# TUPLES ALL THE WAY DOWN?

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We can introduce singular terms for ordered pairs by means of an abstraction principle. Doing so proves useful for a number of projects in the philosophy of mathematics. However there is a question whether we can appeal to the abstraction principle in good faith, since a version of the Caesar Problem can be generated, posing the worry that abstraction fails to introduce expressions which refer determinately to the requisite sort of object. In this short paper I will pose the difficulty, and then propose a solution. Since my solution appeals to a plausible constraint on the introduction of new expressions to a language, it is of interest independently of the particular case of terms for pairs. Since these provide the occasion for discussion we should nonetheless review the use of abstraction for pairs before the argumentative business of the paper commences.

1

When we are first introduced to ordered pairs, we are told that they are governed by a simple principle, often known as *Peano pairing*:[1]

(PAIR) $$(< x_1, x_2 >=< x_3, x_4 >) \leftrightarrow (x_1 = x_3 \land x_2 = x_4)$$

It is tempting to view (PAIR) as satisfying the general form of an abstraction principle:

(ABS) $$\natural a = \natural b \leftrightarrow a \approx b$$

Where '$\natural$' is a term-forming operator on expressions of a kind instanced by '$a$' and '$b$' and $\approx$ is an equivalence relation on entities of the sort denoted by those expressions. Typically '$a$' and '$b$' will be variables bound by quantifiers.

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[1]The usual conventions about taking free variables as bound by implicit prenex universal quantifiers apply. The usage *Peano Pairing* is owing to Kanamori [13].

Note, however, that (PAIR) is not of the form (ABS), since although identity is an equivalence relation the right-hand side of the biconditional consists of the conjunction of two identity statements. We are, as Shapiro puts it, taking terms 'two at a time' [17, 337]. This is not a great problem, though, since the principle is sufficiently alike in 'feel' to instances of (ABS) that it is extremely difficult to imagine how somebody who takes such instances to be in epistemically or semantically privileged positions would justify not taking (PAIR) to be in a similar position. This might be particularly clear if, following Shapiro, we eliminate logical complexity from the RHS of (PAIR) by defining a predicate 'E' such that:[2]

(E-INT)                               $Eabcd \leftrightarrow (a = c \wedge b = d)$

Now we can formulate an (ABS)-like principle which is equivalent to (PAIR),

(PAIR*)                               $(< x_1, x_2 >=< x_3, x_4 >) \leftrightarrow E x_1 x_2 x_3 x_4$

Given an antecedent grasp on the vocabulary on the right-hand side of the biconditional, the principle allows us to introduce expressions for ordered pairs and gives us an understanding of canonical identity conditions. From a neo-Fregean perspective this seems to assure us that the terms have sense. Moreover since it is sufficient for a singular term to refer that it occur in a non-opaque context in a true sentence, we can conclude that ordered pairs exist, given only the assurance of an instance of the RHS. For an in depth account of this approach to abstraction principles the reader is referred to the literature on neo-Fregeanism.[3]

Why might we want to avail ourselves of (PAIR*)? There are a number of reasons. Most obviously, talk of pairs is common in mathematics and we may want to give a satisfying account of the possibility of reference to them, in the spirit of neo-Fregeanism.[4] Shapiro notes the availability of (PAIR*) in his abstractionist treatment of real analysis, and whilst he doesn't make direct use of it himself, he offers it to readers who might be averse to taking objects two at a time in abstraction

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[2]Whilst the detour *via* 'E' here is worthwhile heuristically, it should be flagged that quarternary $E$ as defined in the body is not an equivalence relation in the usual sense. For this we need quasi-reflexivity ($Eabab$) and quasi-symmetry ($Eabcd \rightarrow Ecdab$).

[3][7] is a good introduction. See also [11] and especially the recent overview in [9].

[4]It might be added that talk of pairs is common *outside* of mathematics as well: I am wearing a pair of shoes and own a pair of gloves. I doubt however that this usage of the word 'pair' is reifying in its import, as is the mathematical usage. A similar phenomenon might be thought to occur with the word 'set'.

principles [17, 338]. Tennant also draws attention to the possibility of abstraction for pairs as he develops his verson of logicism[5] (he also introduces the difficulty we will meet in a moment) [18, 21].

Perhaps more ambitious would be the use of pair abstraction to develop a version of neo-Fregean logicism about arithmetic using a plural logic.[6,7] This would be attractive in many respects, since it has the potential to answer some of the worries directed at the neo-Fregean use of second-order logic regarding ontological commitment and tacit appeal to set-theoretic resources [15]. However plural logics can only be used, following Boolos [5], to interpret *monadic* second-order logic and (HP) is irreducibly polyadic, since '≈' abbreviates an existential claim about a dyadic relation. This could be simulated by plural quantification over pairs, affording the plurally minded neo-Fregean something equivalent to (HP).[8]

This last application will be at the forefront of consideration in what follows. Apart from its intrinsic importance within the philosophy of mathematics, this is partly because of my own interest in the matter (I discuss plural logicism in []). Less egocentrically, the use of pair abstraction to permit a plural formulation of (HP) brings into sharp relief the constraints on a genuinely sense-bestowing abstraction principle in a foundational mathematical context. As we will see, navigating these constraints is a central task for those hoping to address a version of the Caesar Problem for pair abstraction. We meet this problem in the next section.[9]

2

For any abstraction principle of the form (ABS), the following challenge must be faced: are the singular terms introduced by the principle possessed of a sense sufficiently determinate to fix the truth value of every sentence in which they occur, given only the determinacy of the sense of the

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[5]Tennant in fact attends to the principle (PAIR).

[6]Boccuni has developed a plural logicism about arithmetic, but this is un-Fregean in several respects, not least in that it doesn't proceed via (HP) or a similar abstraction principle [3] [4] [].

[7]A referee points out that (PAIR*) in combination with any principle of the form $\forall x\, (a \neq < a, x >)$ yields an infinity of objects. It is then straightforward to interpret second-order PA by interpreting succession $Sx =< a, x >$. However, from the perspective of neo-Fregean logicism, the issue is going to be one of whether we are *entitled* to the additional principle asserting the non-identity of any object with the ordered pair containing only itself. It is this question that the present paper addresses.

[8]For details of how this might go see [].

[9]I view the difficulty developed here as a variant of the Caesar Problem in the following sense: the Caesar Problem concerns the apparent failure of an abstraction principle to secure reference to a unique object; the present problem concerns the apparent failure of an abstraction principle to secure reference to a unique kind of object.

other subsentential expressions? For example, does Hume's Principle deliver the falsity of the claim that the number two is identical to Julius Caesar? The threat here is that, since indeterminacy of sense compromises determinacy of reference, the appeal to abstraction principles fails to account satisfactorily for our reference to mathematical objects. Given that a major selling point of neo-Fregeanism is precisely supposed to be its ability to account for this,[10] failure to respond to the challenge amounts to an admission of defeat.

Note in particular that abstraction principles are supposed to introduce expressions referring to a particular *kind* of mathematical object, that is objects falling under a common sortal ('number', 'set', 'direction'...). It is a requirement then that an instance of (ABS), deployed for neo-Fregean purpose, fix the sense of the singular terms on the LHS sufficiently to secure that no such term refers to an entity of a kind membership of which is incompatible with falling under the sortal associated with the abstraction principle. What sortal incompatibility might consist in, and how we might go about testing it are involved questions. Thankfully we don't need a programmatic answer to them in order to raise a problem for (PAIR*).

All that is required is the observation that the sense of 'pair' in which we hope to introduce expressions for pairs using (PAIR*) is one for which pairs are *well-founded*. Intuitively, pairing involves taking two entities and bringing them together and in an order; this requires that the objects be 'given' prior to pairing, and in particular that they don't depend metaphysically on the pair. Indeed, surely the pair depends metaphysically on its co-ordinates, not the co-ordinates on the pair.[11] Moreover, a common thought goes, metaphysical dependence must 'bottom out' somewhere: no entity can depend for its existence on an infinitely descending chain of distinct entities.[12] Tennant presents the issue of well-foundedness for pairs through two questions, each of which demand the answer 'no': (a) could an ordered pair have an infinite 'pedigree'?; and, (b) even if finite, could the pedigree of an ordered pair contain loops? (i.e. might there be a pedigree that is not a finite tree) [18, 26].

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[10]Our capacity to refer to mathematical objects is a live issue in the wake of [2]. Hale and Wright have repeatedly situated their project in response to this epochal paper.

[11]I do not take this assertion of metaphysical dependence to depend on a global theory of 'Grounding' (for scepticism about which see [19]), and of itself I understand it as neutral between platonist and constructivist approaches to the metaphysics of mathematics.

[12]This, I think, is not obvious – and indeed might look like a version of the cosmological argument which many philosophers reject. No matter; so long as you are convinced that it is part of the target concept of pairs that pairs are well-founded, you do not need to be convinced by any particular argument to that conclusion. In any case, in the final section, we will see that there are linguistic reasons why pairs introduced by abstraction have to be well-founded.

Tennant concurs with the present position on (a) and (b) for the reason we have already put forward – the metaphysical dependence of pairs upon their co-ordinates. He offers a reflection on the implications of this for approaches to pairs that do not proceed *via* the standard set-theoretic implementation:

> These ontological intuitions go strictly beyond what is required of the notion of ordered pair in order for it to serve as it does in the set-theoretical reconstruction of mathematics. Given the Axiom of Foundation in set theory, and the Kuratowski definition of ordered pair, it is clear that the 'orderly pairing' pedigrees of Kuratowskian ordered pairs must be finite trees. But note that this is only because the membership relation among sets is in general well-founded. If one is dealing with orderly pairing *sui generis*, outside the context of set theory, then one has to do more in order to ensure that the pedigrees of one's ordered pairs are indeed finite trees.[18, 27]

Alas it is easy to see that (PAIR*) does not ensure this. For it has a model with the domain {$a$}, such that $a = \langle a, a \rangle$, violating Tennant's constraint (b). Similarly (a) falls with the recognition that we can interpret the nodes of an infinite binary tree as pairs, with each node's daughter nodes as its co-ordinates and obtain a model for (PAIR*). Is the hope that we might avail ourselves of pairs by abstraction a vain one?

Certainly the most familiar response to the Caesar Problem is not going to help us. Confronted with that difficulty Hale and Wright propose an appeal to *categories*, where entities *a* and *b* are in different categories just in case *a* possesses different kinds of identity conditions from *b*. So, for example, the identity conditions for the kind under which Caesar falls might be given in terms of spatiotemporal location, or Lockean psychological continuity, or whatever, whereas those for numbers are given in terms of equinumerosity. No entity belongs to two wholly distinct categories, so the thought goes. However, (PAIR*) gets the identity conditions for pairs *right*. This is in essence the condition to which appeal is made day by day in working mathematics (being a reformulation of (PAIR)); particularly if we want the introduction of expressions by abstraction to satisfy what is know as *Frege's constraint* – that the introduction of a class of expressions by abstraction must relate to the canonical applications of the referents of those expressions – conditions along these lines are surely the only game in town. But then well-behaved pairs belong to the same category as their troublesome non-wellfounded cousins, and we can't appeal to categorial distinctions to exclude deviant

interpretations of (PAIR*) in terms of non-wellfounded pairs.

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Isn't this the end for any hope of introducing pairs by abstraction? If the desired abstraction principle doesn't ensure that the expressions it introduces refer to the right kind of entity then doesn't the logicist need to adopt some other approach to pairs? This is the conclusion Tennant draws, and it is also drawn in recent work by Pleitz [16]. Such resignation is too hasty, however. Recall a central thought behind the introduction of expressions by abstraction: *if a language-user understands the expressions occuring on the RHS*, she is in a position to understand those on the LHS (since her understanding of the RHS will afford her the truth-conditions for the LHS, on which the only hitherto unfamiliar lexical item is the introduced term-forming operator). But now suppose (PAIR*) could be used to introduce an expression referring to $a$, where $a = \langle a, a \rangle$. In order to grasp the sense of the RHS a language user would need to understand expressions referring to $a$. But it is this referential capacity that is supposed to be secured by the LHS. So the language user will never be in a position to introduce a singular term for $a$ by abstraction.

Mightn't it be the case, however, that the expressions on the RHS refer to $a$ but do so by means of a sense distinct from that on the LHS? By means of the abstraction principle, this line of objection goes, a language user might come to refer to the pair $a$ under a new guise, much as the ancient astronomer might come to refer to the heavenly body he already knows as Hesperus as Phosphorus. The reason this won't save reference to $a$ by abstraction is that it is a reasonable constraint on attributing understanding of a singular term that a language user have at least an implicit grasp on the (object-level) identity conditions for a sortal associated with the term.[13] In the case of (PAIR*) therefore the person who understands the RHS of the instance, the LHS of which introduces reference to $a$, will have to understand the identity conditions for pairs, given by (PAIR*). In order to assess whether the LHS is true (and therefore whether the reference of its singular terms is secured), she will need first to assess whether the RHS is true. This involves assessing the identity statements in the relevant instances of (E-INT), which since the objects in question are pairs requires her to appeal to (PAIR*). And so on in infinite regress. There is no instance of (PAIR*) with a determinate sense

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[13]In passing: I am conceding more to the objection here than some would. A referee points out that for standard neo-Fregeanism the RHS of an abstraction principle is supposed to be concept-constitutive for the LHS, blocking the possibility of understanding the latter without an understanding of the former. See [20].

such that it introduces reference to *a* by abstraction. Indeed, more than that, the preceding considerations serve as an effective *reductio* that any language user could have a grasp of the RHS of an instance of (PAIR*) purporting to introduce reference to *a*. For that would require her to be able to grasp the sense of singular terms for *a* associated with the identity condition given by (PAIR*) which she cannot.

But might she not grasp a sortal under which *a* falls other than the sortal *pair*, enabling her to understand the LHS without prior understanding of the RHS? Some authors are reluctant to allow that one and the same entity can fall under distinct sortals with divergent identity conditions (witness Hale and Wright's appeal to ontological *categories*). Setting this aside for the sake of charity to the objector, there is a severe problem appealing to alternative sortals within the context of using pairs to develop a version of logicism about arithmetic. Remember that we are interested in appealing to plural quantification over pairs in order to reproduce something like Hume's Principle within plural logic. But now consider the most likely alternative sortals under which pairs fall. These are mathematical objects, since pairs may be implemented in number theory, set-theory, and so on. Thus, for instance, the Cantor pairing function within number theory:

$$< a, b >= \frac{[(a + b)(a + b + 1)]}{2} + b$$

yields pairs subject to the identity conditions for numbers (in spite of being itself non-wellfounded). Clearly nobody who wishes to appeal to pairs for neo-logicist purposes can appeal to Cantor pairing in order to escape the charge of circularity. Such a person simply makes herself victim to a new circularity. She wishes to introduce numerical expressions via an abstraction principle appealing to quantification over pairs. However her understanding of expressions for pairs is mediated by a prior understanding of numerical terms. We are still stuck in a circle; a different circle, for sure, but still a circle for all that.

What though if the sortal under which *a* falls is not a mathematical one at all, but rather something like *person*?[14] Suppose that Buffy=⟨Buffy, Buffy⟩. Surely the pairing properties of Buffy are captured by (PAIR*) and there is no good reason I could not introduce a new singular term for Buffy by applying PAIR* right to left in the light of my antecedent understanding of the expression 'Buffy'. There are two paths of response here. That of least resistance shrugs its shoulders at the Buffy

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[14]Thanks to a referee for *Thought* for pressing this question.

case: it is not *this* kind of deviant pair we are worried about; so long as pairs of mathematicalia are well-founded, that is all we want to secure properly determinate mathematical reference, if concrete objects happen to be the repeated co-ordinate of their own pair, then so be it.[15]. I don't favour this path, since – as we saw above – (PAIR*) is supposed to introduce only expressions referring to a single *sort* of object, and allowing that it may introduce both well-founded and non-wellfounded pairs looks in danger of conceding that it fails in this respect.

What then might be a principled reason for holding that (PAIR*) cannot introduce an expression referring to Buffy? Here, I think, we need to revisit Hale and Wright's response to the Caesar problem.[16] Remember that they rule out transcategorial entities, but this is exactly what Buffy would be, being subject to the identity conditions for both pairs and persons. In order to get a handle on one reason these might be unacceptable, note that admitting the possibility of Buffy entails a strong *metaphysical realism*.[17] There can be, on this view, facts about identity which are in principle inscrutable, which – so to speak – can only be seen to be true from a God's eye perspective. Language users do not have access to any resources that would enable them to decide whether Buffy is identical to her own pair.[18] In virtue of what, then, is this true? One needn't buy into a full-blown Dummettian justificationism to worry about the answer here,[19] since inscrutable identity facts look particularly problematic from a neo-Fregean perspective. For if we claim that what it is to be an object is to be the possible referent of a singular term [10], and the introduction of these terms understood in terms of carving content, then there is something very odd about the thought that facts about the identity of objects can come apart completely from the practice of referring to objects. Hero carves some content in a pair-ish way, and thereby manages to refer to ⟨Buffy, Buffy⟩; he is also familiar with the name 'Buffy', and correctly understands it to be governed by the identity conditions associated with the sortal *person*, which canonically recarve content in an appropriate way. There are two distinct acts of carving here and, by the lights of the background meta-ontology, there ought to be two objects. Take issue with that background meta-ontology by all means, but if you are playing the neo-Fregean game at all – and that you are is a presupposition of the present paper – you oughtn't to

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[15]Compare the similar view of singletons as Quine atoms (the existence of which is consistent with NF) This is countenanced in, amongst other places, Lewis' *Parts of Classes* [14].

[16]A reminder of the dialectic here: up until this point we have not availed ourselves of anything like a *general* response to the Caesar problem along Hale/ Wright lines. So those who are sceptical of such responses can take our path of least resistance with respect to the Buffy case whilst otherwise going along with the position of this paper.

[17]In the sense discussed in [6], following Putnam.

[18]*Objection:* Yes they do. In good Lewisian fashion they can appeal to our best overall theory. *Reply:* At this point we are so far from neo-Fregean method that it is unclear why we'd be worrying about the Caesar problem in the first place.

[19]On which, see [8].

like identity claims of the sort at issue.[20]

In other words, if we are minded to take the Buffy case seriously, then reflection on a neo-Fregean account of what it is to be an object ought to convince us that we *shouldn't* take it seriously (at least if we are to remain neo-Fregeans). The situation described, in which there is an in principle unknowable truth about identity, is not a genuine possibility. Being brought to realise this is a kind of philosophical therapy, our worry being laid to rest by conceptual clarity, as well as a route to embracing Hale and Wright's response to the original Caesar problem, ruling out transcategorical identity claims. As we saw above, however, this response isn't adequate to address the worry that (PAIR*) might introduce reference to non-wellfounded pairs. Doing this requires appeal to considerations about linguistic understanding, of the sort we have examined.

We can rule out models of (PAIR*) containing non-wellfounded pairs as deviant. If a speaker starts with a language all of whose expressions have a well-functioning determinate sense and adds (PAIR*), she will not introduce reference to monsters such as *a*. These only arise as possibilities when one considers the abstraction principle as a formal statement apart from considerations about language use and the practice of reference. Then, of course, one can point to non-wellfounded models, but one has at this point strayed from the point of the neo-Fregean project. Similarly, of course, there is a perfectly legitimate mathematical study of non-wellfounded objects,[21] including non-wellfounded tuples. All that the foregoing establishes is that reference to objects of this sort cannot be explained by abstraction, but is perhaps a more holistic theory-embedded affair.

It is a constraint on the explicit introduction of new expressions to a language that the relevant speakers understand all the expressions used to make the introduction. When thus stated the principle seems obvious. Yet it is in the nature of philosophy that obvious points can get buried beneath details. Once we have cleared up we often get a better view of the subject matter. In particular, we can see that we may introduce ordered pairs by abstraction without difficulty.[22]

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[20]On neo-Fregean meta-ontology see Hale and Wright's [12], as well as the essays in [9].

[21]The canonical text is [1].

[22]Or at least – it need hardly be said – without any difficulties that don't attend to the use of abstraction principles *in general*.

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REFERENCES

[1] Peter Aczel, *Non-well-founded sets*, CSLI, Stanford, CA., 1988, CSLI Lecture Notes Number 14.

[2] Paul Benacerraf, *Mathematical Truth*, Journal of Philosophy **70** (1973), no. 19, 661–79.

[3] Francesca Boccuni, *Plural Grundgesetze*, Studia Logica **96** (2010), no. 2, 315–330.

[4] ———, *On the Consistency of a Plural Theory of Frege's Grundgesetze*, Studia Logica **97** (2011), no. 3, 329–345.

[5] George Boolos, *To be is to be a value of a variable (or to be some values of some variables)*, Logic, Logic and Logic, Harvard University Press, Cambridge, MA., 1998, pp. 54–72.

[6] Tim Button, *The Limits of Realism*, Oxford University Press, Oxford, 2013.

[7] Roy T. Cook, *New waves on an old beach: Fregean philosophy of mathematics today*, New Waves in Philosophy of Mathematics (Ø. Linnebo O. Bueno, ed.), 2009.

[8] Michael Dummett, *Thought and Reality*, Oxford University Press, 2006.

[9] Philip A. Ebert and Marcus Rossberg, *Introduction to abstractionism*, Abstractionism: Essays in Philosophy of Mathematics, Oxford University Press, 2016, pp. 3–34.

[10] Bob Hale, *Necessary Beings*, Oxford University Press, Oxford, 2013.

[11] Bob Hale and Crispin Wright, *Introduction*, The Reason's Proper Study - Towards a neo-Fregean Philosophy of Mathematics, Clarendon Press, Oxford, 2001, pp. 1–30.

[12] ———, *The metaontology of abstraction*, Metametaphysics: New Essays on the Foundations of Ontology (David John Chalmers, David Manley, and Ryan Wasserman, eds.), Oxford University Press, 2009, pp. 178–212.

[13] Akihiro Kanamori, *The empty set, the singleton, and the ordered pair*, Bulletin of Symbolic Logic **9** (2003), 273–289.

[14] David Lewis, *Parts of Classes*, Blackwell, Oxford, 1991.

[15] Fraser MacBride, *Speaking with shadows : A study of neo-logicism*, British Journal for the Philosophy of Science **54** (2003), no. 1, 103–163.

[16] Martin Pleitz, *Two Accounts of Pairs*, The Logica Yearbook 2016 (Pavel Arazim and Tomáš Lávička, eds.), College Publications, London, 2017, pp. 201–221.

[17] Stewart Shapiro, *Frege meets Dedekind: A neologicist treatment of real analysis*, Notre Dame Journal of Formal Logic **41** (2000), no. 4, 335–364.

[18] Neil Tennant, *Natural Logicism via the logic of orderly pairing*, Manuscript, 2010.

[19] Jessica M. Wilson, *No work for a theory of grounding*, Inquiry **57** (2014), no. 5-6, 535–579.

[20] Crispin Wright, *Abstraction and Entitlement : On the Epistemic Status of Hume's Principle*, Abstractionism: Essays in Philosophy of Mathematics (Philip A. Ebert and Marcus Rossberg, eds.), Oxford University Press, 2016, pp. 161–185.

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