When Structural Principles Hold Merely Locally

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Abstract: In substructural logics, structural principles may hold in some fragments of a consequence relation without holding globally. I look at this phenomenon in my preferred substructural logic, in which Weakening and Cut fail but which is supra-intuitionistic. I introduce object language operators that keep track of the admissibility of Weakening and of intuitionistic implications. I end with some ideas about local transitivity.

Keywords: Structural Principles, Nonmonotonic Logic, Nontransitive Logic, Logical Expressivism, Inferentialism, Atomic Systems

1 Introduction

Consequence relations are traditionally thought to obey the following—so-called structural—principles:[2]

WEAKENING	If $\Gamma \vdash A$ and $\Delta \supseteq \Gamma$, then $\Delta \vdash A$.
CUT	If $\Gamma \vdash A$ and $\Gamma, A \vdash B$, then $\Gamma \vdash B$.
CONTRACTION	If $\Gamma, A, A \vdash B$, then $\Gamma, A \vdash B$.
PERMUTATION	If $\Gamma, A, B, \Delta \vdash C$, then $\Gamma, B, A, \Delta \vdash C$.
REFLEXIVITY	If $A \in \Gamma$, then $\Gamma \vdash A$.

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[2]The first two are also known as (i) "monotonicity" or "monotony" or "MO," and (ii) "cumulative transitivity" or "CT." Reflexivity is also known as "identity" or "containment" or "CO."

Substructural logics are logics in which at least one of these five conditions fails. A nonmonotonic logic (denoted by "$\mathrel{|\!\sim}$") is one in which Weakening fails. Even in a nonmonotonic logic, however, there can be a set Θ such that for all $\Delta \supseteq \Theta$ we have $\Delta \mathrel{|\!\sim} A$. In such a case, we may say that Θ implies A "monotonically" or "persistently." We can think of this as monotonicity holding locally at the sequent $\Theta \mathrel{|\!\sim} A$.

Suppose we could say in the object language that A is a monotonic consequence of Θ, e.g., by having an operator $\mathfrak{M}$ such that Θ implies $\mathfrak{M}A$ iff Θ persistently implies A. We could then view monotonic logic as merely a restricted perspective on a shared topic: Monotonic logic looks at the turnstile as if it were always followed by a silent $\mathfrak{M}$. Such an operator lets us say what the monotonic logician is getting right and which consequences she simply ignores. Analogous ideas apply to the other structural rules.

When we look at the matter in this way, a new task for logicians becomes visible, namely the task of investigating and developing the expressive resources that are needed to think about local structural features. If logicians study and develop such expressive resources, they give us the expressive tools to be explicit about the structural features that we think our inferences obey and to disagree about whether we are right about this in a particular case. Acquiring these expressive resources will allow us to see structural logics as being blind to everything but a very special class of consequences and lacking the expressive power to see how special these consequences really are.

These ideas bear some resemblance, e.g., to Girard's (1987) recovery of intuitionistic logic within linear logic. In linear logic Weakening and Contraction fail. Girard introduces an operator "!" (shriek) that, in effect, allows applications of Weakening and Contraction. You can weaken any sequent with $!A$ on the left, and you can contract $!A\,!A$ to $!A$ on the left. With this operator in hand, Girard can define a translation function, θ, such that $A_1, ..., A_n \vdash_{int} B$ holds in intuitionistic logic just in case $!\theta(A_1), ..., !\theta(A_n) \vdash_{linear} \theta(B)$ holds in linear logic. Thus, we have an operator that allows us to use the structural rules that distinguish linear logic from intuitionistic logic in a controlled fashion, and we can thus use the operator to recover intuitionistic logic.

There are three main differences between Girard's recovery of intuitionistic logic and my aims in the remainder of this paper. First and most importantly, I am looking for ways of making something explicit that is there anyway. I start with a material consequence relation over atomic sentences, and I want to keep track of the structural principles that hold locally in this material consequence relation and its logical extension. I don't want to simply create a new part of the language in which certain structural principles hold by introducing a new operator. That is what Girard does by allowing us to weaken any sequent with $!A$ irrespective of the weakening behavior of the sequent before the introduction of shriek. In contrast to this, I want to find ways to express facts about where structural principles hold locally even before we introduce any new expressions. Second, the structural principles with which I will be concerned are Weakening and Cut, rather than Weakening and Contraction. Third, the substructural logic I shall work with is supra-intuitionistic. Hence, the problem of "recovering" intuitionistic logic is really just the problem of singling out all and only the implications that this logic shares with intuitionistic logic.

The paper is structured as follows: In Section 2, I introduce the substructural logic with which I will be working. I then show how to introduce an operator that says, in the object language, that Weakening holds at a sequent (Section 3). In Section 4, I introduce an object language expression that allows us to keep track of intuitionistic implications. I end, in Section 5, with some ideas about the local admissibility of Cut.

2 A Nonmonotonic and Nontransitive Logic

In this section, I introduce the reader to the substructural logic within which I want to keep track of the admissibility of some structural principles and of intuitionistic logic. I will do so very briefly, as I have laid out a very similar logic in more detail elsewhere (Hlobil, 2016).

2.1 Philosophical Background

The logic that I will be concerned with is motivated by semantic inferentialism, i.e. the view (very roughly) that the meaning of a sentence is settled once it is settled what follows from it, what it follows from, and what is incompatible with it.[3] For our current purposes, we can think of inferentialism as the view that the meaning of a sentence is given by its place in a consequence relation and incoherence property defined over the language to which it belongs.

[3] I am ignoring here how causal relations—especially perception and action—play a role in the fixation of meaning. Of course, a plausible semantic inferentialism must ultimately include what Sellars (1954) calls "language-entry transitions" and "language-exit transitions."

Inferentialism is a general claim about content, not just a claim about the meaning of logical vocabulary (Brandom, 1994; Peregrin, 2014). The claim applies to atomic sentences just as much as it applies to logically complex sentences. Now, formal consequence relations are closed under uniform substitutions of atomic sentences. In other words, formal consequence relations assign to all atomic sentences symmetrical inferential roles. Any set of atoms implies all its members and no other atoms. These only minimally different inferential roles can hardly suffice to confer genuinely distinct meanings onto the different atomic sentences. So if we want to capture the inferential roles of atomic sentences, we must reject the idea that all consequence relations are formal. Rather, we must allow that a set of atomic sentences implies another atomic sentence that is not in the set.

Once we accept such material implications, it becomes natural to think of logic as extending a given material consequence relation (and incoherence property) to logically complex sentences.[4] Logic introduces logical vocabulary into an atomic language whose sentences already stand in rich inferential relations.

Now, it seems that the inferential and incompatibility relations between atomic sentences are virtually always defeasible. After all, we can normally infer that the streets are wet from the fact that it is raining, but it can happen that the authorities have covered up the streets to protect it from the rain, etc. Hence, the consequence relation over the atomic language with which we begin must be nonmonotonic. Given that a logic should extend this relation in a conservative fashion, the consequence relation that results from introducing logical vocabulary must also be nonmonotonic.

At this point a second motivating idea behind this logic becomes relevant: logical expressivism (Brandom, 2008). According to logical expressivism, it is the expressive job of logical vocabulary to allow us to express claims about the inferential and incoherence roles of non-logical expressions (where the "about" must not be understood in a representationalist fashion). The paradigm case is the conditional. According to logical expressivism, the conditional lets us claim that the antecedent implies the consequent (in a given context). If we spell this out in a non-representationalist way, we should say something like this: You have reasons to assert a conditional just in case these reasons together with the antecedent are reasons to assert the

[4]Similar ideas have recently been discussed under the heading of "atomic systems" or "atomic bases" and extensions thereof (Sandqvist, 2015). The present approach differs from familiar approaches in that it doesn't enforce transitivity in the atomic base. Thus, we reject what is sometimes called "definitional reflection."

consequent. In other words, a deduction-detachment theorem (DDT) holds:

DDT $\qquad \Gamma \mid\!\sim A \rightarrow B$ iff $\Gamma, A \mid\!\sim B.$

This is a substantive constraint on the inferential behavior of the conditional. It is justified by the claim that it is the job of the conditional to let us express what follows from what. The conditional makes explicit the consequence relation. Together with Reflexivity, DDT implies that our logic must be nontransitive. For a reflexive consequence relation in which DDT holds and that obeys Cut must be monotonic.[5]

The upshot of all this is that if you accept the inferentialism and expressivism I am endorsing, then you need rules for introducing logical connectives that extend a nonmonotonic, material consequence relation in such a way that the resulting consequence relation is not only nonmonotonic but also nontransitive and obeys DDT. Furthermore, the rules governing the logical connectives should be plausible, in the sense that they can naturally be thought to capture the meanings of the logical connectives.

2.2 A Sequent Calculus

In order to construct a system that has the properties just mentioned, we start with a consequence relation and incoherence property over a finite atomic language, $\mathfrak{L}_{0-}$. In order to represent the consequence relation and the incoherence property in a unified fashion, we first introduce the symbol "$\perp$," which functions like an empty right side in Gentzen's sequent calculus for intuitionistic logic in that it cannot be embedded and can only occur on the right. Let $\mathfrak{L}_0 = \mathfrak{L}_{0-} \cup \{\perp\}$. We define the relation $\mid\!\sim_0 \subseteq \mathcal{P}(\mathfrak{L}_{0-}) \times \mathfrak{L}_0$ by saying that $\Gamma \mid\!\sim_0 p$ holds just in case either (i) $p = \perp$ and Γ is materially incoherent or (ii) Γ materially implies p. We require that $\mid\!\sim_0$ obeys Reflexivity and that if $\forall \Delta \subseteq \mathfrak{L}_{0-}$ $(\Gamma, \Delta \mid\!\sim_0 \perp)$, then $\Gamma \mid\!\sim p$ for all atoms, p, in $\mathfrak{L}_{0-}$. The second requirement, which I call *ex falso fixo quodlibet* (ExFF), is a restricted version of explosion, which applies only if a set is not only incoherent but is, as I shall say, persistently incoherent, i.e., all of its supersets are incoherent.

We now give a sequent calculus formulation of the extended consequence relation. I call the resulting system NM (for Non-Monotonic). Our

[5] A mixed-context version of Cut together with Reflexivity implies Weakening even without DDT. After all, suppose that $\Gamma \mid\!\sim A$. By Reflexivity, $\Gamma, \Delta, A \mid\!\sim A$. By mixed-context Cut, $\Gamma, \Delta \mid\!\sim A$. The deduction-detachment theorem is just needed to show that the same holds for shared-context versions of Cut.

axioms are all the sequents in the underlying material consequence relation, $\mid\!\sim_0$. Moreover, we define not only a single snake-turnstile but one for every $X \subseteq \mathcal{P}(\mathfrak{L}_{0-})$. The idea is that $\Gamma \mid\!\sim^{\uparrow X} p$ holds just in case $\forall \Delta \in X\,(\Delta, \Gamma \mid\!\sim p)$. By convention we can write $\Gamma \mid\!\sim^{\uparrow \mathcal{P}(\mathfrak{L}_{0-})} p$ as $\Gamma \mid\!\sim^{\uparrow} p$. Our axioms are:

Axioms of NM:

Ax1: If $\Gamma \mid\!\sim_0 p$, then $\Gamma \mid\!\sim p$ is an axiom.

Ax2: If $X \subseteq \mathcal{P}(\mathfrak{L}_{0-})$ and $\forall \Delta \in X\ (\Delta, \Gamma \mid\!\sim_0 p)$, then $\Gamma \mid\!\sim^{\uparrow X} p$ is an axiom.

We define the extended consequence relation, $\mid\!\sim$, as the smallest relation that closes these axioms under the following rules.

Rules of NM:
Notation: The square brackets mean that what is inside the brackets is optional or, if preceded by a "/", that it is an alternative. The rules are systematically ambiguous: unless a rule explicitly contains a sequent with an upward arrow, the rule applies if we uniformly replace the plain snake-turnstile by a snake-turnstile with an upward arrow with a particular set of atomic sets throughout the rule. In LC, we can uniformly substitute the right top-sequent and the root-sequent for ones with the same upward arrow. In PushUp, the top sequent cannot have an upward arrow.

$$\frac{\Gamma, A \mid\!\sim B}{\Gamma \mid\!\sim A \to B}\ \text{RC} \qquad \frac{\Gamma \mid\!\sim^{\uparrow} A \qquad \Gamma, B \mid\!\sim C}{\Gamma, A \to B \mid\!\sim C}\ \text{LC}$$

$$\frac{\Gamma, A \mid\!\sim \bot}{\Gamma \mid\!\sim \neg A}\ \text{RN} \qquad \frac{\Gamma \mid\!\sim A}{\Gamma, \neg A \mid\!\sim \bot}\ \text{LN}$$

$$\frac{\Gamma \mid\!\sim A \qquad \Gamma \mid\!\sim B}{\Gamma \mid\!\sim A \& B}\ \text{R\&} \qquad \frac{\Gamma, A, B \mid\!\sim C}{\Gamma, A \& B \mid\!\sim C}\ \text{L\&}$$

$$\frac{\Gamma \mathrel{|\!\sim} A\,[/B]}{\Gamma \mathrel{|\!\sim} A \vee B}\,\text{Rv} \qquad \frac{\Gamma, A \mathrel{|\!\sim} C \qquad \Gamma, B \mathrel{|\!\sim} C}{\Gamma, A \vee B, [A], [B] \mathrel{|\!\sim} C}\,\text{Lv}$$

$$\frac{\Gamma \mathrel{|\!\sim}^{\uparrow} A}{\Gamma, B \to C \mathrel{|\!\sim}^{[\uparrow]} A}\,\text{pCW} \qquad \frac{\Gamma \mathrel{|\!\sim}^{\uparrow} A}{\Gamma, \neg B \mathrel{|\!\sim}^{[\uparrow]} A}\,\text{pNW}$$

$$\frac{\Gamma \mathrel{|\!\sim}^{\uparrow X} A \qquad \Gamma \mathrel{|\!\sim}^{\uparrow Y} A}{\Gamma \mathrel{|\!\sim}^{\uparrow X \cup Y} A}\,\text{UN} \qquad \frac{p_1 ... p_n, \Gamma \mathrel{|\!\sim} A}{\Gamma \mathrel{|\!\sim}^{\uparrow\{\{p_1 ... p_n\}\}} A}\,\text{PushUp}$$

$$\frac{\Gamma \mathrel{|\!\sim}^{\uparrow} \bot}{\Gamma \mathrel{|\!\sim}^{[\uparrow]} A}\,\text{ExFF}$$

This gives us a nonmonotonic consequence relation over a language, $\mathfrak{L}_{-}$, that contains conditionals, negations, conjunctions, and disjunctions. All but the last five rules should look more or less familiar from standard sequent calculi. The differences to familiar versions of the connective rules—e.g. that LC requires a persistent sequent as the left premise-sequent—are all such that they cannot even be formulated in a monotonic setting. Note that we are building in permutation and contraction by working with sets on the left.

Among the unfamiliar rules, pCW and pNW allow us to weaken with conditionals and negations respectively if we can weaken with arbitrary sets of atoms. And UN and PushUp allow us to form new upward arrows. ExFF can be viewed as a restricted version of Gentzen's right weakening rule.

Since our axioms contain no logically complex sentences, Reflexivity isn't true by stipulation in NM. It can be shown, however, that the rules preserve Reflexivity. Moreover, it can be shown that the extension is conservative, i.e., if $\Gamma \subseteq \mathfrak{L}_{0-}$ and $p \in \mathfrak{L}_0$, then $\Gamma \mathrel{|\!\sim} p$ iff $\Gamma \mathrel{|\!\sim}_0 p$. This ensures not only that our connectives don't trivialize the consequence relation in a "tonk-like" fashion; it also shows that they don't change the facts that they are meant to make explicit and that they don't force monotonicity. Furthermore, the connectives are well-behaved. The conditional obeys the deduction-detachment theorem. A negation is implied just in case the premises together with the negated sentences are incoherent. A conjunction is implied just in case both conjunctions are implied. And a disjunction is implied just in case at least one of the disjuncts is implied.

In sum, the NM system gives us a surprisingly well-behaved nonmonotonic logic that conservatively extends a nonmonotonic material consequence relation. Moreover, NM is supra-intuitionistic.

Theorem 1 *NM is supra-intuitionistic, i.e., if* $\Gamma \vdash A$ *holds in intuitionistic logic, then* $\Gamma \mathrel{|\!\sim} A$ *holds in NM.*

Proof. Due to limitations of space, I just give a sketch of the proof. First, we notice that Cut and Weakening can both be eliminated in Gentzen's sequent calculus formulation of intuitionistic logic, LJ, if we are allowed to use as our axioms atomic instances of Reflexivity, i.e., if the leaves of the proof-trees can be of the form $p_0, ..., p_n \vdash p_n$.[6] Next, we notice that Weakening is admissible in NM proof-trees all of whose leaves are instances of Reflexivity. This is because we can weaken such leaves with arbitrary atoms. Given these facts, we can show that every proof-tree that uses the rules of LJ but not Cut or Weakening and whose leaves are instances of Reflexivity (with context) can be translated into NM because we can translate all the remaining rules of LJ into NM.

The translation from LJ works as follows: KA and Cut are eliminated. KS translates into ExFF after making the premise-sequent persistent via PushUp and UN. W and C are given by working with sets. FES translates into RC. FEA can be derived from LC after making the left premise-sequent persistent. The translations of the rules for conjunction, disjunction and negation are similarly straightforward. ■

Although NM is supra-intuitionistic, Weakening and Cut are not globally admissible in NM. Outside of the intuitionistic fragment of the consequence relation they may fail. Can we introduce operators that allow us to keep track—in the object language—of the admissibility of Weakening and Cut? And can we introduce an operator that allows us to make explicit, in the object language, that an implication of NM also holds in intuitionistic logic? Those are the questions that I will address in the remainder of this paper.

[6] Here, as elsewhere in this paper, I am assuming that we are only concerned with finite premise sets.

3 Making Monotonicity Explicit

Weakening is not always admissible in NM. However, it is sometimes admissible; e.g., it is admissible to weaken instances of Reflexivity. We want to introduce an operator that allows us to keep track of where Weakening is admissible. In order to do this, it is helpful to think of monotonicity as analogous to a modality. We can think of premise sets as points of evaluation. From each premise set all and only its supersets are accessible. A sentence is verified at a premise set just in case the premise set implies the sentence. A set, Γ, monotonically implies a sentence, A, just in case all points of evaluation accessible from Γ verify A. Looking at monotonicity in this way, it is natural to understand it as a kind of necessity. Hence, it is natural to express it in a similar way in the object language, namely by introducing an operator, $\Box$, such that $\Gamma \mid\!\sim \Box A$ just in case $\forall \Delta \subseteq \mathfrak{L}_{-}\ (\Delta, \Gamma \mid\!\sim A)$.

Fortunately, such an operator can easily be introduced. That is in part because it can be shown that weakening with arbitrary sets of sentences is admissible for a particular sequent just in case the sequent can be weakened with arbitrary sets of atomic sentences. That is, $\forall \Delta \subseteq \mathfrak{L}_{-}\ (\Delta, \Gamma \mid\!\sim A)$ iff $\forall \Delta_0 \subseteq \mathfrak{L}_{0-}\ (\Delta_0, \Gamma \mid\!\sim A)$. Hence, it suffices to have an operator that keeps track of when we can weaken with arbitrary sets of atoms. But we already have a device that does that in the meta-language, namely the upward arrow. All we need to do is to find rules that use the upward arrow in the right way to introduce the box. It can be shown that the following rules do that:

$$\frac{\Gamma \mid\!\sim {}^{\uparrow} A}{\Gamma \mid\!\sim {}^{[\uparrow]} \Box A}\ \text{RB} \qquad\qquad \frac{\Gamma, A \mid\!\sim B}{\Gamma, \Box A \mid\!\sim B}\ \text{LB}$$

In other words, if we introduce the box into NM by adding these two rules, then the box obeys the following principle.[7]

BOX $\quad \Gamma \mid\!\sim \Box A$ iff $\forall \Delta \subseteq \mathfrak{L}_{-}\ (\Gamma, \Delta \mid\!\sim A)$.

This means that the box makes explicit where monotonicity holds locally. We can thus keep track of the local admissibility of Weakening in the object language.

[7] I omit the proof for reasons of space. The strategy is to show that $\Gamma \mid\!\sim \Box A$ iff $\Gamma \mid\!\sim^{\uparrow} A$ iff $\forall \Delta \subseteq \mathfrak{L}_{-}\ (\Gamma, \Delta \mid\!\sim A)$. The left-to-right direction of the first biconditional is shown via induction on proof-height. The left-to-right direction of the second biconditional is shown by induction on the complexity of A. The other direction is straightforward via PushUp, UN, and RB.

4 Making Intuitionism Explicit

Keeping control over Weakening turned out to be relatively easy in NM. Unfortunately, keeping control over Cut is more complicated. However, it is easy to introduce an operator that keeps track of intuitionistic implications. Hence, I will start with that.

In this section, I will introduce an operator, $\boxed{\text{I}}$ (pronounced "I-box"), that allows us to keep track of intuitionistic logic. More precisely, $A_0, ..., A_n \vdash B$ holds in intuitionistic logic just in case $\boxed{\text{I}}A_0, ..., \boxed{\text{I}}A_n \mid\!\sim \boxed{\text{I}}B$ is derivable in NM plus the I-box, a system which I will call NM$\boxed{\text{I}}$. Thus, the I-box is our analog to Girard's shriek.

For ease of exposition, let us write $\boxed{\text{I}}\Gamma$ for a set of sentences that is like Γ except that every sentence is prefaced by an I-box.

How can we introduce the I-box? It can be shown that the sequents of intuitionistic logic are all and only the sequents of NM that can be derived in proof-trees all of whose leaves are atomic instances of Reflexivity. This is easy to see if we realize that the rules of NM and the rules of LJ are intertranslatable if Weakening and Cut hold for the leaves of proof-trees. For in this case, we can push all applications of Weakening and Cut into the leaves. And the rest of the translation is straightforward.

Since intuitionistic sequents are exactly those sequents that can be derived in NM from leaves that are all instances of Reflexivity, we can keep track of intuitionistic logic by keeping track of such squents. The obvious way to do that is to mark atomic instances of Reflexivity and then allow the rules to transfer the mark from the premise sequent(s) to the conclusion sequent. We do this in the meta-language by introducing a new kind of snake-turnstile: $\mid\!\sim_{\mathfrak{I}}$. Since this turnstile can be combined with the upward arrow turnstile to yield $\mid\!\sim_{\mathfrak{I}}^{\uparrow X}$ via PushUp and UN, we double the number of our turnstiles. We introduce $\mid\!\sim_{\mathfrak{I}}$ at the ground level by adding the following axioms:

Additional Axioms of NM$\boxed{\text{I}}$

Ax3: If $p \in \Gamma \subseteq \mathfrak{L}_{0-}$, then $\Gamma \mid\!\sim_{\mathfrak{I}} p$ is an axiom.

We allow all of the rules of NM to be applied to sequents with our new turnstiles, i.e. to sequents of the form $\Gamma \mid\!\sim_{\mathfrak{I}}^{[\uparrow X]} A$. Next we need to introduce $\boxed{\text{I}}$ in such a way that it expresses our new turnstile in much the way in which the box expresses the plain upward arrow. The following rules do that:

Additional Rules of NM$\boxed{\mathrm{I}}$

$$\frac{\Gamma, A \mid\!\sim_{\mathfrak{I}} B}{\Gamma, \boxed{\mathrm{I}}A \mid\!\sim_{[\mathfrak{I}]} B}\ \text{LIB} \qquad\qquad \frac{\Gamma, A \mid\!\sim_{\mathfrak{I}} B}{\Gamma, A \mid\!\sim_{[\mathfrak{I}]} \boxed{\mathrm{I}}B}\ \text{RIB}$$

It is easy to show that NM$\boxed{\mathrm{I}}$ preserves Reflexivity and is a conservative extension of any nonmonotonic material consequence relation that obeys ExFF and Reflexivity.[8] Most importantly, however, the I-box lets us keep track—in the object language—of intuitionistic logic.

Theorem 2 *$\boxed{\mathrm{I}}A_0, ..., \boxed{\mathrm{I}}A_n \mid\!\sim \boxed{\mathrm{I}}B$ holds in NM$\boxed{\mathrm{I}}$, where $A_0, ..., A_n$ and B don't contain any I-boxes, iff $A_0, ..., A_n \vdash B$ holds in intuitionistic logic.*

Proof. ($\Leftarrow$) If $A_0, ..., A_n \vdash B$ is derivable in intuitionistic logic, it can be derived in Gentzen's LJ (Gentzen, 1934). Given a proof-tree in LJ, we can translate it into a proof-tree in NM. In order to do this, we push Cut and Weakening into the leaves. Then we translate all rule applications in accordance with the translation manual given above. All the leaves of the resulting tree are instances of Reflexivity, which are derivable in NM. Moving to NM$\boxed{\mathrm{I}}$, we use $\mid\!\sim_{\mathfrak{I}}^{[\uparrow X]}$-turnstiles in the leaves. The root of the resulting tree is $A_0, ..., A_n \mid\!\sim_{\mathfrak{I}} B$. By RIB and repeated applications of LIB, we get $\boxed{\mathrm{I}}A_0, ..., \boxed{\mathrm{I}}A_n \mid\!\sim \boxed{\mathrm{I}}B$.

($\Rightarrow$) Suppose $\boxed{\mathrm{I}}A_0, ..., \boxed{\mathrm{I}}A_n \mid\!\sim \boxed{\mathrm{I}}B$. This must come by LIB, RIB, or ExFF. If it comes by LIB, the premise is $\boxed{\mathrm{I}}A_0, ..., \boxed{\mathrm{I}}A_{n-1}, A_n \mid\!\sim_{\mathfrak{I}} \boxed{\mathrm{I}}B$. We can show by induction on proof-height that having at least one I-box on the left as a principal operator suffices to ensure that $A_0, ..., A_n \mid\!\sim_{\mathfrak{I}} \boxed{\mathrm{I}}B$. Moreover, $\boxed{\mathrm{I}}B$ on the right can only come from ExFF or RIB. In either case, we can get B instead. Hence, $A_0, ..., A_n \mid\!\sim_{\mathfrak{I}} B$. The same reasoning applies to the cases of RIB and ExFF. We can also show that if there are no I-boxes in $A_0, ..., A_n, B$, then we can derive $A_0, ..., A_n \mid\!\sim_{\mathfrak{I}} B$ just in case $A_0, ..., A_n \vdash B$ holds in intuitionistic logic. To show this, we simply reverse the translation scheme provided above. ■

With this result in hand, we can say that the intuitionist is reasoning as if all sentences were prefaced by a silent "$\boxed{\mathrm{I}}$". From the perspective of NM$\boxed{\mathrm{I}}$, the problem with intuitionistic logic, is that it is blind to the rich structure

[8] For reasons of space, I am omitting the proofs. They work similarly to those for NM, which are sketched in (Hlobil, 2016).

consequence and incoherence that does not derive from Reflexivity. The I-box allows us to restrict our vision 'artificially,' as it were, to single out all and only the sequents that the intuitionist can see.

There is a general strategy behind the techniques that we have used in the last two sections: (a) We want to keep track of a local structural feature of our consequence relation within the object language; this structural feature was the admissibility of Weakening in the first case and the derivability in intuitionistic logic in the second case. (b) We do this by first introducing a new kind of turnstile that filters out all and only the sequents at which the feature in question holds. (c) Next we introduce an operator that marks the result of this filtration in the object language. The result is that we have an object language expression that allows us to mark precisely the region of our logic within which the structural feature in question holds.

5 Making Some Local Transitivity Explicit

Turning to transitivity, everything would be plain sailing if we were able to keep track of when Cut is admissible in a way that is analogous to what we did for Weakening. However, keeping track of local Cut admissibility turns out to be far more difficult than keeping track of local Weakening admissibility or intuitionistic logic.

The problem is that the admissibility of Cut is not preserved by our rules. To see this, let's define $Cn_0(\Gamma)$ as the set of atomic consequences of Γ and $Cn(\Gamma)$ for the consequences of Γ in NM. Suppose Γ obeys Cut in the base consequence relation, i.e., if $\Delta \subseteq Cn_0(\Gamma)$, then $Cn_0(\Gamma) \supseteq Cn_0(\Gamma \cup \Delta)$. Now, suppose that $\Gamma \not{\mid\!\sim}_0\, p$ and $\Gamma, p \not{\mid\!\sim}_0 \bot$ and $\Gamma \not{\mid\!\sim}_0 \bot$. So, $\Gamma \not{\mid\!\sim} \neg p$. However, suppose also that $\Gamma \mid\!\sim_0 q$ and $\Gamma, p, q \mid\!\sim_0 \bot$, which doesn't violate the assumption that Γ obeys Cut because p is not a consequence of Γ. By RN, $\Gamma, q \mid\!\sim \neg p$. Hence, $q \in Cn(\Gamma)$ and $\neg p \notin Cn(\Gamma)$ but $\neg p \in Cn(\Gamma \cup \{q\})$.

Unfortunately, I don't know how to solve this problem in full generality. Here I can only make a first step towards a solution. The step consists in a generalization of the idea behind the I-box. In the case of the I-box, we exploited the fact that the atomic instances of Reflexivity are closed under Cut and Weakening. That makes Cut and Weakening admissible in proof-trees all of whose leaves are instances of Reflexivity. The general lesson that we should learn from this special case is that a set of sequents is closed under Cut if they can all be derived in proof-trees all of whose leaves belong to a set of atomic sequents at are closed under Cut and Weakening. Let's define sets of atomic sequents that have this property:

Definition 1 *A set, $\mathfrak{S}$, of atomic sequents has the T+W property iff (a) if* $\Gamma \mathrel{|\!\sim_0} p \in \mathfrak{S}$*, then* $\forall \Delta \subseteq \mathfrak{L}_0\, (\Delta, \Gamma \mathrel{|\!\sim_0} p \in \mathfrak{S})$*; and (b) if* $\Gamma \mathrel{|\!\sim_0} p \in \mathfrak{S}$ *and* $\Gamma, p \mathrel{|\!\sim_0} q \in \mathfrak{S}$*, then* $\Gamma, \mathrel{|\!\sim_0} q \in \mathfrak{S}$.

Let's focus on maximal sets that have the T+W property, i.e., T+W sets such that no proper superset of them has the property. Let's enumerate these sets and call them $\mathfrak{S}_0, ..., \mathfrak{S}_n$. We can now treat each of these sets like we have treated the atomic instances of Reflexivity when we introduced the I-box. For each $\mathfrak{S}_i$ we can introduce a new kind of turnstile, $|\!\sim_i^{[\uparrow X]}$, and a new operator, $\boxed{\mathrm{T}}^i$ (pronounced "T-box", for transitivity-box). For the new turnstiles, we add axioms with those turnstiles.

Ax3: If $\Gamma \mathrel{|\!\sim_0} p$ is in $\mathfrak{S}_i$, then $\Gamma \mathrel{|\!\sim_i} p$ is an axiom.

We apply the usual NM rules to our new kinds of sequents if they are of the same kind, i.e. if the same subscript on the turnstile occurs in all the sequents in the application of the rule. Moreover, we add rules that introduce an operator $\boxed{\mathrm{T}}^i$ for each $\mathfrak{S}_i$.

$$\frac{\Gamma, A \mathrel{|\!\sim_i} B}{\Gamma, \boxed{\mathrm{T}}^i A \mathrel{|\!\sim_{[i]}} B}\ \text{LTB} \qquad\qquad \frac{\Gamma \mathrel{|\!\sim_i} A}{\Gamma \mathrel{|\!\sim_{[i]}} \boxed{\mathrm{T}}^i A}\ \text{RTB}$$

Let's call the resulting system NM$\boxed{\mathrm{T}}$. In this system we can keep track of some regions in which Cut is admissible, namely regions that are extensions of maximal T+W sets in the base. More precisely, we can show the following:

Theorem 3 *Sequents of the form* $\Gamma \mathrel{|\!\sim_i} A$ *are closed under Cut.*

Proof. First, we note that in order to derive a sequent of the form $\Gamma \mathrel{|\!\sim_i} A$, all the sequents in the proof-tree must be of that form. The rest of the proof follows Gentzen's original Cut-elimination proof very closely. Just as in Gentzen, it is a double induction on rank and degree; and we can divide the cases in the way he did. In effect, we push Cut up into the leaves; and we know that Cut holds among the leaves. The proof is tedious but straightforward. ∎

So we have a way to mark off some regions in which Cut holds locally in the metalanguage of the sequent calculus. But we also have object language operators that allow us to keep track of these regions.

Theorem 4 *If we can derive* $\boxed{T}^i A_0, ..., \boxed{T}^i A_m \mid\!\sim \boxed{T}^i B$ *in NM*$\boxed{T}$ *and we can derive* $\boxed{T}^i A_0, ..., \boxed{T}^i A_m, \boxed{T}^i B \mid\!\sim \boxed{T}^i C$*, then* $\boxed{T}^i A_0, ..., \boxed{T}^i A_m \mid\!\sim \boxed{T}^i C$ *also holds.*

Proof. It suffices to show that if $\boxed{T}^i \alpha_0, ..., \boxed{T}^i \alpha_k \mid\!\sim \boxed{T}^i \beta$, then we also have $\alpha_0, ...\alpha_k \mid\!\sim_i \beta$. The proof is parallel to the one for the I-box. We argue by induction on proof-height and if $\boxed{T}^i \alpha_0, ..., \boxed{T}^i \alpha_k \mid\!\sim \boxed{T}^i \beta$, this must come from $\alpha_0, ...\alpha_k \mid\!\sim_i \boxed{T}^i \beta$. Then we argue that wherever $\boxed{T}^i \beta$ was first introduced into the proof-tree, we can replace it with β. ■

This technique gives us a family of object language operators each of which keeps track of a particular region of our consequence relation that is closed under Cut. The regions we keep track of are extensions of atomic sequents that are closed under Weakening and Cut. It may be worth mentioning that these regions behave intuitionistically, in the sense that they are equivalent to extensions of the underlying atomic sequents via the rules of Gentzen's LJ.

Let me point out some limitations of this technique. (1) It does not give us a way to say that a set, Γ, obeys Cut in the sense that $\Delta \subseteq Cn(\Gamma)$, then $Cn(\Gamma) \supseteq Cn(\Gamma \cup \Delta)$. After all, it can happen that $\Gamma \mid\!\sim_i B$ and $\Gamma, B \mid\!\sim A$ but not $\Gamma \mid\!\sim A$. (2) The technique does not allow us to keep track of regions that are closed under Cut but where Weakening fails. (3) There is no guarantee that the technique allows us to keep track of all the sets of sequents that are closed under Cut and Weakening. For some such regions may not be traceable to monotonic and transitive regions in the atomic consequence relation.

Despite these limitations, the T-boxes allow us to make explicit in the object language a particular class of cases in which Cut is admissible. From the perspective of NM$\boxed{T}$, to insist that Cut holds globally is to insist that we should always think of our premises and conclusions as if they were prefaced by silent T-boxes (with the same superscript). In NM$\boxed{T}$ we can talk about such implications without being blind to all other implications.

6 Conclusion

Let's take stock. The sequent calculus NM extends a nonmonotonic, nontransitive material consequence relation over atomic sentences to the language of propositional logic. The resulting consequence relation is supra-intuitionistic. We can introduce an operator, $\Box$, that keeps track of where monotonicity holds locally. Moreover, we can introduce another operator, $\boxed{\mathrm{I}}$, that keeps track of intuitionistic logic within NM. Finally, we can introduce a family of operators, $\boxed{\mathrm{T}}^i$, that keep track of some of the regions of our consequence relation that are closed under Cut and Weakening.

The results I have presented here are limited in various respects. In particular, we still need better ways to keep track of where Cut is admissible. If we can develop such techniques and do the same for the other structural principles, we will have a general framework in which what used to look like disagreements about the foundations of logic will emerge as disagreements about particular claims that can be formulated in a common logical framework.

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