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CARDINALITY LOGICS. PART II: DEFINABILITY IN LANGUAGES BASED ON 'exactly'

HAROLD HODES

This paper continues the project initiated in [5]: a model-theoretic study of the concept of cardinality within certain higher-order logics. As recommended by an editor of this JOURNAL, I will digress to say something about the project's motivation. Then I will review some of the basic definitions from [5]; for unexplained notation the reader should consult [5].

The syntax of ordinary usage (with respect to the construction of arguments as well as the construction of individual sentences) makes it natural to classify numerals and expressions of the form 'the number of F's' as singular terms, expressions like 'is prime' or 'is divisible by' as predicates of what Frege called "level one", and expressions like 'for some natural number' as first-order quantifier-phrases. From this syntatic classification, it is a short step—so short as to be frequently unnoticed—to a semantic thesis: that such expressions play the same sort of semantic role as is played by the paradigmatic (and nonmathematical) members of these lexical classes. Thus expressions of the first sort are supposed to designate objects (in post-Fregean terms, entities of type 0), those of the second sort to be true or false of tuples of objects, and those of the third sort to quantify over objects. All this may be summed up in Frege's dictum: "Numbers are objects."

As Frege realized, if we buy the above doctrine, then cardinal numbers are objects that somehow intrinsically represent certain quantifiers, those expressed by expressions like 'there are exactly ten x's such that'. In a very revealing passage in his ***Nachlass***, dated July 1919, Frege wrote [2, pp. 256–257]:

> These second-level concepts form a series.... But still we do not have the numbers of arithmetic; we do not have objects, but concepts. How can we get from these concepts to the numbers of arithmetic in a way that cannot be faulted? Or are there simply no numbers in arithmetic? Could the numerals help to form signs for these second-level concepts, and yet not be signs in their own right?

In [4] I argue that we cannot "get from these concepts to the numbers of arithmetic". Instead, Frege's suggestion, that in the relevant sense there are no numbers, is

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correct; equivalently, the semantic thesis cited above should be rejected. (This does not mean that the usual logical syntax assigned to mathematical discourse is faulty; rather its semantics should be reconceived; the model-theory for such a reconception is presented in [6].) [5] and this paper concern several model-theoretic semantics in which cardinal numbers (as objects) are avoided in favor of cardinality-quantifiers. For more extended discussion, see [4] and [6].

Unless otherwise stated, we do not assume the axiom of choice. Let Card be the class of Scott-cardinals; for $\kappa \in$ Card we adopt these definitions:

κ is infinite iff for some (thus any) $x \in \kappa$, x is infinite;

κ is an aleph iff κ is infinite and for some (thus any) $x \in \kappa$, x is well-orderable;

$\bar{\kappa} = \{n : n < \kappa\}$;

$\mathrm{ncb}(\kappa)$ = number of cardinals below $\kappa = \mathrm{card}(\bar{\kappa})$.

As usual, ω is the set of finite von Neumann ordinals. For any set x and class y, ${}^{x}y$ is the class of functions from x into y; if $x \in \omega$ then such functions are identified with x-tuples.

Fix for each $k \in \omega$ sets $\mathbf{Var}(2k)$ of type-k variables and $\mathbf{Var}(1)$ of type-1 variables, all countable. Let **Pred** be a set of predicate-constants and **Funct** a set of function-constants, each j-place for some $j \in \omega$. From the logical lexicon {'$\perp$', '$\supset$', '$\exists$', '$=$', '**exactly**', '$\leq$'} we form several languages. The terms of type 0 are generated from **Funct** $\cup$ $\mathbf{Var}(0)$ as usual; atomic formulae of these languages are generated from those terms and from **Pred** $\cup$ $\mathbf{Var}(1) \cup \{$'$\perp$'$\}$ as usual. The set $\mathrm{Fml}(L^{1,\omega}(\mathbf{exactly}, \leq))$ of formulae of $L^{1,\omega}(\mathbf{exactly}, \leq)$ is defined by the usual rules governing '$\supset$' and '$\exists$' along with these:

If τ, σ are terms of type 0 then $\tau = \sigma$ is a formula.

If $\mu, \mu' \in \mathbf{Var}(2k)$ for $k > 0$ then $\mu \leq \mu'$ is a formula.

If φ is a formula, $v \in \mathbf{Var}(2k)$ and $\mu \in \mathbf{Var}(2k+2)$, then $(\mathbf{exactly}\ \mu\ v)\varphi$ is a formula.

If φ is a formula and $\mu \in \mathbf{Var}(2k)$ then $(\exists\mu)\varphi$ is a formula.

For $k > 0$, form $L^{1,2k}(\mathbf{exactly}, \leq)$ by dropping variables of type greater than $2k$. Form $L^{1,2k*}(\mathbf{exactly}, \leq)$ from $L^{1,2k}(\mathbf{exactly}, \leq)$ by adding this formation-rule:

If φ is a formula and $\mu, \rho \in \mathbf{Var}(2k)$ then $(\mathbf{exactly}\ \mu\ \rho)\varphi$ is a formula.

In any formula of the form $(\mathbf{exactly}\ \mu\ \rho)\varphi$, the indicated occurrence of μ is free, and the indicated occurrence of ρ binds all occurrences of ρ free in φ.

Form $L^{0,2k}(\mathbf{exactly}, \leq)$ from $L^{1,2k}(\mathbf{exactly}, \leq)$ by dropping $\mathbf{Var}(1)$; form $L^{0,2k*}(\mathbf{exactly}, \leq)$ analogously; form $L^{1,2k}(\mathbf{exactly})$ by dropping '$\leq$' from the logical lexicon; form $L^{1,2k*}(\mathbf{exactly})$ analogously. Form $L^{0,-}(\mathbf{exactly}, =)$ from $L^{0,-}(\mathbf{exactly})$ by allowing use of '$=$' between variables of type ≥ 2.

Given $\kappa \in$ Card and a model $\mathscr{A}$ for **Pred**, **Funct**, introduce the type-0 constant $\mathbf{a}$ for each $a \in |\mathscr{A}|$; for each $A \subseteq |\mathscr{A}|$ introduce the type-1 constant $\mathbf{A}$. For each $n < \mathrm{ncb}^k(\kappa)$ we introduce $\mathbf{n}$ as a constant of type $2k + 2$. For each $k < \omega$, constants of type $2k$ shall be substitutable for variables of type $2k$; similarly for type 1. We define $\models_\kappa$ as usual, with the following novel clauses where $n, m \in \bar{\kappa}$, $v \in \mathbf{Var}(0)$, and $\rho \in \mathbf{Var}(2k+2)$:

$\mathscr{A} \models_\kappa \mathbf{n} \leq \mathbf{m}$ iff $n \leq m$;

$\mathscr{A} \models_\kappa (\mathbf{exactly}\ \mathbf{n}\ v)\varphi$ iff $\mathrm{card}(\hat{v}\varphi^{\mathscr{A}}) = n$;

$\mathscr{A} \models_\kappa (\mathbf{exactly}\ \mathbf{n}\ \rho)\varphi$ iff $\mathrm{card}(\hat{\rho}\varphi^{\mathscr{A}}) = n$;

$\mathscr{A} \models_\kappa (\exists\rho)\varphi$ iff, for some $m < \mathrm{ncb}^k(\kappa)$, $\mathscr{A} \models_\kappa \varphi(\rho/\mathbf{m})$.

Here we have used the notation

$$\hat{v}\varphi^{\mathscr{A}} = \{a \in |\mathscr{A}| : \mathscr{A} \vDash_\kappa \varphi(v/\mathbf{a})\};$$
$$\hat{\rho}\varphi^{\mathscr{A}} = \{m < \mathrm{ncb}^k(\kappa) : \mathscr{A} \vDash_\kappa \varphi(\rho/\mathbf{m})\}.$$

Relative to a model, we define satisfaction of a formula by an appropriate finite sequence as usual; we express this using the familiar form: $\mathscr{A} \vDash \varphi[\cdots]$.

For any formula φ of one of these languages and any $\rho \in \mathbf{Var}(2k)$, let $(\mathbf{big}\ \rho)\varphi$ abbreviate $\neg(\exists\mu)(\mathbf{exactly}\ \mu\ \rho)\varphi$, where μ does not occur free in φ, with $\mu \in \mathbf{Var}(2k)$ if we are working with $L^{i,2k*}(\mathbf{exactly}, \leq)$, and otherwise $\mu \in \mathbf{Var}(2k+2)$. Clearly for any model $\mathscr{A}$:

$$\mathscr{A} \vDash_\kappa (\mathbf{big}\ \rho)\varphi \quad \text{iff} \quad \mathrm{card}(\hat{\rho}\varphi^{\mathscr{A}}) \notin \bar{\kappa}.$$

§1. Where L is one of the languages introduced above and $\varphi(\mu_0, \ldots, \mu_{l-1}) \in \mathrm{Fml}(L)$ with free variables among $\mu_0, \ldots, \mu_{l-1}$, $\varphi(\mu_0, \ldots, \mu_{l-1})$ is of type $\vec{t} = \langle 2t_0, \ldots, 2t_{l-1} \rangle \in {}^l\omega$ iff for each $i < l$, $\mu_i \in \mathbf{Var}(2t_i)$. Suppose $\varphi = \varphi(\mu_0, \ldots, \mu_{l-1})$ is of type $\vec{t}$ with $t_i > 0$ for each $i < l$. Let $\kappa \in \mathrm{Card}$ be infinite. We adopt the following definitions:

$${}^{\vec{t}}\kappa = \{\langle n_0, \ldots, n_{l-1} \rangle : \text{for all } i < l,\ n_i < \mathrm{ncb}^{t_i - 1}(\kappa)\}.$$

For a model $\mathscr{A}$, R is κ-defined by φ over $\mathscr{A}$ iff

$$R = \{\langle n_0, \ldots, n_{l-1} \rangle \in {}^{\vec{t}}\kappa : \mathscr{A} \vDash_\kappa \varphi(\mathbf{n}_0, \ldots, \mathbf{n}_{l-1})\}.$$

R is κ-defined by φ iff for any model $\mathscr{A}$ with $\mathrm{card}(\mathscr{A}) \geq \kappa$, R is κ-defined by φ over $\mathscr{A}$. R is κ-definable in L with respect to type $\vec{t}$ iff there is a $\varphi \in \mathrm{Fml}(L)$ of type $\vec{t}$ which κ-defines R. R is κ-definable in L iff, for some $\vec{t}$, R is κ-definable in L with respect to $\vec{t}$. For $R \subseteq {}^l\mathrm{Card}$, R is uniformly defined by φ iff for each infinite $\kappa \in \mathrm{Card}$, $R \cap {}^l\kappa$ is κ-defined by φ; R is uniformly definable in L with respect to $\vec{t}$ iff for some $\varphi \in \mathrm{Fml}(L)$ of type $\vec{t}$, R is uniformly defined by φ; and R is uniformly definable in L iff for some $\vec{t}$, R is uniformly definable with respect to $\vec{t}$.

The inclusions given in §2.1 of [5] generalize to yield the following, where $i \in 2$.

(1) If R is κ-definable in $L^{i,2k}(\mathbf{exactly}, \leq)$ with respect to $\vec{t}$, then R is κ-definable in $L^{i,2k+2}(\mathbf{exactly}, \leq)$ with respect to $\vec{t}$.

(2) If R is κ-definable in $L^{i,\omega}(\mathbf{exactly}, \leq)$ or in $L^{i,2k+2*}(\mathbf{exactly}, \leq)$, then R is κ-definable in $L^{i,2k*}(\mathbf{exactly}, \leq)$.

Thus the sequence of languages $L^{i,-}(\mathbf{exactly}, \leq)$ as '—' is replaced by '2',..., 'ω',..., '2*' is a hierarchy of order-type $\omega + 1 + \omega^*$, yielding nondecreasing classes of κ-definable relations.

Furthermore, if κ is an aleph, Observation 2.3 of [5] easily generalizes to yield the following.

(3) R is κ-definable in $L^{1,2k}(\mathbf{exactly}, \leq)$ with respect to $\vec{t}$ iff R is κ-definable in $L^{1,2k}(\mathbf{exactly})$ with respect to $\vec{t}$.

(4) R is κ-definable in $L^{1,2k*}(\mathbf{exactly}, \leq)$ with respect to $\vec{t}$ iff R is κ-definable in $L^{1,2k*}(\mathbf{exactly})$ with respect to $\vec{t}$.

Let a language be *pure* iff its nonlogical lexica **Pred** and **Funct** are empty; a pure formula is a formula of a pure language. In considering κ-definability in L, we may without loss of generality take L to be pure. Here's why.

Given **Pred**, **Funct**, a nonempty set U and a function $\mathscr{N}$ mapping 0-place members of **Funct** into U, let $\mathscr{M}_{U,\mathscr{N}} = \mathscr{M}$ be the model for **Pred**, **Funct** with $|\mathscr{M}| = U$ and:
if $\mathbf{P}$ is 0-place then $\mathbf{P}^{\mathscr{M}} = \text{False}$;
if $\mathbf{P}$ is j-place for $j > 0$ then $\mathbf{P}^{\mathscr{M}} = \{\ \}$;
if $\mathbf{f}$ is 0-place then $\mathbf{f}^{\mathscr{M}} = \mathscr{N}(\mathbf{f})$;
if $\mathbf{f}$ is j-place for $j > 0$ then $\mathbf{f}^{\mathscr{M}}(a_0, \ldots, a_{j-1}) = a_0$ for all $a_0, \ldots, a_{j-1}$.

Suppose that **Pred**, **Funct** is the nonlogical lexicon of L. Consider $\varphi \in \mathrm{Fml}(L)$ of type $\vec{t}$ with $t_i > 0$ for all $i < l$; suppose $\varphi = \varphi'(v_0/\mathbf{c}_0, \ldots, v_{n-1}/\mathbf{c}_{n-1})$, where φ' contains no 0-place function-constants, $\mathbf{c}_0, \ldots, \mathbf{c}_{n-1}$ are the 0-place function-constants occurring in φ, and $v_0, \ldots, v_{n-1} \in \mathbf{Var}(0)$ and do not occur in φ. Form $\bar{\varphi}$ from φ' as follows: replace all atomic formulae starting with a member of **Pred** by '$\perp$', replace any remaining term of the form $\mathbf{f}(\tau_0, \ldots, \tau_{m-1})$ by τ_0, and iterate that until all members of **Funct**(m) for $m > 0$ are gone. Thus for any model $\mathscr{A}$ for **Pred**, **Funct** and any $\vec{n} \in {}^{\vec{t}}\kappa$:

$$\mathscr{A} \vDash_\kappa (\forall v_0) \cdots (\forall v_{n-1}) \bar{\varphi}[\vec{n}] \quad \text{iff for every } \mathscr{N}\colon \mathbf{Funct}(0) \to |\mathscr{A}|,\ \mathscr{M}_{|\mathscr{A}|,\mathscr{N}} \vDash_\kappa \varphi[\vec{n}].$$

Suppose φ κ-defines R with respect to $\vec{t}$. For any $\mathscr{N}\colon \mathbf{Funct}(0) \to |\mathscr{A}|$,

$$\vec{n} \in R \quad \text{iff} \quad \mathscr{M}_{|\mathscr{A}|,\mathscr{N}} \vDash_\kappa \varphi[\vec{n}].$$

Thus

$$\vec{n} \in R \quad \text{iff} \quad \text{for every } \mathscr{N}\colon \mathbf{Funct}(0) \to |\mathscr{A}|,\ \mathscr{M}_{|\mathscr{A}|,\mathscr{N}} \vDash_\kappa \varphi[\vec{n}].$$

So $(\forall v_0) \cdots (\forall v_{n-1}) \bar{\varphi}$, a pure formula, κ-defines R with respect to $\vec{t}$.

The next two sections address the question "What are the classes of relations κ-definable in above languages?" We will also get results on the Turing degrees of the sets of κ-satisfiable formulae in these languages.

§2. In this section we prove a technical theorem that characterizes the relations κ-definable in $L^{1,2}$(**exactly**) and $L^{1,2*}$(**exactly**) in terms of definability over two specific models using languages that are not quite higher-order. The basic point is this: for κ-definability in $L^{1,2}$(**exactly**) [$L^{1,2*}$(**exactly**)], cardinal addition [and '**exactly**' applied to count cardinals] absorbs quantification over types 0 and 1, permitting us to pull type 2 down to type 0.

Let L_2 be the first-order language with nonlogical lexicon **Pred** $= \{`\leq', `\mathbf{S}'\}$ and **Funct** empty. For $\kappa \in \mathrm{Card}$, let $\mathscr{M}_2(\kappa)$ be the model for L_2 with universe $\bar{\kappa}$, $\leq^{\mathscr{M}_2(\kappa)} = \leq \restriction \bar{\kappa}$, and $\mathbf{S}^{\mathscr{M}_2(\kappa)} =$ cardinal addition on $\bar{\kappa}$ viewed as a three-place relation. Form L_2(**exactly**) by adding '**exactly**' to the logical lexicon of L_2 with the formation-rule:

if $v, \rho \in \mathbf{Var}(0)$ and φ is a formula
then (**exactly** $\rho\, v)\varphi$ is a formula.

L_2(**exactly**) is to be interpreted only in models of the form $\mathscr{M}_2(\kappa)$, with the truth-clause

$$\mathscr{M}_2(\kappa) \vDash (\textbf{exactly } \mathbf{n}\, v)\varphi \quad \text{iff} \quad \mathrm{card}(\hat{v}\varphi^{\mathscr{M}_2(\kappa)}) = n,$$

for any $n \in \bar{\kappa}$.

For $\kappa \in \mathrm{Card}$, let κ be *reasonable* iff $\bar{\kappa}$ is closed under cardinal addition.

THEOREM 1. *For any reasonable infinite* $\kappa \in$ Card:

(i) *R is κ-definable in* $L^{1,2}$(**exactly**) *iff R is definable over* $\mathscr{M}_2(\kappa)$ *in* L_2;

(ii) *R is κ-definable in* $L^{1,2*}$(**exactly**) *iff R is definable over* $\mathscr{M}_2(\kappa)$ *in* L_2(**exactly**).

From right to left Theorem 1 follows from the usual second-order definition of cardinal addition, formulated in L_2. For any $\mu_0, \mu_1, \mu_2 \in \mathbf{Var}(2)$, let $\mathbf{Add}(\mu_0, \mu_1, \mu_2)$ be:

$$(\forall\gamma_0)(\forall\gamma_1)([(\textbf{exactly } \mu_0\, v)\gamma_0(v) \mathbin{\&} (\textbf{exactly } \mu_1\, v)\gamma_1(v) \\ \mathbin{\&} \neg(\exists v)(\gamma_0(v) \mathbin{\&} \gamma_1(v))] \supset (\textbf{exactly } \mu_2 v)(\gamma_0(v) \vee \gamma_1(v))),$$

where $\gamma_0, \gamma_1 \in \mathbf{Var}(1)$ and are distinct. For any model $\mathscr{A}$ with card$(\mathscr{A}) \geq \kappa$ and any $n_0, n_1, n_2 \in \bar{\kappa}$:

$$\mathscr{A} \models_\kappa \mathbf{Add}(\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2) \quad \text{iff} \quad n_0 + n_1 = n_2.$$

If μ_0, μ_1, μ_2 are all distinct, $\mathbf{Add}(\mu_0, \mu_1, \mu_2)$ κ-defines cardinal addition on $\bar{\kappa}$ and uniformly defines cardinal addition *simpliciter*. Furthermore these definitions may plausibly be called "analytic" in that they provide an analysis of what we mean by "cardinal addition". Notice that the reasonableness of κ is not used.

We now prove Theorem 1(ii) from left to right. Our strategy is to find a finite set of simple formulae, each one determining whether an assignment of values (to the free variables of φ) and a choice of cardinalities for the "cells" produced by the free type-1 variables satisfies φ (Lemma 1); then for each such simple formula we construct a formula ψ in our target language asserting the existence of values satisfying φ (Lemma 3).

For any formula θ, let θ^0 be θ and θ^1 be $\neg\theta$. Suppose that

$$\Delta_0 = \{v_0, \ldots, v_{l-1}\} \subseteq \mathbf{Var}(0), \qquad \Delta_1 = \{\gamma_0, \ldots, \gamma_{k-1}\} \subseteq \mathbf{Var}(1).$$

A *profile* for $\Delta_0 \cup \Delta_1$ is a satisfiable conjunction of formulae of the forms

$$(v_i = v_j)^{a(i,j)}, \quad \text{where } a(i,j) \in 2 \text{ for } i < j < l,$$
$$\gamma_j(v_i)^{b(i,j)}, \quad \text{where } b(i,j) \in 2 \text{ for } i < l, j < k.$$

Clearly a and b uniquely determine and are determined by a profile for $\Delta_0 \cup \Delta_1$. Suppose θ is such a profile. For $i, i' \in l$ let $i \simeq_\theta i'$ iff:

$$\text{either } i = i', \text{ or } i < i' \text{ and } a(i,i') = 0, \text{ or } i' < i \text{ and } a(i',i) = 0;$$

clearly $\simeq_\theta$ is an equivalence relation on l. In what follows, let $A_j = A_j^0 \subset |\mathscr{A}|$ and $A_j^1 = |\mathscr{A}| - A_j$.

For each $k > 0$ and $c \in {}^k2$, introduce a distinct type-2 variable μ_c and a distinct type-0 variable v_c; for $z \subseteq {}^k2$ let $\vec{\mu}_z$ and $\vec{v}_z$ be the sequences of these variables with subscripts $c \in z$, ordered according to the position of c in the lexicographic ordering of ${}^k2$. Let $\bigotimes_{k,z}$ be:

$$\mathop{\&}\left\{(\textbf{exactly } \mu_c\, v)\left(\mathop{\&}_{j<k} \gamma_j(v)^{c(j)}\right) : c \in z\right\} \\ \mathbin{\&} \mathop{\&}\left\{(\textbf{big } v)\left(\mathop{\&}_{j<k} \gamma_j(v)^{c(j)}\right) : c \in {}^k2 - z\right\}.$$

Given $\mathscr{A}$ and $\vec{A} \in {}^k\mathscr{P}(|\mathscr{A}|)$, there is a unique z and $\vec{p} \in {}^z\bar{\kappa}$ so that $\mathscr{A} \vDash_\kappa \bigotimes_{k,z}[\vec{A}, \vec{p}]$, that is such that, for each $c \in z$, $p_c = \mathrm{card}(\bigcap_{j<k} A_j^{c(j)})$, and, for each $c \in {}^k2 - z$, $\mathrm{card}(\bigcap_{j<k} A_j^{c(j)}) \notin \bar{\kappa}$.

For κ_0, κ_1, $\kappa \in \mathrm{Card}$, let $\kappa_0 \simeq_\kappa \kappa_1$ iff either $\kappa_0, \kappa_1 \notin \bar{\kappa}$ or $\kappa_0 = \kappa_1 \in \bar{\kappa}$. In what follows, let Δ_0, Δ_1 be as above, let θ be a profile for $\Delta_0 \cup \Delta_1$, and let $\Delta_2 = \{\mu_0, \ldots, \mu_{m-1}\} \subseteq \mathbf{Var}(2)$, all μ_i different from all variables of the form μ_c and all $v_i \in \Delta_0$ different from all variables of the form v_c for $c \in {}^k2$. We write '$(\exists\vec{v})$' for '$(\exists v_0)\cdots(\exists v_{l-1})$', etc. Let $\varphi \in \mathrm{Fml}(L^{1,2*}(\mathbf{exactly}))$ be pure and have free variables in $\Delta_0 \cup \Delta_1 \cup \Delta_2$. For each $\mu_i \in \Delta_2$ fix a distinct $\mu_i' \in \mathrm{Var}(0)$ different from all type-0 variables fixed so far.

LEMMA 1. *Suppose that* $\kappa \in \mathrm{Card}$ *is infinite,* $\mathscr{A}_0$ *and* $\mathscr{A}_1$ *are models with* $\mathrm{card}(\mathscr{A}_0) \simeq_\kappa \mathrm{card}(\mathscr{A}_1)$, $\vec{a}_i \in {}^l|\mathscr{A}_i|$, $\vec{A}_i \in {}^k\mathscr{P}(|\mathscr{A}_i|)$ *for* $i \in 2$, *and* $\vec{q} \in {}^m\bar{\kappa}$. *Suppose that for* $\vec{p} \in {}^z\bar{\kappa}$ *and both* $i \in 2$, $\mathscr{A}_i \vDash_\kappa (\theta \mathbin{\&} \bigotimes_{k,z})[\vec{a}_i, \vec{A}_i, \vec{p}]$, *where* μ_c *is assigned to* p_c *for each* $c \in z$. *Then*

$$\mathscr{A}_0 \vDash_\kappa \varphi[\vec{a}_0, \vec{A}_0, \vec{q}] \quad \text{iff} \quad \mathscr{A}_1 \vDash_\kappa \varphi[\vec{a}_1, \vec{A}_1, \vec{q}].$$

LEMMA 2. $(\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \bigotimes_{k,z} \mathbin{\&} \varphi)$ *is superequivalent to* $(\forall\vec{\gamma})(\forall\vec{v})((\theta \mathbin{\&} \bigotimes_{k,z}) \supset \varphi)$.

(*Note. Superequivalent* means equivalent under $\vDash_\kappa$ for models of cardinality $\geq \kappa$ for every $\kappa \in \mathrm{Card}$.)

LEMMA 3. *From* φ, θ *and* $z \subseteq {}^k2$ *we may construct* $\psi \in \mathrm{Fml}(L_2(\mathbf{exactly}))$ *with free variables among* $\vec{\mu}'$ *and* $\vec{v}_z$ *so that for any infinite* $\kappa \in \mathrm{Card}$, *any model* $\mathscr{A}$ *with* $\mathrm{card}(\mathscr{A}) \geq \kappa$, *any* $\vec{p} \in {}^z\bar{\kappa}$ *and any* $\vec{q} \in {}^m\bar{\kappa}$

$$\mathscr{A} \vDash_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \textstyle\bigotimes_{k,z} \mathbin{\&} \varphi)[\vec{p}, \vec{q}] \quad \textit{iff} \quad \mathscr{M}_2(\kappa) \vDash \psi(\vec{p}, \vec{q});$$

in the latter statement, v_c *is assigned to* p_c *for each* $c \in z$.

The proof of Lemma 1 is by induction on the construction of φ. We consider the case in which φ is $(\mathbf{exactly}\ \mu_{k-1}\, v_l)\varphi_0$. By the induction hypothesis, for all $i \in 2$

$$\mathscr{A}_0 \vDash_\kappa \varphi_0[\vec{a}_0, a_{0,i}, \vec{A}_0, \vec{q}] \quad \text{iff} \quad \mathscr{A}_1 \vDash_\kappa \varphi_0[\vec{a}_1, a_{1,i}, \vec{A}_1, \vec{q}].$$

Suppose that for $j \in 2$, $a_j \notin \{a_{j,0}, \ldots, a_{j,l-1}\}$ and $a_j \in \bigcap_{i\in k} A_{j,i}^{c(i)}$ for $c \in {}^k2$ independent of j; again we have

$$\mathscr{A}_0 \vDash_\kappa \varphi_0[\vec{a}_0, a_0, \vec{A}_0, \vec{q}] \quad \text{iff} \quad \mathscr{A}_1 \vDash_\kappa \varphi_0[\vec{a}_1, a_1, \vec{A}_1, \vec{q}].$$

Thus for some $s \subseteq l$ and $t \subseteq {}^k2$

$$\hat{v}_l\varphi_0[\vec{a}_j, \vec{A}_j, \vec{q}]^{\mathscr{A}_j} = \bigcup\left\{\bigcap_{i<k} A_{j,i}^{c(i)} : c \in t\right\} - \{a_{j,i} : i \in s\},$$

for both $j \in 2$. Thus, by the construction of θ and $\bigotimes_{k,z}$,

$$\mathrm{card}(\hat{v}_l\varphi_0[\vec{a}_0, \vec{A}_0, \vec{q}]^{\mathscr{A}0}) \simeq_\kappa \mathrm{card}(\hat{v}_l\varphi_0[\vec{a}_1, \vec{A}_1, \vec{q}]^{\mathscr{A}1});$$

this yields the desired biconditional. Other cases are left to the reader.

Lemma 2 follows easily from Lemma 1.

The proof of Lemma 3 is by induction on the construction of φ. We may assume that none of the bound variables in φ are among those variables fixed so far. It will be convenient to suppose that negation and disjunction, rather than '$\supset$' and '$\perp$', are primitive.

Suppose φ is $(v_i = v_{i'})$ or $\gamma_j(v_i)$. If φ is a conjunct of θ let ψ be '$\neg\perp$'; otherwise $\neg\varphi$ is a conjunct of θ, in which case let ψ be '$\perp$'.

Suppose φ is $\neg\varphi_0$ and that ψ_0 has been constructed for φ_0 and θ; let ψ be $\neg\psi_0$. Then for any model $\mathscr{A}$ with $\operatorname{card}(\mathscr{A}) \geq \kappa$, the following are equivalent:

$$\begin{aligned}
&\mathscr{A} \models_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \textstyle\bigotimes_{k,z} \mathbin{\&} \varphi)[\vec{p},\vec{q}];\\
&\mathscr{A} \not\models_\kappa (\forall\vec{\gamma})(\forall\vec{v})((\theta \mathbin{\&} \textstyle\bigotimes_{k,z}) \supset \varphi_0)[\vec{p},\vec{q}];\\
&\mathscr{A} \not\models_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \textstyle\bigotimes_{k,z} \mathbin{\&} \varphi_0)[\vec{p},\vec{q}];\\
&\mathscr{M}_2(\kappa) \not\models \psi_0(\vec{p},\vec{q});
\end{aligned}$$

this uses Lemma 2.

Suppose φ is $(\varphi_0 \vee \varphi_1)$, and ψ_i has been constructed for φ_i for $i \in 2$. Let ψ be $(\psi_0 \vee \psi_1)$. Since $(\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \bigotimes_{k,z} \mathbin{\&} \varphi)$ is superequivalent to

$$(\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \textstyle\bigotimes_{k,z} \mathbin{\&} \varphi_0) \vee (\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \textstyle\bigotimes_{k,z} \mathbin{\&} \varphi_1),$$

ψ is as required.

Suppose φ is $(\exists v_l)\varphi_0$. Let $\theta_0,\ldots,\theta_{y-1}$ be the profiles for $\Delta_0 \cup \{v_l\} \cup \Delta_1$, consistent with θ; suppose ψ_i has been constructed for θ_i, z and φ_0, for each $i < l$. Since $\theta \mathbin{\&} \bigotimes_{k,z} \mathbin{\&} \varphi_0$ is superequivalent to $\bigvee\{\theta_i \mathbin{\&} \bigotimes_{k,z} \mathbin{\&} \varphi_0 : i < y\}$ for some y, the following are superequivalent:

$$\begin{aligned}
&(\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \textstyle\bigotimes_{k,z} \mathbin{\&} \varphi);\\
&(\exists\vec{\gamma})(\exists\vec{v})(\exists v_l)(\theta \mathbin{\&} \textstyle\bigotimes_{k,z} \mathbin{\&} \varphi_0);\\
&\textstyle\bigvee\{(\exists\vec{\gamma})(\exists\vec{v})(\exists v_l)(\theta_i \mathbin{\&} \bigotimes_{k,z} \mathbin{\&} \varphi_0) : i < y\}.
\end{aligned}$$

Thus $\bigvee\{\psi_i : i < y\}$ is as desired.

Suppose that φ is $(\exists\gamma_k)\varphi_0$. For each $d \in {}^{(k+1)}2$ we have $\mu_d \in \mathbf{Var}(2)$ and $v_d \in \mathbf{Var}(0)$, all distinct from the other variables under discussion; for $z' \subseteq {}^{k+1}2$, let $\vec{\mu}_{z'}$ and $\vec{v}_{z'}$ be the sequences of these μ_d and v_d respectively, for $d \in z'$ according to the lexicographic order of ${}^{(k+1)}2$. Let $\theta_0,\ldots,\theta_{2^l-1}$ be the listing of all profiles for $\Delta_0 \cup \Delta_1 \cup \{\gamma_k\}$ consistent with θ. Let z' refine z iff for all c

$$c \in z \quad \text{iff} \quad c{*}0 \text{ and } c{*}1 \in z'.$$

(Here $c{*}i = c \cup \{\langle k,i\rangle\}$ for $c \in {}^k2$.) Suppose that $\psi_{j,z'}$ has been constructed for θ_j, z', φ_0, for each $j < 2^l$ and z' refining z. Let $\psi_{z'}$ be

$$\bigvee\left\{(\exists\vec{v}_{z'})\left(\psi_{j,z'} \mathbin{\&} \mathop{\&}_{c\in z} \mathbf{A}(v_{c*0}, v_{c*1}, v_c)\right) : j < 2^l\right\}.$$

Let ψ be $\bigvee\{\psi_{z'} : z' \text{ refines } z\}$. We now show that ψ is as required. Suppose that

$$\mathscr{A} \models_\kappa (\theta \mathbin{\&} \textstyle\bigotimes_{k,z} \mathbin{\&} \varphi)[\vec{a},\vec{A},\vec{p},\vec{q}].$$

Fix $A \subseteq |\mathscr{A}|$ so that $\mathscr{A} \models_\kappa \varphi_0[\vec{a},\vec{A},A,\vec{q}]$. Fix the unique z', $\vec{r} \in {}^{z'}\bar{\kappa}$ and $j < 2^l$ such that

$$\mathscr{A} \models_\kappa (\theta_j \mathbin{\&} \textstyle\bigotimes_{k+1,z'})[\vec{a},\vec{A},A,\vec{r}].$$

Clearly z' refines z, and for $c \in z$ and $i_0 \in 2$

$$r_{c*i_0} = \operatorname{card}\left(\bigcap_{i<k} A_i^{c(i)} \cap A^{i_0}\right);$$

so $p_c = r_{c*0} + r_{c*1}$. By choice of j, $\mathscr{M}_2(\kappa) \vDash \psi_{j,z'}[\vec{r}, \vec{q}]$; so $\mathscr{M}_2(\kappa) \vDash \psi[\vec{p}, \vec{q}]$. Conversely, suppose that $\mathscr{M}_2(\kappa) \vDash \psi[\vec{p}, \vec{q}]$. Fix $j < 2^l$, z' refining z and $\vec{r} \in {}^{z'}\kappa$ so that $\mathscr{M}_2(\kappa) \vDash \psi_{j,z'}[\vec{r}, \vec{q}]$ and $q_c = r_{c*0} + r_{c*1}$ for all $c \in z$. By the induction hypothesis

$$\mathscr{A} \vDash_\kappa (\exists\vec{\gamma})(\exists\gamma_k)(\exists\vec{v})(\theta_j \,\&\, \bigotimes_{k,z} \,\&\, \varphi_0)[\vec{r}, \vec{q}].$$

When $\mathscr{A} \vDash_\kappa (\theta_j \,\&\, \bigotimes_{k+1,z'} \,\&\, \varphi_0)[\vec{a}, \vec{A}, A, \vec{r}, \vec{q}]$, $\mathscr{A} \vDash_\kappa \varphi[\vec{a}, \vec{A}, \vec{q}]$; since $r_{c*i_0} = \operatorname{card}(\bigcap_{i<k} A_i^{c(i)} \cap A^{i_0})$ for $i_0 \in 2$,

$$\mathscr{A} \vDash_\kappa \bigotimes_{k,z}[\vec{A}, \vec{p}];$$

thus

$$\mathscr{A} \vDash_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \,\&\, \bigotimes_{k,z} \,\&\, \varphi)[\vec{p}, \vec{q}],$$

as required.

Suppose that φ is $(\exists\mu_m)\varphi_0$ and that φ_0 has been constructed for φ_0. Then we take ψ to be $(\exists\mu'_m)\psi_0$, where $\mu'_m \in \mathbf{Var}(0)$ is new.

Now suppose φ is (**exactly** $\mu_{m-1}\, v_l)\varphi_0$ for $v_l \in \mathbf{Var}(0)$. Let $\theta_0, \ldots, \theta_{y-1}$ be the profiles for $\Delta_0 \cup \{v_l\} \cup \Delta_1$ consistent with θ and such that θ_j has the form $\theta \,\&\, \bar{\theta}_j$, where $\bar{\theta}_j$ has the form

$$\&\{(v_i \neq v_l): i < l\} \,\&\, \&\{\gamma_i(v_l)^{c(i)}: i < k\};$$

let c_j be that $c \in {}^k2$. For $X \subseteq y$, let φ_X be

$$\&\{(\exists v_l)(\bar{\theta}_j \,\&\, \varphi_0): j \in X\} \,\&\, \&\{\neg(\exists v_l)(\bar{\theta}_j \,\&\, \varphi_0): j \in y - X\}.$$

By the induction hypothesis and the previous cases of the induction step, there is a ψ_X corresponding to θ, z, φ_X meeting the conditions of this lemma.

Claim 1. *For each $\vec{p} \in {}^z\bar{\kappa}$ and $\vec{q} \in {}^m\bar{\kappa}$ there is a unique $X = X_{\vec{p},\vec{q}} \subseteq y$ such that $\mathscr{M}_2(\kappa) \vDash \psi_X[\vec{p}, \vec{q}]$.*

To see this, fix $\mathscr{A}$, $\vec{a} \in {}^l|\mathscr{A}|$ and $\vec{A} \in {}^k\mathscr{P}(|\mathscr{A}|)$ so that

$$\mathscr{A} \vDash_\kappa (\theta \,\&\, \bigotimes_{k,z})[\vec{a}, \vec{A}, \vec{p}, \vec{q}].$$

Let $j \in X$ iff $\mathscr{A} \vDash_\kappa (\exists v_l)(\bar{\theta}_j \,\&\, \varphi_0)[\vec{a}, \vec{A}, \vec{q}]$; therefore $\mathscr{A} \vDash_\kappa \varphi_X[\vec{a}, \vec{A}, \vec{q}]$; so $\mathscr{M}_2(\kappa) \vDash_\kappa \psi_X[\vec{p}, \vec{q}]$. Now suppose that, for $X' \subseteq y$, $\mathscr{M}_2(\kappa) \vDash_\kappa \psi_{X'}[\vec{p}, \vec{q}]$; so

$$\mathscr{A} \vDash_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \,\&\, \bigotimes_{k,z} \,\&\, \varphi_{X'})[\vec{p}, \vec{q}].$$

Fixing witnesses $\vec{B} \in {}^k\mathscr{P}(|\mathscr{A}|)$ and $\vec{b} \in {}^l|\mathscr{A}|$, we have that $j \in X'$ iff $\mathscr{A} \vDash_\kappa (\exists v_l)(\bar{\theta}_j \,\&\, \varphi_0)[\vec{b}, \vec{B}, \vec{q}]$. By Lemma 1, the right-hand side holds iff

$$\mathscr{A} \vDash_\kappa (\exists v_l)(\bar{\theta}_j \,\&\, \varphi_0)[\vec{a}, \vec{A}, \vec{q}];$$

thus $X = X'$, establishing Claim 1.

For $U \subseteq l/\simeq_\theta$, let φ^*_U be

$$\&\{\varphi_0(v_l/v_i): [i] \in U\} \,\&\, \&\{\neg\varphi_0(v_l/v_i): [i] \notin U\}.$$

For $X \subseteq y$ let X^* be

$$\{i < l\colon \text{for some } j \in y - X, \text{ for each } j' \in k \; \gamma(v_i)^{c_j(j')} \text{ is a conjunct of } \theta_j\}.$$

For U and X as above let $\bar{\psi}_{X,U}$ be

$$\mu_{m-1} + c = \sum_{j \in X} v_{c_j} + d,$$

expressed using 'S', where $c = \operatorname{card}(l/\simeq_\theta - U)$ and $d = \operatorname{card}\{[i]\colon i \in X^*\}$. Let ψ be

$$\bigvee\{\psi_X \mathbin{\&} \varphi_U \mathbin{\&} \bar{\psi}_{X,U}\colon X \subseteq y \text{ and } U \subseteq l/\simeq_\theta\}.$$

We must show that ψ is as required.

Suppose that $\mathscr{A} \vDash_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \bigotimes_{k,z} \mathbin{\&} \varphi)[\vec{p},\vec{q}]$, with $\vec{A} \in {}^k\mathscr{P}(|\mathscr{A}|)$ and $\vec{a} \in {}^l|\mathscr{A}|$ as witnesses. Let $Y = \hat{v}_l\varphi_0[\vec{a},\vec{A},\vec{q}]^{\mathscr{A}}$; thus $\operatorname{card}(Y) = p_{m-1}$. Let $X = X_{\vec{p},\vec{q}}$. Fix $U \subseteq l$ so that $\mathscr{A} \vDash_\kappa \varphi_U[\vec{a},\vec{A},\vec{q}]$.

Claim 2. *For $j < y$, if $\mathscr{A} \vDash_\kappa \bar{\theta}_j[\vec{a},a,\vec{A}]$ then $a \in Y$ iff $j \in X$.*

Assume the antecedent. If $a \in Y$ then by the argument for Claim 1, $j \in X$. Suppose that $j \in X$. By the definition of X, $\mathscr{M}_2(\kappa) \vDash \psi_X[\vec{p},\vec{q}]$; so by the construction of ψ_X,

$$\mathscr{A} \vDash_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \bigotimes_{k,z} \mathbin{\&} \varphi_X)[\vec{p},\vec{q}];$$

fix witnesses $\vec{B}$ and $\vec{b}$ for this. Since $j \in X$,

$$\mathscr{A} \vDash_\kappa (\exists v_l)(\bar{\theta}_j \mathbin{\&} \varphi_0)[\vec{b},\vec{B},\vec{q}];$$

fix the witness b for this. By Lemma 1, $\mathscr{A} \vDash_\kappa \varphi_0[\vec{a},a,\vec{A},\vec{q}]$ iff $\mathscr{A} \vDash_\kappa \varphi_0[\vec{b},b,\vec{B},\vec{q}]$; thus $a \in Y$.

Thus we have

$$Y \cup \{a_i \notin Y\colon i < l\}$$
$$= \bigcup_{j \in X} \bigcap_{j' < k} A_{j'}^{c_j(j')} \cup \left\{a_i\colon i < l \text{ and, for some } j \in y - X, a_i \in \bigcap_{j' < k} A_{j'}^{c_j(j')}\right\}.$$

Therefore

$$\operatorname{card}(Y) + \operatorname{card}(l/\simeq_\theta - U)$$
$$= \sum_{j \in X} \operatorname{card}\left(\bigcap_{j' < k} A_{j'}^{c_j(j')}\right) + \operatorname{card}\{[i]\colon i \in X^*\},$$

yielding $\mathscr{M}_2(\kappa) \vDash \bar{\psi}_{X,U}[\vec{p},\vec{q}]$, and thus $\mathscr{M}_2(\kappa) \vDash \psi[\vec{p},\vec{q}]$.

Now suppose that $\mathscr{M}_2(\kappa) \vDash \psi[\vec{p},\vec{q}]$. Fix X and U so that

$$\mathscr{M}_2(\kappa) \vDash (\psi_X \mathbin{\&} \varphi_U \mathbin{\&} \bar{\psi}_{X,U})[\vec{p},\vec{q}].$$

By Claim 1, $X = X_{\vec{p},\vec{q}}$. By the construction of ψ_X,

$$\mathscr{A} \vDash_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \mathbin{\&} \bigotimes_{k,z} \mathbin{\&} \varphi_X)[\vec{p},\vec{q}];$$

fix witnesses $\vec{A}$ and $\vec{a}$. Letting $Y = \hat{v}_l\varphi_0[\vec{a},\vec{A},\vec{q}]^{\mathscr{A}}$, by the construction of $\bar{\psi}_{X,U}$ we have $\operatorname{card}(Y) = q_{m-1}$. Thus $\mathscr{A} \vDash_\kappa \varphi[\vec{a},\vec{A},\vec{q}]$.

For the final case, suppose φ is $(\textbf{exactly}\ \mu_{m-1}\,\mu_m)\varphi_0$, with $\mu_m \in \textbf{Var}(2)$ distinct from the entries in $\vec{\mu}$, and with ψ_0 constructed for φ_0. Let ψ be $(\textbf{exactly}\ \mu_{m-1}\,\mu_m)\psi_0$. Suppose that

$$\mathscr{A} \models_\kappa (\theta \ \&\ \bigotimes_{k,z} \ \&\ \varphi)[\vec{a}, \vec{A}, \vec{p}, \vec{q}].$$

Letting $Y = \hat{\mu}_{m-1}\varphi_0[\vec{a}, \vec{A}, \vec{q}]^{\mathscr{A}}$, $q_{m-1} = \mathrm{card}(Y)$. For each $q \in Y$,

$$\mathscr{A} \models_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \ \&\ \bigotimes_{k,z} \ \&\ \varphi_0)[\vec{p}, \vec{q}, q];$$

by the induction hypothesis, $\mathscr{M}_2(\kappa) \models \psi_0[\vec{p}, \vec{q}, q]$. Conversely, if $\mathscr{M}_2(\kappa) \models \psi_0[\vec{p}, \vec{q}, q]$, by the induction hypothesis, fix $\vec{b}$ and $\vec{B}$ so that $\mathscr{A} \models_\kappa \varphi_0[\vec{b}, \vec{B}, \vec{p}, \vec{q}, q]$. By Lemma 1, $\mathscr{A} \models_\kappa \varphi_0[\vec{a}, \vec{A}, \vec{q}, q]$; so $q \in Y$. Thus $\mathscr{M}_2(\kappa) \models \psi[\vec{p}, \vec{q}]$. Now suppose that $\mathscr{M}_2(\kappa) \models \psi[\vec{p}, \vec{q}]$; letting $Y = \hat{\mu}'_m\psi_0[\vec{p}, \vec{q}]^{\mathscr{M}_2(\kappa)}$, we have $q_{m-1} = \mathrm{card}(Y)$. Suppose that Y is nonempty; fix $q \in Y$; then

$$\mathscr{A} \models_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \ \&\ \bigotimes_{k,z} \ \&\ \varphi_0)[\vec{p}, \vec{q}, q].$$

Fix witnesses $\vec{A}$ and $\vec{a}$ for this. By Lemma 1, for any $q' \in Y$,

$$\mathscr{A} \models_\kappa \varphi_0[\vec{a}, \vec{A}, \vec{q}, q'].$$

For any $q' \notin Y$,

$$\mathscr{A} \not\models_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \ \&\ \bigotimes_{k,z} \ \&\ \varphi_0)[\vec{p}, \vec{q}, q'].$$

Thus for any $q' \in \bar{\kappa}$, $q' \in Y$ iff $\mathscr{A} \models_\kappa \varphi_0[\vec{a}, \vec{A}, \vec{q}, q']$. Thus $\mathscr{A} \models_\kappa \varphi[\vec{a}, \vec{A}, \vec{q}]$. If Y is empty, $q_{m-1} = 0$; fix any $\vec{a}$ and $\vec{A}$ so that $\mathscr{A} \models_\kappa (\theta \ \&\ \bigotimes_{k,z})[\vec{a}, \vec{A}, \vec{p}]$; then for all $q \in \bar{\kappa}$, $\mathscr{A} \not\models_\kappa \varphi_0[\vec{a}, \vec{A}, \vec{q}, q]$; thus $\mathscr{A} \models_\kappa \varphi[\vec{a}, \vec{A}, \vec{q}]$. So

$$\mathscr{A} \models_\kappa (\exists\vec{\gamma})(\exists\vec{v})(\theta \ \&\ \bigotimes_{k,z} \ \&\ \varphi)[\vec{p}, \vec{q}].$$

This completes the proof of Theorem 1(ii). By restricting this construction to $\varphi \in \mathrm{Fml}(L^{1,2}(\textbf{exactly}))$, part (i) also follows.

COROLLARY. *If κ is an aleph then κ-satisfiability for formulae of $L^{1,2}$(**exactly**) is decidable. For $\varphi \in \mathrm{Fml}(L^{1,2}(\textbf{exactly}))$ with free variables among $\vec{v}, \vec{\gamma}, \vec{\mu}$ as above, φ is κ-satisfiable iff for some profile θ for $\vec{v}, \vec{\gamma}$ and some $z \subseteq {}^k2$, $(\exists\vec{\gamma})(\exists\vec{v})(\theta \ \&\ \bigotimes_{z,k} \ \&\ \varphi)$ is κ-satisfiable.*

Since the translation constructed in proving Theorem 1(i) was effective, the corollary follows if satisfiability in $\mathscr{M}_2(\kappa)$ for formulae of L_2 is decidable. Since κ is an aleph, for $n, m \in \bar{\kappa}$, if either n or m is transfinite, their sum is their maximum; thus addition is really no worse than on $\bar{\aleph}_0$; so the last-mentioned problem is decidable; details are left to the reader.

§3. We now consider the special case in which $\kappa = \aleph_0$. For this section, let $\mathscr{M} = \mathscr{M}_2(\aleph_0)$. The next theorem identifies the class of relations $\aleph_0$-definable in $L^{1,2*}$(**exactly**) with a simple and well-understood class of relations on $\bar{\aleph}_0$; see [3] for more information on that class.

THEOREM 2. *R is $\aleph_0$-definable in $L^{1,2*}$(**exactly**) iff R is first-order definable over $\mathscr{M}$, i.e. iff R is definable in Presburger arithmetic.*

LEMMA 4. *If R is definable in L_2(**exactly**) over $\mathscr{M}$ then R is first-order definable over $\mathscr{M}$.*

Using Theorem 1, Lemma 4 will yield Theorem 2 from left to right. The other direction is trivial. To prove Lemma 4, it will be convenient to revise $\mathscr{M}$, L_2, and

L_2(**exactly**). Fix **Pred** $= \{`<'\} \cup \{`E_m': 0 < m < \omega\}$, all 2-place; let **Funct** $= \{`\mathbf{0}', `s', `+'\}$, where '**0**' is 0-place, 's' is 1-place, and '$+$' is 2-place. Let the first-order language based on this **Pred, Funct** be L_2'. Let $|\mathscr{M}'| = \bar{\aleph}_0$, $<^{\mathscr{M}'} = < \restriction \bar{\aleph}_0$, $\mathbf{0}^{\mathscr{M}'} = 0$, $s^{\mathscr{M}'} =$ successor $\restriction \aleph_0$, $+^{\mathscr{M}'} =$ addition $\restriction \bar{\aleph}_0$, and $E_m^{\mathscr{M}'} = \{\langle n_0, n_1\rangle : n_0 \equiv_m n_1\}$. Since there are obvious translations between L_2 interpreted over $\mathscr{M}$ and L_2' interpreted over $\mathscr{M}'$, it suffices in Lemma 4 to replace the former pair by the latter pair. We will construct a translation t from Fml(L_2'(**exactly**)) to the set of quantifier-free formula of L_2' so that for any $\varphi \in$ Fml(L_2'(**exactly**)), $\varphi \equiv t(\varphi)$ is true in $\mathscr{M}'$ for all assignments of values to free variables.

If φ is '$\perp$', $(\tau = \sigma)$, $(\tau < \sigma)$, or $E_m(\tau, \sigma)$, let $t(\varphi) = \varphi$. Let $t(\varphi \supset \psi)$ be $t(\varphi) \supset t(\psi)$. Form $t((\exists v)\varphi)$ by eliminating the prefixed quantifier in $(\exists v)t(\varphi)$, using the procedure due to Presburger and described in [1, pp. 189–192].

We must construct $t((\textbf{exactly}\ \rho\ v)\varphi)$, where $t(\varphi)$ has been constructed. Suppose that $t(\varphi)$ is in disjunctive normal form; say $t(\varphi)$ *is* $\bigvee_{j<z} \psi_j$, where each ψ_j is a conjunction of atomic and negated atomic formulae. If $z = 0$, let $t((\textbf{exactly}\ \rho\ v)\varphi)$ be $(\rho = \mathbf{0})$; suppose that $z > 0$. Without loss of generality let each ψ_j be ψ_j^+ & ψ_j^- with v occurring free in each conjunct of ψ_j^+ and not in any conjunct of ψ_j^-, and suppose that for distinct $j < j' < z$:

either ψ_j^- is $\psi_{j'}^-$ or ψ_j^- and $\psi_{j'}^-$ are incompatible;
if ψ_j^- is $\psi_{j'}^-$ then ψ_j^+ and $\psi_{j'}^+$ are incompatible.

We may also suppose that no conjunct of ψ_j is negated, making these replacements and distributing '&' into '$\vee$':

$$(\tau \neq \sigma) : (\tau < \sigma) \vee (\sigma < \tau);$$
$$\neg(\tau < \sigma) : (\sigma < \tau) \vee (\tau = \sigma);$$
$$\neg E_m(\tau, \sigma) : \bigvee_{0<i<m} E_m(\tau, s^i(\sigma)).$$

We let the left entry below abbreviate the right entry:

$$n\tau : \tau + \cdots + \tau, \text{ with } n \text{ occurrences of } \tau;$$
$$\tau_2 = \tau_0 - \tau_1 : \tau_0 = \tau_1 + \tau_2;$$
$$E_m(\tau_2, \tau_0 - \tau_1) : E_m(\tau_0, \tau_1 + \tau_2).$$

Thus each ψ_j^+ may be viewed as a conjunction of atomic formulae of these forms:

$$nv = \tau_0 - \tau_1, \qquad E_m(nv, \tau_0 - \tau_1), \qquad nv < \tau_0 - \tau_1, \qquad nv > \tau_0 - \tau_1.$$

For $j, j' \in z$, let $j \approx j'$ iff ψ_j^- is $\psi_{j'}^-$; $\approx$ is an equivalence relation; for $d \in z/\approx$, let ψ_d^- be ψ_j^- for any $j \in d$; let ψ_d^+ be $\bigvee_{j \in d} \psi_j^+$. Then $(\textbf{exactly}\ \rho\ v)\varphi$ is equivalent over $\mathscr{M}'$ to

$$\bigvee_{d \in z/\approx} (\psi_d^- \ \&\ (\textbf{exactly}\ \rho\ v)\psi_d^+) \vee \left(\rho = \mathbf{0}\ \&\ \mathop{\&}_{d \in z/\approx} \neg\psi_d^-\right).$$

Fix a $d \in z/\approx$; suppose $d = \{j_0, \ldots, j_{q-1}\}$. Let $\rho_{j_0}, \ldots, \rho_{j_{q-1}} \in \mathbf{Var}(0)$ be distinct, distinct from ρ, and not occurring in ψ_d^+. Then $(\textbf{exactly}\ \rho\ v)\psi_d^+$ is equivalent over $\mathscr{M}'$ to

$$(\exists\rho_{j_0})\ldots(\exists\rho_{j_{q-1}})\left(\rho = \rho_{j_0} + \cdots + \rho_{j_{q-1}}\ \&\ \mathop{\&}_{i<q} (\textbf{exactly}\ \rho_{j_i}\ v)\psi_{j_i}^+\right).$$

We now take a careful look at (**exactly** $\rho_j v)\psi_j^+$ for $j \in d$. If some conjunct of ψ_j^+ has the form $nv = \tau_0 - \tau_1$, let $\bar{\psi}_j$ be

$$((\exists v)(nv = \tau_0 - \tau_1) \mathbin{\&} \rho_j = \mathbf{1}) \vee (\neg(\exists v)(nv = \tau_0 - \tau_1) \mathbin{\&} \rho_j = \mathbf{0}).$$

Suppose now that no conjunct of ψ_j^+ has that form. Transform ψ_j^+ into ψ_j^{++} by uniformizing the coefficients on v; in other words, where p is the least common multiple of all such coefficients, multiply through each atomic conjunct in which v has coefficient n by p/n. Thus in ψ_j^{++} all occurrences of v have coefficient p; clearly ψ_j^{++} is equivalent over $\mathcal{M}'$ to ψ_j^+. Transform ψ_j^{++} into ψ_j^{+++} by replacing each occurrence of pv by v and conjoining $E_p(v, \mathbf{0})$. Then:

$$\mathcal{M}' \models \psi_j^{++}[\alpha_a^v] \quad \text{iff} \quad \mathcal{M}' \models \psi_j^{+++}[\alpha_{pa}^v];$$
$$\text{if } \mathcal{M}' \models \psi_j^{+++}[\alpha_b^v] \text{ then } b = p \cdot a \text{ for some } a.$$

Thus (**exactly** $\rho_j v)\psi_j^+$ is equivalent over $\mathcal{M}'$ to (**exactly** $\rho_j v)\psi_j^{+++}$. ψ_j^{+++} has the form

$$\mathop{\&}_{i<r} (\tau_i - \tau_i' < v) \mathbin{\&} \mathop{\&}_{i<t} (v < \sigma_i - \sigma_i') \mathbin{\&} \mathop{\&}_{i<u} E_{m_i}(v, \eta_i - \eta_i').$$

We may then suppose that $0 < r$, since we could let τ_0 and τ_0' be '$\mathbf{0}$'.

For each nonempty $b \subseteq r$ and $c \subseteq t$ let $\theta_{0,b}$ and $\theta_{1,c}$ be respectively

$$\mathop{\&}\{(\tau_i - \tau_i' = \tau_{i'} - \tau_{i'}'): i, i' \in b\} \mathbin{\&} \mathop{\&}\{(\tau_i - \tau_i' < \tau_{i'} - \tau_{i'}'): i \in b \text{ and } i' \in r - b\};$$
$$\mathop{\&}\{(\sigma_i - \sigma_i' = \sigma_{i'} - \sigma_{i'}'): i, i' \in c\} \mathbin{\&} \mathop{\&}\{(\sigma_i - \sigma_i' < \sigma_{i'} - \sigma_{i'}'): i \in c \text{ and } i' \in t - c\}.$$

We may suppose, without loss of generality, that there is a unique $b \subseteq r$ (let it be i_b), and if $t \neq 0$ there is a unique $c \subseteq t$ (let it be i_c) so that $\theta_{0,b}$ and $\theta_{1,c}$ are conjuncts of each disjunct in ψ_d^+. (If this condition fails, we may select "finer" ψ_j's and a larger z at the beginning and end up with formulae for which this holds.) If $t \neq 0$ let $\hat{\psi}_j$ be

$$(\tau_{i_b} - \tau_{i_b}' < v) \mathbin{\&} (v < \sigma_{i_c} - \sigma_{i_c}') \mathbin{\&} \mathop{\&}_{i<u} E_{m_i}(v, \eta_i - \eta_i').$$

If $t = 0$ let $\hat{\psi}_j$ be

$$(\tau_{i_b} - \tau_{i_b}' < v) \mathbin{\&} \mathop{\&}_{i<u} E_{m_i}(v, \eta_i - \eta_i').$$

Let α be an assignment of type-0 variables to members of $\bar{\aleph}_0$. Suppose that $\mathcal{M}' \models \theta_{0,b}[\alpha]$ and if $t \neq 0$ then $\mathcal{M}' \models \theta_{1,c}[\alpha]$. Clearly, $\mathcal{M}' \models \psi_j^+[\alpha_a^v]$ iff $\mathcal{M}' \models \hat{\psi}_j[\alpha_a^v]$. Now suppose that $t \neq 0$ and a_0 is the least so that $\mathcal{M}' \models \hat{\psi}_j[\alpha_a^v]$. Let m be the least common multiple of $m_0, \ldots, m_{n-1}$. Then, for any a,

$$\mathcal{M}' \models \mathop{\&}_{i<u} E_{m_i}(v, \eta_i - \eta_i')[\alpha_a^v] \quad \text{iff} \quad a \equiv_m a_0.$$

Then $\operatorname{card}(\hat{v}\hat{\psi}_j^{\mathscr{M}',\alpha})$ = the maximum y such that

$$\mathscr{M}' \models (\mathbf{a}_0 + m\rho_j < \sigma_{i_c} - \sigma'_{i_c})[\alpha_y^{\rho_j}].$$

Let $\bar{\psi}_j$ be

$$\begin{aligned}&[\neg(\exists v)\hat{\psi}_j \,\&\, \rho_j = \mathbf{0}]\\ &\quad \vee [(\exists v)\hat{\psi}_j \,\&\, (\forall v')(v' < v \supset \neg\hat{\psi}_j(v/v')) \,\&\, v + m\rho_j < \sigma_{i_c} - \sigma'_{i_c}\\ &\quad \& (\forall v')(\rho_j < v' \supset \neg(v + mv' < \sigma_{i_c} - \sigma'_{i_c}))],\end{aligned}$$

where $v' \in \mathbf{Var}(0)$ does not occur in $\hat{\psi}_j$ and is distinct from ρ_j. Now suppose that $t = 0$. If $\mathscr{M}' \models \hat{\psi}_j[\alpha_{a_0}^v]$ then, for infinitely many a, $\mathscr{M}' \models \hat{\psi}_j[\alpha_a^v]$; so $\operatorname{card}(\hat{v}\hat{\psi}^{\mathscr{M}',\alpha}) \in \{0, \aleph_0\}$. Let $\bar{\psi}_j$ be $\neg(\exists v)\hat{\psi}_j \,\&\, \rho_j = \mathbf{0}$. For either case on t we have, if $\mathscr{M}' \models \psi_d^-[\alpha]$ and $y \in \omega$, then

$$\mathscr{M}' \models \bar{\psi}_j[\alpha_y^{\rho_j}] \quad \text{iff} \quad \mathscr{M}' \models (\mathbf{exactly}\ \rho_j\, v)\psi_j^+[\alpha_y^{\rho_j}].$$

We may now let $t(\varphi)$ *be*

$$\bigvee_{d \in z/\approx} (\exists\rho_{j_0})\cdots(\exists\rho_{j_{q-1}})(\rho = \rho_{j_0} + \cdots + \rho_{j_{q-1}} \,\&\, \bar{\psi}_{j_0} \,\&\cdots\&\, \bar{\psi}_{j_{q-1}}) \vee (\rho = \mathbf{0} \,\&\, \psi^*).$$

By the preceding remarks, $t(\varphi)$ is as required.

Suppose **Pred** and **Funct** are given, all members of **Pred** are at most 1-place, and all members of **Funct** are 0-place. Suppose we are given $\varphi \in \operatorname{Fml}(L^{1,2*}(\mathbf{exactly}))$ with free variables among $v_1, \ldots, v_{l-1} \in \mathbf{Var}(0)$, $\gamma_0, \ldots, \gamma_{n-1} \in \mathbf{Var}(1)$, and $\mu_0, \ldots, \mu_{k-1} \in \mathbf{Var}(2)$. By existentially quantifying out (and replacing 0-place predicates by '$\perp$' or '$\neg\perp$' in all possible ways and taking disjunctions) we obtain a pure $\varphi^* \in \operatorname{Fml}(L^{1,2*}(\mathbf{exactly}))$; let φ' be $(\exists\vec{\gamma})(\exists\vec{v})\varphi^*$. Clearly φ is $\aleph_0$-satisfiable iff φ' is $\aleph_0$-satisfiable. Using the above constructions, φ' may be effectively transformed into a $\psi \in \operatorname{Fml}(L_2)$ so that φ' is $\aleph_0$-satisfiable iff ψ is satisfiable in $\mathscr{M}$; and whether the latter is the case is decidable. Thus:

THEOREM 3. *$\aleph_0$-satisfiability of a formula containing only* 0- *or* 1-*place predicate-constants and* 0-*place function-constants is decidable.*

§4. So far we have been looking at κ-definability in the most powerful of our languages. We will now look at the least powerful ones. For $\rho_0, \rho_2 \in \mathbf{Var}(2)$ and $\mu \in \mathbf{Var}(4)$ let $\mathbf{Plus}'(\rho_0, \mu, \rho_2)$ be

$$(\mathbf{exactly}\ \mu\,\rho)(\rho_0 \le \rho \,\&\, \rho < \rho_2),$$

where $\rho \in \mathbf{Var}(2)$ is distinct from ρ_0 and ρ_2, and $(\rho < \rho_2)$ abbreviates $(\rho \le \rho_2 \,\&\, \neg(\rho_2 \le \rho))$. With $\rho_1 \in \mathbf{Var}(2)$ let $\mathbf{Plus}^*(\rho_0, \rho_1, \rho_2)$ be

$$(\exists\mu)(\mathbf{Plus}'(\rho_0, \mu, \rho_2) \,\&\, \rho_1 =_2 \mu)$$

where, following [5], $(\rho_1 =_2 \mu)$ abbreviates $(\mathbf{exactly}\ \mu\ \rho)\rho < \rho_1$ where $\rho \in \mathbf{Var}(2)$ is distinct from ρ_1. (The latter $\aleph_0$-defines identity on $\bar{\aleph}_0$ with respect to $(2, 4)$.) It is easy to see that with ρ_0, ρ_1, ρ_2 all distinct:

$\mathbf{Plus}'(\rho_0, \mu, \rho_1)$ $\aleph_0$-defines addition on $\bar{\aleph}_0$ with respect to $(2, 4, 2)$;

$\mathbf{Plus}^*(\rho_0, \rho_1, \rho_2)$ $\aleph_0$-defines addition on $\bar{\aleph}_0$ with respect to $(2, 2, 2)$.

Let $\mathbf{Fin}(\rho_2)$ be as in [5], and let $\mathbf{Max}(\rho_0, \rho_1, \rho_2)$ be:

$$(\rho_0 < \rho_1 \supset \rho_2 = \rho_1) \,\&\, (\rho_1 < \rho_0 \supset \rho_2 = \rho_0);$$

let $\mathbf{Plus}(\rho_0, \rho_1, \rho_2)$ abbreviate

$$(\mathbf{Fin}(\rho_2) \supset \mathbf{Plus^*}(\rho_0, \rho_1, \rho_2)) \mathbin{\&} (\neg \mathbf{Fin}(\rho_2) \supset \mathbf{Max}(\rho_0, \rho_1, \rho_2)).$$

For any aleph κ, if ρ_0, ρ_1, ρ_2 are distinct then $\mathbf{Plus}(\rho_0, \rho_1, \rho_2)$ κ-defines addition on $\bar{\kappa}$; in fact, $\mathbf{Plus}(\rho_0, \rho_1, \rho_2)$ uniformly defines cardinal addition. Clearly any R that is first-order definable over $\mathscr{M}_2(\kappa)$ is κ-definable in $L^{0,4}(\mathbf{exactly}, \leq)$ with respect to (2, 2, 2). Together with Theorem 2, the above discussion yields the following.

THEOREM 4. *For any R, R is $\aleph_0$-definable in $L^{0,4}(\mathbf{exactly}, \leq)$ iff R is first-order definable over $\mathscr{M}_2(\aleph_0)$.*

The analysis provided in §3 of definability in $L_2(\mathbf{exactly})$ over $\mathscr{M}_2(\kappa)$ for the case in which $\kappa = \aleph_0$ may be extended to the case in which $\kappa < \aleph_{\omega^2}$. Thus Theorems 2, 3, and 4 also extend to that case; details are left to the reader. Curiously, this is as far as we may go with those theorems: multiplication on $\bar{\aleph}_0$ is $\aleph_{\omega^2}$-definable in $L^{0,4}(\mathbf{exactly}, \leq)$.

The idea is to compute $n \cdot m$ by counting $\bigcup_{i<m}\{\omega i + j : j < n\}$. (Here ωi is an ordinal product.) For $\rho \in \mathbf{Var}(2)$ let $\mathbf{Lim}(\rho)$ be as in [5]; so, for any $\mathscr{A}$ and any $n < \kappa$, $\mathscr{A} \vDash_\kappa \mathbf{Lim}(\mathbf{n})$ iff either $n = 0$ or n is a limit cardinal. For $\mu_0, \mu_1, \mu_2 \in \mathbf{Var}(4)$ let $\mu \in \mathbf{Var}(4)$ be new, and fix distinct $\rho_0, \rho_1, \rho_2, \rho, \rho' \in \mathbf{Var}(2)$. Let $\mathbf{Times}(\mu_0, \mu_1, \mu_2)$ be:

$$\begin{aligned}
&(\exists \rho_0)(\mathbf{Lim}(\rho_0) \mathbin{\&} (\mathbf{exactly}\ \mu_1\, \rho)(\rho \leq \rho_0 \mathbin{\&} \mathbf{Lim}(\rho))\\
&\qquad \mathbin{\&} (\mathbf{exactly}\ \mu_2\, \rho)[\rho < \rho_0 \mathbin{\&} (\exists \rho_1)(\exists \mu)(\mathbf{Lim}(\rho_1)\\
&\qquad\qquad\qquad \mathbin{\&} \neg(\exists \rho_2)(\mathbf{Lim}(\rho_2) \mathbin{\&} \rho_1 < \rho_2 \mathbin{\&} \rho_2 \leq \rho)\\
&\qquad\qquad\qquad \mathbin{\&} (\mathbf{exactly}\ \mu\, \rho')(\rho_1 \leq \rho' \mathbin{\&} \rho' < \rho) \mathbin{\&} \mu < \mu_0)]).
\end{aligned}$$

It is not hard to see that if μ_0, μ_1, μ_2 are distinct then this formula $\aleph_{\omega^2}$-defines multiplication on $\bar{\aleph}_0$. Since multiplication is not first-order definable over $\mathscr{M}_2(\aleph_{\omega^2})$, the analogs of Theorems 2 and 4 fail for $\aleph_{\omega^2}$. Furthermore, arithmetic-truth can be encoded into $\aleph_{\omega^2}$-satisfiability of pure sentences of $L^{0,4}(\mathbf{exactly}, \leq)$, and thus of $L^{1,2*}(\mathbf{exactly})$. So the Turing degree of the set of $\aleph_{\omega^2}$-satisfiable pure sentences of even $L^{0,4}(\mathbf{exactly}, \leq)$ is at least $\mathbf{0}^{(\omega)}$.

§5. We now show that $L^{0,4}(\mathbf{exactly}, \leq)$ is the lowest language in the hierarchy for which the sorts of results proved so far could hold. Let L_1 be the first-order language with nonlogical lexicon $\mathbf{Pred} = \{`\leq'\}$ and $\mathbf{Funct} = \{\mathbf{n} : n < \aleph_0\}$; let L_0 be the first-order language based on $\mathbf{Pred}$ empty and $\mathbf{Funct}$ as just above. For $\kappa \in$ Card let $\mathscr{M}_1(\kappa)$ be the model for L_1 with universe $\bar{\kappa}$, $\leq^{\mathscr{M}_1(\kappa)} = \leq \restriction \bar{\kappa}$ and $\mathbf{n}^{\mathscr{M}_1(\kappa)} = n$; let $\mathscr{M}_0(\kappa)$ be the contraction of $\mathscr{M}_1(\kappa)$ to L_0. Let L_{-1} be the first-order language *without identity* based on the nonlogical lexicon $\mathbf{Pred} = \{I_n : n < \aleph_0\}$, each 1-place, and $\mathbf{Funct}$ empty; let $\mathscr{M}_{-1}(\kappa)$ be the model for L_{-1} with universe $\bar{\kappa}$ and $I_n^{\mathscr{M}_{-1}(\kappa)} = \{n\}$. For this section we will assume that every Dedekind-finite set is finite.

THEOREM 5. *For any $\kappa \in$ Card and any R:*

(i) *R is κ-definable in $L^{0,2}(\mathbf{exactly}, \leq)$ iff R is first-order definable over $\mathscr{M}_1(\kappa)$;*

(ii) *R is κ-definable in $L^{0,2}(\mathbf{exactly}, =)$ iff R is first-order definable over $\mathscr{M}_0(\kappa)$.*

(iii) *R is κ-definable in $L^{0,2}(\mathbf{exactly})$ iff R is first-order definable over $\mathscr{M}_{-1}(\kappa)$.*

Our strategy will be to transform a defining formula into one in a simple normal form. For $\mu \in \mathbf{Var}(2)$ and $n < \aleph_0$, let $(\mathbf{n} = \mu)$ be as in [5], so that for any infinite $\kappa \in$ Card and $m < \kappa$ and any infinite model $\mathscr{A}$, $\mathscr{A} \vDash_\kappa \mathbf{n} = \mathbf{m}$ iff $n = m$. Let a basic formula be of the form $\mu \leq \mu'$, $\mathbf{n} = \mu$, or '$\perp$'. Let $\Delta_0 = \{v_0, \ldots, v_{l-1}\} \subseteq \mathbf{Var}(0)$, all indicated variables distinct. Let a normal formula for Δ_0 have the form $\bigvee_{j<c}(\psi_j \ \& \ \theta_j)$, where each θ_j is a profile for Δ_0, each ψ_j is generated from basic formulae, and for any $j, j' < c$:

either θ_j is $\theta_{j'}$ or $(\theta_j \ \& \ \theta_{j'})$ is not satisfiable;
similarly for ψ_j and $\psi_{j'}$.

To prove Theorem 5(i) it will suffice to prove the following.

LEMMA 5. *For any pure* $\varphi \in \mathrm{Fml}(L^{0,2}(\mathbf{exactly}, \leq))$ *with free variables of type* 0 *belonging to* Δ_0, φ *is superequivalent to a normal formula for* Δ_0.

PROOF (by induction on the construction of φ). Suppose that φ is $(\exists v_l)\varphi'$ for $v_l \in \mathbf{Var}(0) - \Delta_0$, and that $\bigvee_{j<c}(\psi_j \ \& \ \theta'_j)$ has been constructed for φ'. Form θ_j from θ'_j by deleting each conjunct containing v_l. Then φ is superequivalent to $\bigvee_{j<c}(\psi_j \ \& \ (\exists v_l)\theta'_j)$, and the latter is in turn superequivalent to $\bigvee_{j<c}(\psi_j \ \& \ \theta_j)$. Suppose φ is $(\exists \mu)\varphi'$ for $\mu \in \mathbf{Var}(2)$ and that $\bigvee_{j<c}(\psi'_j \ \& \ \theta_j)$ has been constructed for φ'. Then $\bigvee_{j<c}((\exists \mu)\psi'_j \ \& \ \theta_j)$ is superequivalent to φ and may be transformed into the desired formula by Boolean manipulation.

For example, if ψ'_j is not $\psi'_{j'}$, replace $((\exists \mu)\psi'_j \ \& \ \theta_j)$ by

$$((\exists \mu)\psi'_j \ \& \ (\exists \mu)\psi'_{j'} \ \& \ \theta_j) \vee ((\exists \mu)\psi'_j \ \& \ \neg(\exists \mu)\psi'_{j'} \ \& \ \theta_j).$$

If φ is of the form $(\varphi_0 \supset \varphi_1)$, Boolean manipulation produces the desired formula from the corresponding formulae for φ_0 and φ_1. Suppose that φ is $(\mathbf{exactly}\ \mu\ v_l)\varphi'$ for v_l as above, and that $\bigvee_{j<c}(\psi'_j \ \& \ \theta'_j)$ has been constructed for φ'. If $c = 0$, φ' is κ-unsatisfiable for all κ; so $\mathbf{0} = \mu$ is as desired. Suppose that $c > 0$. Let $\bar{\theta}_j$ be the profile for Δ_0 obtained from θ'_j by deleting all equations and inequations containing v_l. Let θ^*_j be the conjunction of all those equations and inequations deleted in forming $\bar{\theta}_j$. For $j, j' < c$, let $j \approx j'$ iff ψ'_j is $\psi'_{j'}$ and $\bar{\theta}_j$ is $\bar{\theta}_{j'}$; $\approx$ is an equivalence relation on c. For $d \in c/\approx$, let ψ_d be ψ_j and $\bar{\theta}_d$ be $\bar{\theta}_j$ for any $j \in d$, and let θ^*_d be $\bigvee_{j \in d}\theta^*_j$. Then φ is superequivalent to

$$\bigvee_{d \in c/\approx} (\psi_d \ \& \ \bar{\theta}_d \ \& \ (\mathbf{exactly}\ \mu\ v_l)\theta^*_d).$$

If for some $j \in d$, $v_i \neq v_l$ is a conjunct of θ^*_j for all $i < l$, then, for any infinite model $\mathscr{A}$ and $\vec{a} \in {}^l|\mathscr{A}|$ satisfying $\bar{\theta}_d$,

$$\mathrm{card}(\hat{v}_l\theta^*_d[\vec{a}]^{\mathscr{A}}) = \mathrm{card}(|\mathscr{A}|),$$

using our assumption that Dedekind-finite sets are finite. For such d, in the preceding disjunction delete the d-disjunct. Suppose for each $j \in d$ there is an $i < l$ with $v_i = v_l$ a conjunct of θ^*_j; from $\bar{\theta}_d$ and θ^*_d we may easily find an $n < \aleph_0$ so that, for $\mathscr{A}$ and $\vec{a}$ as above, $\mathrm{card}(\hat{v}_l\theta^*_d[\vec{a}]^{\mathscr{A}}) = n$. In the above disjunction, replace $(\mathbf{exactly}\ \mu\ v_l)\theta^*_d$ by $\mathbf{n} = \mu$. The resulting formula may easily be transformed into the desired formula.

Similar constructions yield (ii) and (iii) of Theorem 5.

§6. Form $L^{2,2*}(\mathbf{exactly})$ from $L^{1,2*}(\mathbf{exactly})$ by introducing a countable set $\mathbf{Var}((0,0))$ of variables of type $(0,0)$, with the further formation rules:

If $\gamma \in \mathbf{Var}((0,0))$ and τ, σ are terms then $\gamma(\tau, \sigma)$ is a formula.

If φ is a formula and $\gamma \in \mathbf{Var}((0,0))$ then $(\exists\gamma)\varphi$ is a formula.

When $A \subseteq {}^2|\mathscr{A}|$, let $\mathscr{A} \vDash_\kappa \mathbf{A}(\tau,\sigma)$ iff $\langle \tau^{\mathscr{A}}, \sigma^{\mathscr{A}}\rangle \in A$; let $\mathscr{A} \vDash_\kappa (\exists\gamma)\varphi$ iff, for some $A \subseteq {}^2|\mathscr{A}|$, $\mathscr{A} \vDash_\kappa \varphi(\gamma/\mathbf{A})$. Similarly, when L is any first-order language, form L^2 by introducing $\mathbf{Var}((0,0))$ as above; if $\mathscr{A}$ is a model for L, a relation R on $|\mathscr{A}|$ is second-order definable over $\mathscr{A}$ iff R is defined over $\mathscr{A}$ by a formula of L^2.

THEOREM 6. *Suppose that κ is an aleph.*

(i) *If* card$(\bar{\kappa}) = \kappa$ *and R is κ-definable in* $L^{2,2*}$(**exactly**) *then R is second-order definable over $\mathscr{M}_1(\kappa)$.*

(ii) *If R is second-order definable over $\mathscr{M}_1(\kappa)$ then R is κ-definable in* $L^{2,2}$(**exactly**).

PROOF. For (i), suppose that card$(\bar{\kappa}) = \kappa$ and R is κ-defined by $\varphi \in$ Fml($L^{2,2*}$(**exactly**)). Then R is κ-defined by φ over $\mathscr{M}_1(\kappa)$; the variables of type 2 in that definition must be replaced by variables of type 0. For each $n < \kappa$ let $f(n) < \kappa$ be the least m so that card$(\bar{m}) = n$; let the canonical κ-standard be $\{\langle n, m\rangle : m < f(n)\}$. The canonical κ-standard exists, and is second-order definable over $\mathscr{M}_1(\kappa)$, since κ is an aleph. Using the canonical κ-standard along the lines of the argument for Observation 2.2 in [5], we may transform φ to a formula of L_1^2 that defines R over $\mathscr{M}_1(\kappa)$; details are left to the reader.

Suppose that R is defined over $\mathscr{M}_1(\kappa)$ by $\varphi \in \mathrm{Fml}(L_1^2)$. We want to replace variables of types 1 and $(0,0)$, as interpreted over $\mathscr{M}_1(\kappa)$, by variables of type $(0,0)$ interpreted over any model $\mathscr{A}$ with card$(\mathscr{A}) \geq \kappa$. The idea is this. Any $A \subseteq \bar{\kappa}$ can be coded by $B \subseteq {}^2|\mathscr{A}|$, where $A = \{\mathrm{card}(\{b : Bab\}) : a \in |\mathscr{A}|\}$; any $A \subseteq {}^2\bar{\kappa}$ can be coded by $\langle B, C\rangle$ where $B, C \subseteq {}^2|\mathscr{A}|$ and

$$A = \{\langle \mathrm{card}(\{b : Bab\}), \mathrm{card}(\{c : Cac\})\rangle : a \in |\mathscr{A}|\}.$$

Since κ is an aleph and card$(\mathscr{A}) \geq \kappa$, such a B (or B and C) exists. Associate each variable of type 0 with a new variable of type 2, each variable of type 1 with a new variable of type $(0,0)$, and each variable of type $(0,0)$ with two distinct new variables of type $(0,0)$. Suppose that $v_0, v_1 \in \mathbf{Var}(2)$ has been associated with μ_0, μ_1, and $\gamma \in \mathbf{Var}((0,0))$ has been associated with γ_0, γ_1; replace the variables in φ by their new associated variables, and replace any occurrence of $\gamma(v_0, v_1)$ by

$$(\exists v)((\textbf{exactly}\ \mu_0\ v')\gamma_0(v,v')\ \&\ (\textbf{exactly}\ \mu_1\ v')\gamma_1(v,v')).$$

It is easy to see that the resulting formula κ-defines R. Details are left to the reader.

For $\mu_0, \mu_1, \mu_2 \in \mathbf{Var}(2)$ let $\mathbf{Mult}(\mu_0, \mu_1, \mu_2)$ abbreviate

$$\begin{aligned}(\forall\gamma)(\forall\gamma')([(\forall v)(\gamma(v) \supset (\textbf{exactly}\ \mu_0\ v')\gamma'(v,v'))\ \&\ (\textbf{exactly}\ \mu_1\ v)\gamma(v)\\ \&\ (\forall v)(\forall v')((\gamma(v)\ \&\ \gamma(v')) \supset \neg(\exists v'')(\gamma'(v,v'')\ \&\ \gamma'(v',v'')))]\\ \supset (\textbf{exactly}\ \mu_2\ v')(\exists v)(\gamma(v)\ \&\ \gamma'(v,v'))).\end{aligned}$$

Here $\gamma \in \mathbf{Var}(1)$ and $\gamma' \in \mathbf{Var}((0,0))$. If μ_0, μ_1, μ_2 are all distinct then, for any $\kappa \in$ Card, $\mathbf{Mult}(\mu_0, \mu_1, \mu_2)$ κ-defines cardinal multiplication restricted to $\bar{\kappa}$; in fact it uniformly defines cardinal multiplication. Indeed this definition is "analytical". It might seem rather abstruse to be taken as expressing the "man in the street's understanding" of multiplication; but it is really not that bad. Think of the v's falling under γ as indexing the class Z_v of v''s such that $\gamma'(v,v')$; it says that if we take μ_0-

many things (the Z_ν's) exactly μ_1-many times and the Z_ν's are pairwise disjoint then we get exactly μ_2-many things (in $\bigcup\{Z_\nu : \gamma(\nu)\}$).

§7. The introduction of quantification of type-((0, 0)) variables expands the class of relations $\aleph_0$-definable in $L^{1,2*}$(**exactly**) from those definable in Presburger arithmetic to those definable in full second-order arithmetic. This is an enormous leap; where, one might ask, are the other familiar classes of relations on $\bar{\aleph}_0$, those more complex than the former class but less complex than the latter? We will now see that the class of arithmetic relations (those first-order definable in Peano arithmetic) is characterizable in terms of a sort of cardinality logic.

We now consider languages of the forms $L^{1,2}$(**fexactly**) and $L^{1,2*}$(**fexactly**) formed in the obvious ways using '**fexactly**' rather then '**exactly**' in the logical lexicon. Fix an infinite $\kappa \in$ Card. We will have variables of type 2 range over $F(\kappa)$ $=$ the set of finite subsets of $\bar{\kappa}$; so for each $x \in F(\kappa)$, fix a type-2 constant **x**. The novel clause in the definition of $\models_\kappa$ is

$$\mathscr{A} \models_\kappa (\mathbf{fexactly\ x}\ \rho)\varphi \quad \text{iff} \quad \mathrm{card}(\hat{\rho}\varphi^{\mathscr{A}}) \in x.$$

Here if $\rho \in \mathbf{Var}(2)$, $\hat{\rho}\varphi^{\mathscr{A}} \subseteq F(\kappa)$. Clearly '**fexactly**' is a weakening of '**exactly**', since to be given finitely many choices of the number of things meeting a condition is less informative than to be given the exact number; nonetheless, it can strengthen our ability to give κ-definitions. Suppose that $\varphi(\mu_0, \ldots, \mu_{l-1}) \in \mathrm{Fml}(L^{1,2*}(\mathbf{fexactly}))$ is of type $\langle 2, \ldots, 2\rangle$; let φ κ-define R over $\mathscr{A}$ iff

$$R = \{\langle n_0, \ldots, n_{l-1}\rangle : \mathscr{A} \models_\kappa \varphi(\mu_0/\{\mathbf{n_0}\}, \ldots, \mu_{l-1}/\{\mathbf{n_{l-1}}\})\}.$$

THEOREM 7. *For any reasonable infinite $\kappa \in$ Card, R is κ-definable in $L^{1,2}$*(**fexactly**) *iff R is definable in L^1_2 (i.e. is finite-monadic-second-order definable, i.e. type-1 variables range over $F(\kappa)$) over $\mathscr{M}_2(\kappa)$.*

COROLLARY 1. *R is $\aleph_0$-definable in $L^{1,2}$*(**fexactly**) *iff R is Presburger-definable.*

COROLLARY 2. *For any reasonable infinite $\kappa \in$ Card, κ-satisfiability of formulae of $L^{1,2}$*(**fexactly**) *is decidable.*

Form $L^{1,2,3}$(**exactly**) from $L^{1,2}$(**exactly**) by introducing a new countable set **Var**(3) of variables of type 3, with the following formation rules:

If $\xi \in \mathbf{Var}(3)$ and $\mu \in \mathbf{Var}(2)$ then $\xi(\mu)$ is a formula.

If φ is a formula and $\xi \in \mathbf{Var}(3)$ then $(\exists\xi)\varphi$ is a formula.

We extend the definition of $\models_\kappa$ by taking type-3 variables to range over $F(\kappa)$, with the obvious clause $\mathscr{A} \models_\kappa \mathbf{x}(\mathbf{n})$ iff $n \in x$. The following brings out the fourth-order aspect of $L^{1,2}$(**fexactly**). For $\varphi \in \mathrm{Fml}(L^{1,2}(\mathbf{fexactly}))$ and $\psi \in \mathrm{Fml}(L^{1,2,3}(\mathbf{exactly}))$, where both have free variables among $\vec{v} \in {}^l\mathbf{Var}(0)$, $\vec{\gamma} \in {}^m\mathbf{Var}(1)$, $\vec{\mu} \in {}^k\mathbf{Var}(2)$, let φ be κ-equivalent to ψ iff for any model $\mathscr{A}$ of cardinality $\geq \kappa$ and any $\vec{a} \in {}^l|\mathscr{A}|$, $\vec{A} \in {}^m\mathrm{Power}(|\mathscr{A}|)$, $\vec{n} \in {}^k\bar{\kappa}$:

$$\mathscr{A} \models_\kappa \varphi[\vec{a}; \vec{A}; \{n_0\}, \ldots, \{n_{k-1}\}] \quad \text{iff} \quad \mathscr{A} \models_\kappa \psi[\vec{a}; \vec{A}; \vec{n}].$$

LEMMA 6. *For any infinite $\kappa \in$ Card:*

(i) *Each formula of $L^{1,2}$*(**fexactly**) *is κ-equivalent to a formula of $L^{1,2,3}$*(**exactly**) *in which no type-3 variables are free.*

(ii) *Each formula of $L^{1,2,3}$*(**exactly**) *in which no type-3 variables are free is κ-equivalent to a formula of $L^{1,2}$*(**fexactly**).

For $\varphi \in \mathrm{Fml}(L^{1,2}(\mathbf{fexactly}))$, form φ^+ by replacing each $\mu \in \mathbf{Var}(2)$ by a new $\xi \in \mathbf{Var}(3)$ and replacing subformulae of the form $(\mathbf{fexactly}\,\mu\,\nu)\varphi$ by $(\exists\mu)(\xi(\mu)\ \&\ (\mathbf{exactly}\ \mu\ \nu)\psi)$, and iterating this until all occurrences of '**fexactly**' are eliminated; obviously φ^+ is as desired. For $\varphi \in \mathrm{Fml}(L^{1,2,3}(\mathbf{exactly}))$ form φ^- by rewriting each type-3 variable as a new type-2 variable, replacing each occurrence of the form $(\mathbf{exactly}\,\mu\,\nu)\psi$ by $\mathbf{unit}(\mu)\ \&\ (\mathbf{fexactly}\,\mu\,\nu)\psi$, and iterating replacement of each subformula of the form $(\exists\mu)\psi$, where μ was of type-2 to begin with, by $(\exists\mu)(\mathbf{unit}(\mu)\ \&\ \psi)$. Here $\mathbf{unit}(\mu)$ abbreviates

$$(\forall\gamma)(\forall\gamma')([(\mathbf{fexactly}\,\mu\,\nu)\gamma(\nu)\ \&\ (\mathbf{fexactly}\,\mu\,\nu)\gamma'(\nu)] \\ \supset (\forall\mu')[(\mathbf{fexactly}\,\mu'\,\nu)\gamma(\nu) \equiv (\mathbf{fexactly}\,\mu'\,\nu)\gamma'(\nu)]);$$

so for any model $\mathscr{A}$ and $x \in F(\kappa)$

$$\mathscr{A} \vDash_\kappa \mathbf{unit}(\mathbf{x}) \text{ iff } x = \{n\} \text{ for some } n < \kappa.$$

With that fact, it is easy to see that φ^- is as desired.

Form L_2^1 from L_2 by introducing $\mathbf{Var}(1)$ with the obvious formation rules; when interpreted in $\mathscr{M}_2(\kappa)$, we will take the type-1 variables to range only over $F(\kappa)$.

LEMMA 7. *For any reasonable infinite $\kappa \in$ Card, R is κ-definable in $L^{1,2,3}(\mathbf{exactly})$ iff R is definable in L_2^1 over $\mathscr{M}_2(\kappa)$.*

From left to right, this is simply a variant of Theorem 1(i); the type-3 variables in $L^{1,2,3}(\mathbf{exactly})$ become the type-1 variables in L_2^1. From right to left is trivial.

Lemmas 6 and 7 yield Theorem 7. For Corollary 1 it suffices to notice that the usual quantifier-elimination proof for Presburger arithmetic still works in the presence of type-1 variables. Thus a relation is definable in L_2^1 over $\mathscr{M}_2(\aleph_0)$ iff it is Presburger-definable. The argument for the corollary to Theorem 1(i) also yields Corollary 2.

The following is what motivated this section.

THEOREM 8. *R is $\aleph_0$-definable in $L^{1,2*}(\mathbf{fexactly})$ iff R is an arithmetical relation on $\bar{\aleph}_0$.*

PROOF. Enrich $L^{1,2*}(\mathbf{exactly})$ to $L^{1,2*}(\mathbf{exactly}, \mathbf{M})$ by adding the constant '**M**' to the logical lexicon with the formation rule

$$\text{if } \mu_0, \mu_1, \mu_2 \in \mathbf{Var}(2) \text{ then } \mathbf{M}(\mu_0, \mu_1\,\mu_2) \text{ is a formula}$$

and the truth-clause

$$\mathscr{A} \vDash_\kappa \mathbf{M}(\mathbf{n}_0, \mathbf{n}_1, \mathbf{n}_2) \quad \text{iff} \quad n_0 \cdot n_1 = n_2.$$

Code each $x \in F(\aleph_0)$ by $t(x) =$ the sequence number for the increasing sequence of elements of x. Construct $\mathbf{Seq}(\mu_1)$, $\mathbf{Member}(\mu_0, \mu_1) \in \mathrm{Fml}(L^{1,2*}(\mathbf{exactly}, \mathbf{M}))$ so that for any model $\mathscr{A}$ of cardinality $\geq \aleph_0$ and any $n, m < \aleph_0$:

$\mathscr{A} \vDash_\kappa \mathbf{Seq}(m)$ iff m is a sequence-number;

$\mathscr{A} \vDash_\kappa \mathbf{Member}(\mathbf{n}, \mathbf{m})$ iff m is a sequence-number and, for some $i < \mathrm{lh}(m)$, $n = (m)_i$.

LEMMA 8. *Suppose $\varphi \in \mathrm{Fml}(L^{1,2*}(\mathbf{fexactly}))$ has free variables among $\vec{v}, \vec{\gamma}, \vec{\mu}$. There is a $\varphi^+ \in \mathrm{Fml}(L^{1,2*}(\mathbf{exactly}, \mathbf{M}))$ with the same free variables such that for any model $\mathscr{A}$ of cardinality $\geq \aleph_0$, any $\vec{a} \in {}^l|\mathscr{A}|$, $\vec{A} \in {}^k\mathscr{P}(|\mathscr{A}|)$, and $\vec{x} \in {}^mF(\kappa)$,*

$$\mathscr{A} \vDash_{\aleph_0} \varphi[\vec{a}, \vec{A}, \vec{x}] \quad \textit{iff} \quad \mathscr{A} \vDash_{\aleph_0} \varphi^+[\vec{a}, \vec{A}, \overrightarrow{t(x)}].$$

To form φ^+, iterate replacing subformulae of φ of the form $(\exists\mu)\psi$ by

$$(\exists\mu)(\mathbf{Seq}(\mu)\ \&\ \psi),$$

and those of the form $(\mathbf{exactly,}\ \mu\ \rho)\psi$ by

$$(\exists\mu')(\mathbf{Member}(\mu',\mu)\ \&\ (\mathbf{exactly}\ \mu'\ \rho)\psi),$$

where $\mu' \in \mathbf{Var}(2)$ and does not occur free in ψ. It is easy to see that this does it.

Form L_3(**exactly**) from L_2(**exactly**) by adding '**M**' to its nonlogical lexicon. Expand $\mathscr{M}_2(\aleph_0)$ to $\mathscr{M}_3(\aleph_0)$ by interpreting '**M**' as multiplication.

LEMMA 9. *If R is $\aleph_0$-definable in $L^{1,2*}$*(**exactly, M**) *then R is definable in L_3*(**exactly**) *over $\mathscr{M}_3(\aleph_0)$.*

This is just the construction used to prove Theorem 1(ii); '**M**' rides along, causing no trouble.

Any relation definable in L_3(**exactly**) over $\mathscr{M}_3(\aleph_0)$ is really arithmetical (i.e. definable in L_3 over $\mathscr{M}_3(\aleph_0)$), since all uses of '**exactly**' can be eliminated using sequence-numbers. This proves Theorem 8 from left to right.

To prove Theorem 8 from right to left, it suffices to get an $\aleph_0$-definition of multiplication on $\bar{\aleph}_0$ in $L^{1,2*}$(**fexactly**). Form $L^{1,2*,3}$(**exactly**) in the obvious way, with type-3 variables ranging over $F(\kappa)$ for any κ.

LEMMA 10. *For any infinite $\kappa \in$ Card, each formula of $L^{1,2*,3}$*(**exactly**) *containing no free variables of type* 3 *is κ-equivalent to a formula of $L^{1,2*}$*(**fexactly**).

The construction is just like that for Lemma 6(ii).

Thus it will suffice to give an $\aleph_0$-definition of multiplication in $L^{1,2*,3}$(**exactly**). For any $\mu \in \mathbf{Var}(2)$ let $\mathbf{Z}(\mu)$ be $(\forall\mu')\mu \le \mu'$, where μ' is not μ. For $\mu_0, \mu_1, \mu_2 \in \mathbf{Var}(2)$ fix new and distinct $\mu, \mu', \mu'' \in \mathbf{Var}(2)$ and $\xi \in \mathbf{Var}(3)$; let $\mathbf{Max}(\xi,\mu_2)$, $\mathbf{Min}(\xi,\mu_0)$ and $\mathbf{Between}(\xi,\mu_0)$ be the following formulae:

$$\xi(\mu_2)\ \&\ (\forall\mu)(\xi(\mu) \supset \mu \le \mu_2); \qquad \xi(\mu_0)\ \&\ (\forall\mu)(\xi(\mu) \supset \mu_0 \le \mu);$$
$$(\forall\mu')(\forall\mu'')([\mu' < \mu''\ \&\ \neg(\exists\mu)(\mu' < \mu\ \&\ \mu < \mu'')$$
$$\&\ \xi(\mu')\ \&\ \xi(\mu'')] \supset \mathbf{A}(\mu',\mu_0,\mu'')).$$

Let $\mathbf{Mult}(\mu_0,\mu_1,\mu_2)$ be the following:

$$(Z(\mu_1)\ \&\ Z(\mu_2))$$
$$\vee\ (\exists\xi)((\mathbf{exactly}\ \mu_1\ \mu)\xi(\mu)\ \&\ \mathbf{Max}(\xi,\mu_2)\ \&\ \mathbf{Min}(\xi,\mu_0)\ \&\ \mathbf{Between}(\xi,\mu_0)).$$

It is easy to see that if μ_0, μ_1 and μ_2 are all distinct, this formula of $L^{1,2*,3}$(**exactly**) does $\aleph_0$-define multiplication on $\bar{\aleph}_0$.

COROLLARY 3. *The set of $\aleph_0$-satisfiable formulae of $L^{1,2*}$*(**fexactly**) *is of Turing degree* $\mathbf{0}^{(\omega)}$.

Since all translations were effective, this follows from the argument for Theorem 8.

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SAGE SCHOOL OF PHILOSOPHY
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853