$ is in $L_{n+1}(U)$, then it is in $L_n^8(U)$, whence both A and $\neg B$ or both $\neg A$ and B are in $L_n(U)$. Thus both $A \leftrightarrow B$ and $\neg(A \leftrightarrow B)$ cannot be in $L_{n+1}(U)$, for otherwise both A and $\neg A$ or both B and $\neg B$ are in $L_n(U)$, contradicting with (h0).

The above results and the induction hypothesis (h0) imply that for no sentence A of $\mathcal{L}$ both A and $\neg A$ belong to $L_{n+1}(U) = L_n(U) \cup \bigcup_{k=0}^{8} L_n^k(U)$.
Since (h0) is proved when $n = 0$, it is by induction valid for every $n \in \mathbb{N}_0$.

If A and $\neg A$ are in $L(U)$, then A is by (2.6) in $L_{n_1}(U)$ for some $n_1 \in \mathbb{N}_0$, and $\neg A$ is in $L_{n_2}(U)$ for some $n_2 \in \mathbb{N}_0$. Then both A and $\neg A$ are in $L_n(U)$ when $n = \max\{n_1, n_2\}$. This is impossible, because (h0) is proved for every $n \in \mathbb{N}_0$. Thus A and $\neg A$ cannot both be in $L(U)$ for any sentence A of $\mathcal{L}$.

The above result and (3.1) imply that there is no sentence A in $\mathcal{L}$ such that both $\#A$ and $\#[\neg A]$ are in $G(U)$. Thus $G(U)$ is consistent. $\square$

Lemma 3.2. *Assume that U and V are consistent subsets of $\mathcal{D}$, and that $V \subseteq U$. Then $L(V) \subseteq L(U)$ and $G(V) \subseteq G(U)$.*

Proof. As consistent sets V and U are proper subsets of $\mathcal{D}$. We shall first show that $L_0(V) \subseteq L_0(U)$.

If $V = \emptyset$, then $L_0(V) = Z \subseteq L_0(U)$ by (2.3).

Assume next that V is nonempty. Thus also U is nonempty.

Let A be a sentence of L. Definition (2.3) of $L_0(U)$ implies that A is in $L_0(U)$ and also in $L_0(V)$ iff A is in Z.

Let $\mathbf{n}$ be a numeral. If $T(\mathbf{n})$ is in $L_0(V)$, it is in $Z_1(V)$, so that $\mathbf{n} = \lceil A \rceil$, where $\#A$ is in V. Because $V \subseteq U$, then $\#A$ is also in U, whence $T(\mathbf{n})$ is in $Z_1(U)$, and hence in $L_0(U)$.

$\neg T(\mathbf{n})$ is in $L_0(V)$ if it is in $Z_2(V)$, in which case $\mathbf{n} = \lceil A \rceil$, where $\#[\neg A]$ is in V. Since $V \subseteq U$, then $\#[\neg A]$ is also in U, whence $\neg T(\mathbf{n})$ is in $Z_2(U)$, and hence in $L_0(U)$.

Because U and V are nonempty and proper subsets of $\mathcal{D}$, then Z_1, Z_2, Z_3, Z_4 and Z_5 are in $L_0(U)$ and in $L_0(V)$ by (2.3).

The above results imply that $L_0(V) \subseteq L_0(U)$. Make the induction hypothesis:

(h1) $\quad L_n(V) \subseteq L_n(U)$

for some $n \in \mathbb{N}_0$. It follows from (2.4) and (h1) that $L_n^k(V) \subseteq L_n^k(U)$ for each $k = 0, \ldots, 8$. Thus

$$L_{n+1}(V) = L_n(V) \cup \bigcup_{k=0}^{8} L_n^k(V) \subseteq L_n(U) \cup \bigcup_{k=0}^{8} L_n^k(U) = L_{n+1}(U).$$

(h1) is proved when $n = 0$, whence it is by induction valid for every $n \in \mathbb{N}_0$.

If A is in $L(V)$, it is by (2.6) in $L_n(V)$ for some $n \in \mathbb{N}_0$. Thus A is in $L_n(U)$ by (h1), and hence in $L(U)$. Consequently, $L(V) \subseteq L(U)$.

If $\#A$ is in $G(V)$ then A is in $L(V)$ by (3.1). Thus A is in $L(U)$, so that $\#A$ is in $G(U)$ by (3.1). This shows that $G(V) \subseteq G(U)$. $\square$

Denote by $\mathcal{C}$ the family of consistent subsets of $\mathcal{D}$. In the formulation and the proof of Theorem 3.1 transfinite sequences indexed by ordinals are used. A transfinite sequence $(U_\lambda)_{\lambda<\alpha}$ of $\mathcal{C}$ is said to be increasing if $U_\mu \subseteq U_\nu$ whenever $\mu < \nu < \alpha$, and strictly increasing if $U_\mu \subset U_\nu$ whenever $\mu < \nu < \alpha$.

Lemma 3.3. *Assume that $(U_\lambda)_{\lambda<\alpha}$ is a strictly increasing sequence of $\mathcal{C}$. Then*
(a) $(G(U_\lambda))_{\lambda<\alpha}$ is an increasing sequence of $\mathcal{C}$.
(b) The union $\bigcup_{\lambda<\alpha} G(U_\lambda)$ is consistent.

Proof. Since $U_\mu \subset U_\nu$ when $\mu < \nu < \alpha$, it follows from Lemma 3.2 that $G(U_\mu) \subseteq G(U_\nu)$ when $\mu < \nu < \alpha$, whence the sequence $(G(U_\lambda))_{\lambda<\alpha}$ is increasing. Consistency of the sets $G(U_\lambda)$, $\lambda < \alpha$, follows from Lemma 3.1 because the sets U_λ, $\lambda < \alpha$, are consistent. This proves (a).

To prove that the union $\bigcup_{\lambda<\alpha} G(U_\lambda)$ is consistent, assume on the contrary that there exists such a sentence A in $\mathcal{L}$ that both $\#A$ and $\#[\neg A]$ are in $\bigcup_{\lambda<\alpha} G(U_\lambda)$. Thus there exist μ, $\nu < \alpha$ such that $\#A$ is in $G(U_\mu)$ and $\#[\neg A]$ is in $G(U_\nu)$. Because $G(U_\mu) \subseteq G(U_\nu)$ or $G(U_\nu) \subseteq G(U_\mu)$, then both $\#A$ and $\#[\neg A]$ are in $G(U_\mu)$ or in $G(U_\nu)$. But this is impossible, since both $G(U_\mu)$ and $G(U_\nu)$ are consistent. Thus, the set $\bigcup_{\lambda<\alpha} G(U_\lambda)$ is consistent. □

Now we are ready to prove the following Theorem.

Theorem 3.1. *Let W denote the set of Gödel numbers of true sentences of L. We say that a transfinite sequence $(U_\lambda)_{\lambda<\alpha}$ of $\mathcal{C}$ is a G-sequence if it has the following properties.*

(G) $(U_\lambda)_{\lambda<\alpha}$ is strictly increasing, $U_0 = W$, and if $0 < \mu < \alpha$, then $U_\mu = \bigcup_{\lambda<\mu} G(U_\lambda)$.

Then the longest G-sequence exists, and it has the last member. This member is the smallest consistent subset U of $\mathcal{D}$ satisfying $U = G(U)$.

Proof. W is consistent, since L has properties (i)-(iii). We shall first show that G-sequences are nested:
(1) Assume that $(U_\lambda)_{\lambda<\alpha}$ and $(V_\lambda)_{\lambda<\beta}$ are G-sequences. Then $U_\lambda = V_\lambda$ when $\lambda < \min\{\alpha, \beta\}$.
$U_0 = W = V_0$ by (G). Make the induction hypothesis:
(h) There exists an ordinal ν which satisfies $0 < \nu < \min\{\alpha, \beta\}$ such that $U_\lambda = V_\lambda$ for each $\lambda < \nu$.
It follows from (h) and (G) that $U_\nu = \bigcup_{\lambda<\nu} G(U_\lambda) = \bigcup_{\lambda<\nu} G(V_\lambda) = V_\nu$. Since $U_0 = V_0$, then (h) holds when $\nu = 1$. These results imply (1) by transfinite induction.
Let $(U_\lambda)_{\lambda<\alpha}$ be a G-sequence. Defining $f(0) = \min U_0$, $f(\lambda) = \min(U_\lambda \setminus U_{\lambda-1})$, $0 < \lambda < \alpha$, and $f(\alpha) = \min(D \setminus \bigcup_{\lambda<\alpha} U_\lambda)$, we obtain a bijection f from $[0, \alpha]$ to a subset of $\mathbb{N}_0$. Thus α is a countable ordinal. Consequently, the set Γ of those ordinals α for which $(U_\lambda)_{\lambda<\alpha}$ is a G-sequence is bounded from above by the smallest uncountable ordinal. Denote by γ the least upper bound of Γ.
To show that γ is a successor, assume on the contrary that γ is a limit ordinal. Given any $\mu < \gamma$, then $\nu = \mu + 1$ and $\alpha = \nu + 1$ are $< \gamma$. $(U_\lambda)_{\lambda<\alpha}$ is a G-sequence, whence $U_\mu = \bigcup_{\lambda<\mu} G(U_\lambda)$, and $U_\mu \subset U_{\mu+1}$. Thus $(U_\lambda)_{\lambda<\gamma}$ has properties (G) when $\alpha = \gamma$, so that $(U_\lambda)_{\lambda<\gamma}$ is a G-sequence. Denote $U_\gamma = \bigcup_{\lambda<\gamma} G(U_\lambda)$. U_γ is consistent by Lemma 3.3(b). Because $U_\mu \subset U_\nu = \bigcup_{\lambda<\nu} G(U_\lambda) \subseteq U_\gamma$ for each $\mu < \gamma$, then $(U_\lambda)_{\lambda<\gamma+1}$ is a G-sequence. This is impossible, since $(U_\lambda)_{\lambda<\gamma}$ contains all G-sequences.
Thus γ is a successor, say $\gamma = \alpha + 1$. If $\lambda < \alpha$, then $U_\lambda \subset U_\alpha$, so that $G(U_\lambda) \subseteq G(U_\alpha)$. Then $U_\alpha = \bigcup_{\lambda<\alpha} G(U_\lambda) \subseteq \bigcup_{\lambda<\gamma} G(U_\lambda) = G(U_\alpha)$, whence $U_\alpha \subseteq G(U_\alpha)$. Moreover, $U_\alpha = G(U_\alpha)$, for otherwise $U_\alpha \subset G(U_\alpha) = \bigcup_{\lambda<\gamma} G(U_\lambda) = U_\gamma$, and $(U_\lambda)_{\lambda<\gamma+1}$ would be a G-sequence.
Consequently, $(U_\lambda)_{\lambda<\gamma}$ is the longest G-sequence, U_α is its last member, and $U_\alpha = G(U_\alpha)$.
Let U be a consistent subset of $\mathcal{D}$ satisfying $U = G(U)$. Then $U_0 = W = G(\emptyset) \subseteq G(U) = U$. Make the induction hypothesis:
(h2) There exists an ordinal μ which satisfies $0 < \mu < \gamma$ such that $U_\lambda \subseteq U$ for each $\lambda < \mu$.
Then $G(U_\lambda) \subseteq G(U)$ for each $\lambda < \mu$, whence $U_\mu = \bigcup_{\lambda<\mu} G(U_\lambda) \subseteq G(U) = U$. Thus, by transfinite induction, $U_\mu \subseteq U$ for each $\mu < \gamma$. In particular, $U_\alpha \subseteq U$. This proves the last assertion of Theorem. □

4 Language $\mathcal{L}_T$ and its properties

Let L_0, L, $\mathcal{L}$ and $\mathcal{D}$ be as in Section 2, and let the sets $L(U)$ and $G(U)$, $U \subset \mathcal{D}$, be defined by (2.6) and (3.1). Define

$$F(U) = \{A : \neg A \in L(U)\}. \tag{4.1}$$

Recall that a subset U of $\mathcal{D}$ is consistent if there is no sentence A in $\mathcal{L}$ such that both $\#A$ and $\#[\neg A]$ are in U. By Theorem 3.1 the smallest consistent subset of $\mathcal{D}$ which satisfies $U = G(U)$ exists.

Definition 4.1. Let U be the smallest consistent subset of $\mathcal{D}$ which satisfies $U = G(U)$. Denote by $\mathcal{L}_T$ the language formed by the object language L_0, and those sentences of $L(U)$ and $F(U)$ and those symbols, formulas and predicates of $\mathcal{L}_0$ which are not in L_0. T and $\neg T$ are predicates determined by the formulas $T(x)$ and $\neg T(x)$ when their domain and the set of terms are defined by

$$D_T = \{\text{the sentences of } \mathcal{L}_T\} \text{ and } N_T = \{\mathbf{n} : \mathbf{n} = \lceil A \rceil, \text{ where } A \text{ is in } D_T\}. \tag{4.2}$$

A valuation is defined for sentences of $\mathcal{L}_T$ as follows.

(I) A sentence of $\mathcal{L}_T$ is valuated as true iff it is in $L(U)$, and as false iff it is in $F(U)$.

Lemma 4.1. *The language $\mathcal{L}_T$ defined by Definition 4.1 and valuated by (I) is bivalent.*

Proof. The subsets $L(U)$ and $F(U)$ of the sentences of $\mathcal{L}_T$ are disjoint. For otherwise there is a sentence A of $\mathcal{L}_T$ which is in $L(U) \cap F(U)$. Then A is in $L(U)$, and by the definition (4.1) of $F(U)$ also $\neg A$ is in $L(U)$. But this is impossible by Lemma 3.1. Consequently, $L(U) \cap F(U) = \emptyset$.

If A is a sentence of $\mathcal{L}_T$, then it is in $L(U)$ or in $F(U)$. If A is true, it is in $L(U)$, but not in $F(U)$, and hence not false, because $L(U) \cap F(U) = \emptyset$. Similarly, if A is false, it is in $F(U)$, but not in $L(U)$, and hence not true. Consequently, A is either true or false, so that $\mathcal{L}_T$ is bivalent. □

Lemma 4.2. *Let $\mathcal{L}_T$ be defined by Definition 4.1 and valuated by (I). Then a sentence of the basic extension L of L_0 is true (respectively false) in the valuation (I) iff it is true (respectively false) in the valuation of L.*

Proof. Let A denote a sentence of L. A is true in the valuation (I) iff A is in $L(U)$ iff (by the construction of $L(U)$) A is in Z iff A is true in the valuation of L. A is false in the valuation (I) iff A is in $F(U)$ iff (by (4.1)) $\neg A$ is in $L(U)$ iff ($\neg A$ is a sentence of L) $\neg A$ is in Z iff $\neg A$ is true in the valuation of L iff (L has properties (i)–(iii)) A is false in the valuation of L. □

Lemma 4.3. *The language $\mathcal{L}_T$ defined by Definition 4.1 and valuated by (I) satisfies the rules (ii) of classical logic given in Introduction.*

Proof. Unless otherwise stated, 'true' means true in the valuation (I), and 'false' means false in the valuation (I). We shall first derive the following auxiliary rule.

(t0) Double negation: If A is a sentence of $\mathcal{L}_T$, then $\neg(\neg A)$ is true iff A is true.

To prove (t0), assume first that $\neg(\neg A)$ is true. Then it is in $L(U)$, and hence, by (2.6), in $L_n(U)$ for some $n \in \mathbb{N}_0$. If $\neg(\neg A)$ is in $L_0(U)$ then it by (2.3) in Z. Thus $\neg(\neg A)$ is true in the valuation of L. Then (negation rule is valid in L) $\neg A$ is false in the valuation of L, which implies that A is true in the

valuation of L. Thus A is by (2.3) in $Z \subset L_0(U) \subset L(U)$, whence A is true.
Assume next that $n \in \mathbb{N}_0$ is the smallest number for which $\neg(\neg A)$ is in $L_{n+1}(U)$. It then follows from (2.5) that $\neg(\neg A)$ is in $L_n^0(U)$, so that A is by (2.4) in $L_n(U)$, and hence in $L(U)$, i.e., A is true.
Thus A is true if $\neg(\neg A)$ is true.
Conversely, assume that A is true. Then A is in $L(U)$, so that A is in $L_n(U)$ for some $n \in \mathbb{N}_0$. Thus $\neg(\neg A)$ is in $L_n^0(U)$, and hence in $L_{n+1}(U)$. Consequently, $\neg(\neg A)$ is in $L(U)$, whence $\neg(\neg A)$ is true.
This concludes the proof of (t0).
Rule (t0) is applied to prove

(t1) Negation: A is true iff $\neg A$ is false, and A is false iff $\neg A$ is true.

Let A be a sentence of $\mathcal{L}_T$. Then A is true iff (by (t0)) $\neg(\neg A)$ is true iff $\neg(\neg A)$ is in $L(U)$ iff (by (4.1)) $\neg A$ is in $F(U)$ iff $\neg A$ is false.
A is false iff A is in $F(U)$ iff (by (4.1)) $\neg A$ is in $L(U)$ iff $\neg A$ is true. Thus (t1) is satisfied.
Next we shall prove the following rule.

(t2) Conjunction: $A \wedge B$ is true iff A and B are true. $A \wedge B$ is false iff A or B is false.

Let A and B be sentences of $\mathcal{L}_T$. If A and B are true, i.e., A and B are in $L(U)$, there is by (2.6) an $n \in \mathbb{N}_0$ such that A and B are in $L_n(U)$. Thus $A \wedge B$ is in $L_n^2(U)$, and hence in $L(U)$, so that $A \wedge B$ is true.
Conversely, assume that $A \wedge B$ is true, or equivalently, $A \wedge B$ is in $L(U)$. Then there is by (2.6) an $n \in \mathbb{N}_0$ such that $A \wedge B$ is in $L_n(U)$. If $A \wedge B$ is in $L_0(U)$, it is in Z. Thus $A \wedge B$ is true in the valuation of L. Because L has property (ii), then A and B are true in the valuation of L, and hence also in the valuation (I) by Lemma 4.2.
Assume next that $n \in \mathbb{N}_0$ is the smallest number for which $A \wedge B$ is in $L_{n+1}(U)$. Then $A \wedge B$ is in $L_n^2(U)$, so that A and B are in $L_n(U)$, and hence in $L(U)$, i.e., A and B are true.
The above reasoning proves that $A \wedge B$ is true iff A and B are true. This result and the bivalence of $\mathcal{L}_T$, proved in Lemma 4.1, imply that $A \wedge B$ is false iff A or B is false. Consequently, rule (t2) is valid.
The proofs of the following rules are similar to the above proof of (t2).

(t3) Disjunction: $A \vee B$ is true iff A or B is true. $A \vee B$ false iff A and B are false.

(t4) Conditional: $A \rightarrow B$ is true iff A is false or B is true. $A \rightarrow B$ is false iff A is true and B is false.

(t5) Biconditional: $A \leftrightarrow B$ is true iff A and B are both true or both false. $A \leftrightarrow B$ is false iff A is true and B is false or A is false and B is true.

Next we shall show that if $P \in \mathcal{P} \cup \{T, \neg T\}$ then $\exists x P(x)$ and $\forall x P(x)$ have the following properties required in (ii).

(p6) $\exists x P(x)$ is true iff $P(b)$ is true for some $b \in N_P$. $\exists x P(x)$ is false iff $P(b)$ is false for every $b \in N_P$.

(p7) $\forall x P(x)$ is true iff $P(b)$ is true for every $b \in N_P$. $\forall x P(x)$ is false iff $P(b)$ is false for some $b \in N_P$.

If $P \in \mathcal{P}$, then P is a predicate of L_0, so that P is in L. Since L has property (ii), then P has properties (p6) and (p7) in the valuation of L, and hence also in the valuation (I) by Lemma 4.2.
To simplify proofs in the cases when P is T or $\neg T$ we shall derive results which imply that T is a truth predicate and $\neg T$ is a non-truth predicate for $\mathcal{L}_T$.

Let A denote a sentence of $\mathcal{L}_T$. The valuation (I), rule (t1), the definitions of $Z_1(U)$, $Z_2(U)$ and $G(U)$, and the assumption $U = G(U)$ imply that A is true iff A is in $L(U)$ iff $\#A$ is in $G(U) = U$ iff $T(\lceil A \rceil)$ is in $Z_1(U) \subset L(U)$ iff $T(\lceil A \rceil)$ is true iff $\neg T(\lceil A \rceil)$ is false.
A is false iff A is in $F(U)$ iff $\neg A$ is in $L(U)$ iff $\#[\neg A]$ is in $G(U) = U$ iff $\neg T(\lceil A \rceil)$ is in $Z_2(U) \subset L(U)$ iff $\neg T(\lceil A \rceil)$ is true iff $T(\lceil A \rceil)$ is false.
The above results imply that the following results are valid for every sentence $A \in \mathcal{L}_T$.

(T) A is true iff $T(\lceil A \rceil)$ is true iff $\neg T(\lceil A \rceil)$ is false. A is false iff $T(\lceil A \rceil)$ is false iff $\neg T(\lceil A \rceil)$ is true.

Consider the validity of (p6) and (p7) when P is T. Because U is nonempty, then $\exists x T(x)$ is in $L_0(U)$ by (2.2) and (2.3), and hence in $L(U)$ by (2.6). Thus $\exists x T(x)$ is by (I) a true sentence of $\mathcal{L}_T$.
$T(\lceil A \rceil)$ is true iff (by (T)) A is true iff (by (I)) A is in $L(U)$. Thus $T(\mathbf{n})$ is true for some $\mathbf{n} \in N_T$.
The above results imply that $\exists x T(x)$ is true iff $T(\mathbf{n})$ is true for some $\mathbf{n} \in N_T$. In view of this result and the bivalence of $\mathcal{L}_T$, one can infer that $\exists x T(x)$ is false iff $T(\mathbf{n})$ is false for every $\mathbf{n} \in N_T$. This concludes the proof of (p6) when P is T.
$\neg \forall x T(x)$ is in $Z_1 \subset L_0(U)$, and hence in $L(U)$, so that it is true. Thus $\forall x T(x)$ is false by (t1).
$T(\lceil A \rceil)$ is false iff (by (T)) A is false iff (by (I)) A is in $F(U)$. Thus $T(\mathbf{n})$ is false for some $\mathbf{n} \in N_T$.
Consequently, $\forall x T(x)$ is false iff $T(\mathbf{n})$ is false for some $\mathbf{n} \in N_T$. This result and the bivalence of $\mathcal{L}_T$ imply that $\forall x T(x)$ is true iff $T(\mathbf{n})$ is true for every $\mathbf{n} \in N_T$. This proves (p7) when P is T.
To show that (p6) is valid when P is $\neg T$, notice first that $\exists x \neg T(x)$ is in $Z_1 \subset L_0(U)$, and hence in $L(U)$, whence it is true.
$\neg T(\lceil A \rceil)$ is true iff (by (t1)) $T(\lceil A \rceil)$ is false iff (by (T)) A is false iff (by (I)) A is in $F(U)$. Thus $\neg T(\mathbf{n})$ is true for some $\mathbf{n} \in N_T$. Consequently, $\exists x \neg T(x)$ is true iff $\neg T(\mathbf{n})$ is true for some $\mathbf{n} \in N_T$. This result and the bivalence of $\mathcal{L}_T$ imply that $\exists x \neg T(x)$ is false iff $\neg T(\mathbf{n})$ is false for every $\mathbf{n} \in N_T$. This concludes the proof of (p6) when P is $\neg T$.
Next we shall prove (p7) when P is $\neg T$. $\neg \forall x \neg T(x)$ is in $Z_1 \subset L_0(U)$, and hence in $L(U)$, so that it is true. Thus $\forall x \neg T(x)$ is false by (t1).
$\neg T(\lceil A \rceil)$ is false iff (by (t1)) $T(\lceil A \rceil)$ is true iff (by (T)) A is true iff (by (I)) A is in $L(U)$. Thus $\neg T(\mathbf{n})$ is false for some $\mathbf{n} \in N_T$. From these results it follows that $\forall x \neg T(x)$ is false iff $\neg T(\mathbf{n})$ is false for some $\mathbf{n} \in N_T$. This result and bivalence of $\mathcal{L}_T$ imply that $\forall x \neg T(x)$ is true iff $\neg T(\mathbf{n})$ is true for all $\mathbf{n} \in N_T$. Thus (p7) is valid when P is $\neg T$.
It remains to show that the following rules are valid when $Q \in \{T, \neg T\}$ and $P \in \mathcal{P} \cup \{T, \neg T\}$.

(qp6) $\exists x Q(\lceil P(\dot{x}) \rceil)$ is true iff $Q(\lceil P(b) \rceil)$ is true for some $b \in N_P$.

$\exists x Q(\lceil P(\dot{x}) \rceil)$ is false iff $Q(\lceil P(b) \rceil)$ is false for every $b \in N_P$;

(qp7) $\forall x Q(\lceil P(\dot{x}) \rceil)$ is true iff $Q(\lceil P(b) \rceil)$ is true for every $b \in N_P$.

$\forall x Q(\lceil P(\dot{x}) \rceil)$ is false iff $Q(\lceil P(b) \rceil)$ is false for some $b \in N_P$.

Consider first the case when Q is T and $P \in \mathcal{P}_1$. Then $P(b)$ is a true sentence of L, and hence, by Lemma 4.2, a true sentence of $\mathcal{L}_T$ for every $b \in N_P$. This implies by (T) that $T(\lceil P(b) \rceil)$ is a true sentence of $\mathcal{L}_T$ for every $b \in N_P$, and hence also for some $b \in N_P$.
Since U is nonempty and proper subset of $\mathcal{D}$, then $\exists x T(\lceil P(\dot{x}) \rceil)$ and $\forall x T(\lceil P(\dot{x}) \rceil)$ are in $L_0(U)$ by (2.2) and (2.3), and hence in $L(U)$. Thus $\exists x T(\lceil P(\dot{x}) \rceil)$ and $\forall x T(\lceil P(\dot{x}) \rceil)$ are by (I) and Lemma 4.1 true sentences of $\mathcal{L}_T$.

The above results imply the first sentences, and by bivalence of $\mathcal{L}_T$ also the second sentences of properties (qp6) and (qp7) when Q is T and P is in $\mathcal{P}_1$. The proof in the case when Q is T and P is in $\mathcal{P}_2$ is similar.

Assume next that Q is T and P is in $\mathcal{P}_3$. Then $P(b)$ is a true sentence of L for some $b \in N_P$, say $b \in N_P^1$, and a false sentence of L for $b \in N_P^2 = N_P \setminus N_P^1$. Hence, by Lemma 4.2, $P(b)$ is a true sentence of $\mathcal{L}_T$ for $b \in N_P^1$, and a false sentence of $\mathcal{L}_T$ for $b \in N_P^2$. This implies by (T) that $T(\lceil P(b) \rceil))$ is a true sentence of $\mathcal{L}_T$ for $b \in N_P^1$, and a false sentence of $\mathcal{L}_T$ for $b \in N_P^2$. Since U is nonempty, then $\exists x T(\lceil P(\dot{x}) \rceil)$ and $\neg(\forall x T(\lceil P(\dot{x}) \rceil))$ are in Z_4 by (2.2), and hence in $L(U)$ by (2.3) and (2.6). Thus $\exists x T(\lceil P(\dot{x}) \rceil)$ is true, and and $\forall x T(\lceil P(\dot{x}) \rceil)$ is false.

The above results imply the first sentence of (qp6) and the second sentence of (qp7) when Q is T and P is in $\mathcal{P}_3$. The second sentence of (qp6) and the first sentence of (qp7) are also valid because $\mathcal{L}_T$ is bivalent. The proofs in the cases when Q is $\neg T$ and $P \in \mathcal{P}$ are similar to those given above.

The sentences $\exists x Q(\lceil P(\dot{x}) \rceil)$, where Q and P are in $\{T, \neg T\}$ are in Z_4 or in Z_5, whence they are in $L(U)$ and hence true. In every case there exists a $b \in N_T$ so that $Q(\lceil P(b) \rceil)$ is true ($b = \mathbf{n} = \lceil A \rceil$, where A, depending on the case, is in $L(U)$ or in $F(U)$). These results imply truth part of (qp6) when Q and P are in $\{T, \neg T\}$. Falsity part in (qp6) is then valid by bivalence of $\mathcal{L}_T$.

The proof of (qp7) in the case when Q and P are in $\{T, \neg T\}$ is similar. $\square$

5 Consistent theories of truth

We say that a theory of truth is formulated for a language if truth values are assigned to its sentences, and if it contains a predicate T which satisfies

T-rule: $\quad T(\lceil A \rceil) \leftrightarrow A$ is true for every sentence of the language.

A predicate T which satisfies T-rule is called a truth predicate. A theory of truth is said to be consistent if $A \wedge \neg A$ is false for every sentence A. It is called definitional if truth values are defined for sentences, and semantical if truth or falsity of sentences are determined by their meanings.

A definitional theory of truth is formulated as follows.

Theorem 5.1. *Let L_0 be a language which conforms to classical logic. The language $\mathcal{L}_T$, defined in Definition 4.1 and valuated by (I), has properties (i)–(iii) given in Introduction. T is a truth predicate, and $\neg T$ is a non-truth predicate of $\mathcal{L}_T$. The so formulated definitional theory of truth (shortly DTT) for $\mathcal{L}_T$ is consistent.*

Proof. It follows from Definition 4.1 that $\mathcal{L}_T$ has properties (i). Properties (ii) are valid by Lemma 4.3, and Lemma 4.2 proves the validity of (iii).

The results (T) derived in the proof of Lemma 4.3 and biconditional rule (t5) imply that the sentence $T(\lceil A \rceil) \leftrightarrow A$ is true and the sentence $\neg T(\lceil A \rceil) \leftrightarrow A$ is false for every sentence A of $\mathcal{L}_T$. T and $\neg T$ are predicates of $\mathcal{L}_T$, and their domain D_T, the set all sentences of $\mathcal{L}_T$, satisfies the condition presented in [3, p. 7] for the domains of truth predicates. The above results imply that T is a truth predicate and $\neg T$ is a non-truth predicate for $\mathcal{L}_T$.

Properties (ii) and (iii) imply that $A \wedge \neg A$ is false for every sentence A of $\mathcal{L}_T$. Thus the so obtained definitional theory of truth (shortly DTT) for $\mathcal{L}_T$ is consistent. $\square$

Next we shall formulate a semantical theory of truth for $\mathcal{L}_T$, by assuming that a meaning is assigned to every sentence of the object language L_0. Standard meanings are assigned to logical symbols. The sentence $T(\lceil A \rceil)$ means: 'the sentence denoted by A is true'. Thus meanings can be assigned to sentences of the basic extension L of L_0 constructed in Section 2, and to sentences of $\mathcal{L}_T$ defined in Definition 4.1.

We shall first prove preliminary Lemmas.

Lemma 5.1. *Assume that L_0 is a language whose every sentence is valuated by its meaning either as true or as false, i.e., what a sentence means is either true or false. Then the basic extension L of L_0 has properties (i)–(iii) when its sentences are valuated by their meanings, and L_0 conforms to classical logic.*

Proof. The object language L_0 is bivalent, since every sentence of it is valuated by its meaning either as true or as false. This bivalence remains valid for the basic extension L of L_0 when its sentences are valuated by their meanings. Moreover, L has syntactical properties (i), and rules (ii) are valid because of standard meanings assigned to logical symbols. Thus the object language L_0 has an extension L whose sentences are valuated by their meanings, which is without a truth predicate, and which has properties (i)–(iii). This implies that L_0 conforms to classical logic. □

Lemma 5.2. *Let L_0 be as in Lemma 5.1, and assume that sentences of both the basic extension L of L_0 and the language $\mathcal{L}_T$ given by Definition 4.1 are valuated by their meanings. Let W be the set of Gödel numbers of true sentences of L, and let U be the smallest subset of $\mathcal{D}$ for which $U = G(U)$. Given a consistent subset V of $\mathcal{D}$ which satisfies $W \subseteq V \subseteq U$, assume that every sentence of $\mathcal{L}_T$ whose Gödel number is in V is true and not false by its meaning. Then every sentence of $L(V)$ is true and not false by its meaning.*

Proof. Because $V \subseteq U = G(U)$, then every sentence whose Gödel number is in V, is in $\mathcal{L}_T$. We shall first prove that every sentence of $L_0(V)$ is true and not false by its meaning.

Since $W \subseteq V$, then every true sentence of L, i.e., every sentence of Z is true and not false by its meaning.

A sentence of $\mathcal{L}_T$ is in $Z_1(V)$ iff it is of the form $T(\lceil A \rceil)$, where A denotes a sentence of $\mathcal{L}$ whose Gödel number is in V. A is by an assumption true and not false by its meaning. $T(\lceil A \rceil)$ means that 'the sentence denoted by A is true', whence it is true iff A is true and false iff A is false. Thus the sentence $T(\lceil A \rceil)$, and hence the given sentence, is true and not false by its meaning. By the standard meaning of negation the sentence $\neg T(\lceil A \rceil)$ is then false and not true by its meaning. Replacing A by $T(\lceil A \rceil)$, it follows from the above results that $T(\lceil T(\lceil A \rceil) \rceil)$ is true and not false by its meaning, and $T(\lceil \neg T(\lceil A \rceil) \rceil)$ is false and not true by its meaning, whence $\neg T(\lceil \neg T(\lceil A \rceil) \rceil)$ is true and not false by its meaning.

A sentence of $\mathcal{L}_T$ is in $Z_2(V)$ iff it is of the form $\neg T(\lceil A \rceil)$, where A denotes a sentence of $\mathcal{L}$, and the Gödel number of the sentence $\neg A$ is in V. $\neg A$ is by a hypothesis true and not false by its meaning, so that A is false and not true by its meaning since V is consistent. Thus the sentence $T(\lceil A \rceil)$ is false and not true by its meaning. Replacing A by $T(\lceil A \rceil)$, we then obtain that $T(\lceil T(\lceil A \rceil) \rceil)$ is false and not true by its meaning. Consequently, by the standard meaning of negation, the sentences $\neg T(\lceil A \rceil)$, $\neg T(\lceil T(\lceil A \rceil) \rceil)$ and $T(\lceil \neg T(\lceil A \rceil) \rceil)$ are true and not false by their meanings.

The set N_T of numerals, defined by (4.2), is formed by numerals $\lceil A \rceil$, where A goes through all the sentences of $\mathcal{L}_T$. Thus, by results proved above $T(\mathbf{n})$, $T(\lceil T(\mathbf{n}) \rceil)$, $\neg T(\lceil \neg T(\mathbf{n}) \rceil)$, $\neg T(\mathbf{n})$, $\neg T(\lceil T(\mathbf{n}) \rceil)$ and $T(\lceil \neg T(\mathbf{n}) \rceil)$ are for some $\mathbf{n} \in N_T$ true and not false by their meanings and for some $\mathbf{n} \in N_T$ false and not true by their meanings. These results and the standard meanings of quantifiers and negation imply that $\exists x T(x)$, $\exists x T(\ulcorner T(\dot{x})\urcorner)$, $\exists x \neg T(\ulcorner \neg T(\dot{x})\urcorner)$, $\exists x \neg T(x)$, $\exists x \neg T(\ulcorner T(\dot{x})\urcorner)$ and $\exists x T(\ulcorner \neg T(\dot{x})\urcorner)$ are true and not false by their meanings, and their negations are false and not true by their meanings, whereas $\forall x T(x)$, $\forall x T(\ulcorner T(\dot{x})\urcorner)$, $\forall x \neg T(\ulcorner \neg T(\dot{x})\urcorner)$, $\forall x \neg T(x)$, $\forall x \neg T(\ulcorner T(\dot{x})\urcorner)$ and $\forall x T(\ulcorner \neg T(\dot{x})\urcorner)$ are false and not true by their meanings, and their negations are true and not false by their meanings.

In particular, the sentences of Z_1, and those sentences of Z_4 and Z_5, where P is T or $\neg T$, are true and not false by their meanings.

The proof that those sentences of Z_4 and Z_5 where P is in $\mathcal{P}_3$ are true and not false by their meanings is similar to that given above for the corresponding sentences where P is T. The proof that the sentences of Z_2 and Z_3 are true and not false by their meanings is even simpler and is left to the reader.
The above results and (2.3) imply that every sentence of $L_0(V)$ is true and not false by its meaning. Thus the following property holds when $n = 0$.

(h3) Every sentence of $L_n(V)$ is true and not false by its meaning.

Make the induction hypothesis: (h3) holds for some $n \in \mathbb{N}_0$.

Given a sentence of $L_n^0(V)$, it is of the form $\neg(\neg A)$, where A is in $L_n(V)$. A is by (h3) true and not false by its meaning. Thus, by standard meaning of negation, its double application implies that the sentence $\neg(\neg A)$, and hence the given sentence, is true and not false by its meaning.

A sentence is in $L_n^1(V)$ iff it is of the form $A \vee B$, where A or B is in $L_n(V)$. By (h3) at least one of the sentences A and B is true and not false by its meaning. Thus, by the the standard meaning of disjunction, the sentence $A \vee B$, and hence given sentence, is true and not false by its meaning.

Similarly it can be shown that if (h3) holds, then every sentence of $L_n^k(V)$, where $2 \le k \le 8$, is true and not false by its meaning.
The above results imply that under the induction hypothesis (h3) every sentence of $L_n^k(V)$, where $0 \le k \le 8$, is true and not false by its meaning.
It then follows from the definition (2.5) of $L_{n+1}(V)$ that if (h3) is valid for some $n \in \mathbb{N}_0$, then every sentence of $L_{n+1}(V)$ is true and not false by its meaning.

The first part of this proof shows that (h3) is valid when $n = 0$. Thus, by induction, it is valid for all $n \in \mathbb{N}_0$. This result and (2.6) imply that every sentence of $L(V)$ is true and not false by its meaning. $\square$

Lemma 5.3. *Let L_0 be a language whose every sentence is valuated by its meaning either as true or as false, and has not a truth predicate. Then the language $\mathcal{L}_T$ given in Definition 4.1 and valuated by meanings of its sentences has the following properties.*
(a) If a sentence of $\mathcal{L}_T$ is true in the valuation (I), it is true and not false by its meaning.
(b) If a sentence of $\mathcal{L}_T$ is false in the valuation (I), it is false and not true by its meaning.

Proof. By Theorem 3.1 the smallest consistent subset U of $\mathcal{D}$ which satisfies $U = G(U)$ is the last member of the transfinite sequence $(U_\lambda)_{\lambda<\gamma}$ constructed in the proof of that Theorem. We prove by transfinite induction that the following result holds for all $\lambda < \gamma$.

(H) Every sentence of $\mathcal{L}_T$ whose Gödel number is in U_λ is true and not false by its meaning.

Make the induction hypothesis: There exists a μ which satisfies $0 < \mu < \gamma$ such that (H) holds for all $\lambda < \mu$.

Let $\lambda < \mu$ be given. Because U_λ is consistent and $W \subseteq U_\lambda \subseteq U$ for every $\lambda < \mu$, it follows from the induction hypothesis and Lemma 5.2 that every sentence of $L(U_\lambda)$ is true and not false by its meaning. This implies by (3.1) that (H) holds when U_λ is replaced $G(U_\lambda)$, for every $\lambda < \mu$. Thus (H) holds when U_λ is replaced by the union of those sets. But this union is U_μ by Theorem 3.1 (G), whence (H) holds when $\lambda = \mu$.
When $\mu = 1$, then $\lambda < \mu$ iff $\lambda = 0$. $U_0 = W$, i.e., the set of Gödel numbers of true sentences of L. Since L, valuated by meanings of its sentences, is bivalent by Lemma 5.1, the sentences of U_0 are true and not false by their meanings. This proves that the induction hypothesis is satisfied when $\mu = 1$.

The above proof implies by transfinite induction properties assumed in (H) for U_λ whenever $\lambda < \gamma$. In particular the last member of $(U_\lambda)_{\lambda<\gamma}$ satisfies (H), which is by Theorem 3.1 the smallest consistent subset U of $\mathcal{D}$ for which $U = G(U)$. Thus every sentence of $\mathcal{L}_T$, which is true in the valuation (I), has its Gödel number in U, and is by the above proof true and not false by its meaning. This proves (a).

To prove (b), let A denote a sentence which is false in the valuation (I). Negation rule implies that $\neg A$ is true in the valuation (I). Thus, by (a), $\neg A$ is true and not false by its meaning, so that by the standard meaning of negation, A is false and not true by its meaning. This proves (b). $\square$

The next result is a consequence of Lemma 5.1, Lemma 5.3 and Theorem 5.1.

Theorem 5.2. *Assume that L_0 is a language whose every sentence is valuated by its meaning either as true or as false, and which has not a truth predicate. A semantical theory of truth (shortly STT) is formulated for the extension $\mathcal{L}_T$ of L_0 defined in Definition 4.1, when valuation (I) is replaced in Theorem 5.1 with the valuation of the sentences of $\mathcal{L}_T$ by their meanings. This valuation is equivalent to valuation (I), and the results of Theorem 5.1 are valid for STT.*

Proof. Let A denote a sentence of $\mathcal{L}_T$. A is by Lemma 4.1 either true or false in the valuation (I). If A is true in the valuation (I), it is by Lemma 5.3 (a) true and not false by its meaning. If A is false in the valuation (I), it is by Lemma 5.3 (b) false and not true by its meaning. Consequently, A is either true or false by its meaning. Thus every sentence of $\mathcal{L}_T$ is either true or false by its meaning. In particular, $T(\lceil A \rceil) \leftrightarrow A$ is true by its meaning and the sentence $\neg T(\lceil A \rceil) \leftrightarrow A$ is false by its meaning for every sentence A of $\mathcal{L}_T$. These results imply that T is a truth predicate and $\neg T$ is a non-truth predicate for $\mathcal{L}_T$. Valuation of $\mathcal{L}_T$ by meanings of its sentences is by the above proof equivalent to the valuation (I). This equivalence implies the last conclusion of Theorem. $\square$

6 An extension of language $\mathcal{L}_T$

Let L_0 be a language which conforms to classical logic. Assume that P is a predicate of L_0 with arity $m > 1$, i.e., P has m free variables. The domain of P is denoted by $D_P = D_P^1 \times \cdots \times D_P^m$, and the set of terms $b = (b_1, \dots, b_m)$ which name objects of D_P by $N_P = N_P^1 \times \cdots \times N_P^m$. Denote for each $m > 1$

$$\mathcal{P}^m = \{P : P \text{ is a predicate of } L_0 \text{ with arity } m\}. \tag{6.1}$$

Assume that the following rule of of classical logic is satisfied whenever $P \in \mathcal{P}^m$ for some $m > 1$.
(iv) For every m-tuple $(q_1, \dots, q_m)$ of quantifiers, where each q_i is either $\forall$ or $\exists$, the sentence $q_1 x_1 \dots q_m x_m P(x_1, \dots$ is true iff the sentence $P(b_1, \dots, b_m)$ is true for all choices of $b_i \in N_P^i$ when q_i is $\forall$, and for some choices of $b_i \in N_P^i$ when q_i is $\exists$. $q_1 x_1 \dots q_m x_m P(x_1, \dots, x_m)$ is false iff its negation is true, i.e.,

$p_1x_1 \dots p_mx_m\neg P(x_1,\dots,x_m)$ is true, where p_i is $\exists$ if q_1 is $\forall$ and vice versa. $(p_1,\dots,p_m)$ is said to be the complement tuple of $(q_1,\dots,q_m)$.

Let P be a predicate of $\mathcal{P}^m$. Denote by $T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ the predicate of sentences $T(\lceil P(b_1,\dots,b_m)\rceil)$, $(b_1,\dots,b_m) \in N_P$. Similarly, the predicate of sentences $\neg T(\lceil P(b_1,\dots,b_m)\rceil)$, $(b_1,\dots,b_m) \in N_p$, is denoted by $\neg T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$. Their domain is the domain D_P of P. Those predicates are not in the language $\mathcal{L}_T$ defined in Definition 4.1. In this section we extend the language $\mathcal{L}_T$ so that these predicates are in it, and that they satisfy rule (iv) whenever P is in $\mathcal{P}^m$ for some $m > 1$.

The extension of $\mathcal{L}_T$ is constructed as follows.

For every predicate P of L_0 which has arity m for some $m > 1$ add to the language $\mathcal{L}_0$ constructed in Section 2 predicates $T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ and $\neg T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ and sentences
$q_1x_1 \dots q_mx_mT(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ and $q_1x_1 \dots q_mx_m\neg T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$
for every m-tuple $(q_1,\dots,q_m)$, where q_i's are either $\forall$ or $\exists$.

Construct then a language $\mathcal{L}$ as in Section 2, choose a Gödel numbering to its sentences, and denote by $\mathcal{D}$ the set of those Gödel numbers.

Denote for every $m > 1$

$$Z_1^m = \{q_1x_1 \dots q_mx_mT(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil) : P \in \mathcal{P}^m \text{ and } q_1x_1 \dots q_mx_mP(x_1,\dots,x_m) \text{ is true}\}; \tag{6.2}$$

$$Z_2^m = \{q_1x_1 \dots q_mx_m\neg T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil) : P \in \mathcal{P}^m \text{ and } q_1x_1 \dots q_mx_m\neg P(x_1,\dots,x_m) \text{ is true}\}. \tag{6.3}$$

When U is a nonempty and proper subset of $\mathcal{D}$, add to the set $L_0(U)$ defined by (2.3) all the sentences which are in Z_1^m or in Z_2^m for $m = 2, 3, \dots$.

The equivalences
$q_1x_1 \dots q_mx_mT(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ is in $L_0(U)$ iff (by (6.2)) $q_1x_1 \dots q_mx_mP(x_1,\dots,x_m)$ is true iff (by (iv)) its negation $p_1x_1 \dots p_mx_m\neg P(x_1,\dots,x_m)$ is false iff (by (6.3)) $p_1x_1 \dots p_mx_m\neg T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ is not in $L_0(U)$ iff (by (iv)) the negation of $q_1x_1 \dots q_mx_mT(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ is not in $L_0(U)$,
and
$q_1x_1 \dots q_mx_m\neg T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ is in $L_0(U)$ iff (by (6.3) $q_1x_1 \dots q_mx_m\neg P(x_1,\dots,x_m)$ is true iff (by (iv)) its negation $p_1x_1 \dots p_mx_mP(x_1,\dots,x_m)$ is false iff (by (6.2) $p_1x_1 \dots p_mx_mT(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ is not in $L_0(U)$ (by (iv)) the negation of $q_1x_1 \dots q_mx_m\neg T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ is not in $L_0(U)$
imply that for every sentence added to $L_0(U)$ its negation is not in $L_0(U)$. Thus, for every sentence A of $\mathcal{L}$ not both A and $\neg A$ are in the extended $L_0(U)$, because the original $L_0(U)$ has this property by the first part of the proof of Lemma 3.1. As in the proof of that lemma, it can be shown by induction that A and $\neg A$ cannot both be in $L(U)$ for any sentence A of $\mathcal{L}$ if U is consistent. The result of Theorem 3.1 is valid, i.e., there exists the smallest consistent subset U of $\mathcal{D}$ which satisfies $U = G(U)$, where $G(U)$ is the set of Gödel numbers of sentences of $L(U)$. Let $F(U)$ be defined by (4.1), let $\mathcal{L}_T$ be defined by Definition 4.1, and let the sentences of $\mathcal{L}_T$ be valuated by (I) or by their meanings if the sentences of L_0 are so valuated.

$\mathcal{L}_T$ has properties (i), because it has these properties before its extension. Since L_0 is bivalent, then as in the proof of Lemma 4.1 it can be shown that $\mathcal{L}_T$ is bivalent, i.e., (iii) is valid. The proof of Lemma 4.3 implies that $\mathcal{L}_T$ has properties (ii) in the extended case. Consequently, the extended language $\mathcal{L}_T$ has properties (i)–(iii).

Lemma 6.1. *Let L_0 be a language which conforms to classical logic. Then for every predicate P of L_0 with arity $m > 1$ the predicates $T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ and $\neg T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$ satisfy rule (iv) in the valuation of $\mathcal{L}_T$.*

Proof. To prove (iv) for predicates $T(\lceil P(\dot{x}_1,\ldots,\dot{x}_m)\rceil)$, let P be a predicate of L_0 with arity $m>1$. It follows from the result (T) derived in the proof of Lemma 4.3 that
(a) $P(b_1,\ldots b_m)$ is true in L_0, and hence also in $\mathcal{L}_T$ iff $T(\lceil P(b_1,\ldots,b_m)\rceil)$ is true in $\mathcal{L}_T$, and $P(b_1,\ldots b_m)$ is false in $\mathcal{L}_T$ iff $T(\lceil P(b_1,\ldots,b_m)\rceil)$ is false in $\mathcal{L}_T$.
If $(q_1,\ldots,q_m)$ is any m-tuple of quantifiers $\forall$ and $\exists$, then the sentence $q_1x_1\ldots q_mx_mT(\lceil P(\dot{x}_1,\ldots,\dot{x}_m)\rceil)$ is true iff it is in $L_0(U)$ iff it is in Z_1^m iff (by (6.2)) the sentence $q_1x_1\ldots q_mx_mP(x_1,\ldots,x_m)$ is true iff (by (iv)) the sentence $P(b_1,\ldots,b_m)$ is true in L_0, and hence also in $\mathcal{L}_T$ for all choices of $b_i\in N_P^i$ when q_i is $\forall$, and for some choices of $b_i\in N_P^i$ when q_i is $\exists$ iff (by (a)) the sentence $T(\lceil P(b_1,\ldots,b_m)\rceil)$ is true for all choices of $b_i\in N_P^i$ when q_i is $\forall$, and for some choices of $b_i\in N_P^i$ when q_i is $\exists$.

The above equivalences imply the following result.
(b) The sentence $q_1x_1\ldots q_mx_mT(\lceil P(\dot{x}_1,\ldots,\dot{x}_m)\rceil)$ true iff the sentence $T(\lceil P(b_1,\ldots,b_m)\rceil)$ is true for all choices of $b_i\in N_P^i$ when q_i is $\forall$, and for some choices of $b_i\in N_P^i$ when q_i is $\exists$.
By the above proof (b) is valid for all 2^m different m-tuples of quantifiers $\forall$ and $\exists$. Thus the predicate $T(\lceil P(\dot{x}_1,\ldots,\dot{x}_m)\rceil)$, where P is a predicate of L_0 with arity $m>1$, satisfies the truth part of rule (iv).

The result (T) derived in the proof of Lemma 4.3 ensures that
(c) $\neg P(b_1,\ldots b_m)$ is true in L_0, and hence also in $\mathcal{L}_T$ iff $\neg T(\lceil P(b_1,\ldots,b_m)\rceil)$ is true in $\mathcal{L}_T$, and $\neg P(b_1,\ldots b_m)$ is false in $\mathcal{L}_T$ iff $\neg T(\lceil P(b_1,\ldots,b_m)\rceil)$ is false in $\mathcal{L}_T$.
In the proof that $T(\lceil P(\dot{x}_1,\ldots,\dot{x}_m)\rceil)$ satisfies falsity part of (iv) we use the following equivalences.
The sentence $q_1x_1\ldots q_mx_mT(\lceil P(\dot{x}_1,\ldots,\dot{x}_m)\rceil)$ is false iff ($\mathcal{L}_T$ is bivalent) its negation is true iff (proved above) it is not in $L_0(U)$ iff (by (6.2)) $q_1x_1\ldots q_mx_mP(x_1,\ldots,x_m)$ is not true iff (by bivalence of L_0) $q_1x_1\ldots q_mx_mP(x_1,\ldots,x_m)$ is false iff (by (iv)) the sentence $p_ix_1\ldots p_mx_m\neg P(x_1,\ldots,x_m)$ is true iff (by (iv) $\neg P(b_1,\ldots,b_m)$ is true in L_0, and hence also in $\mathcal{L}_T$ for all choices of $b_i\in N_P^i$ when p_i is $\forall$, and for some choices of $b_i\in N_P^i$ when p_i is $\exists$ iff (by (c)) $\neg T(\lceil P(b_1,\ldots,b_m)\rceil)$ is true for all choices of $b_i\in N_P^i$ when p_i is $\forall$, and for some choices of $b_i\in N_P^i$ when p_i is $\exists$.
The equivalence of the first and the last sentences of the above chain of equivalences proves that $T(\lceil P(\dot{x}_1,\ldots,\dot{x}_m)\rceil)$ satisfies falsity part of (iv).
The proof that $\neg T(\lceil P(\dot{x}_1,\ldots,\dot{x}_m)\rceil)$ satisfies rule (iv) is similar to the above one. □

The above results imply that Theorem 5.1 can be extended as follows.

Theorem 6.1. *Let L_0 be a language which conforms to classical logic, and has predicates which have several free variables. The language $\mathcal{L}_T$ constructed above and valuated by (I) has properties (i)–(iii) given in Introduction. To every predicate P of L_0 with arity $m>1$ the predicates $T(\lceil P(\dot{x}_1,\ldots,\dot{x}_m)\rceil)$ and $\neg T(\lceil P(\dot{x}_1,\ldots,\dot{x}_m)\rceil)$ satisfy rule (iv). T is a truth predicate, and $\neg T$ is a non-truth predicate for $\mathcal{L}_T$. The so formulated definitional theory of truth (shortly DTT) for $\mathcal{L}_T$ is consistent.*

Theorem 5.2 can be extended similarly if L_0 is a language whose every sentence is valuated by its meaning either as true or as false. Thus a semantical theory of truth (shortly STT) can be formulated for the extended $\mathcal{L}_T$ if L_0 has predicates with arity >1.

7 On compositionality of truth in theories DTT and STT

One of the norms presented in [7] for theories of truth is that truth should be compositional. In this section we shall present some logical equivalences which theories DTT and STT of truth prove.

Lemma 7.1. *Theories DTT and STT formulated in Theorems 5.1 and 5.2 prove the following logical equivalences when A and B are sentences of $\mathcal{L}_T$, and P is a predicate of L_0 or P is T.*

(a0) $T(\lceil T(\lceil A\rceil)\rceil) \leftrightarrow T(\lceil A\rceil) \leftrightarrow A.$

(a1) $\neg T(\lceil A\rceil) \leftrightarrow T(\lceil \neg A\rceil) \leftrightarrow \neg A.$

(a2) $T(\lceil A\rceil) \vee T(\lceil B\rceil) \leftrightarrow T(\lceil A \vee B\rceil) \leftrightarrow A \vee B.$

(a3) $T(\lceil A\rceil) \wedge T(\lceil B\rceil) \leftrightarrow T(\lceil A \wedge B\rceil) \leftrightarrow A \wedge B.$

(a4) $(T(\lceil A\rceil) \rightarrow T(\lceil B\rceil)) \leftrightarrow T(\lceil A \rightarrow B\rceil) \leftrightarrow (A \rightarrow B).$

(a5) $(T(\lceil A\rceil) \leftrightarrow T(\lceil B\rceil)) \leftrightarrow T(\lceil A \leftrightarrow B\rceil) \leftrightarrow (A \leftrightarrow B).$

(a6) $\neg T(\lceil A \vee B\rceil) \leftrightarrow \neg(A \vee B) \leftrightarrow \neg A \wedge \neg B \leftrightarrow T(\lceil \neg A\rceil) \wedge T(\lceil \neg B\rceil) \leftrightarrow \neg T(\lceil A\rceil) \wedge \neg T(\lceil B\rceil).$

(a7) $\neg T(\lceil A \wedge B\rceil) \leftrightarrow \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B \leftrightarrow T(\lceil \neg A\rceil) \vee T(\lceil \neg B\rceil) \leftrightarrow \neg T(\lceil A\rceil) \vee \neg T(\lceil B\rceil).$

(a8) $\forall x T(\lceil P(\dot{x})\rceil) \leftrightarrow T(\lceil \forall x P(x)\rceil) \leftrightarrow \forall x P(x) \leftrightarrow \neg\exists x \neg P(x) \leftrightarrow T(\lceil \neg\exists x \neg P(x)\rceil) \leftrightarrow \neg\exists x \neg T(\lceil P(\dot{x})\rceil).$

(a9) $\exists x T(\lceil P(\dot{x})\rceil) \leftrightarrow T(\lceil \exists x P(x)\rceil) \leftrightarrow \exists x P(x) \leftrightarrow \neg\forall x \neg P(x) \leftrightarrow T(\lceil \neg\forall x \neg P(x)\rceil) \leftrightarrow \neg\forall x \neg T(\lceil P(\dot{x})\rceil).$

(a10) $\neg T(\lceil \forall x P(x)\rceil) \leftrightarrow T(\lceil \neg\forall x (P(x)\rceil) \leftrightarrow \neg\forall x P(x) \leftrightarrow \exists x \neg P(x) \leftrightarrow T(\lceil \exists x \neg P(x)\rceil).$

(a11) $\neg T(\lceil \exists x P(x)\rceil) \leftrightarrow T(\lceil \neg\exists x P(x)\rceil) \leftrightarrow \neg\exists x P(x) \leftrightarrow \forall x \neg P(x) \leftrightarrow T(\lceil \forall x \neg P(x)\rceil).$

Proof. T-rule implies equivalences of (a0).
The first equivalences in (a1)–(a5) are easy consequences of rules (t1)–(t5) and T-rule (cf. [5, Lemma 4.1]). Their second equivalences are consequences of T-rule.
The first and third equivalences of (a6) and (a7) follow from T-rule. Their second equivalences are DeMorgan laws of classical logic (cf. [1]), and their last equivalences are consequences of (a1).
The first equivalences of (a8) and (a9) are easy consequences of rules (tp6) and (tp7) and T-rule (cf. [5, Lemma 4.2]). T-rule implies their second equivalences. The third equivalences are DeMorgan laws for quantifiers (cf. [1]). The fourth ones follow from T-rule. DeMorgan laws with $P(x)$ replaced by $T(\lceil P(\dot{x})\rceil)$ imply equivalence of the last and the first ones. (a10) and (a11) are negations to some equivalences of (a8) and (a9). □

If L_0 has predicates of several variables, the extended theories DTT and STT prove the logical equivalences
$T(\lceil q_1x_1 \dots q_mx_mP(x_1,\dots,x_m)\rceil) \leftrightarrow q_1x_1 \dots q_mx_mP(x_1,\dots,x_m) \leftrightarrow q_1x_1 \dots q_mx_mT(\lceil P((\dot{x}_1,\dots,\dot{x}_m)\rceil)$
and
$T(\lceil q_1x_1 \dots q_mx_m\neg P(x_1,\dots,x_m)\rceil) \leftrightarrow q_1x_1 \dots q_mx_m\neg P(x_1,\dots,x_m) \leftrightarrow q_1x_1 \dots q_mx_m\neg T(\lceil P(\dot{x}_1,\dots,\dot{x}_m)\rceil)$
for predicates P of $\mathcal{P}^m$ for every $m > 1$ when $(q_1,\dots q_m)$ is any of the 2^m different m-tuples which can be formed from quantifiers $\forall$ and $\exists$. T-rule implies the first equivalences, and the second equivalences are consequences of (6.2), (6.3) and bivalence of $\mathcal{L}_T$.

Let L_0 be a bivalent first-order language with or without equality. L_0 conforms to classical logic, since it has properties (i)–(iii) presented in Introduction. Moreover, if P and Q are predicates of L_0 with the same domain D, then $\neg P$, $P \vee Q$, $P \wedge Q$, $P \rightarrow Q$ and $P \leftrightarrow Q$ are predicates of L_0 with domain

D. Replacing P and/or Q by some of them we obtain new predicates with domain D, and so on. Thus P in (a8) and (a9) can be replaced by anyone of these predicates Their universal and existential quantifications are sentences of L. They are also sentences of $\mathcal{L}_T$. Anyone of them can be the sentence A and/or the sentence B in results (a1)–(a7) derived above. Moreover, P can be replaced by anyone of those predicates in (a8)–(a11). Take a few examples.
$\forall xT(\lceil P(\dot{x}) \to Q(\dot{x})\rceil) \leftrightarrow T(\lceil\forall x(P(x) \to Q(x))\rceil) \leftrightarrow \forall x(P(x) \to Q(x))$.
$\exists xT(\lceil P(\dot{x}) \wedge Q(\dot{x})\rceil) \leftrightarrow T(\lceil\exists x(P(x) \wedge Q(x)\rceil) \leftrightarrow \exists x(P(x) \wedge Q(x))$.
$\forall xT(\lceil P(\dot{x}) \to \neg Q(\dot{x})\rceil) \leftrightarrow T(\lceil\forall x(P(x) \to \neg Q(x))\rceil) \leftrightarrow \forall x(P(x) \to \neg Q(x))$.
$\exists xT(\lceil P(\dot{x}) \wedge \neg Q(\dot{x})\rceil) \leftrightarrow T(\lceil\exists x(P(x) \wedge \neg Q(x)\rceil) \leftrightarrow \exists x(P(x) \wedge \neg Q(x))$.

These equivalences correspond to the four Aristotelian forms: 'All P's are Q's', 'some P's are Q's', 'no P's are Q's' and 'some P's are not Q's' (cf. [1]).

If P is a predicate of L_0 with arity m for any $m > 1$, then $T(\lceil P((\dot{x}_1, \ldots, \dot{x}_m)\rceil) \leftrightarrow P(x_1, \ldots, x_m)$ is a predicate of $\mathcal{L}_T$ having the domain of P as its domain. An application of T-rule proves the universal T-schema:
(UT) $\quad \forall x_1 \ldots \forall x_m\big(T(\lceil P((\dot{x}_1, \ldots, \dot{x}_m)\rceil) \leftrightarrow P(x_1, \ldots, x_m)\big)$.

Example 7.1. Assume that L_0 is the language of arithmetic with its standard interpretation. Let $R(x, y)$ be formula $2x = y$, and let R be the corresponding predicate with domain $D_R = \mathbb{N}_0 \times \mathbb{N}_0$. Then the truth theories DTT and STT formulated for the extension $\mathcal{L}_T$ of L_0 prove the logical equivalences
(q1) $q_1xq_2yT(\lceil R(\dot{x}, \dot{y})\rceil) \leftrightarrow T(\lceil q_1xq_2yR(x, y)\rceil) \leftrightarrow q_1xq_2yR(x, y)$
and
(q2) $q_1xq_2y\neg T(\lceil R(\dot{x}, \dot{y})\rceil) \leftrightarrow \neg T(\lceil q_1xq_2yR(x, y)\rceil) \leftrightarrow q_1xq_2y\neg R(x, y)$,
and the universal T-schema
(q3) $\forall x\forall y\big(T(\lceil R((\dot{x}, \dot{y})\rceil) \leftrightarrow R(x, y)\big)$.

The sentences in (q1) are true iff q_1q_2 is $\forall\exists$ or $\exists\exists$, and false iff q_1q_2 is $\forall\forall$ or $\exists\forall$. In (q2) the sentences are true iff q_1q_2 is $\forall\forall$ or $\exists\forall$, and false iff q_1q_2 is $\forall\exists$ or $\exists\exists$.

8 Remarks

Results of Theorems 5.1 and 5.2 imply that theories DTT and STT of truth together contain the theory DSTT of truth formulated in [5, Theorem 4.1]. In particular, they conform by [5, Theorem 4.2] to the norms presented in [7] for theories of truth.

The languages $\mathcal{L}_T$ for which theories DTT and STT of truth are formulated extend languages $\mathcal{L}^0$ for which theory DSTT is formulated in [5]. The amount of predicates and compositional sentences are multiplied by means of the added predicate $\neg T$. While $\neg T$ is used in first-order languages to construct a Liar sentence (cf. [3, p. 185]), it is here a non-truth predicate. Moreover, in Section 7 it is shown how to extend the language $\mathcal{L}_T$ so that it both T and $\neg T$ compose with every predicate of the object language L_0 which has several free variables.

The family of those languages which conform to classical logic is considerably larger than the families of those object languages considered in [5]. For instance, the object language L_0 can be any language whose every sentence is valuated by its meaning either as true or as false. Every language L_0 which has properties (i) – (iii), e.g., every bivalent first-order language with or without equality, conforms to classical logic. In such a case L_0 coincides with its basic extension L.

Object languages may have only a finite number of sentences. For example, let L_0 be a language formed by a sentence and its negation. If one of the sentences of L_0 is valuated as true and the other one as false, then L_0 conforms to classical logic. But if both sentences are valuated as true, or both sentences are valuated as false, then this valuation contradicts with the negation rule. Thus L_0 does not conform to classical logic although it is bivalent.

The set U in the above formulated theory is the smallest consistent set for which $U = G(U)$, where $G(U)$ is the set of Gödel numbers of sentences of $L(U)$. Thus U is the minimal fixed point of $G : \mathcal{C} \to \mathcal{C}$, where $\mathcal{C}$ is the set of consistent sets of Gödel numbers of sentences of $\mathcal{L}$. Because every true sentence of $\mathcal{L}_T$ is in $L(U)$ and other sentences are false, and $\mathcal{L}_T$ is bivalent, the sentences of $\mathcal{L}_T$ are grounded in the sense defined by Kripke in [6, p. 18]. The language $\mathcal{L}_\sigma$ determined by the minimal fixed point in Kripke's construction contains also sentences which don't have truth values. For instance, the sentence $A \leftrightarrow T(\lceil A \rceil)$ has not a truth value for every sentence A of $\mathcal{L}_\sigma$. Thus a three-valued logic is needed in [6], as well as in [3]. The only logic used in this paper is classical.

In the metalanguage used in the above presentation some concepts dealing with predicates and their domains are revised from those used in [5] so that they agree better with the corresponding concepts in informal languages of first-order logic (cf. [1]). The circular reasoning used in [5] to show that $G(U)$ is consistent if U is consistent is corrected in the proof of Lemma 3.1.

Mathematics, especially ZF set theory, plays a crucial role in this paper. Metaphysical necessity of pure mathematical truths is considered in [8].

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