Recursive Function Theory: Newsletter

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The *NEWSLETTER* is an informal means of circulating information amongst recursion theorists and others interested in recursive function theory. Results, queries, letters, news and other announcements should be kept fairly brief, and should be sent to

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in every w.s. of player 2.

Remark. The essential point of these results is that, as $\Delta^1_1 = \bigcup_{\alpha\ \mathrm{rec}} \Sigma^0_\alpha$, winning strategies for clopen games with recursive code are simply generalized Skolem functions for infinitary $L_{\omega_1^{ck},\omega}$ formulae. The most obvious evaluation of the complexity of these Skolem functions (basis result) is in general the best possible (anti-basis result). The height of the clopen set corresponds to a generalized way of counting the number of quantifier alternations.

1. A. Blass "Complexity of Winning Strategies" Discrete Mathematics Volume 3 Number 4 1972

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275. AN EXACT PAIR FOR THE ARITHMETIC DEGREES WHOSE JOIN IS NOT A WEAK UNIFORM UPPER BOUND

Where I is a set of Turing degrees, $(\underline{a}, \underline{b})$ is I-exact iff $I = \{\underline{c} \mid \underline{c} \leq \underline{a}\} \cap \{\underline{c} \mid \underline{c} \leq \underline{b}\}$. Where $F \subseteq {}^\omega\omega$, an $F \in {}^\omega\omega$ parametrizes (sub-parametrizes) F iff $F = \{(f)_x \mid x \in \omega\}$ ($F \subseteq \{(f)_x \mid x \in \omega\}$). $\underline{a}$ is a weak uniform upper bound (weak sub-u.u.b.) on I iff for some $A \in \underline{a}$, A parametrizes (sub-parametrizes) $\cup I \cap {}^\omega 2$. AR is the set of arithmetic degrees. Theorems 1 and 2 of [2] point to this analogy between joins of exact pairs and weak u.u.b.'s on any countable ideal I which is closed under jump: for any $\underline{a}$, $\underline{a}$ u.u.b. on I iff (1) there is an I-exact pair $(\underline{b},\underline{c})$ with $\underline{b} \vee \underline{c} < \underline{a}$ and $(\underline{b} \vee \underline{c})^{(2)} \leq \underline{a}^{(1)}$ iff (2) there is a weak u.u.b. $\underline{d} < \underline{a}$ with $\underline{d}^{(2)} \leq \underline{a}^{(1)}$. This makes the relation between exact pairs and weak u.u.b's interesting.

(Note: $\underline{a}$ is a u.u.b. on I iff $\underline{a}$ is the degree of some parametrizing of I.) In this note we prove the following.

(i) There is an AR-exact pair $(\underline{a},\underline{b})$ such that $\underline{a} \vee \underline{b}$ computes no weak sub-u.u.b. on AR (and thus no weak u.u.b. on AR).

(ii) There are continuum many such pairs.

(iii) Any u.u.b. on AR computes the join of such a pair.

We use forcing for 2-quantifier sentences in the language of arithmetic supplemented by uninterpreted predicates '$\underline{A}$' and '$\underline{B}$'. $\langle P,Q\rangle$ is a condition iff $P \equiv_T Q$, P, Q, ε AR, P and Q are uniformly recursively pointed perfect trees. We assume familiarity with forcing in this context; see [1] or [3] for details and notation.

Lemma 1: Let $\phi(z,v,\underline{A},\underline{B})$ represent a recursive relation. Suppose that for any $\langle P_0,Q_0\rangle$ extending $\langle P,Q\rangle$ there are $e \in \omega$ and $\langle P_1,Q_1\rangle \Vdash \forall v\, \phi(\underline{e},v,\underline{A},\underline{B})$. Then there are $c \in \omega$ and $\langle R,S\rangle$ extending $\langle P,Q\rangle$, $\langle R,S\rangle \Vdash \forall v \phi(\underline{e},v,\underline{A},\underline{B})$ and $R \oplus S \leq_T P \oplus Q$.

This lemma is implicit in [5, lemma 3.1]. Let $\Vdash^{P,Q}$ represent Cohen forcing restricted to Range (P) x Range (Q).

Claim: For some $e \in \omega$, $\delta_0,\delta_1 \in$ Str,

$\langle P(\delta_0), Q(\delta_1)\rangle \Vdash^{P,Q} \forall v\phi(\underline{e},v,\underline{A},\underline{B})$. Suppose not: then for every such e, δ_0,δ_1 there are n,τ_0, τ_1, $\delta_0 \subseteq \tau_0$, $\delta_1 \subseteq \tau_1$,

$\langle P(\tau_1),Q(\tau_1)\rangle \Vdash^{P,Q} \neg\phi(\underline{e},\underline{n},\underline{A},\underline{B})$. Imitating the construction of [5, lemma 3.1], build $\langle P_0,Q_0\rangle$ extending $\langle P,Q\rangle$ and forcing $\forall x \exists v \neg\phi(x,v,\underline{A},\underline{B})$. But then our supposition yields an e and $\langle P_1,Q_1\rangle$ extending $\langle P_0,Q_0\rangle$ and forcing $\forall v\phi(\underline{e},v,\underline{A},\underline{B})$, for a contradiction. Now let $R(\delta) = P(\delta_0 \wedge \delta)$, $S(\delta) = Q(\delta_1 \wedge \delta)$ for all $\delta \in$ Str. $\langle R,S\rangle$ and e are as desired. Here $\delta_i \wedge \delta$ = the concatenation of δ_i and δ.

Lemma 2: There is a Π^0_1 relation $\phi(n,f)$ such that for any $n \in \omega$ there is a unique $f \in {}^\omega\omega$ such that $\phi(n,f)$; for this f, for any i and x: if $i \leq n$ then ($(f)_i(x) \neq 0$ iff $x \in 0^{(i)}$); if $i > n$, $(f)_i(x) = 0$. Thus $f \equiv_T 0^{(n)}$.)

Proof: As with [4, problem 16.98], except even easier. To prove (i), it suffices to show that <u>if</u> $\langle P,Q\rangle \Vdash$ "$\{\underline{e}\}^{\underline{A} \oplus \underline{B}}$ is total" then for some n and $\langle R,S\rangle$ extending $\langle P,Q\rangle$, $\langle R,S\rangle \Vdash \neg\exists x\phi(\underline{n}, (\{\underline{e}\}^{\underline{A} \oplus \underline{B}})_x)$. For we just construct a generic sequence, at odd stages coding members of $AR \cap {}^{\omega}2$ into our conditions to ensure that $(A,B) = \bigcap_i [P_i] \times [Q_i]$ is a pair of u.b.'s on AR, making sure that if some $\langle P_i,Q_i\rangle$ forces "$\{\underline{e}\}^{\underline{A} \oplus \underline{B}}$ is total", some $\langle P_j,Q_j\rangle$ forces "$\neg\exists x\phi(\underline{n},(\{\underline{e}\}^{\underline{A} \oplus \underline{B}})_x)$" for some n. Then $\{e\}^{A \oplus B}$ will fail to sub-parametrize AR.

Suppose $\langle P,Q\rangle \Vdash$ "$\{\underline{e}\}^{\underline{A} \oplus \underline{B}}$ is total", $P \oplus Q \leq_T 0^{(n)}$, but for every $\langle P_0,Q_0\rangle$ extending $\langle P,Q\rangle$ there are $k \in \omega$ and $\langle P_1,Q_1\rangle$ extending $\langle P_0,Q_0\rangle$, $\langle P_1,Q_1\rangle \Vdash \phi(\underline{n+1},(\{\underline{e}\}^{\underline{A} \oplus \underline{B}})_k)$. By lemma 1, some $\langle R,S\rangle$ extends $\langle P,Q\rangle$, forces $\phi(\underline{n+1},(\{\underline{e}\}^{\underline{A} \oplus \underline{B}})_{\underline{k}})$ for some k, and $R \oplus S \leq_T P \oplus Q$. We may now construct $(A,B) \in [R] \times [S]$, $A \oplus B \leq_T R \oplus S$, so that $\{e\}^{A \oplus B}$ is total, and $\phi\ (n+1,(\{e\}^{A \oplus B})_k)$. But $0^{(n+1)} \equiv_T (\{e\}^{A \oplus B})_k \leq_T A \oplus B \leq_T R \oplus S \leq_T P \oplus Q \leq_T 0^{(n)}$, for a contradiction.

Applying the technique of [6] we may modify the previous construction to prove (ii). Embedding the previous construction in that of [2, Theorem 1], we may prove (iii).

We note that an AR exact pair $(\underline{a},\underline{b})$ such that $\underline{a} \vee \underline{b}$ is not the jump of an u.b. on AR may be easily obtained from known results. (Select $(\underline{a},\underline{b})$ so that $(\underline{a} \vee \underline{b})^{(2)} = \underline{0}^{(\omega)}$; for any u.b. $\underline{c}$ on AR $\underline{0}^{(\omega)} \leq \underline{c}^{(2)}$; we can't have $\underline{a} \vee \underline{b} = \underline{c}'$.) But a construction like the previous one shows that there are continuum many such $(\underline{a},\underline{b})$, and that any u.u.b. on AR computes the join of such a pair.

All these results hold with HYP = the set of hyperarithmetic degrees in place of AR. We would like to know whether they hold for all countable ideals which are closed under jump. We'd like to know whether all weak sub-u.u.b's on AR are actually weak u.u.b.'s. (In [3] it is shown that all sub-u.u.b.'s on AR are u.u.b.'s; this fails for HYP.) Most of all, we'd like to know whether every weak u.u.b. on AR computes the join of an exact pair.

[1] H. Hodes, "Jumping Through the Transfinite," Journal of Symbolic Logic, Vol. 45 (1980), pp. 204-220.

[2] H. Hodes, "More About Uniform Upper Bounds," forthcoming in the Journal of Symbolic Logic.

[3] C. Jockusch and S. Simpson, "A Degree-Theoretic Characterization of the Ramified Analytical Hierarchy," Annals of Mathematical Logic, Vol. 10 (1976).

[4] H. Rogers, The Theory of Recursive Functions and Effective Computability, McGraw Hill, 1967.

[5] G. Sacks, "Forcing with Perfect Closed Sets," Proceedings of Symposia in Pure Mathematics, Vol. 13 (1971), American Mathematical Society, Providence, Rhode Island.

[6] L. Sasso, "A Cornucopia of Minimal Degrees," Journal of Symbolic Logic, Vol. 35 (1970), pp. 383-388.

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276 NONCOMPUTABILITY IN ANALYSIS AND PHYSICS: A COMPLETE DETERMINATION OF THE CLASS OF NONCOMPUTABLE LINEAR OPERATORS

The following abstract will appear in the Abstracts of the American Mathematical Society:

With mild side conditions we prove: bounded operators preserve computability, unbounded operators do not. The side conditions are: a) the operator is closed and b) the operator acts effectively on the sequence $x^0, x^0, x^2, \ldots, x^n \ldots$ or instead on the functions in some "effective generating set". The theorem is formulated axiomatically - in terms of an axiomatic "computability theory" on an arbitrary Banach Space. An intrinsic definition of L^p-computability is also given.

The above result is applied to the wave, heat, and potential equations, and to Fourier series and transforms, as well as to a variety of other topics.

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