Factive knowability and the problem of possible omniscience*

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Abstract

Famously, the Church-Fitch paradox of knowability is a deductive argument from the thesis that all truths are knowable to the conclusion that all truths are known. In this argument, knowability is analyzed in terms of having the possibility to know. Several philosophers have objected to this analysis, because it turns knowability into a nonfactive notion. In addition, they claim that, if the knowability thesis is reformulated with the help of factive concepts of knowability, then omniscience can be avoided. In this article we will look closer at two proposals along these lines (Edgington 1985; Fuhrmann 2014a), because there are formal models available for each. It will be argued that, even though the problem of omniscience can be averted, the problem of possible or potential omniscience cannot: there is an accessible state at which all (actual) truths are known. Furthermore, it will be argued that possible or potential omniscience is a price that is too high to pay. Others who have proposed to solve the paradox with the help of a factive concept of knowability should take note (Fara 2010; Spencer 2017).

Keywords Factive knowability; Actuality Potential knowledge; Knowability thesis; Church-Fitch paradox of knowability Possible omniscience Potential omniscience

1 Introduction: factive knowability and the paradox of knowability

How to conceptualise knowability? In the philosophical literature the most common conceptualisation of knowability is in terms of having the possibility to know, i.e. there being an accessible possible world in which one knows. This

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concept does not entail truth or truth at the actual world: there are (actual) falsehoods that nevertheless are known in an accessible possible world. For instance, it does not (actually) rain but it could have been and, if someone had looked out of the window, that person would have known that it rained. However, our primary interest is in what (actual) truths are knowable. In other words, we need a *factive* concept of knowability:

> Everything that is knowable is true.

There are alternative conceptualisations of knowability that do entail truth or truth at the actual world: having the possibility to know that something is actually true (Edgington, 1985); actually having the capacity to know that something is actually true (Fara, 2010); having the potential to know (Fuhrmann, 2014a,b); having the ability to know (Spencer, 2017). Note that, if knowability entails (actual) truth, then we cannot have the following:

> If something is possibly known, then it is knowable

For otherwise we would also have the following:

> If something is possibly known, then it is (actually) true.

The latter is not in general case, as we have seen.

One of the most important issues in theory of knowledge pertains to the limits of knowledge: what is knowable? The *knowability thesis* is the following:

> Every truth is knowable

The knowability thesis is a "is a weak consequence of verificationism and a weak thesis of idealism" (Hart, 1979, p. 156) – see also (Hart and McGinn, 1976). The kind of verificationism that we are talking about here is the one of "[p]hilosophers like Bolzano or logical empiricists [who] took verificationism seriously but [...] still kept the law of excluded middle" (Rabinowicz and Segerberg, 1994, Section 1). We are not talking here about intuitionistic verificationism – see (Dummett, 1977). The Church-Fitch paradox of knowability (Fitch, 1963) starts from the knowability thesis with knowability conceptualised as having the possibility to know:

> Every truth is possibly known.

The *conclusion* of the paradox is the following:

> Every truth is known.

This is an expression of *omniscience*. The *deductive argument* that leads from the knowability thesis to the conclusion of omniscience builds on the following independent result:

> It is not possible to know an unknown truth.

The deductive argument relies on classical logic. For a discussion of the Church-Fitch paradox against the background of intuitionistic logic, see (Williamson, 1982, 1988, 1992, 1994) and (Percival, 1990). Because of the transition from a thesis that is plausible to some to a conclusion that is implausible to all, the argument is seen by some as paradoxical. Originally, the premise that all truths are knowable was formulated with the help of the concept of having the possibility to know.

A prominent solution strategy consists in reformulating the knowability thesis with the help of factive concepts of knowability (Edgington, 1985; Fara, 2010; Fuhrmann, 2014a; Spencer, 2017). A crucial success condition is that these factive concepts of knowability cannot entail the concept of having the possibility to know. Otherwise one can argue that, even for this factive concept of knowability, it holds that:

> It is unknowable that a particular truth is unknown.

Furthermore, one can block the specific *argument* if the above is underivable.

Even if the specific Church-Fitch argument can be blocked, this does not preclude that there could be other arguments from the knowability thesis to the omniscience scheme. What we would need to show then is that there is no *valid* argument from the knowability thesis to omniscience. In other words, there is no *truth-preservation* from the knowability thesis to the omniscience scheme, i.e. that there are models in which a particular instance of the knowability thesis is true but the relevant instance of the omniscience is false. In fact, one would need to show something stronger. The knowability thesis functions not just as (hypothetical) truth, but as an axiom – see (Fischer, 2013). Axioms are supposed to be valid relative to a certain class of models, i.e. truth in all the models belonging to that class. What we would need to show then is that the following *admissibility* statement is false (Maffezioli et al, 2013, p. 2697): if the knowability thesis is valid, then so is omniscience.[1] In other words, there is no *validity-preservation* from the knowability thesis to the omniscience scheme, i.e. that there is a class of models in each of which the knowability thesis is true (valid with respect to that class of models) but in at least one of which an instance of the omniscience scheme is false (not valid with respect to that class of models). Clearly, a failure of validity-preservation entails a failure of truth-preservation, but not *vice versa*.

Rabinowicz and Segerberg (1994) provide models for Edgington's concept of knowability and Fuhrmann (2014a) does the same for his own concept. Therefore, we check the admissibility statements for those concepts.[2] For that reason we will focus on those two factive concepts of knowability: Section 2 is dedicated to Edgington's concept and Section 3 is dedicated to Fuhrmann's concept.

Before we can discuss the above mentioned concepts of factive knowability, it is important to introduce the concept of a *relational frame* and the related concept

[1] Maffezioli et al (2013) show the inadmissibility of omniscience in an intuitionistic knowability logic.

[2] If we had analytic proof systems for factive notions of knowability, then we could also use those. Maffezioli et al (2013) do this for intuitionistic knowability logic.

of *frame validity*. The concept of a relational frame goes back to Kripke (1963) and is the subject of many textbook expositions, e.g. (Hughes and Cresswell, 1996). A relational frame is a tuple $\langle W, R\rangle$, with W a non-empty set of worlds and R a two-place relation on W. Models $\mathcal{M}$ based on relational frames are tuples containing a relational frame and a function from sentence letters and worlds belonging to W to truth-values. Truth at a world in a model ($\mathcal{M}, w \models \phi$) is defined inductively as follows: if $\phi = p$, then $\mathcal{M}, w \models \phi$ if and only if $V(p, w) = 1$; if $\phi = \Box\psi$, then $\mathcal{M}, w \models \phi$ if and only if $\mathcal{M}, w' \models \psi$ for all $w' \in W$ such that wRw'; all other clauses are as expected. A formula is valid on a frame if and only if it true at all worlds in all the models based on the frame.

As will become clear, models for factive concepts of knowability are based on kinds of frames that are different from but related to relational frames. In particular, we will be looking into bi-relational frames and two-dimensional frames (Section 2) and hyperrelational frames (Section 3). The bi-relational frames and two-dimensional frames will be used to formalize Edgington's concept of knowability, while the hyperrelational frames have been used by Fuhrmann to formalize his concept of knowability. We will later see that the bi-relational frames and two-dimensional frames are connected (Subsection 2.1) and the same for the bi-relational frames and hyperrelational frames (Subsection 3.3).

The main conclusion of the paper will be the following. First, by reformulating the knowability thesis with the help of either Edgington's or Fuhrmann's notion of knowability, one can avoid omniscience even if one assumes the validity of the reformulated knowability theses. Second, if one assumes the validity of the reformulated knowability theses, one also has the accept what will be called 'possible or potential omniscience'.

2 Edgington's knowability thesis and the problem of possible omniscience

Edgington (1985)'s key idea was to reformulate the knowability thesis as follows:

> Every *actual* truth is possibly known to be *actually* true.

She proceeded with arguing that a Church-Fitch argument against the reformulated knowability thesis is blocked. A major issue with her proposal is that it is not clear how one can acquire non-trivial knowledge about the actual world in a non-actual world (Williamson, 1987, 2000; Edgington, 2010). But here we will assume for the sake of the argument that it is possible to do so, since in this article we are more interested in problems common to different kinds of models for different concepts of factive knowability, so we ignore problems that are specific to models for Edgington's conception of factive knowability. In Subsection 2.1 we will look at models that have been developed to formally study Edgington's idea. In Subsection 2.2 we will look at the application of those models to (the relevant variation on) the Church-Fitch paradox of knowability. Finally, in Subsection 2.3

we will introduce a new problem, viz. the problem of possible omniscience. As will be argued in Subsection 3.4, a very similar problem applies to a different kind of models for factive knowability.

2.1 The models

Let us use $\Diamond$ for the possibility operator, K for the knowledge operator and A for actuality operator. Then one can express Edgington's knowability thesis as follows:

$$A\phi \rightarrow \Diamond KA\phi. \tag{1}$$

The standard models for a formal language in which one can express Edgington's thesis are based on *bi-relational* frames.[3]

Definition 1 (Bi-relational frames)**.** A bi-relational frame is a tuple $\langle W, R_M, R_E \rangle$, with

1. W a non-empty set of worlds;
2. R_M a two-place relation on W;
3. R_E a two-place reflexive relation on W.

It is straightforward to define the truth at world in the model of formulas of the form $\Box\phi$ and $K\phi$. The canonical semantical theory of actuality goes back to (Kaplan, 1977, p. 545). Against the background of (bi-)relational frames, one can formulate the key idea as follows: given a 'reference world' w_0, $A\phi$ is true at a world w if and only if ϕ is true at w_0. The following definitions spell this out.

Definition 2 (Simple Kaplan models)**.** A simple Kaplan model $\mathcal{M}$ based on a bi-relational frame is a tuple $\langle \mathcal{F}, V, w_0 \rangle$, with $\mathcal{F}$ a multi-modal frame, V a function from proposition letters and elements of W to truth-values, and w_0 an element from W.

Definition 3 (Truth at a world in simple Kaplan models)**.** The relation

$$\mathcal{M}, w \models \phi$$

(with $\mathcal{M}$ a simple Kaplan model based on a bi-relational frame and ϕ a well-formed formula) is defined inductively as follows:

1. if $\phi = p$, then $\mathcal{M}, w \models \phi$ iff $V(p, w) = 1$;
2. if $\phi = \Box\psi$, then $\mathcal{M}, w \models \phi$ iff $\mathcal{M}, w' \models \psi$ for all $w' \in W$ such that wR_Mw';
3. if $\phi = K\psi$, then $\mathcal{M}, w \models \phi$ iff $\mathcal{M}, w' \models \psi$ for all $w' \in W$ such that wR_Ew';

[3]See (Hughes and Cresswell, 1996, pp. 217–220) for more information on 'multi-modal' (incl. 'bi-modal') logics.

4. if $\phi = A\psi$, then $\mathcal{M}, w \models \phi$ iff $\mathcal{M}, w_0 \models \psi$;

5. all other clauses are as expected.

There are two ways one could define truth in a simple Kaplan model.

Definition 4 (General truth in a simple Kaplan model)**.** For every formula ϕ, ϕ is generally true in a simple Kaplan model ($\mathcal{M} \models \phi$) iff, for every world $w \in W$, $\mathcal{M}, w \models \phi$.

Definition 5 (Weak truth in a simple Kaplan model)**.** For every formula ϕ, ϕ is weakly true in a simple Kaplan model ($\mathcal{M} \models_{weak} \phi$) iff $\mathcal{M}, w_0 \models \phi$.

A formula is valid ($\models \phi$) if and only if it is true in all models. Given the two different ways of defining truth in a model, there are two flavours of validity: general and weak validity. All generally valid formulas are also weakly valid.

Rabinowicz and Segerberg (1994) quickly noted that the straightforward combination will not do, because one can obtain the (relevant variation on the) *conclusion* of the Church-Fitch paradox. In particular, they showed that the following is a theorem:

$$A\phi \rightarrow KA\phi. \tag{2}$$

Indeed, if $\mathcal{M}, w \models A\phi$, then $\mathcal{M}, w_0 \models \phi$ and, hence, $\mathcal{M}, w' \models A\phi$ for all $w' \in W$, including all $w' \in W$ such that wR_Ew'. Therefore, $A\phi \rightarrow KA\phi$ is generally true in the arbitrarily chosen model $\mathcal{M}$.

The problem is due to the fact that the reference world (w_0), i.e. the world at which sentences within the scope of the actuality operator are evaluated, remains fixed. Rabinowicz and Segerberg (1994)'s solution to this problem was to reconceptualise the epistemic accessibility relation as a relation between states, i.e. pairs of worlds $\langle w, w' \rangle$ that contain both evaluation worlds (w) and reference worlds (w'), which allows the reference worlds to shift. The following definitions spell this out:[4]

Definition 6 (Two-dimensional frames)**.** A two-dimensional frame is a tuple

$$\langle W, R_E, R_M \rangle,$$

with

[4]These definitions are epistemic variations on the definitions for belief and actuality models in (Heylen, 2016, pp. 1654–1655), which are essentially the models in (Rabinowicz and Segerberg, 1994) with one exception. In (Rabinowicz and Segerberg, 1994, Section 3) each model contains a set Π of propositions (i.e. subsets of $W \times W$), which (i) contains the truth sets that correspond to atomic formulas and (ii) is closed under complement, finite intersection and operations that correspond to the knowledge operator, the necessity operator and the actuality operator. Each formula corresponds in their models to a proposition, but in principle there can be propositions that do not correspond to a formula. Adding a set of propositions is for the purposes of this article a needless complication, since we will be using *schemes* in the object language and quantify over formulas in the meta-language rather than quantify over propositions in the object language.

1. W a non-empty set of worlds;

2. R_E a two-place reflexive relation on $W \times W$ such that, if $\langle w, w' \rangle R_E \langle v, v' \rangle$ and $w = w'$, then $v = v'$;

3. R_M a two-place relation on W.

Definition 7. A (Rabinowicz and Segerberg, 1994)-model based on a two-dimensional frame is a tuple $\langle \mathcal{F}, V \rangle$, with $\mathcal{F}$ a two-dimensional frame and with V a function from proposition letters and worlds to truth-values.

Definition 8. The relation $\mathcal{M}, \langle w, w' \rangle \models \phi$ (with $\mathcal{M}$ a (Rabinowicz and Segerberg, 1994)-model based on a two-dimensional frame and ϕ a formula) is defined inductively as follows:

1. if $\phi = p$, then $\mathcal{M}, \langle w, w' \rangle \models \phi$ iff $V(p, w) = 1$;

2. if $\phi = K\psi$, then $\mathcal{M}, \langle w, w' \rangle \models \phi$ iff $\mathcal{M}, \langle v, v' \rangle \models \psi$ for every v, v' such that
$$\langle w, w' \rangle R_E \langle v, v' \rangle;$$

3. if $\phi = \Box\psi$, then $\mathcal{M}, \langle w, w' \rangle \models \phi$ iff $\mathcal{M}, \langle v, w' \rangle \models \psi$ for all $v \in W$ such that $wR_M v$;

4. if $\phi = A\psi$, then $\mathcal{M}, \langle w, w' \rangle \models \phi$ iff $\mathcal{M}, \langle w', w' \rangle \models \psi$;

5. all other clauses are as expected.

A quick glance at Definition 8 reveals that the first world in a world pair is the evaluation world, while the second world in the world pair is the reference world.

States of the form $\langle w, w \rangle$, i.e. states where the evaluation world is the same as the reference world are self-centered states. One can evaluate formulas at *all* states or only at self-centered states. Correspondingly, there are two ways one can define validity and satisfiability.

Definition 9 (General validity and satisfiability)**.** For every formula ϕ, ϕ is generally valid ($\models \phi$) iff for every model $\mathcal{M}$ and for all $w, w' \in W$, $\mathcal{M}, \langle w, w' \rangle \models \phi$, and ϕ is generally satisfiable iff $\not\models \neg\phi$.

Definition 10 (Weak validity and satisfiability)**.** For every formula ϕ, ϕ is weakly valid ($\models_{weak} \phi$) iff $\models A\phi$, and ϕ is weakly satisfiable iff $\not\models_{weak} \neg\phi$.

In order to ensure that the results to be proved are the strongest it is best to use general validity for theorems about what is *valid* and weak validity for theorems about what is *invalid*, because general validity entails weak validity and weak invalidity entails general invalidity.

It is easy to show that (2) is not weakly valid according to Definitions 7-8 and Definition 10. Here is a model:

- $W = \{w_1, w_2\}$;
- $\langle w_1, w_1 \rangle R_E \langle w_2, w_2 \rangle$, and for all $v, v' \in W$, $\langle v, v' \rangle R_E \langle v, v' \rangle$;
- $R_M = W \times W$;
- $V(p, w_1) = 1$, $V(p, w_2) = 0$.

Then $\mathcal{M}, \langle w_1, w_1 \rangle \models p$ and, hence, $\mathcal{M}, \langle w_1, w_1 \rangle \models Ap$. But $\mathcal{M}, \langle w_2, w_2 \rangle \not\models p$, because $V(p, w_2) = 0$. Therefore, $\mathcal{M}, \langle w_2, w_2 \rangle \not\models Ap$. Consequently, $\mathcal{M}, \langle w_1, w_1 \rangle \not\models KAp$, since $\langle w_1, w_1 \rangle R_E \langle w_2, w_2 \rangle$.

The relation between simple Kaplan models based on relational models on the one hand and (Rabinowicz and Segerberg, 1994)-models based on two-dimensional frames on the other hand is straightforward: for every formula ϕ that does not contain the actuality operator (but may contain the knowledge or necessity operator), if there is a simple Kaplan model $\mathcal{M}$ and a world w such that $\mathcal{M}, w \models \phi$, then there is a (Rabinowicz and Segerberg, 1994)-model $\mathcal{M}'$ and a pair of worlds $\langle v, v' \rangle$ such that $\mathcal{M}', \langle v, v' \rangle \models \phi$, and vice versa.[5]

2.2 The application to the Church-Fitch paradox of knowability

It is a good thing that (2) is not a *theorem*, given Definitions 7-8. However, what still needs to be checked is whether scheme (2) can be *inferred* from scheme (1). To do this, it is convenient to know under which conditions (1) is true. The following lemma gives us that information.[6]

Lemma 1. *Edgington's knowability thesis*

$$A\phi \to \Diamond KA\phi$$

corresponds to the following condition on two-dimensional frames:

$$\forall w, w' \exists v \left(wR_M v \land \forall u, u' \left(\langle v, w' \rangle R_E \langle u, u' \rangle \to u' = w' \right) \right).$$

Proof. Left-to-right: Suppose that $A\phi \to \Diamond KA\phi$ is valid on a class of two-dimensional frames. Assume for a *reductio ad absurdum* that

$$\exists w, w' \forall v \left(wR_M v \to \exists u, u' \left(\langle v, w' \rangle R_E \langle u, u' \rangle \land u' \neq w' \right) \right).$$

[5] See (Heylen, 2016, p. 1655, Theorem 3.7).

[6] In (Rabinowicz and Segerberg, 1994, Section 3) the following condition for the weak validity of $\phi \to \Diamond KA\phi$ is put forward: for every $w \in W$ and every proposition $\pi \in \Pi$ (see 4), if $\langle w, w \rangle \in \pi$, then there exists some $v \in W$, such that $wR_M v$ and, for every $w', v' \in W$, if $\langle v, w \rangle E \langle v', w' \rangle$, then $\langle w', w' \rangle \in \pi$. This condition is more general than is needed for the (weak) validity of the *scheme* $\phi \to \Diamond KA\phi$, since there can be propositions that are not expressed by formulas, although every formula expresses a proposition (see footnote 4). It is easy to transform the model used in the left-to-right direction of the proof of Lemma 1 into a full-fledged (Rabinowicz and Segerberg, 1994)-model. The proposition π is the singleton $\{\langle w', w' \rangle\}$ and it has to be included in the set of propositions Π, because the latter by stipulation includes the truth set of atomic formula p.

Now consider a model $\mathcal{M}$ based on the above condition and such that $V(p, w',) = 1$ and $V(p, u',) = 0$ for all $u' \neq w'$. It is easily verified that $\mathcal{M}, \langle w, w' \rangle \models Ap$. Yet, $\mathcal{M}, \langle w, w' \rangle \not\models \Diamond KAp$, since for every modally accessible world v there is an epistemically accessible state $\langle u, u' \rangle$ such that $V(p, u') = 0$ and, hence, $\mathcal{M}, \langle u', u' \rangle \not\models p$ and, consequently, $\mathcal{M}, \langle u, u' \rangle \not\models Ap$, which in turn implies that $\mathcal{M}, \langle v, w' \rangle \not\models KAp$. This contradicts the assumption.

Right-to-left: Suppose that the following condition applies to a class of two-dimensional frames:

$$\forall w, w' \exists v \left(wR_M v \wedge \forall u, u' \left(\langle v, w' \rangle R_E \langle u, u' \rangle \rightarrow u' = w' \right) \right).$$

Consider any model $\mathcal{M}$ and any worlds w, w' such that $\mathcal{M}, \langle w, w' \rangle \models A\phi$ and, hence, $\mathcal{M}, \langle w', w' \rangle \models \phi$. The condition implies that there is a $v \in W$ such that $wR_M v$. For any state $\langle u, u' \rangle$ that is epistemically accessible from $\langle v, w' \rangle$, it is the case that $u' = w'$. But then $\mathcal{M}, \langle u', u' \rangle \models \phi$, which implies that $\mathcal{M}, \langle u, u' \rangle \models A\phi$. Therefore, $\mathcal{M}, \langle v, w' \rangle \models KA\phi$, which entails that $\mathcal{M}, \langle w, w' \rangle \models \Diamond KA\phi$. □

Returning to the model given above (see p. 7), it is easy to see that it is a model based on the frame condition that corresponds to (1). Consider the four states $\langle w_1, w_1 \rangle$, $\langle w_1, w_2 \rangle$, $\langle w_2, w_1 \rangle$ and $\langle w_2, w_2 \rangle$. The last three cases are straightforward: each world is modally accessible from itself and the only states that are epistemically accessible from $\langle w_1, w_2 \rangle$, $\langle w_2, w_1 \rangle$ and $\langle w_2, w_2 \rangle$ are $\langle w_1, w_2 \rangle$, $\langle w_2, w_1 \rangle$ and $\langle w_2, w_2 \rangle$ respectively, and the reference world remains the same in each case. For the first state it is important that w_2 is modally accessible from w_1 and the only state that is epistemically accessible from $\langle w_2, w_1 \rangle$ is $\langle w_2, w_1 \rangle$, so the reference world remains the same. Then Lemma 1 tells us that the model makes all instances of (1) true. Yet, as we have seen, (2) is false in the model. So, the (relevant variation on) the conclusion of the Church-Fitch paradox is not admissible.

2.3 The problem of possible omniscience

To appraise the philosophical significance of what is to come, it is important to pay attention to two things. First, there is the difference between scheme (2) on the one hand and the following scheme on the other hand:

$$\Diamond \left(A\phi \rightarrow KA\phi \right). \tag{3}$$

The first expresses that something is true, whereas the second expresses that it is possibly true. Assuming the validity of (1), one cannot derive the validity of (2), as we have seen in Subsection 2.2. In contrast, assuming (1), one can derive (3). First, given Definition 8, $\Box A\phi \rightarrow A\phi$ is valid, whence it follows from (1) that

$$\Box A\phi \rightarrow \Diamond KA\phi. \tag{4}$$

Second, $\Box\psi \rightarrow \Diamond\theta$ entails $\Diamond(\psi \rightarrow \theta)$ in any normal modal logic (i.e., extension of system **K**) – see (Hughes and Cresswell, 1996, p. 35). Therefore, it follows from (4) that (3).

Second, there is the difference between the following two claims:

1. for all formulas ϕ, there is a state in which $A\phi \to KA\phi$ is true;

2. there is a state in which, for all formulas ϕ, $A\phi \to KA\phi$ is true.

The difference between the two claims is in the order of the quantifiers. The first claim is nothing but scheme (3). I will now show that the second claim is also derivable.

Theorem 1. *If $A\phi \to \Diamond KA\phi$ is valid on a two-dimensional frame, then for every model $\mathcal{M}$ based on the frame, for all worlds w, w', there is a world v such that $wR_M v$ and, for every formula ϕ, $\mathcal{M}, \langle v, w' \rangle \models A\phi \to KA\phi$.*

Proof. Suppose that $A\phi \to \Diamond KA\phi$ is valid on a two-dimensional frame. Then by Lemma 1:

$$\forall w, w' \exists v \left(wR_M v \wedge \forall u, u' \left(\langle v, w' \rangle R_E \langle u, u' \rangle \to u' = w' \right) \right).$$

Consider v and any formula ϕ such that $\mathcal{M}, \langle v, w' \rangle \models A\phi$. Then it follows that

$$\mathcal{M}, \langle w', w' \rangle \models \phi.$$

Since $w' = u'$ for any $u, u' \in W$ such that $\langle v, w' \rangle R_E \langle u, u' \rangle$, it follows that

$$\mathcal{M}, \langle u', u' \rangle \models \phi.$$

Therefore, $\mathcal{M}, \langle u, u' \rangle \models A\phi$ for any $u, u' \in W$ such that $\langle v, w' \rangle R_E \langle u, u' \rangle$. Consequently, $\mathcal{M}, \langle v, w' \rangle \models KA\phi$. □

To sum up, whereas omniscience of actual truths can be avoided, *possible omniscience* of actual truths cannot.

The philosophical significance of this may become clear with the help of examples. The first example will make use of an open question that has not been definitely answered by science yet. The second example will make use of a common presupposition of our best scientific theories. The third example will make use of theorems. Throughout the examples we will only be concerned with finite knowers such as humans. In particular, I will assume that knowers that can have only finitely many observations (at any given point in time). After all, the paradox of knowability is supposed to target verificationism, and verficationists such as Carnap (1931) were only considering verification by finitely many observation statements (or finite conjunctions thereof). In addition, we will assume that knowers can only have finitely many proofs or can only carry out finitely many steps of a decision procedure (at any given point in time).[7]

For a first example, let p_n stand for: there is liquid water on planet n. Now consider the following two claims:

[7] It might be objected that in these models one has *logical omniscience* and, since there are infinitely many logical truths, an infinite numbers of truths is known. In response one could, for instance, change Definition 1 by adding a set W^* of sets of formulas of the language ('impossible worlds'), with R_M a two-place relation on W and with R_E a two-place reflexive relation on $W \cup W^* \times W \cup W^*$ such that, if $\langle w, w' \rangle R_E \langle v, v' \rangle$ and $w = w'$, then $v = v'$. The relation of truth relative

1′. for all p_n, there is a state in which $Ap_n \rightarrow KAp_n$ is true;

2′. there is a state in which, for all p_n, $Ap_n \rightarrow KAp_n$ is true.

The possibility of acquiring knowledge about actual truths about *a particular planet* may involve serious technological and scientific advances and it may also involve a lot of resources. For instance, the particular planet may be very far away and the liquid water on the planet may be under the surface. But let us assume that it is indeed possible to acquire knowledge about the actual presence of liquid water on a particular planet. The possibility of omniscience with regard to all the actual truths about liquid water on all of the planets is incredibly much more demanding. The larger the number of planets is, the more demanding it is. According to some estimates there are billions or even trillions of planets in our galaxy alone, and there is evidence for the existence of hundreds of galaxies in the part of the universe that is visible from Earth. It is currently an *open question* whether the universe is infinitely large or not. If there happen to be infinitely many planets, then it is metaphysically *impossible* for finite knowers or finite groups of finite knowers to acquire all the actual truths about the presence of liquid water on planets.

For a second example, assume that space is continuous, as is presupposed in relativity theory, which is our best theory about space. Let the distance between two points a and b be a non-computable non-negative irrational number r, with $r = z_0.z_1z_2\ldots$ with z_0 a non-negative integer and $0 \leq z_i \leq 9$ for $1 \leq i$. (Because of cardinality considerations we know that most real numbers are irrational and most real numbers are non-computable, so this is a typical situation.) Let p_n stand for: the distance between a and b is at least $z_0.z_1\ldots z_n$. For each p_n at least up to some very large decimal expansion, it is conceivable that its actual truth can be known. But there are infinitely many p_n that are actually true, so they *cannot* collectively be known to be actually true, as long as we are talking about finite knowers. Moreover, r is by assumption not computable. Finally, it is not clear how an indefinite empirical investigation would proceed, if we restrict ourselves to agents with finite powers of perceptual discrimination (De Clercq and Horsten, 2004).

For a third and final example consider arithmetical statements. There are infinitely many true arithmetical statements. For instance, let p_n express that n is a prime number. Euclid proved that there are infinitely many prime numbers. For each p_n, we can come to know that p_n is true, if p_n is true. E.g., we can use the sieve of Eratosthenes, an algorithm to find all prime numbers smaller than an ar-

to a model and a state with an evaluation world belonging to W remains the same. The relation of truth relative to a model and a state with an evaluation world belonging to W^* is determined by membership of the elements of W^*. Logical truth is defined relative to states where both the evaluation and the reference world are from W. It is easy to check that logical omniscience no longer holds. Moreover, Lemma 1 still holds, although at one point in the proof of the left-to-right direction a slight generalisation, viz. $V(p, u') = 0$ or $p \notin u'$, is needed. I leave it to the reader to check this.

bitrarily chosen natural number. However, there are infinitely many p_n, so they *cannot* collectively be known to be true, as long as we restrict ourselves to finite knowers. Moreover, there is not even a general decision procedure for arithmetical truth. Indeed, arithmetic is essentially undecidable (Boolos et al, 2003, p. 223). Williamson (2016) argues that, for every arithmetical truth, there is a finite mind that can know it. He bases this on the claim that, for every arithmetical truth, there can be a finite mind that finds it 'primitively compelling'. One may share Williamson's optimism, but even Williamson (2016, p. 249) describes the scenario in which some finite mind knows every mathematical truth by having some 'non-recursive pattern recognition capacity' as 'wild speculation'.

The conclusion is that, even if omniscience of actual truths does not follow, it may still be the case that one still ends up with possible omniscience of actual truths, which is a price that is too high to pay, at least if one wants to have a notion of knowability for finite knowers. The latter condition is not only in spirit with the verificationism of Carnap (1931), it is a condition that fits knowers such as humans. Of course, as Yap (2014) points out, models of epistemic logic may contain all kinds of idealizing features. Possible omniscience may be one those, just as logical omniscience is one such feature of standard Kripke models for epistemic logic (i.e. if ϕ is a logical truth, then it is known). If one is only interested in one particular aspect, i.c. (non-)omniscience, then one may make simplifying assumptions with respect to another aspect, e.g. whether the knowers are finite or not.[8] Yap (2014, p. 3365) stresses that one can take a programmatic stance towards unrealistic features of epistemic logic, viz. one can develop a research programme in which one progressively drops those unrealistic features. For those who want to take this stance, let this be an invitation to develop models for factive notions of knowability that avoid the problem of possible omniscience.

3 Fuhrmann's thesis and the problem of potential omniscience

Fuhrmann (2014a,b) introduces and develops the notion of *potential knowledge*. According to Fuhrmann (2014a, p. 1630, 1635) there are at least four desiderata with respect to the different concepts of knowability:

D1 Knowability entails truth.

D2 Knowability does not follow from possible knowledge, and

D3 Knowability does not entail possible knowledge.

[8]One the one hand, for the purpose of checking that the validity of the knowability thesis does not entail omniscience, it is actually an advantage that the models have logical omniscience built into them, for it makes the inadmissibility claim stronger. On the other hand, the admissibility of possible omniscience is stronger, if one does not assume logical omniscience. See footnote 7 for more on this.

D4 What is knowable about a world a must be determined in a itself.

The first three desiderata have already been discussed in Section 1. Desideratum D4 is based on the worry about non-trivial knowledge about the actual world in non-actual worlds. Fuhrmann claims that the concept of *potential knowledge* meets the above desiderata. The intuitive idea is that, when confronted with evidence that a subject is prepared to accept, the subject will update its knowledge states in accordance with its epistemic preferences. Any epistemic state that can be reached in a preferred response to evidence one is prepared to accept is a potential knowledge state. With the help of this notion the knowability thesis can be reformulated as follows:

> Every truth is potentially known to be true.

The above version of the knowability thesis is not under threat by the (relevant variation on) the Church-Fitch argument:

> Take a Fitch-sentence, F = (P and X does not know that P). It is not possible that X knows that F (by the Fitch-argument). But F is knowable to X: The course of evidence in the actual world may plausibly run such that under improved evidential conditions X knows that P while recognising, i.e. knowing, that under the less favourable conditions X cannot count as knowing that P. Under the potential, favourable conditions F would be known to X and thus under the given, less favourable conditions F is knowable for X. (Fuhrmann, 2014a, p. 1638)

In Subsection 3.1 we will look at models for the concept of potential knowledge. In Subsection 3.2 we will look at the application of those models to (the relevant variation on) the Church-Fitch paradox of knowability. Next, in Subsection 3.3 we are going to have a closer look at the relation between bi-relational frames and models on the one hand and hyperrelational frames and models on the other hand. This will be connected to the knowability paradox. Finally, in Subsection 3.4 we will introduce a problem, viz. the problem of potential omniscience, which is very similar to the problem introduced in Subsection 2.3.

3.1 The models

Let us use $\langle K\rangle\phi$ for the potential knowledge operator. With it one can express the knowability thesis as follows:

$$\phi \rightarrow \langle K\rangle\phi. \tag{5}$$

Fuhrmann (2014b,a) has developed models for the formal language of (5). The basis for those models are a new type of frames. The following definitions spell this out:

Definition 11 (Hyperrelational frames)**.** A hyperrelational frame $\mathcal{F}$ is a tuple

$$\langle W, S, \leq \rangle,$$

with W a non-empty set, S a binary relation in $W \times W$ and $\leq$ a binary relation in $W^2 \times W^2$.

Definition 12. A model $\mathcal{M}$ based on a hyperrelational frame is a tuple $\langle \mathcal{F}, V \rangle$, with $\mathcal{F}$ a hyperrelational frame and V a function mapping atomic formulas at worlds to truth-values.

Definition 13. The relation $\mathcal{M}, w \models \phi$ (with $\mathcal{M}$ a model based on a hyperrelational frame, $w \in W$ and ϕ a well-formed formula) is defined inductively as follows:

1. $\mathcal{M}, w \models K\psi$ iff $\forall v \in W$, if wSv, then $\mathcal{M}, v \models \psi$;
2. $\mathcal{M}, w \models \langle K \rangle \psi$ iff $\exists S' : S \leq S'$ and $\forall v \in W$, if $wS'v$, then $\mathcal{M}, v \models \psi$;
3. the clauses for atomic formulas, negated formulas and conjoined formulas are as usual.

It is convenient to use the following notation in what follows: for any S' such that $S \leq S'$, let S'_w be $\{v \mid wS'v\}$.

The logic that corresponds to the class of models based on hyperrelational frames is not very interesting. It consists of the following principles:

RN If $\vdash \phi$, then $\vdash K\phi$.

K $K(\phi \to \psi) \to (K\phi \to K\psi)$.

RM$_\mathbf{p}$ If $\vdash \phi \to \psi$, then $\vdash \langle K \rangle \phi \to \langle K \rangle \psi$.

The first two principles tell us that the K-operator behaves as a normal modal operator, which is not a surprise since the truth of $K\phi$ depends on a Kripke-style accessibility relation on possible worlds. The third principle is a rule of monotonicity, which does not entail the closure of potential knowledge under theoremhood and material implication. The $\langle K \rangle$-operator does not behave as normal modal operator.

The ensuing logic is inadequate in that it does not represent K as a *knowledge* operator, because K is not factive. Morevoer, it does not represent $\langle K \rangle$ as a *knowability* operator, since $K\phi$ does not entail $\langle K \rangle \phi$. Finally, it does not represent $\langle K \rangle$ as a *factive* knowability operator, since $K\phi$ does not entail ϕ. Fuhrmann fixes this by adding the following frame conditions:

S-reflexivity $\forall w\,(wSw)$.

Preservation $\forall X \forall w\,\big(S_w \subseteq X \to \exists S'\,\big(S \leq S' \wedge S'_w \subseteq X\big)\big)$.[9]

[9]In (Fuhrmann, 2014a, p. 1644) the w-index is missing, but X is supposed to be a subset of W, whereas S' is a subset in $W \times W$.

p-reflexivity $\forall S' \forall w (S \leq S' \rightarrow wS'w)$.

The above conditions correspond to:

T $K\phi \rightarrow \phi$.

C $K\phi \rightarrow \langle K \rangle \phi$.

$\mathbf{T_p}$ $\langle K \rangle \phi \rightarrow \phi$.

Fuhrmann (2014a, pp. 1643–1644) thinks it important to turn the potential knowledge operator into a normal modal operator, so that it is closed under theoremhood and material implication. To that end he adds the following frame conditions:

Continuation $\exists S' (S \leq S')$.

Combination

$$\forall S' \forall S'' \forall X \forall Y \big((S \leq S' \wedge S'_w \subseteq X \wedge S \leq S'' \wedge S''_w \subseteq Y\big) \\ \rightarrow \exists R (S \leq R \wedge R_w \subseteq X \cap Y)).$$

The above conditions correspond to:

$\mathbf{RN_p}$ If $\vdash \phi$, then $\vdash \langle K \rangle \phi$.

$\mathbf{M_p}$ $(\langle K \rangle \phi \wedge \langle K \rangle \psi) \rightarrow \langle K \rangle (\phi \wedge \psi)$.

Whether it is a good idea to add the frame conditions guaranteeing 'normal' closure properties will be discussed in Subsection 3.3.

Finally, Fuhrmann (2014a, p. 1643) also adds the following frame condition:

S-transitivity $\forall w \forall w' \forall w'' ((wSw' \wedge w'Sw'') \rightarrow wSw'')$.

The above condition corresponds to:

4 $K\phi \rightarrow KK\phi$.

This may give some pause, because principle 4 has been the subject of much discussion in epistemology – see e.g. (Williamson, 2000). However, it makes perfect sense to add principle 4 in the dialectical context of an investigation into whether the knowability thesis (5) entails omniscience, since principle 4 is a direct consequence of omniscience. If the knowability thesis does not entail omniscience, even if omniscience with respect to knowledge itself is assumed, then that is a stronger result than if the knowability thesis does not entail omniscience, with no assumption of omniscience with respect to knowledge. Fuhrmann (2014a, p. 1645) refrains from adding a frame condition that corresponds to $\langle K \rangle \phi \rightarrow \langle K \rangle \langle K \rangle \phi$. This makes again sense in the dialectical context of an investigation into the consequences of the knowability thesis (5), because it is a consequence of the knowability thesis (5), so there is no need to add it.

Hyperrelational frames that satisfy S-reflexivity, S-transitivity, Preservation, p-reflexivity, Continuation and Combination are called *potential knowledge structures* (or, alternatively, potential knowledge frames). In the next subsection we will look at the application of potential knowledge models to the knowability thesis (5).

3.2 The application to the Church-Fitch paradox of knowability

The starting point is the knowability thesis (5) and the background consists of potential knowledge models. We will consider two issues. First, can the (relevant variation on the standard) Church-Fitch *argument* be blocked? Second, is the *conclusion*, i.e.

$$\phi \to K\phi, \tag{6}$$

inadmissible given the assumption that (5) is valid on certain hyperrelational frames?

The first question can be answered affirmatively. In a Church-Fitch-style argument one considers the following instantiation of (5):

$$(\phi \land \neg K\phi) \to \langle K \rangle (\phi \land \neg K\phi). \tag{7}$$

But $\langle K \rangle (\phi \land \neg K\phi)$ is satisfiable. Consider a model with two worlds, w_1 and w_2, such that p is true at w_1 but false at w_2. Let the extension of the epistemic accessibility relation S be the following:

$$\{\langle w_1, w_1 \rangle, \langle w_2, w_2 \rangle, \langle w_1, w_2 \rangle\}.$$

Furthermore, let there be one and only one alternative epistemic accessibility relation S' with $S \leq S'$, namely the relation with the following extension:

$$\{\langle w_1, w_1 \rangle, \langle w_2, w_2 \rangle\}.$$

Then at w_1 $\langle K \rangle (p \land \neg Kp)$ is satisfied, since p is true at w_1, while p is false at the epistemically accessible w_2. We still need to check whether the model is a potential knowledge model, i.e. whether it is based on a frame that satisfies S-reflexivity and -transitivity, Preservation, p-reflexivity, Continuation and Combination. This can easily be done by the reader, but let us briefly discuss Preservation and Combination. Preservation is satisfied, because $S_{w_2} = S'_{w_2}$ and $S'_{w_1} \subset S_{w_1}$, so $S'_w \subseteq S_w$ and, hence, for every w, if $S_w \subseteq X$, then $S'_w \subseteq X$. Combination is satisfied, because there is only one S' such that $S \leq S'$. Consequently, Combination reduces to:

$$\forall S' \forall X \forall Y \left(\left(S \leq S' \land S'_w \subseteq X \land S'_w \subseteq Y \right) \to \exists R \left(S \leq R \land R_w \subseteq X \cap Y \right) \right),$$

which is trivial.

The second question can also be answered affirmatively. For this it is profitable to consider the frame conditions for knowability and omniscience.

Theorem 2. *Fuhrmann's knowability thesis, i.e.*

$$\phi \to \langle K \rangle \phi,$$

corresponds to the following condition on hyperrelational frames:

$$\forall X \forall w \left(w \in X \to \exists S' \left(S \leq S' \land S'_w \subseteq X \right) \right). \tag{8}$$

Proof. This has been claimed by Fuhrmann (2014a, p. 1646, fn. 17). □

One can verify that the model described above is based on a frame that satisfies the condition of Theorem 2. Indeed, $S'_w = \{w\}$, for all $w \in W$. So, whenever $w \in X$, $\{w\} \subseteq X$ or, equivalently, $S'_w \subseteq X$. Yet, $p \to Kp$ is false at w_1, since $w_1 S w_2$ and $V(p, w_2) = 0$. So, we have proved inadmissibility.

3.3 Hyperrelational frames, bi-relational frames, and the knowability paradox

The main formal innovation in (Fuhrmann, 2014a) consists in the introduction of hyperrelational frames, which differ from relational or bi-relational frames. As we have seen in Subsection 3.1, Fuhrmann focused on those hyperrelational frames that satisfy Continuation and Combination. Those frame conditions correspond to closure properties of the potential knowledge operator that turn the latter into a normal modal operator. In this subsection I will argue that hyperrelational frames that satisfy the Combination condition are much closer to bi-relational frames than they at first sight appear to be, unless the former are infinitary in the following sense: there are infinitely many alternative epistemic accessibility relations and, for every world, there are no alternative epistemic accessibility relations such that the intersection of worlds accessible from the given world via those alternative epistemic accessibility relations is finite. Note that the foregoing implies that there are infinitely many epistemic possibilities.

Lemma 2. *For any hyperrelational frame that satisfies Combination and at least one of the following conditions:*

1. *there are finitely many S' such that $S \leq S'$, or*
2. *for any $w \in W$, there are S', S'' such that $S \leq S'$ and $S \leq S''$ and $S'_w \cap S''_w$ is finite,*

it is the case that

$$\forall w \exists R \left(S \leq R \land \forall S' \left(S \leq S' \to R_w \subseteq S'_w \right) \right).$$

Proof. First, suppose that there are finitely many S' such that $S \leq S'$. Enumerate them: $S^{\dagger}, S^{\dagger\dagger}, \ldots$. By Combination and by the fact that $S^{\dagger}_w \subseteq S^{\dagger}_w$ and $S^{\dagger\dagger}_w \subseteq S^{\dagger\dagger}_w$, it is the case that

$$\exists R \left(S \leq R \land R_w \subseteq S^{\dagger}_w \cap S^{\dagger\dagger}_w \right).$$

Note that $R_w \subseteq S_w^{\dagger}$ and $R_w \subseteq S_w^{\dagger\dagger}$. By Combination and by the fact that $R_w \subseteq R_w$ and $S_w^{\dagger\dagger\dagger} \subseteq S_w^{\dagger\dagger\dagger}$, it is the case that

$$\exists R' \left(S \leq R' \wedge R'_w \subseteq R_w \cap S_w^{\dagger\dagger\dagger} \right).$$

Note that $R'_w \subseteq S_w^{\dagger}$ and $R'_w \subseteq S_w^{\dagger\dagger}$ and $R'_w \subseteq S_w^{\dagger\dagger\dagger}$. Continue move until you used every one of the finitely many S' such that $S \leq S'$. In the end you will have established that

$$\exists R^* \left(S \leq R^* \wedge \forall S' \left(S \leq S' \rightarrow R^*_w \subseteq S'_w \right) \right).$$

Second, suppose that, for any $w \in W$, there are S', S'' such that $S \leq S'$ and $S \leq S''$ and $S'_w \cap S''_w$ is finite. By Combination and by the fact that $S'_w \subseteq S'_w$ and $S''_w \subseteq S''_w$, it is the case that

$$\exists R \left(S \leq R \wedge R_w \subseteq S'_w \cap S''_w \right).$$

Since R_w is a subset of a finite set, R_w is itself also finite. Next, assume for a *reductio ad absurdum* that

$$\exists w \forall R \left(S \leq R \rightarrow \exists S' \left(S \leq S' \wedge R_w \nsubseteq S'_w \right) \right).$$

Given the above reasoning, there is an R with $S \leq R$ and R_w is finite. Let us say that there are n elements in R_w. By the *reductio* assumption, there is an S' with $S \leq S' \wedge R_w \nsubseteq S'_w$. By Combination and by the fact that $R_w \subseteq R_w$ and $S'_w \subseteq S'_w$,

$$\exists R' \left(S \leq R' \wedge R'_w \subseteq R_w \cap S'_w \right).$$

Note that $R'_w \subseteq R_w$, because $R'_w \subseteq R_w \cap S'_w$. We will now prove that $R'_w \subset R_w$. Assume that $R'_w = R_w$. But then $R_w \subseteq R_w \cap S'_w$, which can only be the case if $R_w \subseteq S'_w$, which contradicts the *reductio* assumption. Therefore, $R'_w \neq R_w$, which in combination with $R'_w \subseteq R_w$ entails that $R'_w \subset R_w$. This means that there is at least one element in R'_w that is not in R_w. By the *reductio* assumption, there is an S'' with $S \leq S'' \wedge R'_w \nsubseteq S''_w$. By Combination and by the fact that $R'_w \subseteq R'_w$ and $S''_w \subseteq S''_w$,

$$\exists R'' \left(S \leq R'' \wedge R''_w \subseteq R'_w \cap S''_w \right).$$

Note that $R''_w \subseteq R'_w$. Moreover, one can by analogous reasoning as before prove that $R''_w \subset R'_w$. That implies that there are at least two elements in R''_w that are not in R_w. Continue this reasoning entail you have established that there is some R^* with $S \leq R^*$ and there are at least n elements in R^*_w that are not in R_w, which contains n elements. In other words, R^*_w is the empty set. By the *reductio* assumption, there is then as S^* with $S \leq S^*$ and $R^*_w \nsubseteq S^*_w$. But the empty set is a subset of any set. We have reached a contradiction. □

Theorem 3. *For every model $\langle W, S, \leq, V \rangle$ that is based on a hyperrelational frame that satisfies Combination and at least one of the following conditions:*

1. *there are finitely many S' such that $S \leq S'$, or*

2. *for any $w \in W$, there are S', S'' such that $S \leq S'$ and $S \leq S''$ and $S'_w \cap S''_w$ is finite,*

there is a Kripke-style knowledge model $\langle W, S, R, V \rangle$ based on a bi-relational frame, with W, S and V the same as in the given model and with R a relation in $W \times W$, such that for every $w \in W$:

1. $\exists S' \left(S \leq S' \wedge S'_w = R_w \right)$*;*

2. *and for every well-formed formula ϕ,*

$$\langle W, S, \leq, V \rangle, w \models \phi \Leftrightarrow \langle W, S, R, V \rangle, w \models \phi.$$

Proof. By Lemma 2 and the assumption of the two frame conditions, the model $\langle W, S, \leq, V \rangle$ is based on a hyperrelational frame that satisfies the following condition:

$$\forall w \exists R \left(S \leq R \wedge \forall S' \left(S \leq S' \rightarrow R_w \subseteq S'_w \right) \right).$$

Note that, if

$$S \leq R \wedge \forall S' \left(S \leq S' \rightarrow R_w \subseteq S'_w \right),$$

and

$$S \leq R' \wedge \forall S' \left(S \leq S' \rightarrow R'_w \subseteq S'_w \right),$$

then $R_w = R'_w$. For each $w \in W$, let R^*_w be the unique R_w such that

$$S \leq R \wedge \forall S' \left(S \leq S' \rightarrow R_w \subseteq S'_w \right).$$

We will now prove that: $\langle W, S, \leq, V \rangle, w \models \langle K \rangle \psi$ if and only if $\langle W, S, \leq, V \rangle, w' \models \psi$ for all w' such that wR^*w'. Since there is an R such that $S \leq R$ and $R_w = R^*_w$, the right-to-left direction is trivial. The other direction follows from the fact that $R^*_w \subseteq S'_w$ for any S' such that $S \leq S'$, because if ψ is true everywhere in S'_w, then it is also true everywhere in R^*_w.

Consider the Kripke-style knowledge model $\langle W, S, R^\circ, V \rangle$, with W, S and V the same as in the model based on a hyperrelational frame. Let R° be the relation on W such that $R^\circ_w = R^*_w$. The truth of a formula ϕ at a world w in $\langle W, S, R^\circ, V \rangle$ is defined inductively as follows: if $\phi = \langle K \rangle \psi$, then $\langle W, S, R^\circ, V \rangle, w \models \phi$ if and only if $\langle W, S, R^\circ, V \rangle, w' \models \psi$ for all $w' \in W$ such that $wR^\circ w'$; all other clauses are as expected.

One can prove that, for every $w \in W$ and for every well-formed formula ϕ,

$$\langle W, S, \leq, V \rangle, w \models \phi \Leftrightarrow \langle W, S, R^\circ, V \rangle, w \models \phi.$$

The proof is by induction on the complexity of formulas. The only non-trivial case is the case of $\langle K \rangle \psi$, which can be proved on the basis of the definition of R° and the fact that $\langle W, S, \leq, V \rangle, w \models \langle K \rangle \psi$ if and only if $\langle W, S, \leq, V \rangle, w' \models \psi$ for all w' such that wR^*w'. □

So, if hyperrelational frames satisfy Combination and if they are not infinitary in the sense explained earlier, then models based on those frames are elementarily equivalent to models based on bi-relational frames that are based on the same set of worlds, the same epistemic accessibility relation and the same valuation function. The only difference between the models consists in the fact that the hyperrelational frames contain alternative epistemic accessibility relations S', whereas the bi-relational frame contains just a second epistemic accessibility relation, R, which has been constructed out of the alternative epistemic accessibility relations. The second epistemic accessibility relation of the bi-relational frame can in a sense be seen as that what determines what a second agent knows, although that second agent may be ideal or hypothetical in a way.[10] To put it more colourfully, for every model based on a hyperrelational frame there is a standard Kripke-style knowledge model based on a bi-relational frame 'hidden' in the former such that something is potentially known by some agent at a world according to the first model if and only if it is simply known by some hypothetical agent at that world in the second model. Even more colourfully, the 'hidden' Kripke-style models are the engines that do all the work.

If hyperrelational frames have to be workhorses in their own right, then the solution is to drop Combination or to provide good reasons for why infinitary hyperrelational frames are needed for modelling purposes. However, in the present dialectical context the first option is a non-starter, since $\mathrm{M_p}$ is valid if (5) and $\mathrm{T_p}$ are. Suppose that $\langle K\rangle\phi$ and $\langle K\rangle\psi$ are true. Then by $\mathrm{T_p}$ it is also true that $\phi \wedge \psi$. Hence, by (5) it is true that $\langle K\rangle(\phi \wedge \psi)$. Let us put aside the second option for a moment.

Note that in the model based on a relational frame $\langle K\rangle$ is an ordinary knowledge operator, because its truth clause is based on the epistemic accessibility relation, R. One might as well replace K by K_S and $\langle K\rangle$ by K_R to reflect this fact in the object language. This mere notational change may help to realize two facts about models based on relational frames that are elementarily equivalent to models based on hyperrelational frames that make (5) valid. First, we obtain *omniscience* with respect to K_R: $\phi \rightarrow K_R\phi$. Second, the reason one can block the (relevant variation on) the Church-Fitch argument for omniscience with respect to K_S boils down to the satisfiability of $K_R(\phi \wedge \neg K_S)$, which is fairly trivial. The solution to the paradox is then not a whiff more mysterious than the following:

> Annie knows that something is true but unknown to Peter.

Similarly, Moore (1942) noted that, while an assertion of 'I went to the movies, but I don't believe it' is paradoxical, whereas an assertion of 'I went to the movies, but he doesn't believe it' is not. So, Lemma 2 and Theorem 3 have also a bearing on Fuhrmann's solution to the knowability paradox (Subsection 3.2). That being

[10] Here both accessibility relations are interpreted as epistemic accessibility relations, whereas in Subsection 2.1 one of the accessibility relations is glossed as a modal one and the other accessibility relation is glossed as an epistemic one.

said, there are two qualifications. First, we still have as an assumption that the hyperrelational frames are not infinitary in the sense explained above. Second, the omniscience with respect to K_R and the satisfiability of $K_R\,(\phi \land \neg K_S)$ are to some extent visible only from an outside perspective, even though the models based on relational frames are in some way 'hidden' in the models based on hyperrelational frames. In the next subsection we will make improvements on both fronts.

3.4 The problem of potential omniscience

Fuhrmann (2014a, p. 1646) provides the following frame condition, which is intended to be a sufficient but unneccessary condition for (5):

Perfectibility $\forall w \forall v \exists S'\,(S \leq S' \land (wS'v \to w = v))\,.$

Fuhrmann (2014a, pp. 1645–1646, incl. fn. 17) writes the following about Perfectibility:

> The schema [(5)] is valid in potential knowledge structures satisfying, for example, the condition that knowledge in a world can be *perfected* to an omniscient state. This is to assume that among the potential knowledge states accessible from what is known about a world [w] is one in which *everything* is known about [w] [i.e. Perfectibility] The condition of perfectibility is stronger than what is needed for the validity of [(5)]. We adopt it here for the sake of simplicity. For a sufficient *and* necessary condition we need to switch the order of the quantifiers: Instead of there being a single state in which every truth is known, it suffices (and is also necessary for [(5)]) that for every truth there is a state in which it is known, i.e. [(8)].

Since Fuhrmann explicitly connects Perfectibility with a state of omniscience, it is worth a closer look. For that purpose let us introduce a related but different frame condition:

Perfectibility*

$$\forall w \exists S'\,(S \leq S' \land \forall v\,(wS'v \to w = v))\,.$$

Fuhrmann's Perfectibility condition is a logical consequence of Perfectibility*, because the only difference between the two is that in Perfectibility* the existential quantification over S' comes before the universal quantification over v, whereas in Perfectibility it is the other way around, whence one can choose the same S' for any v. It will now be shown that Perfectibility* is a necessary and sufficient condition for (5), which by the above argument shows that Perfectibility is necessary after all.

Theorem 4. *Fuhrmann's knowability thesis*

$$\phi \to \langle K \rangle \phi$$

corresponds to the Perfectibility condition on hyperrelational frames, i.e.*

$$\forall w \exists S' \left(S \leq S' \wedge \forall v \left(wS'v \rightarrow w = v \right) \right).$$

Proof. Left-to-right: Suppose that $A\phi \rightarrow \Diamond KA\phi$ is valid on a hyperrelational frame. Assume for a *reductio ad absurdum* that

$$\exists w \forall S' \left(S \leq S' \rightarrow \exists v \left(wS'v \wedge w \neq v \right) \right).$$

Now consider a model $\mathcal{M}$ based on the above condition and such that $V(p, w) = 1$ and $V(p, v) = 0$ for all $v \neq w$. The first valuation guarantees that $\mathcal{M}, w \models p$. Next, take any S' such that $S \leq S'$. Then there is a v such that $wS'v$ and $w \neq v$, which entails that $V(p, v) = 0$ and, therefore, $\mathcal{M}, v \not\models p$. By generalisation on S' and the truth clause for $\langle K \rangle$-formulas, one can deduce that $\mathcal{M}, w \not\models \langle K \rangle p$. This contradicts the assumption.

Right-to-left: Suppose that the following condition applies to a class of bi-relational frames:

$$\forall w \exists S' \left(S \leq S' \wedge \forall v \left(wS'v \rightarrow w = v \right) \right).$$

Consider any model $\mathcal{M}$ and any world w such that $\mathcal{M}, w \models \phi$. For this world w it is the case that $\exists S' \left(S \leq S' \wedge \forall v \left(wS'v \rightarrow w = v \right) \right)$. Suppose that there are v such that $wS'v$. For any of those v, $w = v$ and, consequently, $\mathcal{M}, v \models \phi$. Therefore, $\mathcal{M}, w \models \langle K \rangle \phi$. If there are no v such that $wS'v$, then the conclusion follows trivially. □

As is clear from the opening quote of this subsection, Fuhrmann connects Perfectibility with potential omniscience. It is clear that *Perfectibility** does yield potential omniscience. It is not just that, for every truth at a world w, there is a potential knowledge state in which it is known. It is also the case that there is a potential knowledge state in which every truth at w is known. What is conspicuously absent from the opening quote is any insight in how very implausible the perfection of knowledge to omniscience is. The examples at the end of Subsection 2.3 can help to drive this point home. This is the problem of *potential omniscience.*

Compare the problem of potential omniscience with the fact that the bi-relational frame $\langle W, S, R \rangle$ built from a hyperrelational frame $\langle W, S, \leq \rangle$ that makes (5) valid makes $\phi \rightarrow K_R\phi$ valid (Subsection 3.3). First, to prove the existence of a state of potential omniscience in hyperrelational frames that make the knowability thesis valid one does not have to make a detour via other kinds of frames, unlike the fact mentioned above. Second, the problem of potential omniscience shows up even without the assumption that the hyperrelational frames are not infinitary in a particular way, whereas the fact mentioned above is conditional on that assumption.

Theorem 4 can also be used to strengthen the criticism from the 'external' point of view of the 'hidden' Kripke-style models.

Corollary 1. *For every model $\langle W, S, \leq, V\rangle$ that is based on a hyperrelational frame that makes $\phi \rightarrow \langle K\rangle\phi$ valid and that satisfies p-reflexivity, there is a Kripke-style model $\langle W, S, R, V\rangle$, with W, S, V the same as in the model based on a hyperrelational frame and with R a relation in $W \times W$ such that for every $w \in W$:*

1. $\exists S' \left(S \leq S' \wedge S'_w = R_w\right)$;
2. *for every well-formed formula ϕ,*

$$\langle W, S, \leq, V\rangle, w \models \phi \Leftrightarrow \langle W, S, R, V\rangle, w \models \phi.$$

Proof. For any hyperrelational frame $\langle W, S, \leq\rangle$ that makes $\phi \rightarrow \langle K\rangle\phi$ valid and that satisfies p-reflexivity and for any $w \in W$, there is a S' such that $S \leq S'$ and S'_w is finite, namely $\{w\}$. That follows from Theorem 4 and p-reflexivity:

$$\forall w \exists S' \left(S \leq S' \wedge wS'w \wedge \forall v \left(wS'v \rightarrow v = w\right)\right).$$

Moreover, any hyperrelational frame that makes $\phi \rightarrow \langle K\rangle\phi$ valid is a frame that satisfies Combination. Take any S', S'', X, Y such that $S \leq S'$, $S \leq S''$, $S' \subseteq X$, $S'' \subseteq Y$. Note that, given *p*-reflexivity, $\{w\} \subseteq S'$ and $\{w\} \subseteq S''$. Consequently, $\{w\} \subseteq S'_w \cap S''_w$. As we have seen, it follows from Theorem 4 and p-reflexivity that there is a R with $S \leq R$ and $R_w = \{w\}$, whence it follows that $R_w \subseteq S'_w \cap S''_w$. Therefore, $R_w \subseteq X \cap Y$.

Then Lemma 2 and Theorem 3 kick in. □

Again, if one replaces K by K_S and $\langle K\rangle$ by K_R, then we obtain omniscience with respect to K_R and the Church-Fitch argument for omniscience with respect to K_S is blocked in an uninteresting way. But this time no 'anti-infinitary' assumptions have to be made.

4 Conclusion

We have looked at factive concepts of knowability, in particular the notion of having the possiblity to know that something is actually the case (Edgington) and having the potential to know (Fuhrmann). Both notions have been used to circumvent the Church-Fitch paradox of knowability. But even if one can block the possibility of deriving *omniscience*, it does not mean one can block the derivation of *possible or potential omniscience*. Edgington's and Furhmann's knowability theses have this unwanted consequence.

Others who have sought to solve the paradox of knowability by reformulating the knowability thesis with the help of other factive concepts of knowability should take note. This applies to Fara (2010), who advances the concept of actually having the capacity to know that something is actually true, and Spencer (2017), who puts forward the concept of being able to know. Neither of them provides models for their concepts, which is understandable because their primary task was to carve out a new niche in conceptual space. But at some point they need models to show that, even when the particular Church-Fitch argument is blocked, there are no alternative arguments from the knowability thesis to the conclusion of omniscience. And if those models are provided, we now know that there is another task on the list: to check whether the problem of possible or potential omniscience shows up.

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