The Case Against Closure of A Priori Knowability under Modus Ponens

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April 29, 2015

Abstract

The topic of this article is the closure of *a priori* knowability under *a priori* knowable material implication: if a material conditional is *a priori* knowable and if the antecedent is *a priori* knowable, then the consequent is *a priori* knowable as well. This principle is arguably correct under certain conditions, but there is at least one counterexample when completely unrestricted. To deal with this, Anderson proposes to restrict the closure principle to necessary truths and Horsten suggests to restrict it to formulas that belong to less expressive languages. In this article it is argued that Horsten's restriction strategy fails, because one can deduce that knowable ignorance entails necessary ignorance from the closure principle and some modest background assumptions, even if the expressive resources do not go beyond those needed to formulate the closure principle itself. It is also argued that it is hard to find a justification for Anderson's restricted closure principle, because one cannot deduce it even if one assumes very strong modal and epistemic background principles. In addition, there is an independently plausible alternative closure principle that avoids all the problems without the need for restriction.

1 Introduction

Ever since the work of Dretske (1970) and Nozick (1981), there has been an intensive debate in epistemology about the principle of closure of knowledge under known material implication, which can be formalized as follows:

$$K(\phi \to \psi) \to (K\phi \to K\psi). \tag{1}$$

The discussion is often framed in terms of general, so-called 'subjunctivist' theories about knowledge. Recently, Holliday (2014) formalized the theories of knowledge proposed by Dretske (1970), Nozick (1981), Heller (1989, 1999) and Sosa (1999) in a single framework and he showed that (1) fails on all of them.[1]

What is perhaps less known is that Dretske (2005) has also implicitly challenged the principle of closure of *knowability*, i.e. possible knowledge, under known material implication, which can be formalized as follows:

$$\Diamond K(\phi \to \psi) \to (\Diamond K\phi \to \Diamond K\psi). \tag{2}$$

Dretske (2005) claims that we face a choice between: first, accepting that we have lots of easily obtainable knowledge of light-weight propositions (e.g., there are cookies in the jar); second, accepting that it is very hard or perhaps impossible to obtain knowledge about certain heavy-weight propositions (e.g., there is no evil demon that deceives us); third, accepting that one often knows that light-weight propositions have

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[1] If one takes into account the basis for a belief, then the outcome is different (Holliday, 2012, appendix 2.D).

heavy-weight implications; fourth, accepting closure of knowledge under known material implication. For discussion of this see the exchange between Dretske (2005) and Hawthorne (2005). Since Dretske thinks that some of the heavy-weight propositions are not simply unknown but also unknowable, one can replace the fourth option, namely closure of *knowledge* under known material implication, with closure of *knowability* under material implication as well.

If one focuses not so much on knowledge in *general* but on *a priori* knowledge in particular, then it seems initially plausible that *a priori knowability* is closed under known material implication, which can be expressed as follows:

$$\Diamond K_a(\phi \to \psi) \to (\Diamond K_a\phi \to \Diamond K_a\psi)\,. \tag{3}$$

Assessing the material soundness of (3) is the main goal of this paper. Fritz (2013) has formalized so-called epistemic two-dimensionalism and closure of *a priori* knowability under modus ponens is a theorem in his framework. Yet it is not entirely uncontroversial. Assume the principle that all tautologies $\top$ are *a priori* knowable, which can be formalized as follows:

$$\Diamond K_a\top. \tag{4}$$

It follows from (3) and (4) that *a priori* knowability is closed under conjunction introduction,[2] which can be formalized as follows:

$$(\Diamond K_a\phi \wedge \Diamond K_a\psi) \to \Diamond K_a(\phi \wedge \psi)\,. \tag{5}$$

Anderson (1993) and Horsten (2000) each offer a counterexample to (5). They each also suggest a way of dealing with these counterexamples. This is the subject of Section 2. I will argue against Horsten's proposal in Section 3. The justification of Anderson's proposal is the subject of Section 4. In Section 5 I will present an alternative principle that is more resilient to the problems encountered in this article and that has good, independent justification.

2 Anderson and Horsten on Closure

Anderson (1993, p. 8-9) presents the following counterexample. Assume that (*a priori*) knowledge is factive, which can formalised as follows:

$$K_a\phi \to \phi. \tag{6}$$

Let ϕ be of the form $p \leftrightarrow @p$, with @ the actuality operator. Let p mean that I don't have *a priori* knowledge of any conjunction.[3] Furthermore, suppose that p is actually true. Next, assume that I know *a priori* that $p \leftrightarrow @p$ and, therefore, it is *a priori* knowable for me. Given (4), it is *a priori* knowable that $p \vee \neg p$. However, it cannot be the case that I know *a priori* that $p \leftrightarrow @p$ *and* $p \vee \neg p$, since in that case it would have to be true that $p \leftrightarrow @p$, which can only the case if p is true, because $@p$ is true. But then in that case it is true that I don't know *a priori* any conjunction and, consequently, I don't know *a priori* that $p \leftrightarrow @p$ *and* $p \vee \neg p$.

Let us consider how strong the counterexample really is. In the example it is assumed that it is actually the case that I don't have *a priori* knowledge of conjunctions, which is manifestly not the case. Anderson also mentions the possibility that p means that I have made no inferences. On that reading of p, it again is actually *false*. A third and final option Anderson considers is that p means that everything I know is of a bounded complexity. This is more promising; it could actually very well be the case. Let us delve a bit deeper.

[2] Interestingly, Holliday (2014) shows that *this* closure principle is a consequence of the various subjunctivist theories of knowledge.

[3] As will become clear, this can only be the case if material equivalence is not analysed in terms of the *conjunction* of two material implications. Alternatively, let p mean that I don't have *a priori* knowledge of any conjunction of the form $(\theta \to @\theta) \wedge (\theta \vee \neg\theta)$.

Anderson does not say which complexity measure he has in mind, but a natural choice here is the following: the complexity of a formula is the number of steps needed to construct the formula on the basis of the recursive definition of well-formed formulas. E.g., the complexity of $(p \leftrightarrow @p) \wedge (p \vee \neg p)$ is higher than the complexity of $(p \leftrightarrow @p)$. Strictly speaking, the complexity of p is then of the lowest complexity. However, this may be deemed somewhat misleading. After all, the complexity of formulas is defined in a metalanguage, so p expresses something that is properly formulated in a metalanguage. This might be the reason why Horsten (2000, p. 65, fn. 9) says that '[Anderson] gives a counterexample based on self-reference'. In order to make sure that there is nothing paradoxical going on, it may be wiser to make to use arithmetization to make explicit in the object language what p means in the metalanguage. However, this raises the question how it is possible to say that I don't know anything above a certain complexity without exceeding that complexity. An arithmetized expression of the claim that nothing above a certain complexity, denoted by the natural number n, is known to be true can be expressed as follows:

$$\neg\exists x \left(K_a T(x) \wedge x > \overline{n}\right), \tag{7}$$

with $T(x)$ a truth predicate and with $\overline{n}$ the Peano numeral of the number n. The Gödel code of the above formula is higher than n, since even the Gödel code of $\overline{n}$ is considerably greater than n. The Gödel code of

$$\neg\exists x \left(K_a T(x) \wedge x > \overline{n}\right) \leftrightarrow @\neg\exists x \left(K_a T(x) \wedge x > \overline{n}\right) \tag{8}$$

is higher still. But in that case (8) is not actually known, because nothing above complexity n is actually known. Moreover, it is not knowable, since (7) is false in any world in which (8) is known, while @ (7) is true in that world, thereby making (8) false. It is not my intention to claim that the above considerations are a definite refutation of (Anderson, 1993)-style counterexamples. However, I think that the burden of proof has shifted to those who want to provide such a counterexample.[4]

Horsten (2000, p. 50-51) also targets (5). Consider the following two sentences:

$$\Diamond K_a \forall x \left(K_a \left(x \in \{x \mid \exists y \in \mathbb{N}\left(x = 2 \times y\right)\}\right) \leftrightarrow \left(x = 2 \vee x = 4\right)\right); \tag{9}$$

$$\Diamond K_a \forall x \left(K_a \left(x \in \{x \mid \exists y \in \mathbb{N}\left(x = 2 \times y\right)\}\right) \leftrightarrow \left(x = 2 \vee x = 4 \vee x = 6\right)\right). \tag{10}$$

The first one says that is is *a priori* knowable that every number of which it is *a priori* known that it is an even number is identical to two or to four. The second one says that it is *a priori* knowable that every number of which it is *a priori* known that it is an even number is identical to two or to four or to six. Combining the above with (5) and using elementary first-order, arithmetical and modal reasoning, one can derive a contradiction via

$$\Diamond \left(K_a \left(6 \in \{x \mid \exists y \in \mathbb{N}\left(x = 2 \times y\right)\}\right) \wedge \neg K_a \left(6 \in \{x \mid \exists y \in \mathbb{N}\left(x = 2 \times y\right)\}\right)\right).$$

Deeming the assumptions plausible, Horsten (2000) rejects (5).

Both Anderson (1993) and Horsten (2000) reject (5), which is entailed by (3) and (4). They both think that on that basis (3) is to be rejected. Notwithstanding their rejection of (3), they both think that a restricted version of (3) can be upheld, although they each favour a different restriction.

Anderson (1993, p. 7) accepts the principle when the *a priori* knowable propositions are also necessary, which can be formalized as follows:

$$\left(\Diamond K_a \left(\phi \rightarrow \psi\right) \wedge \Box \left(\phi \rightarrow \psi\right)\right) \rightarrow \left(\left(\Diamond K_a \phi \wedge \Box \phi\right) \rightarrow \Diamond K_a \psi\right). \tag{11}$$

The counterexample he used against (3) involved a *contingent* proposition, viz. $p \leftrightarrow @p$ and, hence, it does not work against (11). Furthermore, Horsten (2000)'s counterexample also involved contingent

[4] Hawthorne (2005, p. 40, fn. 4) mentions a similar example: 'Suppose I know P and know necessarily P [→] I will never perform a deductive inference again.' Without specifying what *P* is and without explaining how one could get *a priori* knowledge of this, the example is underdeveloped.

truths, viz. truths of the form $K_a\phi$ and $\neg K_a\phi$. Moreover, (11) still has a large class of applications. Analytic truths (e.g. all vixens are foxes) and mathematical truths (e.g. $2+3=5$) are all thought to be necessary. So far, the negative justification for (11). The positive justification offered by Anderson (1993, p. 9) is, however, not relevant. He says the following:

> Our knower has the potential to have both of the necessities p and q in mind and to put them together into a conjunction. If both of them are known a priori, and he believes the conjunction as a result of his inference, then he surely has a priori knowledge of that conjunction.
>
> [...] We know by proof the necessary truths theorem A and theorem B. Can it be that if we infer their conjunction and believe it because of the inference we have made, we still may not necessarily have a priori knowledge of that conjunction? Surely this is not a possibility.

The problem with these considerations is that that they do not directly support (5) or rather

$$((\Diamond K_a\phi \wedge \Diamond K_a\psi) \wedge (\Box\phi \wedge \Box\psi)) \to \Diamond K_a(\phi \wedge \psi).^5 \tag{12}$$

Instead, they support something along the following lines:

$$((K_a\phi \wedge K_a\psi) \wedge (\Box\phi \wedge \Box\psi)) \to \Diamond K_a(\phi \wedge \psi). \tag{13}$$

I will later return to the logical difference between (12) on the one hand and (13) on the other hand.

Horsten (2000, p. 57, 60) adopts a closure principle that, unlike Anderson (1993)'s (12), has non-trivial applications to *contingent* formulas. His closure principle says that, if two conjuncts are each *a priori* knowable, then it is *a priori* knowable that each conjunct is possibly true, which can formalized as follows:

$$(\Diamond K_a\phi \wedge \Diamond K_a\psi) \to \Diamond K_a(\Diamond\phi \wedge \Diamond\psi). \tag{14}$$

It can easily be seen that both Anderson (1993)'s counterexample and Horsten (2000)'s counterexample fail if (5) is replaced by (14). Unfortunately, the only reason given to accept (14) is that, if $\Diamond K_a\phi \wedge \Diamond K_a\psi$, then it is *true* that $\Diamond\phi \wedge \Diamond\psi$ (Horsten, 2000, p. 56). This follows in modal system **K** from the assumption and (6). It is acknowledged that the 'argument does not establish the soundness of this aggregation principle beyond all possible doubt' (Horsten, 2000, p. 66, fn. 15). Note that there is to some extent agreement with Anderson in the case of necessary formulas. For suppose that ϕ and ψ are necessary. Then it is *true* that $\phi \wedge \psi$. Analogous reasoning then leads one to the conclusion that $\Diamond K_a(\phi \wedge \psi)$.

Horsten (2000, p. 51) is also prepared to go beyond (14). He thinks that the problem is due to the fact that his counterexample is formulated in a very expressive language. He notes that the systems of intensional mathematics found in (Shapiro, 1985) and (Horsten, 1994) each contain a version of the relevant closure principle, but the languages in which these closure principles are formulated is less expressive than the language that is used to give the counterexample to the closure principle. In particular, the language used by Shapiro (1985) is the extension of the language of Peano Arithmetic (**PA**), $\mathscr{L}_{\mathbf{PA}}$, extended with a *single* operator expressing complex *a priori* knowability, which will be expressed here with $\langle K_a \rangle$, whereas the language used by Horsten (1994, 1998) is the extension of $\mathscr{L}_{\mathbf{PA}}$ with both a possibility operator, $\Diamond$, and an *a priori* knowledge operator, K_a. Call the first language $\mathscr{L}_{\mathbf{EA}}$ and the second language $\mathscr{L}_{\mathbf{MEA}}$. Neither language contains the set membership relation, $\in$, or the set-abstraction notation, $\{x|\ldots\}$. So, neither (9) nor (10) can be formulated in $\mathscr{L}_{\mathbf{EA}}$ or $\mathscr{L}_{\mathbf{MEA}}$. Similarly, the actuality operator does not belong to either $\mathscr{L}_{\mathbf{EA}}$ or $\mathscr{L}_{\mathbf{MEA}}$, so Anderson (1993)'s counterexample is also inexpressible in those languages.

The positive proposals are related to one another. To see this, it is useful to define a translation from $\mathscr{L}_{\mathbf{EA}}$ to $\mathscr{L}_{\mathbf{MEA}}$ that systematically replaces $\langle K_a \rangle$ with $\Diamond K_a$.

Definition 2.1. Let F be a translation from $\mathscr{L}_{\mathbf{EA}}$ to $\mathscr{L}_{\mathbf{MEA}}$ as follows (Heylen, 2013, p. 93):

[5] From this and closure of *a priori* knowability under tautological equivalence one can deduce (11).

1. if ϕ is an atomic formula, then $\phi^F = \phi$;
2. if ϕ is $\neg\psi$, then $\phi^F = \neg\psi^F$;
3. if ϕ is $\psi \to \theta$, then $\phi^F = \left(\psi^F \to \theta^F\right)$;
4. if $\phi = \exists x\psi$, then $\phi^F = \exists x\psi^F$;
5. if ϕ is $K\psi$, then $\phi^F = \Diamond K_a\psi^F$.

For instance, $(\exists y\exists z\,(y = z+z))^F$ is just $\exists y\exists z\,(y = z+z)$ but $(\langle K_a\rangle\exists y\exists z\,(y = z+z))^F$ is $\Diamond K_a\exists y\exists z\,(y = z+z)$. Shapiro's closure principle can then be expressed in $\mathscr{L}_{\mathbf{MEA}}$ as follows:

$$\Diamond K_a\left(\phi^F \to \psi^F\right) \to \left(\Diamond K_a\phi^F \to \Diamond K_a\psi^F\right). \tag{15}$$

If one assumes **PA** and quantified modal system **S5**, then one can prove that all formulas in the range of the F-translation are true if possibly true, which can be formalized as follows:

$$\Diamond\phi^F \to \phi^F. \tag{16}$$

The details can be found in (Horsten, 1994, p. 287) and (Heylen, 2013, p. 95). If one combines this result with (6), then one can prove that (15) is deducible from Anderson (1993)'s (11). *Ergo*, Shapiro (1985)'s version of the closure principle satisfies both Anderson (1993)'s restriction to necessary truths and Horsten (2000)'s restriction to less expressive languages. The same does not hold for Horsten (1994, 1998)'s version of the closure principle, which allows instantiation with contingent truths, e.g. $K_a\phi$ and $\neg K_a\phi$.

The main goal of this paper is to further assess the material soundness of the principle of closure of *a priori* knowability under *a priori* knowable material implication. First, Horsten's restriction strategy will be examined. It will be shown that the relevant closure principle in combination with a few very plausible other principles leads to a highly implausible consequence. No expressive resources other than the ones needed to formalize the relevant closure principle are used. It is however crucial for obtaining this result that one can instantiate the closure principle with certain contingent formulas. This takes Horsten (1994, 1998)'s version off the table but it still leaves Anderson (1993)'s and Shapiro (1985)'s versions on the table. This is the subject of Section 3.

Second, Anderson's restriction strategy will be examined. In particular, I will look at the closure principle restricted to one kind of necessary truths, namely arithmetical ones. It will be shown that if the arithmetical complexity of the consequent is low, the closure principle is provable, provided some fairly weak background assumptions are made. However, it is *not* provable when also arithmetically more complex sentences are allowed in the consequent, even if one throws in a substantial number of modal, epistemic and modal-epistemic principles. This does not amount to a refutation of the closure principle restricted to necessary truths, but it does raise the question of how exactly to justify it.

Third, I will present an alternative principle, which epistemic logicians and mainstream epistemologists have good reason to accept. Furthermore, it manages to avoid both the counterexamples offered by Anderson (1993) and Horsten (2000) and it is immune to the Socratic objection. Moreover, it is not restricted to necessary truths. This is the subject of Section 5.

3 Closure and the Socratic objection

Horsten (2000, p. 61-62) shows that (3) is *not* a theorem of his system. In fact, one needs only a subtheory of his system. The argument makes use of three principles that have not been introduced before. The first is a principle of weak positive introspection and the second is a principle of weak negative introspection, which can be formalized as follows:

$$K_a\phi \to \Diamond K_a K_a\phi; \tag{17}$$

$$\neg K_a\phi \to \Diamond K_a \neg K_a\phi. \tag{18}$$

The third is what Horsten calls a 'density' principle:

$$\Diamond\neg K_a\phi. \tag{19}$$

The above is claimed to be the 'opposite' of logical omniscience (Horsten, 2000, p. 51), but this claim is too strong, since even logically omniscient agents may be ignorant about truths that are not theorems of the system, e.g. proposition letters. Given these principles, one can reconstruct his arguments as a mutual inconsistency result.[6] One advantage is that the focus shifts from whether or not (3) is a theorem of the system in (Horsten, 2000)) to whether or not (3) is compatible with a number of other principles that may belong to other theories as well. An additional benefit is that one no longer has to consider the plausibility of certain factual assumptions such as (9) and (10), which were used in Horsten (2000)'s counterexample to (3).

Theorem 3.1. *If* **K4** $\subset T$, $\vdash_T$ (3), $\vdash_T$ (4) $\vdash_T$ (6), $\vdash_T$ (17), $\vdash_T$ (18), $\vdash_T$ (19), *then T is inconsistent.*

Proof. Let θ be $p \vee \neg p$. By (4), it follows that $\Diamond K_a\theta$. It is a **K4**-consequence of the latter and (17) that $\Diamond K_a K_a\theta$. By (19), it is also the case that $\Diamond\neg K_a\theta$. It is a **K4**-consequence of the latter and (18) that $\Diamond K_a\neg K_a\theta$. By (5), it follows that

$$\Diamond K_a (K_a\theta \wedge \neg K_a\theta).$$

It is a **K**-consequence of the latter and (6) that $\Diamond(K_a\theta \wedge \neg K_a\theta)$, which contradicts a theorem of extensions of **K**. □

A philosopher wishing to uphold (3), could choose to drop modal system **K4**, (4), (6), (17), (18) or (19). I will first show that dropping (18) and (19) is not solution.

Lemma 3.2. *If* **K** $\subset T$, $\vdash_T$ (3), $\vdash_T$ (4) $\vdash_T$ (6), *then*

$$\vdash_T \neg(\Diamond K_a\phi \wedge \Diamond K_a\neg\phi).$$

Proof. Suppose that $\Diamond K_a\phi \wedge \Diamond K_a\neg\phi$. It follows that $\Diamond K_a(\phi \wedge \neg\phi)$. Given modal system **K** and (6), it follows that $\Diamond(\phi \wedge \neg\phi)$. This contradicts a theorem of extensions of modal system **K**. □

Theorem 3.3. *If* **K4** $\subset T$, $\vdash_T$ (3), $\vdash_T$ (4), $\vdash_T$ (6), *then:*

1. *if* $\vdash_T$ (17), *then* $\vdash_T \Diamond K_a\neg K_a\phi \to \Box\neg K_a\phi$;
2. *if* $\vdash_T$ (18), *then* $\vdash_T \Diamond K_a K_a\phi \to \Box K_a\phi$.

Proof. By Lemma 3.2, it is the case that:

1. $\Diamond K_a\neg K_a\phi \to \neg\Diamond K_a K_a\phi$;
2. $\Diamond K_a K_a\phi \to \neg\Diamond K_a\neg K_a\phi$.

Next, note that the following are **K4**-consequences of (17) and (18) respectively:

[6] In Horsten's proof the density principle is invoked to infer $\Diamond K_a\neg K_a\theta$, where that principle by itself only allows the inference of $\Diamond\neg K_a\theta$. However, one can give a charitable interpretation of the proof in which the desired conclusion follows from (19) *and* (18), which is a trivial consequence of the axiom schemes of his theory.

1. $\neg\Diamond K_a K_a \phi \to \neg\Diamond K_a \phi$;

2. $\neg\Diamond K_a \neg K_a \phi \to \neg\Diamond\neg K_a \phi$.

Finally, the following are tautological consequences of the above:

1. $\Diamond K_a \neg K_a \phi \to \neg\Diamond K_a \phi$;

2. $\Diamond K_a K_a \phi \to \neg\Diamond\neg K_a \phi$.

$\square$

Corollary. *If* **K4** $\subset T$, $\vdash_T$ (3), $\vdash_T$ (4), $\vdash_T$ (6), $\vdash_T$ (17), $\vdash_T$ (18), *then:*

1. $\vdash_T K_a \phi \to \Box K_a \phi$;

2. $\vdash_T \neg K_a \phi \to \Box\neg K_a \phi$.

Dropping (19) is not a solution, since one could then still prove a highly undesirable result (Corollary 3), namely that knowledge and ignorance are necessary. Dropping (19) *and* (18) is also not a solution, since one could then still prove that a priori knowable ignorance entails necessary ignorance (Theorem 3.3).[7] Suppose that one has a list of proofs that one has obtained. Next, one wants to check whether one has proved some conjecture. After having gone through the entire list and not finding the conjecture on it, one knows that one does not have a proof of it. It would be absurd to conclude that one consequently cannot have a proof of the conjecture. Similarly, the result that known ignorance entails necessary ignorance is too strong. Knowing about one's ignorance does not preclude learning. Since Socrates famously stated that knowledge of one's ignorance is the beginning of wisdom, I call this the *Socratic objection* to (3).

Theorem 3.3 is bad news for Horsten (1994, 1998), because his system of Modal-Epistemic Arithmetic (**MEA**) contains all of (3), (4), (6) and (17). Still, someone who would want to uphold (3) could choose to drop (17). This is to no avail either, if one accepts that (*a priori*) knowledge is closed under conjunction elimination, which can be formalized as follows:

$$K_a(\phi \wedge \psi) \to (K_a \phi \wedge K_a \psi). \tag{20}$$

With only (6), (4) and (20) and modal system **K** in the background, one can still level the Socratic objection to (3).

Lemma 3.4. *If* **K** $\subset T$, $\vdash_T$ (3), $\vdash_T$ (4), $\vdash_T$ (6), $\vdash_T$ (20), *then*

$$\vdash_T \neg(\Diamond K_a \phi \wedge \Diamond K_a \neg K_a \phi).$$

Proof. Suppose that $\Diamond K_a \phi \wedge \Diamond K_a \neg K_a \phi$.[8] It follows that $\Diamond K_a(\phi \wedge \neg K_a \phi)$. Given (20), it is **K**-consequence of the latter that $\Diamond(K_a \phi \wedge K_a \neg K_a \phi)$. Assuming (6), it is a **K**-consequence of the latter that $\Diamond(K_a \phi \wedge \neg K_a \phi)$, which contradicts a theorem of extensions of **K**. $\square$

[7] The principle of strong negative introspection for (*a priori*) knowledge says that ignorance entails known ignorance, which can be formalized as $\neg K_a \to K_a \neg K_a \phi$. This is not equal to the conclusion that *knowable* ignorance entails *necessary* ignorance. Moreover, Theorem 3.3 does not assume *weak* negative introspection, let alone strong negative introspection. It is perhaps also worth pointing out that, if strong negative introspection of (*a priori*) knowledge is combined with the principle of factivity of (*a priori*) knowledge, then one can deduce that falsity entails known ignorance (Hendricks, 2006, p. 87). This is a particularly strong consequence.

[8] The reader familiar with the paradox of Fitch (1963) will recognize this sentence, albeit that it contains an *a priori* knowledge operator rather than a (general) knowledge operator. However, the argument does not start from the assumption of weak verificationism, namely that all truths are knowable. Indeed, it would hardly be credible that all truths are *a priori* knowable. Rather, my argument starts from (5), which is derivable from (3) and (4). For discussion of the paradox of Fitch, see (Williamson, 2000, ch. 12).

Theorem 3.5. *If* **K** $\subset T$, $\vdash_T$ (3), $\vdash_T$ (4), $\vdash_T$ (6), $\vdash_T$ (20), *then* $\vdash_T \Diamond K_a \neg K_a \phi \rightarrow \neg \Diamond K_a \phi$.

So, philosophers who want to uphold (3) and avoid the Socratic objection have to reject modal system **K**, (4), (6) or (20). (Horsten, 2000, p. 54, 60) is willing to accept even **S5** for the diamond operator. Anderson (1993, p. 2) is more cautious,[9] but he is also willing to go beyond **K**, since he accepts **T**. Tautologies are paradigm examples of *a priori* knowable truths. It is the standard view in epistemology that *knowledge* in general is factive (Williamson, 2000, p. 33-41). That knowledge distributes over conjunction is a very reasonable assumption (Williamson, 2000, p. 275-283). Given the high plausibility of (4), (6), (20) and the weakness of modal system **K**, the only reasonable option seems to be to reject (3). Nothing hinges on the expressive power of the language: it suffices that the language contains the modal possibility operator $\Diamond$, the *a priori* knowledge operator K_a and the usual logical connectives. Therefore, Horsten (2000)'s suggestion that (3) is safe when formulated in a less expressive language, viz. $\mathscr{L}_{\mathbf{MEA}}$, is false.

The next question is whether the Socratic objection can be raised against Anderson (1993)'s (11) and Horsten (2000)'s (14) as well, if modal system **K4**, (4), (6), (17) and (20) are in the background. The answer is no.

Theorem 3.6. *There is an awareness model of* $\mathscr{L} \cup \{\Diamond, K_a\}$ *such that*

$$\mathscr{M} \models \mathbf{K4}, (4), (6), (17), (20)$$

and

$$\mathscr{M} \models (11), (14),$$

but $\mathscr{M} \not\models \Diamond K_a \neg K_a p \rightarrow \neg \Diamond K_a p$.

Proof. Let $\mathscr{M}$ be an awareness model, with the the following characteristics:

1. $W = \{w_1, w_2\}$;
2. $R_M = W \times W$;
3. $R_E = \{\langle w_1, w_1 \rangle, \langle w_2, w_2 \rangle\}$;
4. V is a function from proposition letters q and possible worlds $w \in W$ such that:
   (a) $V(p, w_1) = V(p, w_2) = 1$;
   (b) $V(q, w_1) = V(q, w_2)$, for every other proposition letter;
5. $A(w_1) = \mathscr{L} \cup \{\Diamond, K_a\}$ and $A(w_2)$ is the set of formulas of $\mathscr{L} \cup \{\Diamond, K_a\}$ recursively defined as follows:
   (a) $\neg K_a p$ is in $A(w_2)$;
   (b) if $\phi \in A(w_2)$, then $K_a \phi \in A(w_2)$;
   (c) nothing else is in $A(w_2)$.

The key clauses in the definition of $\mathscr{M}, w \models \phi$ are the one for $\phi = \Diamond \psi$ and the one for $\phi = K_a \psi$:

- $\mathscr{M}, w \models \Diamond \psi$ if and only if $\mathscr{M}, w' \models \psi$ for all w' such that $w R_M w'$;
- $\mathscr{M}, w \models K_a \psi$ if and only if $\psi \in A(w)$ and $\mathscr{M}, w' \models \psi$ for all w' such that $w R_E w'$.

[9]He refers to Salmon (1989), who argues that modal axiom scheme **4** is not correct for the logic of metaphysical or counterfactual possibility. Williamson (2007) argues that even **S5** is correct for metaphysical or counterfactual possibility.

Given the definition of R_E, the last conjunct of the last clause can be simplified to: $\mathscr{M},w \models \psi$. Moreover, given the definition of $A(w_1)$, the case of w_1 can be simplified to: $\mathscr{M},w_1 \models K_a\psi$ if and only if $\mathscr{M},w_1 \models \psi$.

Since R_M is an equivalence relation, $\mathscr{M} \models$ **S5** and, hence, $\mathscr{M} \models$ **K4**. Since R_E is a reflexive relation, $\mathscr{M} \models (6)$. Clearly, $\mathscr{M} \models (4)$, because for every $w \in W$, there is a $w' \in W$, namely w_1, such that wR_Mw' and $A(w') = \mathscr{L} \cup \{\Diamond, K_a\}$. It follows that $\phi \in A(w_1)$. For any tautology ϕ, it is the case that $\mathscr{M} \models \phi$, i.e. $\mathscr{M},w' \models \phi$ for every $w' \in W$ and, a fortiori, $\mathscr{M},w_1 \models \phi$. Therefore, it follows that $\mathscr{M},w_1 \models K_a\phi$. Consequently, $\mathscr{M} \models \Diamond K_a\phi$, because $w'R_Mw_1$ for every $w' \in W$. Hence, $\mathscr{M} \models (4)$. It is also the case that $\mathscr{M} \models (17)$. If $\mathscr{M},w_1 \models K_a\phi$, then $\mathscr{M},w_1 \models K_aK_a\phi$ and, given the reflexivity of R_M, it is also the case that $\mathscr{M},w_1 \models \Diamond K_aK_a\phi$. The case of w_2 can be proved on the basis of the construction of $A(w_2)$ and the reflexivity of R_M. Finally, $\mathscr{M} \models (20)$. The case of w_1 is trivial as before. The case of w_2 follows from the fact that no formula of the form $\phi \wedge \psi$ belongs to $A(w_2)$ and, therefore, $\mathscr{M},w_2 \not\models K_a(\phi \wedge \psi)$, which makes (20) trivially true.

Next, $\mathscr{M} \models (11)$. The reason is that, if $\mathscr{M},w' \models \Diamond K_a(\phi \to \psi) \wedge \Box(\phi \to \psi)$ and $\mathscr{M},w' \models \Diamond K_a\phi \wedge \Box\phi$, then $\mathscr{M},w' \models \Box\psi$. Since R_M is an equivalence relation, the latter entails that $\mathscr{M},w_1 \models \psi$. Hence, $\mathscr{M},w_1 \models K_a\psi$ and, given the properties of R_M, it is also the case that $\mathscr{M},w_1 \models \Diamond K_a\psi$ and $\mathscr{M},w_2 \models \Diamond K_a\psi$. Furthermore, $\mathscr{M} \models (14)$. Suppose that $\mathscr{M},w' \models \Diamond K_a\phi \wedge \Diamond K_a\psi$. Since the model makes (6) and **S5** true, it follows that $\mathscr{M},w_1 \models \Diamond\phi \wedge \Diamond\psi$. Therefore, $\mathscr{M},w_1 \models K_a(\Diamond\phi \wedge \Diamond\psi)$.

Finally, let us check that the model is a counterexample to $\Diamond K_a\neg K_ap \to \neg\Diamond K_ap$. First, note that $\mathscr{M},w_2 \models K_a\neg K_ap$, since $\neg K_ap \in A(w_2)$ and $\mathscr{M},w_2 \models \neg K_ap$, because $K_ap \notin A(w_2)$. Second, $\mathscr{M},w_2 \models \Diamond K_ap$, since $\mathscr{M},w_1 \models K_ap$ (because $\mathscr{M},w_1 \models p$) and $w_2R_Mw_1$.

□

In this section it was shown that (3) entails that *a priori* knowable ignorance entails necessary ignorance, which was called the Socratic objection to (3). The consequence was deduced with the help of principles that Horsten (1994, 1998) accepts (Theorem 3.3) or with the help of principles that are highly plausible on independent grounds (Theorem 3.5). Apart from the usual logical connectives, neither result depends on expressive resources beyond those needed to express (3). Therefore, Horsten (2000)'s restriction strategy to salvage (3) from counterexamples, which was discussed in Section 2, fails. A hopeful conclusion of the investigations in this section is that both Anderson (1993)'s (11) and Horsten (2000)'s (14) are secure against the Socratic objection (Theorem 3.6). In the next section it is considered whether these principles can also be positively justified.

4 Closure and the justification challenge

The good news about Anderson (1993)'s (11) and Horsten (2000)'s (14) is that they both block Anderson's and Horsten's counterexamples (Section 2) and that they are immune to the Socratic objection (Section 3). Unfortunately, neither principle was adequately justified by Anderson and Horsten (Section 2). In this section I will investigate the prospects of giving such a justification. In Subsection 4.1 it will be argued that there is partial justification for (11). However, it will also be argued that it is very challenging to find an adequate justification for (11) or (14) in their full generality – see Subsection 4.2.

4.1 Partial justification

Let us start with the partial justification for (11). One could reason as follows. If a material implication and its antecedent are each *a priori* knowable, then the material implication and its antecedent are each possibly true. Suppose that they each belong to a category of formulas that are such that, if they are possibly true, then they are true. This category is, of course, a subset of the category of strict implications (i.e. necessary material implications) with necessary antecedents. In that case the material implication and its antecedent are true and, therefore, its consequent is true as well. Suppose furthermore that the consequent belongs to a category of formulas that are such that, if they are true, then they are *a priori* knowable. In that case the consequent is *a priori* knowable as well. The natural question to ask is whether there are examples of material implications, antecedents and consequents with the mentioned properties. The answer is yes.

Let us consider material implications with arithmetical antecedents and consequents. First one needs a definition of a particular class of arithmetical formulas. It is a consequence of (16) that these are true if possibly true. Moreover, let us consider not just any arithmetical consequent, but so-called $\exists$-rudimentary formulas. These are defined as follows.

Definition 4.1. A formula $\phi \in \mathscr{L}_{\mathbf{PA}}$ is a rudimentary formula if and only if:

1. if ϕ is an atomic formula of $\mathscr{L}_{\mathbf{PA}}$, then ϕ is a rudimentary formula;
2. if ψ is a rudimentary formula, then so is $\neg\psi$;
3. if ψ and θ are rudimentary formulas, then so is $\psi \wedge \theta$;
4. if ψ is a rudimentary formula and if t is a term of $\mathscr{L}_{\mathbf{PA}}$, then $\forall x\,(x < t \rightarrow \psi)$ and $\exists x\,(x < t \wedge \psi)$ are rudimentary formulas.

An $\exists$-rudimentary formula is a formula of the form $\exists x\psi$, with ψ a rudimentary formula. A $\forall$-rudimentary formula is a formula of the form $\forall x\psi$, with ψ a rudimentary formula. (Boolos et al., 2003, p. 204)

It is a an important theorem that an $\exists$-rudimentary sentence ϕ is true in the standard interpretation of arithmetic if and only if $\vdash_{\mathbf{Q}} \psi$ (Boolos et al., 2003, p. 199, 208), with **Q** the system of minimal arithmetic.

Let $\mathbf{MEA}^{\dagger}$ be the extension of **PA** with modal system **S5**, (6) and the closure of *a priori* knowability under provability in **Q**, which can be formalized as follows:

$$\vdash_{\mathbf{Q}} \phi \quad \Rightarrow \quad \vdash_{\mathbf{Q}} \Diamond K_a \phi. \tag{21}$$

Theorems of **Q** constitute another class of paradigm examples of *a priori* knowable truths.

Theorem 4.1. *For any formula $\phi \in \mathscr{L}_{\mathbf{PA}}$, for any $\exists$-rudimentary sentence ψ, it is the case that $\vdash_{\mathbf{MEA}^{\dagger}} \Diamond K_a(\phi \rightarrow \psi) \rightarrow (\Diamond K_a \phi \rightarrow \Diamond K_a \psi)$.*

Proof. Note that $\vdash_{\mathbf{MEA}^{\dagger}} \Diamond\phi \rightarrow \phi$ (Heylen, 2013, p. 95). It follows that

$$\vdash_{\mathbf{MEA}^{\dagger}} \Diamond K_a \phi \rightarrow \phi$$

(Horsten, 1994, p. 287). Suppose that $\Diamond K_a(\phi \rightarrow \psi)$. Therefore, $(\phi \rightarrow \psi)$. Suppose that $\Diamond K_a \phi$. Hence, ϕ. By modus ponens, it follows that ψ. Suppose that ψ is an $\exists$-rudimentary formula. Either ψ is true in the standard interpretation of arithmetic or not. If it is true, then $\vdash_{\mathbf{Q}} \psi$. If it is false, then $\vdash_{\mathbf{Q}} \neg\psi$. By (21), $\vdash_{\mathbf{Q}} \Diamond K_a \psi$ or $\vdash_{\mathbf{Q}} \Diamond K_a \neg\psi$. The second option is incompatible with the hypothetical fact that ψ, given that $\Diamond K_a \neg\psi \rightarrow \neg\psi$. Hence, $\Diamond K_a \psi$.

□

Corollary. *For any formula $\phi \in \mathscr{L}_{\mathbf{PA}}$, for any $\exists$-rudimentary sentence ψ, it is the case that $\vdash_{\mathbf{MEA}^{\dagger}} (\Diamond K_a(\phi \rightarrow \psi) \wedge \Box(\phi \rightarrow \psi)) \rightarrow ((\Diamond K_a \phi \wedge \Box\phi) \rightarrow \Diamond K_a \psi)$.*

So, it is possible to find justification for a restricted version of Anderson (1993)'s (11), namely the version in which the antecedent is an arithmetical formula and the consequent is an arithmetical formula of a certain low complexity. This raises the question what happens when the consequent is an arithmetical formula of a higher complexity. For these formulas it is not provable that they are *a priori* knowable if true, given (21).

4.2 No more justification

It will be argued that that one cannot similarly justify (11) or (14) for all arithmetical cases, even if one makes very strong modal and epistemic assumptions. The argumentation strategy is as follows. I will introduce a theory that contains very strong modal and epistemic principles. Then I am going to show that neither Anderson (1993)'s (11) nor Horsten (2000)'s (14) are theorems in this theory. This means that even with some very strong modal, epistemic and modal-epistemic principles there is no deductive justification for (11) or (14). The challenge is then to find some other (perhaps non-deductive or informal) justification for (11) or (14). The thought is that, if it is very challenging to find justification for (11) and (14) when restricted to arithmetic, it is also very challenging to find justification for those principles when they are unrestricted.

Let $\mathbf{MEA}^{\dagger\dagger}$ be the extension of **PA** with modal system **S5**, (6) and with the following other epistemic principles. First, add the principle that all theorems of $\mathbf{MEA}^{\dagger\dagger}$ are *a priori* known, which can be formalized as follows:

$$\vdash_{\mathbf{MEA}^{\dagger\dagger}} \phi \quad \Rightarrow \quad \vdash_{\mathbf{MEA}^{\dagger\dagger}} K_a\phi. \tag{22}$$

Second, add the principle that *a priori knowledge* is closed under *a priori known* material implication, which can be formalized as follows:

$$K_a(\phi \to \psi) \to (K_a\phi \to K_a\psi). \tag{23}$$

Third, add the principle that all *a priori* knowledge is *a priori known*, which can be formalized as follows:

$$K_a\phi \to K_aK_a\phi. \tag{24}$$

All three principles are surely too strong, but this only makes the negative result stronger. I will show that neither (11) nor (14) is a theorem of $\mathbf{MEA}^{\dagger\dagger}$.

The proof will make use of a couple of facts about about undecidable arithmetical sentences (Boolos et al., 2003, ch. 17, 18). Gödel's first incompleteness says that there is no consistent, complete, axiomatizable extension of **Q**. If **PA** is consistent, then it is not complete. In particular, there is a sentence, $G_{\mathbf{PA}}$, i.e. the sentence such that $\vdash_{\mathbf{PA}} G_{\mathbf{PA}} \leftrightarrow \neg\exists y Prov_{\mathbf{PA}}(\ulcorner G_{\mathbf{PA}} \urcorner, y)$, and $\nvdash_{\mathbf{PA}} G_{\mathbf{PA}}$. Moreover, if **PA** is ω-consistent, then $\nvdash_{\mathbf{PA}} G_{\mathbf{PA}}$. It follows that both $\mathbf{PA} \cup \{G_{\mathbf{PA}}\}$ and $\mathbf{PA} \cup \{\neg G_{\mathbf{PA}}\}$ are consistent. Let us refer to the first theory as $\mathbf{PA}^*$ and to the second theory as $\mathbf{PA}^{**}$. What will be needed is that neither theory proves $G_{\mathbf{PA}^*}$. It follows from the first incompleteness theorem that $\nvdash_{\mathbf{PA}^*} G_{\mathbf{PA}^*}$. It is true that, if $\vdash_{\mathbf{PA}} \phi$, then $\vdash_{\mathbf{PA}^*} \phi$. What we need, is that this can be formalized in **PA**, i.e. $\vdash_{\mathbf{PA}} \exists y Prov_{\mathbf{PA}}(\ulcorner \phi \urcorner, y) \to \exists y Prov_{\mathbf{PA}^*}(\ulcorner \phi \urcorner, y)$. If this is the case, then $\vdash_{\mathbf{PA}} \neg\exists y Prov_{\mathbf{PA}^*}(\ulcorner \phi \urcorner, y) \to \neg\exists y Prov_{\mathbf{PA}}(\ulcorner \phi \urcorner, y)$. In particular, $\vdash_{\mathbf{PA}} G_{\mathbf{PA}^*} \to G_{\mathbf{PA}}$. But then $\nvdash_{\mathbf{PA}^{**}} G_{\mathbf{PA}^*}$. So, the question is whether one can prove that $\vdash_{\mathbf{PA}} \exists y Prov_{\mathbf{PA}}(\ulcorner \phi \urcorner, y) \to \exists y Prov_{\mathbf{PA}^*}(\ulcorner \phi \urcorner, y)$. The details depend on the details of arithmetization (Boolos et al., 2003, ch. 15) and they will be omitted here, but the definition of $Prov_{\mathbf{PA}^*}(\ulcorner \phi \urcorner, y)$ is identical to the definition of $Prov_{\mathbf{PA}}(\ulcorner \phi \urcorner, y)$, except that the clause $axiom(e)$ is defined as

$$logicalaxiom(e) \lor PAaxiom(e)$$

in the second case and as

$$logicalaxiom(e) \lor PAaxiom(e) \lor Gödelsentence(e)$$

in the first case. Given these definitions, the theorem is essentially a case of disjunction introduction. Finally, note that both $G_{\mathbf{PA}}$ and $G_{\mathbf{PA}^*}$ are true in the standard model of arithmetic and that neither $G_{\mathbf{PA}}$ nor $G_{\mathbf{PA}^*}$ are $\exists$-rudimentary formulas.

Theorem 4.2. *There is an interpretation* $\mathscr{M}$ *of* $\mathscr{L}_{\mathbf{MEA}}$ *such that* $\mathscr{M} \models \mathbf{MEA}^{\dagger\dagger}$ *but* $\mathscr{M} \not\models (\Diamond K_a(G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}) \land \Box(G_{\mathbf{PA}} \to G_{\mathbf{PA}^*})$ $((\Diamond K_a G_{\mathbf{PA}} \land \Box G_{\mathbf{PA}}) \to \Diamond K_a G_{\mathbf{PA}^*})$ *and* $\mathscr{M} \not\models (\Diamond K_a G_{\mathbf{PA}} \land \Diamond K_a(G_{\mathbf{PA}} \to G_{\mathbf{PA}^*})) \to \Diamond K_a(\Diamond G_{\mathbf{PA}} \land \Diamond(G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}))$.

Proof. Let $\mathbf{MEA}^*$ be identical to $\mathbf{MEA}^{\dagger\dagger}$, extended with $\phi \to K_a\phi$, for any $\phi \in \mathscr{L}_{\mathbf{MEA}}$. Let M be an awareness model, with the following characteristics:[10]

1. $W = \{w_1, w_2\}$;
2. $D = \mathbb{N}$;
3. $R_M = W^2$;
4. $R_E = \{\langle w_1, w_1\rangle, \langle w_2, w_2\rangle\}$;
5. V is $\mathscr{N}$, the standard interpretation of the language of arithmetic;
6. S the set of variable assignment functions;
7. A is a function from W and S to $\mathscr{L}_{\mathbf{MEA}}$, defined as follows:
   (a) for any variable assignment a, $A(w_1, a) = \{\phi \mid \Gamma_1(a) \vdash_{\mathbf{MEA}^*} \phi\}$, with $\Gamma_1(a) = \{G_{\mathbf{PA}}\} \cup \{t = t' \mid den_{\mathscr{M},a}(t) = den_{\mathscr{M},a}(t')\}$;
   (b) for any variable assignment a, $A(w_2, a) = \{\phi \mid \Gamma_2(a) \vdash_{\mathbf{MEA}^*} \phi\}$, with $\Gamma_2(a) = \{G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}\} \cup \{t = t' \mid den_{\mathscr{M},a}(t) = den_{\mathscr{M},a}(t')\}$.

The key clause in the recursive definition of $\mathscr{M} \models \phi$ is the clause for $\phi = K_a\psi$: $\mathscr{M}, w, a \models K_a\psi$ if and only if $\psi \in A(w, a)$ and $\mathscr{M}, w', a, \models \psi$ for all w' such that wR_Ew'. Simplifying, this means that $\mathscr{M}, w_i, a \models K_a\psi$ if and only if $\Gamma_i(a) \vdash_{\mathbf{MEA}^*} \psi$ and $\mathscr{M}, w_i, a, \models \psi$.

We need to check that, if $\vdash_{\mathbf{MEA}^{\dagger\dagger}} \phi$, then $\mathscr{M} \models \phi$. The proof is by induction on the complexity of proof in $\mathbf{MEA}^{\dagger\dagger}$.

Since R_M is an equivalence relation, $\mathscr{M} \models \mathbf{S5}$. Since R_E is a reflexive relation, $\mathscr{M} \models (6)$. Now consider the case of (22). Suppose that $\vdash_{\mathbf{MEA}^{\dagger\dagger}} \Diamond K_a\phi$, inferred from $\vdash_{\mathbf{MEA}^{\dagger\dagger}} \phi$. It follows from the latter by construction of $A(w,a)$ that $\phi \in A(w)$, for any $w \in W$. By the induction hypothesis, $\mathscr{M} \models \phi$. Therefore, $\mathscr{M}, w, a \models K_a\phi$ and, by the reflexivity of R_M, it is also the case that $\mathscr{M}, w, a \models \Diamond K_a\phi$. Next, consider the case of (23). Suppose that $\mathscr{M}, w, a \models K_a(\phi \to \psi) \wedge K_a\phi$. It follows by reflexivity of R_E and the clauses for conjunction and material implication that $\mathscr{M}, w, a \models \psi$. It is also the case that $\phi \to \psi, \phi \in A(w,a)$. Since $A(w,a)$ is by its construction closed under modus ponens, it is also the case that $\psi \in A(w,a)$. Therefore, $\mathscr{M}, w, a \models K_a\psi$. Finally, consider the case of (24). Suppose that $\mathscr{M}, w, a \models K_a\phi$ for some w, a. Then $\phi \in A(w,a)$ and, by construction, $K_a\phi \in A(w,a)$. Since $\{w' \mid wR_Ew'\} = \{w\}$, it follows that $\mathscr{M}, w, a \models K_aK_a\phi$. As for the first-order and arithmetical part, I will leave the proof to the reader, except for noting that the $\{t = t' \mid den_{\mathscr{M},a}(t) = den_{\mathscr{M},a}(t')\}$ part of $A(w,a)$ is important for proving a Principle of Replacement and a Principle of Agreement, which are used in the proofs of universal instantiation and universal generalisation.

The model provides us with a counterexample to (3). First, note that $G_{\mathbf{PA}}$ and $G_{\mathbf{PA}^*}$ are true in the standard model of arithmetic and, therefore, $\mathscr{M} \models G_{\mathbf{PA}}$ and $\mathscr{M} \models G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}$. Second, note that, by definition, $G_{\mathbf{PA}} \in A(w_1, a)$ and $G_{\mathbf{PA}} \to G_{\mathbf{PA}^*} \in A(w_2, a)$. Consequently, $\mathscr{M}, w_1, a \models K_aG_{\mathbf{PA}}$ and $\mathscr{M}, w_2, a \models K_a(G_{\mathbf{PA}} \to G_{\mathbf{PA}^*})$. Therefore, $\mathscr{M}, w_1, a \models \Diamond K_aG_{\mathbf{PA}}$ and $\mathscr{M}, w_1, a \models \Diamond K_a(G_{\mathbf{PA}} \to G_{\mathbf{PA}^*})$. Since $\mathscr{M}, w, a \models \phi$ for all ϕ that are true in the standard model of arithmetic and for all $w \in W$, it is also the case that $\mathscr{M}, w_1, a \models \Box G_{\mathbf{PA}}$ and $\mathscr{M}, w_1, a \models \Box(G_{\mathbf{PA}} \to G_{\mathbf{PA}^*})$. The question then is whether there

[10] The model that will be constructed is of the type defined in (Heylen, 2013, Section 4, especially Definitions 4.10, 4.11). Definition 4.10 (Heylen, 2013, p. 103) hard codes the Principle of Replacement, but it ought to hard code the Principle of Agreement as well. The first is needed for proving the soundness of Universal Instantiation. The second is needed to prove the soundness of Universal Generalisation. In the model constructed here the Principle of Agreement and the Principle of Replacement come out true by construction of the awareness sets.

is a $w \in W$ such that $\mathscr{M}, w, a \models K_a G_{\mathbf{PA}^*}$. It is sufficient for a negative answer that $G_{\mathbf{PA}^*} \notin A(w_1, a)$ and $G_{\mathbf{PA}^*} \notin A(w_2, a)$. For this it needs to be shown that $\Gamma_1(a) \nvdash_{\mathbf{MEA}^*} G_{\mathbf{PA}^*}$ and $\Gamma_2(a) \nvdash_{\mathbf{MEA}^*} G_{\mathbf{PA}^*}$.

I will demonstrate that the two cases under consideration can be reduced to $G_{\mathbf{PA}} \nvdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$ and $G_{\mathbf{PA}} \to G_{\mathbf{PA}^*} \nvdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$ respectively. First, for every ϕ, if $\Gamma_i(a) \vdash_{\mathbf{MEA}^*} \phi$, then $\Gamma_i(a)^E \vdash_{\mathbf{PA}} \phi^E$, with E the eraser translation from $\mathscr{L}_{\mathbf{MEA}}$ to $\mathscr{L}_{\mathbf{PA}}$ that erases all occurrences of $\Diamond$ and K_a. Note that the eraser translation of any identity sentence belonging to $\Gamma_i(a)$ belongs to $\mathscr{L}_{\mathbf{PA}}$ and that $(G_{\mathbf{PA}})^E = G_{\mathbf{PA}}$, $(G_{\mathbf{PA}} \to G_{\mathbf{PA}^*})^E = G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}$ and $(G_{\mathbf{PA}^*})^E = G_{\mathbf{PA}^*}$. Therefore, the case of $\Gamma_1(a) \vdash_{\mathbf{MEA}^*} G_{\mathbf{PA}}$ reduces to the case of $\Gamma_1(a) \vdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$ and the case of $\Gamma_2(a) \vdash_{\mathbf{MEA}^*} G_{\mathbf{PA}^*}$ reduces to the case of $\Gamma_2(a) \vdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$. Next, if t_1 and t_2 are closed terms and if $t_1 = t_2 \in \Gamma_i(a)$, then $\vdash_{\mathbf{PA}} t_1 = t_2$, since all true rudimentary sentences are theorems of $\mathbf{Q}$. Therefore, one can take them out of $\Gamma_i(a)$ without affecting the deducibility. Call the reduced set $\Gamma_i(a)'$. If $\Gamma_i(a)' \vdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$, then there is a finite subset $\Gamma_i(a)''$ of $\Gamma_i(a)'$ such that $\Gamma_i(a)'' \vdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$. Let $t_i = t_j$, …, $t_k = t_l$ be the identity statements belonging to $\Gamma_1(a)''$. If $\Gamma_1(a)'' \vdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$, then $G_{\mathbf{PA}}, t_i = t_j \wedge \cdots \wedge t_k = t_l \vdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$. Let $x_1, \ldots, x_n$ be the free variables in the assumption. By the instantiation rule for the existential quantifier, if $G_{\mathbf{PA}}, t_i = t_j \wedge \cdots \wedge t_k = t_l \vdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$, then $G_{\mathbf{PA}}, \exists x_1 \ldots \exists x_n (t_i = t_j \wedge \cdots \wedge t_k = t_l) \vdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$. Indeed, $G_{\mathbf{PA}}$ and $G_{\mathbf{PA}^*}$ are sentences, so they do not contain any free variables. Note that $t_i = t_j \wedge \cdots \wedge t_k = t_l$ is satisfied in the standard model relative to a and, therefore, $\exists x_1 \ldots \exists x_n (t_i = t_j \wedge \cdots \wedge t_k = t_l)$ is true in the standard model as well. The latter sentence is a $\exists$-rudimentary sentence and, consequently, it is a theorem of $\mathbf{PA}$. Therefore, one can take it out without affecting the deducibility. Analogous reasoning can be used in the second case. Hence, one can further reduce the two cases to $G_{\mathbf{PA}} \vdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$ and $G_{\mathbf{PA}} \to G_{\mathbf{PA}^*} \vdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$ respectively.

The essential point is that neither $\mathbf{PA}^*$ nor $\mathbf{PA} \cup \{G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}\}$ entail $G_{\mathbf{PA}^*}$. The first point was made in the introduction to the proof. The second point follows from the fact that $\nvdash_{\mathbf{PA}^{**}} G_{\mathbf{PA}^{**}}$ or $\neg G_{\mathbf{PA}} \nvdash_{\mathbf{PA}} G_{\mathbf{PA}^*}$, which was also noted in the introduction to the proof.

The second question is whether there is a $w \in W$ such that

$$\mathscr{M}, w, a \models K_a (\diamond G_{\mathbf{PA}} \wedge \diamond (G_{\mathbf{PA}} \to G_{\mathbf{PA}^*})),$$

which is answered in the negative if

$$\diamond G_{\mathbf{PA}} \wedge \diamond (G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}) \notin A(w_1, a)$$

and

$$\diamond G_{\mathbf{PA}} \wedge \diamond (G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}) \notin A(w_2, a).$$

The latter is the case if

$$\Gamma_i(a) \nvdash_{\mathbf{MEA}^*} \diamond G_{\mathbf{PA}} \wedge \diamond (G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}).$$

This can again be reduced to

$$G_{\mathbf{PA}} \nvdash_{\mathbf{PA}} G_{\mathbf{PA}} \wedge (G_{\mathbf{PA}} \to G_{\mathbf{PA}^*})$$

and

$$G_{\mathbf{PA}} \to G_{\mathbf{PA}^*} \nvdash_{\mathbf{PA}} G_{\mathbf{PA}} \wedge (G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}),$$

which follow from the two facts noted in the introduction to the proof.

□

Let us recapitulate. It was proved that neither Anderson (1993)'s (11) nor Horsten (2000)'s (14) are provable even when they are restricted to arithmetic and *very* strong modal and epistemic assumptions are made (Theorem 4.2). This is in a way disappointing, since it was also proved that, if the antecedent in (11) is an arithmetical formula and if the consequent is an arithmetical formula of a certain low complexity, then (14) follows from modest modal and epistemic assumptions (Theorem 4.1). Of course, it is open to defenders of (11) or (14) to suggest additional principles, e.g. (18), which do the trick and which are independently plausible. However, I think that it is fair to say that at this stage of the debate it is up to them to make a move.

The foregoing highlights the difficulty of finding a positive justification for (11) or (14). It was argued in Section 2 that the attempts by Anderson (1993) and Horsten (2000) to justify those principles fail. In this section it has been argued that (11) is justified on two conditions. First, the material implication and its antecedent have to be true if possibly true. Second, the consequent has to be *a priori* knowable if true. When the material implication is arithmetical and the consequent is an arithmetical formula of a certain low complexity, then those two conditions are satisfied. The challenge is to justify (11) when the second assumption is not made or is not provable given some background assumptions. This should not be mistaken for the claim that the second assumption is false. The case when the material implication is arithmetical but the consequent is of a higher arithmetical complexity, is a nice case in point. Again, it is not assumed that the formula in question, viz. $G_{\mathbf{PA}^*}$, is really *a priori* unknowable.

In fact, Myhill (1960, p. 463) has made an interesting case for the *a priori* knowability of the arithmetical statement that says that **PA** is consistent. The consistency statement for **PA** is equivalent with the Gödel sentence for **PA** (Boolos et al., 2003, p. 233-234). Gödel (1995, p. 290 ff) claimed that the following disjunction is true: either the mind is not a Turing Machine or there exist absolutely unknowable diophantine equations, which are $\forall$-rudimentary formulas (Gödel, 1995, p. 156 ff). Ever since Gödel advanced his disjunction, attempts have been made to argue for one of the disjuncts. In particular, Leitgeb (2009) and Horsten (2009) have investigated the limits of knowledge about arithmetic.[11] But to repeat, the argument developed here does not depend on the assumption that Gödel's second disjunct is true or on the more general assumption that there exist *a priori* unknowable propositions, although the argument does assume that this has not been ruled out.

In Section 1 it was pointed out that Dretske (2005) has explicitly criticized (1) and that he has implicitly challenged (2). In addition, his counterexample also affects logical competence, since it is easy to know that you have hands and it is easy to know that, if you have hands, you are not a handless brain-in-a-vat, but it is impossible to know that you are not a handless brain-in-a-vat. Of course, this does not concern the notion of *a priori* knowledge. Interesting from our current perspective is that there is an analogy between Dretske's argument and the proof of Theorem 4.2: $G_{\mathbf{PA}}$ is an easily knowable sentence of lower arithmetical complexity; $G_{\mathbf{PA}^*}$ is an arithmetically more complex sentence that is not knowable; $G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}$ is knowable. A disanalogy is that, unlike the statement that connects having hands with not being a handless brain-in-a-vat, $G_{\mathbf{PA}} \to G_{\mathbf{PA}^*}$ is not a logical truth. To turn the proof in a Dretske-style argument against (3) it is necessary but not sufficient to argue for the existence of *a priori* unknowable arithmetical sentences of a certain complexity. This falls outside the scope of this article.[12]

5 Closure and logical competence

If it is already challenging to find a deductive justification for (11) and (14) when they are restricted to arithmetic, it is also very challenging to find a deductive justification for those principles when they are unrestricted. As I will argue next, the justification challenge is even stronger than one might think, since there is a good alternative, which in one form or another is upheld by both epistemic logicians and mainstream epistemologists. Since the work of Kuhn (1962) and Lakatos (1970), philosophers of science are very much aware of the fact that scientists only leave a paradigm or research program if there is an alternative paradigm or research program available.

It is an important task for epistemic logicians to steer between the Scylla of *logical omniscience* and the Charybdis of *logical incompetence*: epistemic agents should be represented as agents that do not *know*

[11] For a critical reply to Horsten (2009), see (Heylen, 2010).

[12] The paradox of Fitch (1963) is not of any help here, since even if p is an arithmetical statement, the formula $p \wedge \neg K_a p$ is not an arithmetical statement.

all logical truths or all logical consequences of what they already know, but for whom those logical truths and logical consequences are *knowable*. Besides (1), some examples of logical omniscience are:

$$K(\phi \wedge \psi) \rightarrow (K\phi \wedge K\psi),$$

$$(K\phi \wedge K\psi) \rightarrow K(\phi \wedge \psi),$$

$$K\phi \rightarrow K(\phi \vee \psi),$$

$$K\neg\neg\phi \rightarrow K\phi.$$

It can easily be seen that all the examples of logical omniscience follow from the closure of knowledge under logical theoremhood and (23). Together these two principles are equivalent to the following generalized logical omniscience rule:

$$\Gamma \vdash \psi \quad \Rightarrow \quad (K)\Gamma \vdash K\psi, \tag{25}$$

with $(K)\Gamma = \{K\phi \mid \phi \in \Gamma\}$. Some examples of logical competence are:

$$K(\phi \wedge \psi) \rightarrow (\Diamond K\phi \wedge \Diamond K\psi),$$

$$(K\phi \wedge K\psi) \rightarrow \Diamond K(\phi \wedge \psi),$$

$$K\phi \rightarrow \Diamond K(\phi \vee \psi),$$

$$K\neg\neg\phi \rightarrow \Diamond K\phi.$$

The examples of logical competence can easily be seen to subsume under the following generalized rule of logical competence:

$$\Gamma \vdash \psi \quad \Rightarrow \quad (K)\Gamma \vdash \Diamond K\psi. \tag{26}$$

Logical omniscience has been widely criticized, because it presents human knowers as having divine cognitive capabilities. However, one should also avoid presenting human knowers as being cognitively handicapped.[13] If you know a material conditional and you know its antecedent and if you are logically competent, then you *can* expand your knowledge by applying modus ponens, although you may not *have* done so yet. Given enough time, attention and other resources, logically competent reasoners should *be able* to know the consequent of a conditional if they already know the antecedent of that conditional and the conditional itself, even if they have not devoted the time, attention and other resources to it.

A different position between logical omniscience and logical incompetence is the claim that epistemic agents have the *possibility to know* all the logical consequences of what they *can know*. Examples of this general principle are:

$$\Diamond K(\phi \wedge \psi) \rightarrow (\Diamond K\phi \wedge \Diamond K\psi),$$

$$(\Diamond K\phi \wedge \Diamond K\psi) \rightarrow \Diamond K(\phi \wedge \psi),$$

$$\Diamond K\phi \rightarrow \Diamond K(\phi \vee \psi),$$

$$\Diamond K\neg\neg\phi \rightarrow \Diamond K\phi.$$

The above examples all subsume under the following rule of closure of *knowability* under logical deducibility: which can be formalized as follows:

$$\Gamma \vdash \psi \quad \Rightarrow \quad (\Diamond K)\Gamma \vdash \Diamond K\psi. \tag{27}$$

Williamson (2000, p. 117) and Hawthorne (2005, p. 29-30) defend a closure principle that is closely related to the logical competence principle (26). In the words of Hawthorne (2005, p. 29): ‘If one knows

[13] See (Heylen, 2013, p. 99) for the problem of logical *in*competence that presents itself for syntactical models, awareness models and impossible worlds models, which were introduced to avoid the problem of logical omniscience.

some premises and competently deduces Q from those premises, thereby coming to believe Q, while retaining one's knowledge of those premises throughout, one comes to know that Q'. There seem to be four differences between what Williamson calls 'inuitive closure' on the one hand and (26) on the other hand. The first difference is that between a temporal conception of knowability, expressed by 'coming to know', and a modal conception of knowability, expressed by 'being able to know'. Based only on this distinction and taking into account that metaphysical or counterfactual possibility is weaker than temporal possibility, the modal version of the closure principle is weaker and, therefore, more easily justified than the temporal version. It is no surprise then that this is compensated by the other three differences, which are all about restricting the temporal version of the closure principle. The second difference is that Hawthorne says that the conclusion of the deduction should come to be believed. The third difference is that one should retain knowledge of the premises throughout. The reason he gives for this is that one may get counterevidence to the premises in the meantime. The fourth difference is that only deductions with at least one premise are considered. Whether one should also impose these restrictions on the logical competence principle or not, the crucial point is that the 'intuitive closure' principle defended by mainstream epistemologists is a version of (26) rather than a version of (27), since *knowledge* of the premises rather than *possible knowledge* of the premises is required.

The general rules (25), (26) and (27) are related to the closure principles with *a priori* knowledge. Clearly, (23) is an instantiation of (25) and, moreover, (25) is equivalent to (23) and closure of *a priori* knowledge under logical theoremhood. Horsten (2000, p. 60) endorses an axiom scheme from which an instantiation of (26) follows:

$$K_a(\phi \rightarrow \psi) \rightarrow (K_a\phi \rightarrow \diamond K_a\psi). \quad (28)$$

In Section 2 it was pointed out that, when Anderson (1993, p. 9) tries to give a justification for his closure principle, he actually ends up supporting (13), which can now be seen to be a special case of (26). Finally, it can easily be seen that (27) entails (3) and is, in fact, equivalent to (3) and closure of *a priori* knowability under logical theoremhood. Interestingly, neither (25) nor (26) justify either (11) or (14). It is easily checked that (25) and (26), when restricted to *a priori* knowledge and when formulated in terms of provability-in-$\mathbf{MEA}^{\dagger\dagger}$, are derivable rules in $\mathbf{MEA}^{\dagger\dagger}$. Hence, one can use Theorem 4.2 to back up this claim.

Let us summarize the situation. There is an alternative closure principle, called a 'logical competence' principle, for which a reasonable case can be made that is based on the aims of epistemic logic and based on the work of mainstream epistemologists. In addition, Anderson (1993) and Horsten (2000) endorse some instantiations of the principle already. Moreover, the logical competence principle does not entail (3), even if one adds some strong modal and epistemic assumptions. So, it looks like there is an alternative to (3). This leaves us with checking whether it is a *good* alternative.

It can be proved that (26) is immune to the Socratic objection. This can also be proved with the help of the model used in the proof of Theorem 26, if one replaces $G_{\mathbf{PA}} \rightarrow G_{\mathbf{PA}^*}$ in the definition of $A(w_2, a)$ with $\neg G_{\mathbf{PA}}$, which does not affect the proof. It is then case that $\mathscr{M}, w_2, a \models K_a \neg K_a G_{\mathbf{PA}}$. First, $\neg K_a G_{\mathbf{PA}} \in A(w_2, a)$, because $\phi \leftrightarrow K_a\phi \in A(w_2, a)$ for every ϕ and $\neg G_{\mathbf{PA}} \in A(w_2, a)$. Second, $\mathscr{M}, w_2, a \models \neg K_a G_{\mathbf{PA}}$, since $G_{\mathbf{PA}} \notin A(w_2, a)$. But $\mathscr{M}, w_1, a \models K_a G_{\mathbf{PA}}$. I leave it to the reader to check that (17) and (20) are true on the model as well. Furthermore, (26) blocks Horsten (2000)'s counterexample, since (9) and (10) are of the form $\Diamond K_a\phi$, not of the form $K_a\phi$. It was pointed out in Section 2 that there are reasons to doubt the efficacy of Anderson's counterexample, so I will ignore that example here. It should also be emphasized that (26) does all this without being restricted, unlike Anderson's (11) or Horsten (2000)'s (14).

To sum up, there is a good, if not better, alternative to (3). This strengthens the justification challenge made in Section 4.

6 Summary

The subject of this article is the closure of *a priori* knowability under *a priori* knowable material implication. In a restricted version it is arguably correct: if the material implication and the antecedent are true if possibly true, and if the consequent is *a priori* knowable if true, then the principle is correct. These conditions are fulfilled when the material implication is arithmetical and when the consequent has a certain low arithmetical complexity. See Subsection 4.1. Philosophers and logicians have endorsed the closure principle even when not all of the conditions mentioned earlier are satisfied. For instance, Fritz (2013) accepts the principle full stop. Anderson (1993) and Horsten (2000) are more cautious. They each present counterexamples, although it was pointed out that Anderson's counterexample is not unequivocally successful. In reaction to their own counterexamples they each put forward a certain restriction strategy. Horsten suggests that the closure principle is safe when the language is not highly expressive and Anderson proposes that the closure principle is secure when restricted to necessary truths. See Section 2.

Against Horsten's restriction strategy I presented the so-called Socratic objection: even with very few expressive resources and with modest background assumptions, one can deduce from the closure principle that knowable ignorance implies necessary ignorance, which is a highly implausible consequence. See Section 3. Against Anderson's restriction strategy I put the so-called justification challenge: if even with very strong modal and epistemic background assumptions, one cannot deduce the closure principle restricted to necessary, arithmetical truths, the question arises how exactly to justify it. See Subsection 4.2. To raise the stakes, I introduced a so-called logical competence principle, which says that if all the premises of a logical deduction are known, then the conclusion is knowable. It is claimed that an adequate epistemic logic, which steers between the horn of logical omniscience and the horn of logical incompetence, should adopt the logical competence principle. Moreover, it was pointed out that there is support in mainstream epistemology for versions of the logical competence principle. Finally, it was shown that the logical competence principle is not hit by the Socratic objection or Horsten's counterexample and that it accomplishes this without any restrictions. Taking into consideration the existence of a superior alternative, the question arises why one should try to find a justification for Anderson's restricted closure principle at all.

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