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A consistent theory of truth for languages which conform to classical logic

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Abstract. Languages which conform to classical logic have extensions for which a consistent theory of truth can be formulated so that it satisfies the norms presented in Hannes Leitgeb's paper 'What Theories of Truth Should be Like (but Cannot be)'.

1 Introduction

Based on 'Chomsky Definition' (cf. [1]) a language is assumed to be a countable set of sentences, each finite in length, and constructed out of a finite set of elements. A language is assumed also to have a theory of syntax, consisting of symbols and rules to construct well-formed sentences.

A language is said to conform to classical logic, if it has, or if it can be extended to have the following properties:

(i) It contains logical symbols $\neg$ (not), $\vee$ (or), $\wedge$ (and), $\rightarrow$ (implies), $\leftrightarrow$ (if and only if), $\forall$ (for all) and $\exists$ (exist), and the following sentences: If A and B are (denote) sentences, so are $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$ and $A \leftrightarrow B$. If $P(x)$ is a formula of the language, and X_P is a set of terms, then P is called a predicate with domain X_P if $P(x)$ is a sentence of that language for each assignment of a term of X_P into x (shortly, for each $x \in X_P$). $\forall x P(x)$ and $\exists x P(x)$ are then sentences of the language.

(ii) The sentences of that language are so interpreted that the following rules of classical logic hold ('iff' means 'if and only if'): If A and B denote sentences of the language, then A is true iff $\neg A$ is false, and A is false iff $\neg A$ is true; $A \vee B$ is true iff A or B is true, and false iff A and B are false; $A \wedge B$ is true iff A and B are true, and false iff A or B is false; $A \rightarrow B$ is true iff A is false or B is true, and false iff A is true and B is false; $A \leftrightarrow B$ is true iff A and B are both true or both false, and false iff A is true and B is false or A is false and B is true. If P is a predicate with domain X_P, then $\forall x P(x)$ is true iff $P(x)$ is true for every $x \in X_P$, and false iff $P(x)$ is false for some $x \in X_P$; $\exists x P(x)$ is true iff $P(x)$ is true for some $x \in X_P$, and false iff $P(x)$ is false for every $x \in X_P$.

(iii) Principle of bivalence: Every sentence is interpreted either as true or as false.

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Results of [2] are shown to imply that every language which conforms to classical logic has an extension for which a consistent theory of truth can be formulated so that it satisfies the norms presented in [3]. This result is shown in Proposition 3.1 to hold for every language whose sentences have meanings which make them either true or false.

2 An extended language and its properties

Let L be a language which conforms to classical logic, and is without a truth predicate. Construct a language $\mathcal{L}_0$ as follows: Its base language is formed by L extended, if necessary, so that the properties (i) – (iii) are valid, an extra formula $T(x)$ and its assignments when x goes through all numerals, which are also added, if necessary, to symbols of L. Fix a Gödel numbering to the base language. The Gödel number of a sentence (denoted by) A is denoted by #A, and the numeral of #A by $\lceil A \rceil$. The construction of $\mathcal{L}_0$ is completed by adding to it sentences $\forall x T(x)$, $\exists x T(x)$, $\forall x T(\lceil T(x) \rceil)$ and $\exists x T(\lceil T(x) \rceil)$, and $\forall x T(\lceil P(x) \rceil)$ and sentences $\exists x T(\lceil P(x) \rceil)$ for every predicate P of L.

When a language $\mathcal{L}_n$, $n \in \mathbb{N}_0 = \{0,1,2,\dots\}$, is defined, let $\mathcal{L}_{n+1}$ be a language which is formed by adding to $\mathcal{L}_n$ those of the following sentences which are not in $\mathcal{L}_n$: $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$ and $A \leftrightarrow B$, where A and B go through all sentences of $\mathcal{L}_n$. The language $\mathcal{L}$ is defined as the union of languages $\mathcal{L}_n$, $n \in \mathbb{N}_0$. Extend the Gödel numbering of the base language to $\mathcal{L}$, and denote by D the set of those Gödel numbers.

Denote by $\mathcal{P}$ the set of all predicates of L. Divide $\mathcal{P}$ into three disjoint subsets.

$$\begin{cases} \mathcal{P}_1 = \{P \in \mathcal{P} : P(x) \text{ is a true sentence of } L \text{ for every } x \in X_P\}, \\ \mathcal{P}_2 = \{P \in \mathcal{P} : P(x) \text{ is a false sentence of } L \text{ for every } x \in X_P\}, \\ \mathcal{P}_3 = \{P \in \mathcal{P} : P(x) \text{ is a true sentence of } L \text{ for some but not for all } x \in X_P\}. \end{cases} \tag{2.1}$$

Given a proper subset U of D, define

$$\begin{cases} C_1(U) = \{T(x) \colon x = \lceil A \rceil, \text{ where } A \text{ is a sentence of } \mathcal{L} \text{ and } \#A \text{ is in } U\}, \\ C_2(U) = \{\neg T(x) \colon x = \lceil A \rceil, \text{ where } A \text{ is a sentence of } \mathcal{L} \text{ and } \#[\neg A] \text{ is in } U\}, \\ C_1 = \{\neg \forall x T(x), \exists x T(x), \neg(\forall x T(\lceil T(x) \rceil)), \exists x T(\lceil T(x) \rceil)\}, \\ C_2 = \{\forall x T(\lceil P(x) \rceil), \exists x T(\lceil P(x) \rceil) : P \in \mathcal{P}_1\}, \\ C_3 = \{\neg(\forall x T(\lceil P(x) \rceil)), \neg(\exists x T(\lceil P(x) \rceil)) : P \in \mathcal{P}_2\}, \\ C_4 = \{\neg(\forall x T(\lceil P(x) \rceil)), \exists x T(\lceil P(x) \rceil) : P \in \mathcal{P}_3\}. \end{cases} \tag{2.2}$$

Subsets $L_n(U)$, $n \in \mathbb{N}_0$, of $\mathcal{L}$ are defined recursively as follows.

$$L_0(U) = \begin{cases} E = \{A : A \text{ is a true sentence of } L\} \text{ if } U = \emptyset \text{ (the empty set)}, \\ E \cup C_1(U) \cup C_2(U) \cup C_1 \cup C_2 \cup C_3 \cup C_4 \text{ if } \emptyset \subset U \subset D. \end{cases} \tag{2.3}$$

When $n \in \mathbb{N}_0$, and a subset $L_n(U)$ of $\mathcal{L}$ is defined, define

$$\begin{cases} L_n^0(U) = \{\neg(\neg A) : A \text{ is in } L_n(U)\}, \\ L_n^1(U) = \{A \vee B : A \text{ or } B \text{ is in } L_n(U)\}, \\ L_n^2(U) = \{A \wedge B : A \text{ and } B \text{ are in } L_n(U)\}, \\ L_n^3(U) = \{A \to B : \neg A \text{ or } B \text{ is in } L_n(U)\}, \\ L_n^4(U) = \{A \leftrightarrow B : \text{both } A \text{ and } B \text{ or both } \neg A \text{ and } \neg B \text{ are in } L_n(U)\}, \\ L_n^5(U) = \{\neg(A \vee B) : \neg A \text{ and } \neg B \text{ are in } L_n(U)\}, \\ L_n^6(U) = \{\neg(A \wedge B) : \neg A \text{ or } \neg B \text{ is in } L_n(U)\}, \\ L_n^7(U) = \{\neg(A \to B) : \#A \text{ and } \neg B \text{ are in } L_n(U)\}, \\ L_n^8(U) = \{\neg(A \leftrightarrow B) : A \text{ and } \neg B, \text{ or } \neg A \text{ and } B \text{ are in } L_n(U)\}, \end{cases} \tag{2.4}$$

and

$$L_{n+1}(U) = L_n(U) \cup \bigcup_{k=0}^{8} L_n^k(U). \tag{2.5}$$

The above constructions imply that $L_n(U) \subseteq L_{n+1}(U) \subset \mathcal{L}$ and $L_n^k(U) \subseteq L_{n+1}^k(U)$ for all $n \in \mathbb{N}_0$ and $k = 0, \ldots, 8$. Define subsets $L(U)$ and $Q(U)$ of $\mathcal{L}$ by

$$L(U) = \bigcup_{n=0}^{\infty} L_n(U) \text{ and } Q(U) = \{A : \neg A \in L(U)\}. \tag{2.6}$$

The subsets $G(U)$ and $F(U)$ of D, defined by

$$G(U) = \{\#A : A \in L(U)\} \text{ and } F(U) = \{\#A : \neg A \in L(U)\}, \tag{2.7}$$

coincide to those defined in [2].

A subset U of D is said to be consistent if there is no sentence A in $\mathcal{L}$ such that both $\#A$ and $\#[\neg A]$ are in U. The existence of the smallest consistent subset U of D which satisfies $U = G(U)$ is proved in [2, Theorem 6.1] by a transfinite recursion method.

Definition 2.1. Let U be the smallest consistent subset of D which satisfies $U = G(U)$. Denote by $\mathcal{L}^0$ a language which is formed by symbols of $\mathcal{L}_0$ and all the sentences of $L(U)$ and $Q(U)$.

(I) A theory of syntax for $\mathcal{L}^0$ consists of its symbols, and rules to form the sentences of L and to construct those sentences of $\mathcal{L}^0$ which are not in L.

An interpretation of $\mathcal{L}^0$ is defined as follows.

(II) A sentence of $\mathcal{L}^0$ is interpreted as true iff it is in $L(U)$, and as false iff it is in $Q(U)$.

It follows from (2.6) and (2.7) that the above definition of $\mathcal{L}^0$ and its interpretation (II) coincide to the corresponding definitions of [2]. Thus the results of that paper are available.

The following properties are verified in [2, Section 3]:

The language $\mathcal{L}^0$ interpreted by (II) conforms to classical logic.
A sentence of L is true (resp. false) in the interpretation of L iff it is true (resp. false) in the interpretation (II).
T is a predicate of $\mathcal{L}^0$ when its domain is defined by

$$X_T = \{x : x = \lceil A \rceil, \text{ where } A \text{ is a sentence of } \mathcal{L}^0\}. \tag{2.8}$$

If all sentences of the object language L are equipped with meanings, then meanings of the sentences of the language $\mathcal{L}^0$ are determined by meanings of the sentences of L, by meaning of $T(\lceil A \rceil)$, i.e., 'The sentence denoted by A is true', and by standard meanings of logical symbols.

The following result is proved in [2, Proposition 3.3].

> If L is interpreted by meanings of its sentences, and if principle of bivalence holds, then $\mathcal{L}^0$ is interpreted by meanings of its sentences, and this interpretation is equivalent to that given in (II).

3 A theory of truth and its properties

The next theorem, proved in [2], provides a theory of truth for the language $\mathcal{L}^0$ defined in Definition 2.1. Because the interpretation of $\mathcal{L}^0$ can be definitional or semantical, we call, as in [2], that theory definitional/semantical theory of truth, shortly DSTT.

Theorem 3.1. Assume that an object language L is without a truth predicate and conforms to classical logic. Then the language $\mathcal{L}^0$ defined by Definition 2.1 and interpreted by (II), or by meanings of its sentences if L is so interpreted, conforms to classical logic. Moreover, $A \leftrightarrow T(\lceil A \rceil)$ is true and $A \leftrightarrow \neg T(\lceil A \rceil)$ is false for every sentence A of $\mathcal{L}^0$, and T is a truth predicate for $\mathcal{L}^0$.

Hannes Leitgeb formulated in his paper [3] the following norms for theories of truth:

(n1) Truth should be expressed by a predicate (and a theory of syntax should be available).
(n2) If a theory of truth is added to mathematical or empirical theories, it should be possible to prove the latter true.
(n3) The truth predicate should not be subject to any type restrictions.
(n4) T-biconditionals should be derivable unrestrictedly.
(n5) Truth should be compositional.
(n6) The theory should allow for standard interpretations.
(n7) The outer logic and the inner logic should coincide.
(n8) The outer logic should be classical.

The next Theorem, proved in [2], shows that theory DSTT satisfies these norms.

Theorem 3.2. The theory of truth DSTT formulated for $\mathcal{L}^0$ in Theorem 3.1 satisfies the norms (n1)–(n8) and is consistent, i.e. free from contradiction.

The proof of the next Proposition shows that if a language is interpreted by meanings of its sequences, and if the principle of bivalence holds, then it conforms to classical logic.

Proposition 3.1. Every language whose sentences have meanings which make them either true or false has an extension possessing the theory DSTT.

Proof. Let L_0 be a language whose sentences have meanings which make them either true or false. This principle of bivalence remains valid when the sentences $\neg A$, $A \vee B$, $A \wedge B$, $A \rightarrow B$, $A \leftrightarrow B$, $\forall x P(x)$ and $\exists x P(x)$, where A and B go through all sentences of L_0 and P its predicates, are added if they are not in L_0, and interpreted by their standard meanings. Denote by L_1 the so extended language.

Replacing L_0 by L_1 and so on, we obtain a sequence of languages L_n, $n \in \mathbb{N}_0$, whose sentences have meanings which make them either true or false. This holds also for the language L which is union of languages L_n, $n \in \mathbb{N}_0$.

If A and B denote sentences of L, there exist n_1 and n_2 such that A is in L_{n_1} and B is in L_{n_2}. Denoting $n = \max\{n_1, n_2\}$, then A and B are sentences of L_n. Thus the sentences $\neg A$, $A \vee B$, $A \wedge B$, $A \to B$ and $A \leftrightarrow B$ are in L_{n+1}, and hence in L. If P is a predicate of L_0, then the sentences $\forall x P(x)$ and $\exists x P(x)$ are in L_1, and hence in L. Since L is interpreted by meanings of its sentences, then the rules of classical logic presented in (ii) hold. Moreover, the syntactic properties and principle of bivalence presented in (i) and (iii) are satisfied. Thus L, and hence also L_0, conforms to classical logic, whence the conclusion follows from Theorems 3.1 and 3.2.

Remark 3.1. The family of those languages having the theory DSTT of truth is extended in this note considerably from that presented in [2]. For instance, object languages can have only a finite number of sentences.

The result of Proposition 3.1 does not necessarily hold for languages which are not interpreted by meanings of its sentences, although they satisfy principle of bivalence. For instance, a language which is formed by a sentence and its double negation conforms to classical logic iff both sentences are interpreted either as true or as false.

Mathematics, especially set theory, plays a crucial role in this note, as well as in [2]. Metaphysical necessity of pure mathematical truths is considered in [4].

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