CHAPTER 1

How to Prove the Consistency of Arithmetic

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1. Has Hilbert's project been shown to be unrealizable?

In the nineteen-twenties, one of the most important developments in the foundations of mathematics was the project by the prominent German mathematician David Hilbert.[1] This program is often called formalistic, and Hilbert's 'formalism' is still routinely listed as one of the three main traditional approaches to the foundations of mathematics, besides logicism and intuitionism. This is nevertheless a seriously oversimplified view. What Hilbert wanted to do was to free the foundations of mathematics from the doubts and uncertainties that had been surfaced partly as a consequence of set-theoretical paradoxes and partly as a result of the criticisms by Brouwer and Brouwer's followers, notably Hermann Weyl.[2] Hilbert proposed to eliminate all these doubts in one fell swoop by proving the consistency of different mathematical theories, in the first place analysis. The success of the gradual process of the arithmetization of analysis in the nineteenth century suggested that the crucial part of this program was to prove the consistency of arithmetic.[3] Needless to say, the consistency proof had to be carried out by means that were not subject to the doubts that affected set theory and—at least according to Brouwer—analysis and even logic. The main stumbling-block for Brouwer in classical mathematics was the use of infinitistic methods, in the first place of the law of excluded middle applied in infinite domains.[4]

So far there is nothing in Hilbert's enterprise that could properly be called formalistic.[5] Indeed, it would be much more apt to call Hilbert an axiomatist.[6] Now the way an axiom system operates is by specifying a class of models. What a mathematician does is to study those models by deriving theorems from the axioms and by establishing metatheorems about the axiom system. Hence the crucial presupposition of mathematical activity was the existence of models, in other words, the model-theoretical consistency of the axiom system. The goal of Hilbert's program was therefore to prove the model-theoretical consistency of different axiom systems.

This Hilbert proposed to do by proving the deductive consistency of the relevant systems. This was supposed to be done by formalizing the axiom systems and the logic they used and by then showing purely combinatorially ('formally')[7] that the formalized rules of inference could not lead to inconsistencies. It is this tactic that has given the misleading impression that Hilbert was a 'formalist'. Now according to the usual rules of logic, if any inconsistent proposition is provable, then any proposition is provable and any proposition is disprovable. Hence it suffices to show the existence of a single proposition that is not disprovable in order to show the deductive consistency of the system in question.

In 1931 Kurt Gödel demonstrated that the formal consistency of as basic a mathematical theory as elementary arithmetic cannot be proved by means of elementary arithmetic itself. This proof was a by-product of Gödel's famous incompleteness theorem. This theorem says that if any formal axiom system AX of elementary arithmetic is consistent, then there are propositions G that are true but unprovable in AX. Gödel's proof thus assumed that AX is consistent. As Gödel (and John von Neumann)[8] quickly realized, his proof could be formulated in AX itself. Hence, if the consistency of AX could be proved in AX, G could likewise be proved in AX, which would contradict its unprovability.

Gödel's result, known as his second incompleteness theorem, has virtually universally been taken to imply a failure of Hilbert's program in its original form.[9] This construal of the consequences of Gödel's results is nevertheless mistaken. For one thing, it will be shown in this paper that the consistency of a suitable first-order system of elementary arithmetic can be proved in the same system. As a consequence, we have to reconsider the entire question of the prospects of Hilbert's program.

This claim might easily seem puzzling and even paradoxical. Gödel's second incompleteness theorem is a valid metatheoretical result. In view of its indisputable validity, it seems hopeless to try to get around it. Some of the sources of this puzzlement will be examined and dispelled in this essay. The most general perspective in any case is easily explained. Gödel's argument does not rest on any assumptions that can be challenged in any literal sense. However, the framework Gödel is operating in, including the logic used in one's arithmetic, is not the only possible one, and is turning out not to be the happiest one, either. Gödel assumes that the logic that is used in his elementary arithmetic is the ordinary first-order logic. Or, perhaps, we should rather speak of the received Frege-Russell logic of quantifiers of any order, for the first-order part was separated from it only slowly under Hilbert's influence.[10] It has by now turned out that this logic is too weak in its expressive power, and has to be replaced by a logic that is richer in this respect.[11] This new logic is known as the independence-friendly (IF) first-order logic. Since it is in certain respects richer than ordinary first-order logic, we cannot exclude, sight unseen, the possibility that, when it is used as a basis of elementary (first-order) arithmetic, the consistency of this arithmetic should become provable in the same arithmetic (with the same stronger logic, of course). Such a resurrection of Hilbert's project would thus be a part of a general revolution in the foundations of logic and mathematics brought about by IF logic.[12] Or perhaps the term 'revolution' is inappropriate. Unfortunately, it is not easy to find an appropriate term, even a metaphoric one, that would capture the double development of deepening the foundations of logic and ipso facto extending the range of its applications.

However, it is far from obvious that enriching one's basic logic facilitates a Hilbert-style project. Three prima facie reasons seem to discourage rather than encourage any attempt to salvage Hilbert's project with the help of a richer logic. For one thing, enriching the underlying logic makes one's arithmetic stronger and hence more prone to contradictions than before, and presumably by the same token more difficult to prove consistent.

A closely related indication of the difficulties here is the fact that Gödel's incompleteness proofs apparently turn on reasoning closely related to the liar paradox, even though in Gödel's hands it results in an incompleteness result rather than a contradiction. Enriching the underlying logic thus might, sight unseen, heighten the danger of liar-type paradoxes.

Most importantly, it seems to become more difficult to carry out Hilbert's strategy of proving the semantical (model-theoretical, intuitive) consistency of a formal system of elementary arithmetic by proving its deductive (syntactical, formal) consistency. This strategy apparently presupposes that the formalized logic is semantically complete in the sense that every logical truth in the semantical (model-theoretical) sense can be formally proved in it. For otherwise hidden contradictions may lurk at the bottom of one's axiom system that cannot be brought out to the open by the deductive methods used. The received first-order logic was proved complete in this sense by Gödel in his 1930 dissertation. But IF first-order logic is known to be semantically incomplete.[13] The set of its valid formulas is not recursively enumerable in the sense in which fully formalized rules of inference effect an enumeration of the theorems that can be proved by their means from formalized axioms. Hence at first sight a reliance on IF logic might seem to make consistency proofs in Hilbert's style totally impracticable.

2. Consistency proofs by means of IF logic

This pessimistic impression is nevertheless unfounded. The reasoning used to motivate the pessimism uses a tacit appeal to the law of excluded middle. The question is whether contradictory conclusion can be derived in an axiom system. If we assume *tertium non datur*, then we can show that a proposition C is inconsistent if and only if we can prove its contradictory negation $\neg$C. But there is no contradictory negation in (unextended) IF first-order logic.[14] Hence for an axiom system using IF logic we must have available a disproof procedure over and above a proof procedure. And for Hilbertian purposes what is needed is indeed not a complete *proof procedure* in the sense of complete method of demonstrating logical truth. What is needed is a complete *disproof procedure*, that is, a recursive enumeration of contradictory formulas. For if we have such a method and if we can show that not everything can be disproved in the axiom system by this method, then we can conclude in virtue of its completeness that the axiom system is consistent also in the model-theoretical sense. And since there is no *tertium non datur* in IF first-order logic, it is perfectly possible to have complete disproof procedure without a complete proof procedure. Indeed, a small modification in the well-known tree method yields such a complete disproof procedure for IF first-order logic.[15] And that is all that is in principle needed for Hilbert's project.

But can we show that this disproof procedure (as used in a suitable elementary arithmetic based on IF logic) cannot disprove every proposition? Answering this question becomes a little easier to formulate if we first extend IF logic by allowing sentence-initial contradictory negation $\neg$ into the language. The resulting logic is called extended IF logic. Within it there obtains a duality (symmetry) between IF sentences and their contradictory negations. For the former, there exists a complete disproof procedure but not a complete proof procedure. For the latter, there exists a complete proof procedure but not a complete disproof procedure.

What can be done in an elementary arithmetic based on extended IF logic can now be seen by a comparison with Gödel's original incompleteness argument. Gödel constructs a predicate $\mathrm{prov}[x]$ in a self-applied number theory which expresses the provability of the sentence S with the Gödel number $g(S) = x$. Then he applies a diagonal argument to the predicate $\neg\mathrm{prov}[x]$ to find a sentence

$$(1) \qquad \neg\mathrm{prov}[\mathbf{n}]$$

with the Gödel number n such that $\mathbf{n}$ is the numeral expressing n. In familiar but somewhat misleading terms (1) says 'I am not provable'. Now if (1) is false the sentence with the Gödel number n is provable. But if the system of number theory is consistent in the strong sense that whatever is provable is true, then (1) is true. This contradicts the assumption of its falsity, wherefore it must be true. Consequently, it is not provable, because that is what it says.

Likewise, we can in our number theory form the IF predicate $\mathrm{disp}[x]$ which says that the sentence with the Gödel number x is disprovable. A diagonal argument then produces a sentence of the form

$$(2) \qquad \mathrm{disp}[\mathbf{n}]$$

where $\mathbf{n}$ is the numeral representing n and where the Gödel number of (2) is n. If (2) is true, it is disprovable. Assuming that all disprovable sentences are false, (2) is then false. (We shall call this assumption the soundness of the disproof procedure.) This contradicts the hypothesis that it is true and shows that (2) is not true. This means that the sentence with the Gödel number n is not disprovable. Hence there is at least one sentence that is not disprovable, which was to be proved.

This argument does not depend on the consistency of the system of arithmetic in question in the sense that all provable sentences should be true. Instead, what is required is the soundness of the disproof procedure, that is, the assumption that all disprovable sentences are false. This can be proved by induction on the length of the disproof in the elementary arithmetic itself.

There are variants to this argument. For instance, instead of (2) we might consider

$$\neg \mathrm{disp}[\mathbf{n}] \supset \mathrm{true}[\mathbf{n}], \tag{3}$$

where n is the Gödel number of (3) and $\mathbf{n}$ the numeral representing n. This is possible, for a truth predicate $\mathrm{true}[x]$ can be defined in the extended IF logic. Here (3) is equivalent to

$$\mathrm{disp}[\mathbf{n}] \vee \mathrm{true}[\mathbf{n}]. \tag{4}$$

Now (4) is either disprovable or not. If it is assumed to be disprovable, then both its disjuncts are disprovable, including the first one. Assuming soundness this disjunct is hence false, which contradicts the assumption. Hence (4) is not disprovable, proving the deductive consistency. Accordingly, (4) is either true or neither true nor false.

This shows the deductive consistency of our IF elementary number theory, assuming the soundness of our disproof procedure. But in what system can this consistency be demonstrated? Ultimately, we would like it to be demonstrated in the IF-logic-based number theory itself. Now the soundness that was assumed in our argument can be demonstrated in elementary number theory. For instance, we can assume that the disproof procedure uses the tree method, and then carry out the argument by induction on the length of the tree procedure.

We are making some progress here. For in the original Gödel case, he had to assume that the system of number theory (including the proof procedure it uses) that he was using is consistent in the sense that each provable sentence is true. For in his argument we had to show that if the critical sentence $\neg\mathrm{prov}[\mathbf{n}]$ is false and $\mathrm{prov}[\mathbf{n}]$ is therefore true, then the sentence with the Gödel number $\mathbf{n}$ is in fact true. This presupposes both consistency and the law of excluded middle. Otherwise we cannot eliminate the possibility that the critical sentence is false. In contrast, we now have to assume only the soundness of the disproof procedure in the sense that each disprovable sentence is false, which does not depend on the *tertium non datur* and is hence possible to prove elementarily, at least if we can assume that it is the use of *tertium non datur* that makes an argument nonelementary.[16]

This turns out to be a crucial advance. For in the same way as in the case of Gödelian incompleteness proof, the argument presented above about (2) can be carried out in elementary number theory itself. Since it does not involve the law of excluded middle, it can even be carried out in IF logic. Since in the extended IF logic there is a complete proof method for contradictory negations of IF sentences, $\neg\mathrm{disp}[\mathbf{n}]$ can be proved formally in the extended IF logic. This shows that the deductive consistency of the kind of elementary number theory we are considering can be proved in that number theory itself.

3. From deductive consistency to model-theoretical consistency

But there is a major catch here. What can be proved in this way is the deductive consistency of the number theory in question in the sense that the disproof procedure does not refute all formulas. But for Hilbert *deductive* consistency was not an end but a means. It was a step in his attempted proof of *model-theoretical* consistency. For this purpose, we would have to prove not only the soundness but also the model-theoretical completeness (usually known as semantical completeness) of our disproof procedure. We would have to prove that if a sentence of our number theory is not disprovable, it has a model. Moreover, and crucially, such a proof must be conducted in our elementary number theory itself.

Now it is immediately obvious that such a proof cannot possibly be carried out in an elementary number theory based on conventional first-order logic. For can a completeness proof be carried out there? A typical proof, for instance a proof using the tree method, relies essentially on König's lemma. Now what does König's lemma say? It says that if a tree branches finitely, then it is either finite or has an infinite branch. Or, to put the same theorem differently, either there is an infinitely branching node or there is an upper bound to the length of branches or there is an infinite branch. Now the notion of infinity cannot be expressed in the received first-order logic, and accordingly König's lemma cannot be expressed in an elementary number theory based on such a logic. Indeed, appeals to König's lemma are often thought of as being the source of the infinitary character of completeness proofs for first-order logic. But infinity can be expressed in IF logic. For instance, an infinite number of individuals satisfies a non-empty predicate $A(x)$ if and only if the following is true:

(5) $(\forall x)(\forall y)((A(x) \wedge A(y)) \supset (\exists z/\forall y)(\exists u/\forall x)(x \neq z \wedge y \neq u \wedge A(z) \wedge A(u) \wedge ((x = y) \leftrightarrow (z = u))))$.

By way of explanation, it may be pointed out that in (5), z is a function of x alone and u of y. Furthermore, if $x = y$, then $z = u$. Hence z is the same function f of x as u is of y, where f satisfies

$$(6) \quad (\forall x)(\forall y)(x \neq f(x) \wedge (A(x) \supset A(f(x)) \wedge ((x = y) \leftrightarrow (f(x) = f(y)))).$$

Hence for any a satisfying $A(a)$, $a, f(a), f(f(a)), \ldots$ are all different individuals.

Since the existence of an upper bound on the lengths of branches is easily expressible in IF first-order logic, the entire König's lemma can be so expressed. It is a known truth of IF logic, and can be assumed to be one of the deductive axioms of our elementary number theory. And if so, the completeness of the disproof procedure that IF logic yields can be proved in our elementary number theory.

In order to show in what a palpable sense this shows the existence of models for sentences that cannot be disproved, the following observations can be made. What König's lemma implies when applied to the attempted model set construction that the tree method is, is that if the procedure does not yield a disproof, there exists an infinite branch which is a model set containing the sentence under scrutiny. By using the clever idea of Henkin's (1949), we can interpret model sets (as sets of symbol combinations) as being their own models in the most concrete sense possible, of course modulo isomorphism.[17]

Thus we can prove by means of IF logic that proof-theoretical consistency of a sentence S implies the existence of a model in which S is not false. Applied to our modified Gödel formula (1), we can conclude that it has a model in which it is not false. And an argument to this effect can be carried out in our IF elementary number theory, which can therefore be proved model-theoretically consistent. The proof actually gives a recipe for constructing a concrete model for the specific sentence (1) albeit only as a limit of an infinite construction. This model consists of Hilbert's preferred building-blocks for such models, symbolic expressions. This completes a significant part of the task that Hilbert took on in his project.

In a sufficiently broad sense of Hilbert's second problem in his 1900 list of important open mathematical problems, what has been found amounts to a solution of this problem. What needs to be argued for is that this sense of solution is one that he would have (or at least should have) approved of.

4. Is non-falsity enough for Hilbertian consistency?

In trying to do so, we are still faced with several problems. For one thing, elementary number theory based on IF logic is likely to be considered by many philosophers and logicians far too strong to be elementary. Even an elementary number theory based on the received first-order logic might at first sight seem dangerously strong to sundry intuitionists and constructivists. And a switch to IF logic as the foundation of number theory might seem even more dubious. Can IF logic even be considered first-order logic in the last analysis? Indeed, IF logic might easily seem to be far too strong to have any claim to the status of an elementary logic in a philosophically respectable sense. Quine has been suspicious of higher-order logic, including second-order logic, calling it 'set theory in sheep's clothing'. Several logicians have in effect tried to brand IF logic a second-order logic in disguise undeterred that it cannot be that by the only clear criterion of first-order logic, namely, the categorial status of the entities quantified over.[18] Indeed, IF first-order logic is equivalent with the Σ^1_1-fragment of second-order logic.[19]

These objections can be dealt with, even though a fully adequate discussion would have to be too extensive to be carried out in one paper.[20] It will have to suffice here to point out the most important indications of the elementary character of IF logic. First and foremost, IF first-order logic is a *first-order* logic in the crucial sense that all quantification is over individuals. It is hence independent of all problems concerning the existence of higher-order entities, including sets. This means that one important desideratum of Hilbert's is satisfied (cf. Hilbert 1922).

Second, even though IF first-order logic has a much greater expressive capacity than the received first-order logic, it is weaker deductively when the negation-sign is taken to express contradictory negation. This sense can be explained by considering the extended IF logic obtained by adding a contradictory negation $\neg$ to IF logic over and above its regular (strong) negation $\sim$. Then a true sentence may become false when $\neg$ is replaced in it by $\sim$. The simplest case is constituted by sentences of the form $(S \vee \neg S)$.

This is highly significant, for the crucial source of nonelementarity for Hilbert as for Brouwer was the principle of *tertium non datur*. This principle is not assumed in IF logic.[21] As a result, IF first-order logic is closely related to intuitionistic logic, even though their precise relation remains to be examined. This relationship becomes especially close when we allow the primitive nonlogical predicates of an IF first-order language to have truth-value gaps.[22] It will not be studied here, however.

Here another important question will be discussed instead. Would our proof have really satisfied Hilbert substantially? It might be objected that all that has been done is to show that in our IF elementary number theory there are sentences which have models in which they are not false. We have not shown how to find sentences true in some models. Would this have satisfied Hilbert? The answer is fairly clearly yes. Hilbert wanted an axiom system to have models which a mathematician can explore. Does this not require that there must be true sentences about them? Yet all that has been strictly proved here is that there are sentences with models in which they are not false. In order to see whether this makes any difference we have to go back to the nature of game-theoretical semantics. What does the truth of a quantificational sentence S mean? It means that the initial verifier has a winning strategy in the correlated game G(S). This is a concrete combinatorial (some people would misleadingly say, set-theoretical)[23] property of the model W in which S is being evaluated and on which G(S) is supposed to be played. This combinatorial fact is tantamount to the existence of certain functions on W. Likewise, the fact that S is false means that the initial falsifier has a winning strategy in G(S), and is an objective fact about W, in other words, it amounts to another combinatorial fact about W.

By the same token, knowing that S is not false amounts to knowledge about the combinatorial structure of W. It is tantamount to knowing that certain kinds of functions do not exist on W. There is no reason whatsoever why Hilbert or anybody else should not consider establishing such a fact about W as a part of the axiomatic exploration of the model in question. Hence the fact that the result reached above only shows that the critical sentence is not false in the model that a failed attempt at a disproof yields (in the limit) does not cast any shadow on the concrete meaning of the consistency proof envisaged here. An example is offered by the logical study of the foundations of quantum theory where it turns out that this substantial (informative) role of the third ('indefinite') truth-value has important interpretational consequences.[24] Perhaps it would be wisest not to call the third truth-value 'indefinite' but to call it instead the intermediate truth-value.

Hence the consistency proof that has been carried out here for IF elementary number theory shows that this theory is model-theoretically consistent in the relevant sense, namely in the sense that there exists a model for it about which substantial facts can be discovered. The robustness of such a model can be seen for instance from the tree method in which a logician ostensively trying to disprove in fact literally constructs for it a model (or at least an isomorphic replica of a model) à la Henkin step by step (cf. Henkin 1949). This fulfills Hilbert's aims concerning arithmetic in spirit and in letter as fully as anyone can hope. The crucial objective for him was all along the existence of models to be examined, not what we initially know about them.[25]

One can also relate the result reached here to later developments. The notion of 'not false' which has been used and in the sense of which the existence of a model in which a specific proposition is not false has another name. It is what is called truth on the no-counterexample interpretation.[26] The important role which this interpretation plays in logicians' theorizing is a testimony to the significance of this notion of 'not true' for actual mathematical practice.

Hence there are good reasons to consider the argument given above as a solution to Hilbert's second problem. This does not suffice to carry out Hilbert's overall program, however, which included a consistency proof for analysis. It turns out that a much stronger logic than first-order IF logic is needed for the whole of analysis and that the kind of consistency proof presented in this paper does not work there. It remains to be studied

whether, and if so in what sense, Hilbert's program can be carried
entirety.

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Notes

1. Hilbert expressed his views in a series of addresses dating 1921 (Hilbert, 1922), 1925 (Hilbert, 1926), 1927 (Hilbert, 1928), 1928 (Hilbert, 1929), among which Hilbert (1926) is usually taken to be the most comprehensive presentation of what is known today as Hilbert's program. Yet the origin of Hilbert's program can be traced back to Hilbert's earlier work in the axiomatic tradition, especially *Grundlagen der Geometrie* (1899). In 1905 Hilbert gave a series of lectures on the axiomatic method and its application to arithmetic. He developed his ideas through the 1910s. In 1918 his 'Axiomatische Denken' was published. In that paper Hilbert emphasizes two main topics to be studied further, namely the independence and consistency of the axioms in an axiomatized mathematical theory. After 1920 Hilbert's axiomatic project grew as a more extensive research program on these two topics; especially on the latter.
2. In 1920 Hermann Weyl joined the intuitionists. After the conversion of his best student, Hilbert's reaction against Brouwerian intuitionism took a more determinate form. In 1922 Hilbert started to arm his program and criticize Brouwer and Weyl with a stronger voice and motivation: "[Weyl and Brouwer] seek to save mathematics by throwing overboard all that is troublesome. ... They would chop up and mangle the science. If we would follow such a reform as the one they suggest, we would run the risk of losing a great part of our most valuable treasures!" (Reid, 1970, p. 155). In his words, Hilbert's metamathematics was proposed to "safeguard [mathematics] by protecting it from the terror of unnecessary prohibitions [arguably Brouwer's] as well as from the difficulty of paradoxes" (Hilbert, 1922, p. 212). It turns out, however, that a consistency proof for elementary arithmetic does not amount to a consistency proof for the whole of analysis, only to significant parts of it.
3. As Hilbert saw the situation in his 1926 paper, Weierstrass' arithmetization freed analysis from all sorts of vague methods and created a firm foundation for it. However, for Hilbert this did not bring foundational issues to an end. The concept of the infinite was still in need of clarification. Hilbert hoped to take care of this problem by means of consistency proofs, among which proof of the consistency of arithmetical axioms was a major one. The appeal to arithmetic in geometry created the need for a consistency proof for the arithmetical axioms. This can be seen as the next big step of Hilbert's overall axiomatic project first of which is his *Grundlagen der Geometrie* where he gave a consistency proof of the axioms of geometry. Zach (2003) gives a firm presentation of the historical development of consistency proofs in Hilbert's program.
4. Brouwer's thesis, as he summarizes it, is that "[the] axiom of *solvability of all problems* as formulated by Hilbert in 1900[4] is equivalent to the logical Principle of the Excluded Middle; therefore, since there are no sufficient grounds for this axiom and since logic is based on mathematics—and not vice versa—the use of the Principle of the Excluded Middle is *not permissible* as part of a mathematical

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proof. The Principle of the Excluded Middle has only a scholastic and heuristic value, so that theorems that in their proof cannot avoid the use of this principle lack all mathematical content" (Brouwer, 1921, p. 23). He adds in a footnote (n. 4): "In my opinion both the axiom of solvability and the Principle of the Excluded Middle are *incorrect*; they are dogmas that have their origin in the practice of first abstracting the system of classical logic from the mathematics of subsets of a definite set, and then attributing to this system an a priori existence independent of mathematics, and finally applying it wrongly—on the basis of its reputed a priori nature—to the mathematics of the infinite sets" (Brouwer, 1921, p. 27). These formulations by Brouwer show that the logic he had in mind was an epistemic logic of mathematics. However, quite apart from the epistemic angle, the use of *tertium non datur* turns out to be an important watershed in the logical foundations of mathematics, as emerges particularly clearly from (Hintikka, forthcoming, b).

5. For a perceptive discussion of Hilbert's project that is free from the false attribution of formalism to him, see Kreisel (1983).

6. In particular, Brouwer's criticism of Hilbert's alleged 'formalism' is misleading in that Hilbert never claims that mathematical activity is (or can be interpreted as being) restricted to the manipulation of formal symbols. What Hilbertian 'formalization' amounts to is a reduction of all derivation of theorems from axioms to purely logical deduction. Such deduction is formal only in the innocent sense of being independent of the interpretation of the basic concepts of the axiom system. (Cf. Hilbert & Bernays 1934, pp. 1–5.) By the same token, all the proofs are independent of the domain of objects (universe of discourse) that is being considered, which implies that we might for instance think in terms of domains consistency of formal sign uses and other expressions. This is eminently compatible with Hilbert's insistence that the choice of axioms is guided by the intuitive content of the concepts involved. Thus in Hilbert & Bernays (1934), Vol. 1, two kinds of axiomatics are discussed: (i) formal axiomatics, and (ii) contentual axiomatics. As Hilbert and Bernays observe, "Formal axiomatics requires contentual axiomatics as a supplement, because only in terms of this supplement can one give instruction in the choice of formalisms and, moreover, in the case of a given formal theory, give an instruction of its applicability to some domain of reality. On the other hand we cannot just stay at the level of contentual axiomatics, since in science we are if not always, so nevertheless predominantly concerned with such theories that get their significance from a simplifying idealization of an actual state of affairs rather than from a complete reproduction of it". Hilbert (1922, p. 212) describes the practice of his metamathematics as follows: "In contrast to the purely formal modes of inference in mathematics proper—we apply contentual inference; in particular, to the proof of the consistency of axioms".

7. Hilbert had the idea of using the combinatorial properties of a completely formal language as the basis of his foundational approach. Arguably, this supports a realistic conception of truth and is not formalistic in any anti-realistic sense. Yet the sense of 'combinatorial' comes close to first-order reasoning, which involves quantification only over individuals; hence sets and general concepts are excluded from the logic of mathematics. (For more on this point, see Hintikka (1996, Chapter 9; esp. pp. 199–202). Also see Hintikka (1997b).)

8. For an account of what happened when Gödel first announced his first incompleteness theorem, see Dawson (1997, pp. 69–71).

9. Although Gödel's incompleteness results led some logicians (such as von Neumann) to give up hope about Hilbert's program, Gödel himself thought that his proof did not contradict Hilbert's metamathematics. His remarks on the implications of his theorem are worth noting here. Reid quotes him: "Viewing the situation from a purely *mathematical* point of view, consistency proofs on the basis of suitably chosen stronger metamathematical presuppositions ... are just as interesting, and they lead to highly important insights into the proof theoretic structure of mathematics. Moreover, the question remains open, whether, or to what extent, it is possible, on the basis of the formalistic approach, to prove 'constructively' the consistency of classical mathematics" (Reid, 1970, p. 199). Here note also that suitably chosen metamathematical presuppositions possibly bring in stronger semantic presuppositions, which may translate what Gödel refers as "purely mathematical point of view" to the semantical plane. This is, for example, the case in game-theoretical semantics and IF logic.
10. The separation and development of first-order logic under Hilbert's influence finds its definitive shape in Hilbert & Ackermann (1928), *Grundzüge der theoretischen Logik.*
11. See Hintikka (1996, Chapter 9), for further discussion.
12. Hintikka (1997a) gives a brief overview of this revolution. For a comprehensive study of the different aspects of it, see Hintikka (1996).
13. IF logic is semantically incomplete in the sense that it does not admit a complete axiomatization of all its truths. See Hintikka (1996, pp. 65–68 and Chapters 5 and 7), for a discussion of the reasons and implications of such incompleteness.
14. Contradictory negation expresses the non-existence of a winning strategy for the verifier in a semantic game. In the unextended IF logic, it is used only in front of an entire formula, and is not used inside any formula, i.e. there are no game rules for the contradictory negation. See Hintikka (1996, Chapter 7), and Hintikka, forthcoming (b) for further investigation of the issue.
15. The existence of a complete disproof procedure is seen from the fact that in the tree construction of a model set we do not need the law of excluded middle.
16. By the same token, we can have a complete proof procedure for the contradictorily negated sentences of the extended IF logic—extended, that is to say, by adding the basic IF logic a sentence-initial contradictory negation.
17. See note 6 above. In his completeness proof for first-order logic, Leon Henkin used as models for certain kinds of formulas (symbol combinations) those very same sets of symbols themselves. Later Hintikka and Smullyan generalized Henkin's argument. Henkin's idea obviously belongs to the same ballpark as the idea of Gödel numbering and indeed Hilbert's formalization of metamathematics. The leading idea is that a correct symbolism constitutes an isomorphic replica of what it represents. Hilbert was maintaining that the only thing that matters in an axiom system is the structure it imposes on its models. When this idea is combined with the assumption of an isomorphim between language (symbolism) and what it represents, there cannot be any objection to using the expressions of a language as their own models, any more than there according to Hilbert can be any objection to imagining that the models of the axioms of geometry consists of tables, chairs and beermugs.
18. Admittedly, other criteria have been proposed and presupposed in the literature. See the illuminating paper by Väänänen (2001).
19. See Hintikka (1996, pp. 61–63).

20. More has to be said—and even to be figured out—concerning the relation of IF first-order logic to higher-order logics. The basic new perspective is sketched in Hintikka, forthcoming (b).

21. This point is easily understandable from the point of view of game-theoretical semantics. There *tertium non datur* amounts to the requirement that semantical games be determinate. Even though that happens to be the case with games associated with the sentences of the received first-order logic, determinacy cannot be expected to hold in general. See Hintikka (1996, Chapter 7), for more discussion.

22. Cf. Hintikka, forthcoming (a).

23. See here Hintikka, forthcoming (b).

24. See here Hintikka, forthcoming (b).

25. Here Brouwer's rather epistemic approach seems to fit into an objective which is interested in and gives the priority to what we initially know about the models to be examined (or, whatever content of mathematics is considered).

26. Kreisel introduced the no-counterexample interpretation in his 1951 and 1952 papers, and developed the idea in Kreisel (1958). For a brief discussion of the crucial aspects of Kreisel's interpretation see Feferman (1996).

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