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CARDINALITY LOGICS, PART I: INCLUSIONS BETWEEN LANGUAGES BASED ON '<u>exactly</u>'

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A connection between higher-order logics and the concept of cardinality has been long recognized; but (as far as I know) it has not been a subject of model-theoretic investigation. This paper begins such an investigation, which is continued in [3]. The philosophical motivation for this project may be culled from [2] and [4]; it seems related to ideas in [1].

To avoid reliance on the Axiom of Choice, we will take cardinals to be Scott-cardinals; that is,

$$\text{card}(x) = \{y: y \subseteq V_\alpha \text{ and } x \text{ is equinumerous with } y\},$$

where α is the least ordinal such that the above set is non-empty. κ is a cardinal iff for some x $\kappa = \text{card}(x)$; Card is the class of cardinals. Card is partially ordered by the injective ordering: for $n,m \in \text{Card}$, $n \leqslant m$ iff for some $x \in n$ and $y \in m$ there is a one–one function from x into y. Let $\kappa \in \text{Card}$ be infinite iff some (thus every) $x \in \kappa$ is infinite. For κ, $\kappa' \in \text{Card}$, let:

$$[\kappa, \kappa') = \{n \in \text{Card}: \kappa \leqslant n < \kappa'\},$$

$$(\kappa, \kappa') = \{n \in \text{Card}: \kappa < n < \kappa'\},$$

$$[\kappa, \kappa'] = \{n \in \text{Card}: \kappa \leqslant n \leqslant \kappa'\}.$$

Let κ be an aleph iff κ is infinite and some (thus every) $x \in \kappa$ is well-orderable. Recall these facts:

(1) If $\kappa' \leqslant \kappa$ and κ is an aleph, then κ' is either finite or an aleph.

(2) These are equivalent: Choice; all infinite cardinals are alephs; $\leqslant$ linearly orders Card.

(3) These are equivalent: all Dedekind-finite sets are finite; for any infinite $\kappa \in \text{Card}$, $\kappa \geqslant \aleph_0$.

For more on Scott-cardinals, see [5].

For $\kappa \in \text{Card}$, let $\bar{\kappa} = \{n \in \text{Card}: n < \kappa\}$, $\text{ncb}(\kappa) = \text{card}(\bar{\kappa}) =$ the Number of Cardinals Below κ. As usual, an ordinal is the set of its predecessors; so $\text{ncb}(\aleph_\xi) = \text{card}(\xi \cup \omega)$.

Remarks on notation. Where convenient, I will ignore the use/mention distinction. Let $x * y$ be the concatination of x and y in that order. Where φ is a formula, v a variable, and τ a term of the same type as v, $\varphi(v/\tau)$ is the result of replacing all free occurrences of v in φ by τ, relettering bound variables in φ if necessary to insure substitutibility. Distinct Greek letters ranging over variables in our object-languages are always assumed to take distinct values: when I say "Consider variables $\mu_0, \ldots, \mu_{p-1}, \eta_0, \ldots, \eta_{q-1}, v$" it is understood that these $p + q + 1$ variables are all distinct.

1. Cardinality languages and their semantics

Fix the basic logical lexicon {'$\perp$', '$\supset$', '$\exists$', '$=$'} and for each $i \in \{1\} \cup \{2n : n < \omega\}$ fix a countable set $Var(i)$ of type-i variables, all sets mutually disjoint. Let *Pred* and *Funct* be given disjoint sets of predicate-constants and function-constants respectively, both disjoint from the other lexical categories; for each $n < \omega$ $Pred(n)$ and $Funct(n)$ are, respectively, the set of n-place members of *Pred* and *Funct*. The set of terms based on *Funct*, Term(*Funct*), is generated from *Funct* and $Var(0)$ as usual. The class of models for *Pred, Funct,* is defined as usual. For $\tau \in \text{Term}(Funct)$, $\text{den}(\tau)$ is defined as usual, relative to such a model.

1.1. To form $\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leqslant, Pred, Funct)$ add '$\underline{\text{exactly}}$' and '$\leqslant$' to the basic logical lexicon, with '$\leqslant$' $\notin Pred$. Hereafter we will omit explicit mention of *Pred* and *Funct*. The formulae of $\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leqslant)$ are defined by the usual formation rules together with the following:

(a) If $\tau \in \text{Term}(Funct)$ and $Y \in Var(1)$, then $Y\tau$ is a formula.

(b) If (and only if) $\rho, \mu \in Var(2i)$ for $i > 0$, then $\rho \leqslant \mu$ is a formula.

(c) If φ is a formula, $\rho \in Var(2i)$ and $\mu \in Var(2i + 2)$, then $(\underline{\text{exactly}}\ \mu\ \rho)\varphi$ is a formula.

Note. Any free occurrence of ρ in φ is bound in $(\underline{\text{exactly}}\ \mu\ \rho)\varphi$ by the indicated occurrence of ρ; the indicated occurrence of μ is free and binds nothing. Let $\text{Fml}(\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leqslant))$ and $\text{Sent}(\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leqslant))$ be respectively the set of formulae and sentences of $\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leqslant)$. Standard abbreviations are in effect, e.g. $\neg\varphi$ for $(\varphi \supset \perp)$.

For each $n \in \text{Card}$, fix a distinct constant $\mathbf{n}$ not belonging to any of our lexical classes. Given a model $\mathscr{A}$ for *Pred, Funct,* form the language $\mathscr{L}^{1,\omega}_{\mathscr{A},\kappa}(\underline{\text{exactly}}, \leqslant)$ by introducing:

– a new individual constant $\mathbf{a}$ for each $a \in |\mathscr{A}|$,

– a new 1-place predicate-constant $\mathbf{A}$ for each $A \subseteq |\mathscr{A}|$,

and counting $\mathbf{n}$ as a constant of type $2j + 2$ if $n < \text{ncb}^j(\kappa)$. For $\varphi \in \text{Sent}(\mathscr{L}^{1,\omega}_{\mathscr{A},\kappa}(\underline{\text{exactly}}, \leqslant))$, we define $\mathscr{A} \vDash_\kappa \varphi$ as usual, with these novel clauses:

$$\mathscr{A} \vDash_\kappa \mathbf{n} \leqslant \mathbf{n}' \quad \text{iff} \quad n \leqslant n',$$

$$\mathscr{A} \vDash_\kappa (\underline{\text{exactly}}\ \mathbf{n}\ v)\varphi \quad \text{iff} \quad \text{card}(\hat{v}\varphi) = n,$$

$$\mathscr{A} \vDash_\kappa (\exists\mu)\varphi \quad \text{iff} \quad \text{for some } n < \text{ncb}^j(\kappa)\ \ \mathscr{A} \vDash_\kappa \varphi(\mu/\mathbf{n}),$$

where $\mu \in Var(2j+2)$ and:

$$\text{if } v \in Var(0), \quad \hat{v}\varphi = \{a \in |\mathscr{A}|\colon \mathscr{A} \vDash_\kappa \varphi(v/\mathbf{a})\},$$
$$\text{if } v \in Var(2j+2), \quad \hat{v}\varphi = \{n < \text{ncb}^j(\kappa)\colon \mathscr{A} \vDash_\kappa \varphi(v/\mathbf{n})\}.$$

Let $\mathscr{A} \vDash \varphi$ iff $A \vDash_{\text{card}(\mathscr{A})} \varphi$.

Where the free variables in φ of type-0 are among $v_0, \ldots, v_{m-1}$, the free variables in φ of type ≥ 2 are among $\mu_0, \ldots, \mu_{l-1}$, $\vec{a} \in |\mathscr{A}|^m$ and $\vec{n} \in \bar{\kappa}^l$, we will write $\varphi[\vec{a}, \vec{n}]$ for $\varphi(v_0/\mathbf{a}_0, \ldots, v_{m-1}/\mathbf{a}_{m-1}, \mu_0/\mathbf{n}_0, \ldots, \mu_{l-1}/\mathbf{n}_{l-1})$ provided that for $\mu_j \in Var(2k+2)$, $n_j < \text{ncb}^k(\kappa)$, for all $j < l$.

For $k < \omega$, let $\mathscr{L}^{1,2k}(\underline{\text{exactly}}, \leq)$ be the sublanguage of $\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leq)$ resulting from dropping all variables of type-$2j$ for $j > k$. Let $\mathscr{L}^{0,\omega}(\underline{\text{exactly}}, \leq)$ be the sublanguage of $\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leq)$ formed by dropping use of type-1 variables. For $i < 2$ and $k < \omega$, let $\mathscr{L}^{0,2k}(\underline{\text{exactly}}, \leq)$ be the sublanguage of $\mathscr{L}^{0,\omega}(\underline{\text{exactly}}, \leq)$ formed by dropping use of all variables of type $> 2k$. Since '$\underline{\text{exactly}}$' and '$\leq$' do not occur in formulae of $\mathscr{L}^{i,0}(\underline{\text{exactly}}, \leq)$, let that language be $\mathscr{L}^i$.

1.2. The model-theoretic semantics just presented may be thought of as a fragment of higher-order logic in which, for a given model, variables of type ≥ 2 range over certain quantifiers over that model. More precisely, we could have introduced languages in which formulae of the form $(\underline{\text{exactly}}\, \mu\, \rho)\varphi$ were replaced by $(\mu\, \rho)\varphi$, and defined satisfaction so that in $\mathscr{A}$, variables of type $2i+2$ ranged over ${}^{2i+2}\text{EXACTLY}(\kappa)$, letting:

$$^{2}Q(n)^\kappa = \{A \subseteq |\mathscr{A}|\colon \text{card}(A) = n\},$$
$$^{2}\text{EXACTLY}(\kappa) = \{^{2}Q(n)^\kappa\colon n < \kappa \quad \text{and} \quad {}^{2}Q(n)^\kappa \text{ is non-empty}\},$$
$$^{2i+2}Q(n)^\kappa = \{Q \subseteq {}^{2i}\text{EXACT}(\kappa)\colon \text{card}(Q) = n\},$$
$$^{2i+2}\text{EXACT}(\kappa) = \{^{2i+2}Q(n)^\kappa\colon n < \text{ncb}^i(\kappa) \quad \text{and} \quad {}^{2i+2}Q(n)^\kappa \text{ is non-empty}\},$$
$$\mathscr{A} \vDash {}^{2i+2}\mathbf{Q}(\mathbf{n}) \leq {}^{2i+2}\mathbf{Q}(\mathbf{n}') \quad \text{iff} \quad n \leq n',$$
$$\mathscr{A} \vDash ({}^{2i+2}\mathbf{Q}\, \rho)\varphi \quad \text{iff} \quad \hat{\rho}\varphi \in {}^{2i+2}Q.$$

(Here if $i \geq 1$ then $\hat{\rho}\varphi = \{^{2i}Q\colon \mathscr{A} \vDash \varphi(\rho/{}^{2i}\mathbf{Q})\}$.)

If $\text{card}(\mathscr{A}) \geq \kappa$, then for every $j > \omega$ the map $n \mapsto {}^{2j+2}Q(n)^\kappa$ is a 1–1 correspondence between $\text{ncb}^j(\kappa)$ and ${}^{2j+2}\text{EXACT}(\kappa)$; therefore truth in $\mathscr{A}$ under $\vDash_\kappa$ for sentences of $\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leq)$ is an alternative representation of truth in $\mathscr{A}$ under the semantics just sketched. The semantics just presented carries a type-structure, since if $0 < i < j$, and ${}^{2i}Q(n)^\kappa$ and ${}^{2j}Q(n)^\kappa$ are both non-empty, ${}^{2i}Q(n)^\kappa \neq {}^{2j}Q(n)^\kappa$; this semantic typing is erased in the semantics for $\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leq)$, under which variables of type $2i$ are assigned simply to $n < \text{ncb}^{i-1}(\kappa)$, rather than to ${}^{2i}Q(n)^\kappa$. The semantic type-structure in the former semantics does no work; so it is more convenient to work with $\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leq)$.

1.3. We will now consider another hierarchy of languages in which final parts of the syntactic type-structure of $\mathcal{L}^{1,\omega}(\underline{\text{exactly}}, \leq)$ are collapsed. For $0<k<\omega$, the categories of variables for $\mathcal{L}^{1,2k*}(\underline{\text{exactly}}, \leq)$ shall be $Var(1)$ and $Var(2j)$ for $j \leq k$. The formation-rules for $\mathcal{L}^{1,2k*}(\underline{\text{exactly}}, \leq)$ are like those for $\mathcal{L}^{1,2k}(\underline{\text{exactly}}, \leq)$, with this addition:

– if $\mu, \rho \in Var(2k)$ and φ is a formula, then $(\underline{\text{exactly}}\, \mu\, \rho)\varphi$ is a formula.

Given $\mathcal{A}$ and $\kappa \in$ Card, form $\mathcal{L}^{1,2k*}_{\mathcal{A},\kappa}(\underline{\text{exactly}}, \leq)$ as before.

The definition of $\mathcal{A} \models_\kappa \varphi$ for $\varphi \in \text{Sent}(\mathcal{L}^{1,2k*}_{\mathcal{A},\kappa}(\underline{\text{exactly}}, \leq))$ proceeds as before. Notice that for $\mu \in Var(2k)$, $(\underline{\text{exactly}}\, \mu\, \mu)\varphi$ is well-formed; the semantics makes the two indicated occurrences of μ function as if they were occurrences of distinct variables; the left-most occurrence of μ is free in $(\underline{\text{exactly}}\, \mu\, \mu)\varphi$; the second occurrence is not free and binds all occurrences of μ free in φ.

We form $\mathcal{L}^{0,2k*}(\underline{\text{exactly}}, \leq)$ from $\mathcal{L}^{1,2k*}(\underline{\text{exactly}}, \leq)$ by dropping $Var(1)$.

1.4. For languages $\mathcal{L}, \mathcal{L}'$ as above and $\varphi \in \text{Sent}(\mathcal{L})$, $\psi \in \text{Sent}(\mathcal{L}')$, we adopt these definitions:

(a) φ is κ-equivalent to ψ iff for all models $\mathcal{A}$ for *Pred, Funct* with card$(\mathcal{A}) \geq \kappa$: $\mathcal{A} \models_\kappa \varphi$ iff $\mathcal{A} \models_\kappa \psi$.

(b) φ is equivalent to ψ iff for all infinite models $\mathcal{A}$ for *Pred, Funct*: $\mathcal{A} \models \varphi$ iff $\mathcal{A} \models \psi$.

(c) φ is equivalent$_\kappa$ to ψ iff for all models $\mathcal{A}$ for *Pred, Funct* of cardinality κ: $\mathcal{A} \models \varphi$ iff $\mathcal{A} \models \psi$.

(d) φ is super-equivalent to ψ iff for all infinite κ, φ is κ-equivalent to ψ.

We now define these inclusion relations:

$\mathcal{L} \overset{\kappa}{\prec} \mathcal{L}'$ iff for each $\varphi \in \text{Sent}(\mathcal{L})$ there is a κ-equivalent $\psi \in \text{Sent}(\mathcal{L}')$;

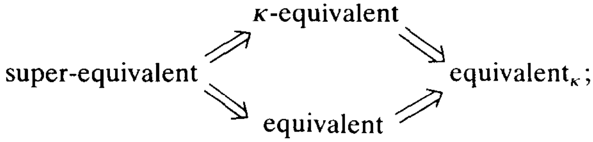
$\mathcal{L} \prec \mathcal{L}'$ iff for each $\varphi \in \text{Sent}(\mathcal{L})$ there is an equivalent $\psi \in \text{Sent}(\mathcal{L}')$;

$\mathcal{L} \underset{\kappa}{\prec} \mathcal{L}'$ iff for each $\varphi \in \text{Sent}(\mathcal{L})$ there is an equivalent$_\kappa$ $\psi \in \text{Sent}(\mathcal{L}')$;

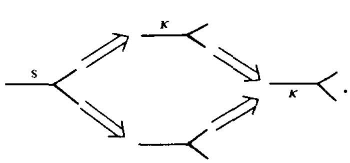
$\mathcal{L} \overset{s}{\prec} \mathcal{L}'$ iff for each $\varphi \in \text{Sent}(\mathcal{L})$ there is a super-equivalent $\psi \in \text{Sent}(\mathcal{L}')$;

$\mathcal{L} \overset{\kappa}{\succ\prec} \mathcal{L}'$ iff $\mathcal{L} \overset{\kappa}{\prec} \mathcal{L}'$ and $\mathcal{L}' \overset{\kappa}{\prec} \mathcal{L}$;

similarly for $\mathcal{L} \succ\prec \mathcal{L}'$, $\mathcal{L} \underset{\kappa}{\succ\prec} \mathcal{L}'$, and $\mathcal{L} \overset{s}{\succ\prec} \mathcal{L}'$. Then:



and so:



2. The basic inclusions

2.1. For $i<2$, clearly:

(1) $\mathscr{L}^{i} \overset{s}{\precsim} \mathscr{L}^{i,2}(\underline{\text{exactly}}, \leqslant) \overset{s}{\precsim} \mathscr{L}^{i,4}(\underline{\text{exactly}}, \leqslant) \overset{s}{\precsim} \cdots \overset{s}{\precsim} \mathscr{L}^{i,\omega}(\underline{\text{exactly}}, \leqslant)$.

This hierarchy of order-type $\omega+1$ continues with order-type of converse$(\omega) = \omega^*$:

(2) $\mathscr{L}^{i,\omega}(\underline{\text{exactly}}, \leqslant) \overset{s}{\precsim} \cdots \overset{s}{\precsim} \mathscr{L}^{i,4*}(\underline{\text{exactly}}, \leqslant) \overset{s}{\precsim} \cdots \overset{s}{\precsim} \mathscr{L}^{i,2*}(\underline{\text{exactly}}, \leqslant)$.

To see this, for $0<k<\omega$ we adopt these abbreviations:

$$\underline{\text{ncb}^k = \rho}\text{:} \quad (\underline{\text{exactly}}\rho\ \eta)\neg\bot,$$

$$\underline{\text{ncb}^k \geqslant \mu}\text{:} \quad (\forall\rho)(\underline{\text{ncb}^k = \rho} \supset \mu \leqslant \rho),$$

for μ, η, $\rho \in Var(2k)$. If $k=1$, we will omit the superscript. Clearly $\underline{\text{ncb}^k \geqslant \mu}$ is a formula of $\mathscr{L}^{0,2k*}(\underline{\text{exactly}}, \leqslant)$, and for every $n < \text{ncb}^{k-1}(\kappa)$ and any model $\mathscr{A}$: $\mathscr{A} \vDash_\kappa \underline{\text{ncb}^k \geqslant \mathbf{n}}$ iff $n \leqslant \text{ncb}^k(\kappa)$. Claim: for $1 \leqslant k < \omega$:

$$\mathscr{L}^{i,2k+2*}(\underline{\text{exactly}}, \leqslant) \overset{s}{\precsim} \mathscr{L}^{i,2k*}(\underline{\text{exactly}}, \leqslant).$$

Given $\varphi \in \text{Sent}(\mathscr{L}^{i,2k+2*}(\underline{\text{exactly}}, \leqslant))$, for each $\mu \in Var(2k+2)$ occurring in φ introduce a distinct $\mu' \in Var(2k)$ not occurring in φ; form φ' as follows: first replace all occurrences of each μ as above by μ'; then restrict all prefexes binding μ' by $(\underline{\text{ncb}^k \geqslant \mu'})$, i.e. replace subformulae of the form $(\exists\mu')\psi$ and $\underline{\text{exactly}}\,\rho\,\mu')\psi$ by $(\exists\mu')(\underline{\text{ncb}^k \geqslant \mu'}\ \&\ \psi)$ and $(\underline{\text{exactly}}\,\rho\,\mu')$ $(\underline{\text{ncb}^k \geqslant \mu'}\ \&\ \psi)$ respectively. This establishes our claim. If $\varphi \in \text{Sent}(\mathscr{L}^{i,\omega}(\underline{\text{exactly}}, \leqslant))$, fix k so that $\varphi \in \text{Sent}(\mathscr{L}^{i,2k+2}(\underline{\text{exactly}}, \leqslant))$. If $1 \leqslant j \leqslant k$ we have $\psi \in \text{Sent}(\mathscr{L}^{i,2k+2*}$ $(\underline{\text{exactly}}, \leqslant))$; also:

$$\mathscr{L}^{i,2k+2*}(\underline{\text{exactly}}, \leqslant) \overset{s}{\precsim} \cdots \overset{s}{\precsim} \mathscr{L}^{i,2j*}(\underline{\text{exactly}}, \leqslant).$$

So φ is expressible in $\mathscr{L}^{i,2j*}(\underline{\text{exactly}}, \leqslant)$, yielding the desired inclusion hierarchy.

Form $\mathscr{L}^{i,2k}(\underline{\text{exactly}})$ from $\mathscr{L}^{i,2k}(\underline{\text{exactly}}, \leqslant)$ by dropping '$\leqslant$' from the logical lexicon. Form $\mathscr{L}^{i,2k}(\underline{\text{exactly}}, =)$ from $\mathscr{L}^{i,2k}(\underline{\text{exactly}})$ by changing the formation-rule for formulae by permitting '$=$' to occur between all variables of type-$2i$ for $i \geqslant k$, giving such atomic formulae the obvious satisfaction conditions. Form $\mathscr{L}^{i,\omega}(\underline{\text{exactly}})$, $\mathscr{L}^{i,2k*}(\underline{\text{exactly}})$, $\mathscr{L}^{i,\omega}(\underline{\text{exactly}}, =)$, and $\mathscr{L}^{i,2k*}(\underline{\text{exactly}}, =)$ in the same way.

It should be obvious that an inclusion-hierarchy like (1) also holds for languages of the form $\mathscr{L}^{i,2k}(\underline{\text{exactly}}, =)$ and $\mathscr{L}^{i,2k}(\underline{\text{exactly}})$. It seems that one like (2) does not hold for languages of the form $\mathscr{L}^{i,2k*}(\underline{\text{exactly}}, =)$ and $\mathscr{L}^{i,2k*}(\underline{\text{exactly}})$. The rub is that '$\leqslant$' is needed in $\underline{\text{ncb}^k \geqslant \mu}$. However it should be clear that for any

$k<\omega$: if $\mathrm{ncb}^k(\kappa)=\mathrm{ncb}^{k+1}(\kappa)$, then:

$$\mathscr{L}^{i,\omega}(\underline{\text{exactly}},=)\overset{\kappa}{\precsim}\cdots\overset{\kappa}{\precsim}\mathscr{L}^{i,2k+4*}(\underline{\text{exactly}},=)\overset{\kappa}{\precsim}\mathscr{L}^{i,2k+2*}(\underline{\text{exactly}},=),$$

$$\mathscr{L}^{i,\omega}(\underline{\text{exactly}})\overset{\kappa}{\precsim}\cdots\overset{\kappa}{\precsim}\mathscr{L}^{i,2k+4*}(\underline{\text{exactly}})\overset{\kappa}{\precsim}\mathscr{L}^{i,2k+2*}(\underline{\text{exactly}}).$$

These observations will be strengthened in §4.6 and §4.7.

2.2. There is no fragment of the semantics for higher-order logic related to the semantics given in §1.3 as that from §1.1 is related to that sketched in §1.2. But the Axiom of Choice makes the semantics from both §1.1 and §1.3 into fragments of the semantics of second-order logic. Fix a countable set $Var((0,0))$ of type-(0, 0) variables. For $x\in\{2k:k<\omega\}\cup\{\omega\}\cup\{2k*:k>\omega\}$, we form $\mathscr{L}^{(0,0),x}(\underline{\text{exactly}},\leqslant)$ by adding $Var((0,0))$ to the lexicon $\mathscr{L}^{i,x}(\underline{\text{exactly}},\leqslant)$ with the obvious new formation rules and the obvious semantics in which, relative to $\mathscr{A}$, type-(0, 0) variables range over $\mathscr{P}(|\mathscr{A}|^2)$. As in §2.1, all the languages from §1.1 and §1.3 are super-included in (i.e. bear $\overset{s}{\precsim}$ to) $\mathscr{L}^{(0,0),2*}(\underline{\text{exactly}},\leqslant)$. In fact, these languages are no stronger than $\mathscr{L}^{(0,0)}$ in the following sense.

Observation. *For* $\varphi\in\mathrm{Sent}(\mathscr{L}^{(0,0),2*}(\underline{\text{exactly}},\leqslant))$ *there is a* $\varphi'\in\mathrm{Fml}(\mathscr{L}^{(0,0)})$ *containing exactly one free type-*1 *variable so that for any model* $\mathscr{A}$ *and* $A\subseteq|\mathscr{A}|$ *with* $\mathrm{card}(A)=\kappa$:

$$\mathscr{A}\vDash_\kappa\varphi \quad \textit{iff} \quad \mathscr{A}\vDash\varphi'(\mathbf{A}).$$

Thus $\mathscr{L}^{(0,0),2*}(\underline{\text{exactly}},\leqslant)\rightarrowtail\mathscr{L}^{(0,0)}$, since where Y is the type-1 variable free in φ', we may replace each subformula of φ' of the form Yv by '$\neg\bot$'.

Let a κ-standard for $\mathscr{A}$ have the form $\langle R,a_0\rangle$ where $R\subseteq|\mathscr{A}|^2$, $a_0\in|\mathscr{A}|$, $a_0\notin\mathrm{RightFld}(R)$, and:

for each $n\in\bar{\kappa}-\{0\}$ there is a unique $a_n\in|\mathscr{A}|$ so that $\mathrm{card}\{a\colon\langle a,a_n\rangle\in R\}=n$.

By choice, if $\mathrm{card}(\mathscr{A})\geqslant\kappa$, then there is a κ-standard for $\mathscr{A}$. For $Y'\in Var(0,0)$, $v_0\in Var(0)$ and $Y\in Var(1)$ there is a $Std(Y',v_0,Y)\in\mathrm{Fml}(\mathscr{L}^{(0,0)})$ so that for any $\mathscr{A}$ and $A\subseteq|\mathscr{A}|$ with $\mathrm{card}(A)=\kappa$:

$$\mathscr{A}\vDash Std(\mathbf{R},\mathbf{a},\mathbf{A}) \quad \text{iff} \quad \langle R,a\rangle \text{ is a } \kappa\text{-standard for } \mathscr{A}.$$

Given $\varphi\in\mathrm{Sent}(\mathscr{L}^{(0,0),2*}(\underline{\text{exactly}},\leqslant))$ not containing Y', v_0 or Y, it is not hard to form $\hat{\varphi}\in\mathrm{Fml}(L^{(0,0)})$ in which Y', v_0 and Y are free and so that for any model $\mathscr{A}$:

if $\langle R,a_0\rangle$ is a κ-standard for $\mathscr{A}$:

$\mathscr{A}\vDash_\kappa\varphi$ iff $\mathscr{A}\vDash\hat{\varphi}(\mathbf{R},\mathbf{a}_0)$.

Let φ' be $(\exists Y')(\exists v_0)(Std(Y',v_0,Y)\,\&\,\hat{\varphi})$.

As is well known, adding variables of type $(0,0)$ gives the expressive power of full second-order logic: any variables of type $(0,\ldots,0)$, with $1<n$ occurrences of '0', can be replaced by variables of type 1 applied to n-tuples formed set-theoretically; then the '$\in$' used in specifying n-tuples can be quantified out by a type $(0,0)$ variable restricted by the axioms of pairs and extensionality; this preserves equivalence.

2.3. Type-1 variables can render '$\leqslant$' superfluous.

Observation. (i) $\mathscr{L}^{1,2}(\underline{\text{exactly}}, \leqslant) \overset{s}{\succ\!\prec} \mathscr{L}^{1,2}(\underline{\text{exactly}})$,

(ii) $\mathscr{L}^{1,2*}(\underline{\text{exactly}}, \leqslant) \overset{s}{\succ\!\prec} \mathscr{L}^{1,2*}(\underline{\text{exactly}})$.

Furthermore, if κ is an aleph, then for $1<k<\omega$:

(iii) $\mathscr{L}^{1,2k}(\underline{\text{exactly}}, \leqslant) \overset{\kappa}{\succ\!\prec} \mathscr{L}^{1,2k}(\underline{\text{exactly}})$,

(iv) $\mathscr{L}^{1,2k*}(\underline{\text{exactly}}, \leqslant) \overset{\kappa}{\succ\!\prec} \mathscr{L}^{1,2k}(\underline{\text{exactly}})$.

Proof. For $\mu, \rho \in Var(2)$, replace $(\mu \leqslant \rho)$ by:

$$(\exists Y_0)(\exists Y_1)((\underline{\text{exactly}}\,\mu\, v)Y_0v \,\&\, (\underline{\text{exactly}}\,\rho\, v)Y_1v \,\&\, (\forall v)(Y_0v \supset Y_1v)),$$

where $Y_0, Y_1 \in Var(1)$ are distinct. This suffices for (i) and (ii). For $\mu, \rho \in Var(2j+2)$, $0<j<\omega$, let $(\mu \leqslant_j \rho)$ abbreviate:

$$(\exists\mu')(\exists\rho')((\underline{\text{exactly}}\,\mu\,\eta)\,\eta \leqslant \mu' \,\&\, (\underline{\text{exactly}}\,\rho\,\eta)\,\eta \leqslant \rho' \,\&\, \mu' \leqslant \rho').$$

If κ is an aleph for any model $\mathscr{A}$ and $n, m < \text{ncb}^j(\kappa)$:

$$\mathscr{A} \vDash_\kappa \mathbf{n} \leqslant_j \mathbf{m} \quad \text{if } n \leqslant m.$$

Given $\varphi \in \text{Sent}(\mathscr{L}^{1,2k+2}(\underline{\text{exactly}}, \leqslant))$ for $0<k$, replace all subformulae of φ of the form $(\mu \leqslant \rho)$ for $\mu, \rho \in Var(2k+2)$ by $(\mu \leqslant_k \rho)$. If $1<k$, then replace subformulae of the form $(\mu \leqslant \rho)$ for $\mu, \rho \in Var(2k)$ by $(\mu \leqslant_{k-1} \rho)$; repeat until this for all subformulae of the form $(\mu \leqslant \rho)$, $\mu, \rho \in Var(2)$; replace these as we did for (i). This establishes (iii); the same construction also yields (iv). □

This procedure is independent of κ; so the axiom of choice entails that for $1<k<\omega$:

(v) $\mathscr{L}^{1,2k}(\underline{\text{exactly}}, \leqslant) \overset{s}{\succ\!\prec} \mathscr{L}^{1,2k}(\underline{\text{exactly}})$,

(vi) $\mathscr{L}^{1,2k*}(\underline{\text{exactly}}, \leqslant) \overset{s}{\succ\!\prec} \mathscr{L}^{1,2k*}(\underline{\text{exactly}})$.

2.4. For variables μ and μ' of type $\geqslant 2$ let $\mu = \mu'$ and $\mu < \mu'$ abbreviate the obvious formulae. For $k < \omega$ it is easy to construct formulae $\boldsymbol{Lim}^k(\mu)$, $\boldsymbol{G}_0(\mu)$, $\boldsymbol{E}_0(\mu)$ and $\boldsymbol{Fin}(\mu)$ meeting these conditions for any model $\mathscr{A}$. For any aleph κ and $n < \kappa$:

$$\mathscr{A} \vDash_\kappa \boldsymbol{Lim}^k(\mathbf{n}) \quad \text{iff} \quad \text{either } n = 0 \text{ or for some } \alpha \text{ and } \beta, \; n = \aleph_\alpha \text{ and } \alpha = \omega^k \cdot \beta;$$

$$\mathscr{A} \vDash_\kappa \boldsymbol{G}_0(\mathbf{n}) \quad \text{iff} \quad \aleph_0 \leqslant n \text{ (iff } n \text{ is Dedekind-infinite);}$$

and if $\aleph_0 \leqslant \kappa$, then:

$$\mathscr{A} \vDash_\kappa \boldsymbol{E}_0(\mathbf{n}) \quad \text{iff} \quad \aleph_0 = n;$$

$$\mathscr{A} \vDash_\kappa \boldsymbol{Fin}(\mathbf{n}) \quad \text{iff} \quad n \text{ is finite.}$$

For $k < \aleph_0$ we want a contextually defined quantifier-expression $\underline{\text{exactly } k}$ and a 'predicate' $\underline{k =}$ so that:

$$\mathscr{A} \vDash_\kappa (\underline{\text{exactly } k}\, v)\varphi \quad \text{iff} \quad \text{card}(\hat{v}\varphi) = k;$$

$$\text{for any } n < \kappa, \quad \mathscr{A} \vDash_\kappa \underline{k =} \mathbf{n} \quad \text{iff} \quad k = n.$$

Here is one way to do this. Where $v, v' \in Var(2j)$ and v' does not occur free in φ, adopt these abbreviations:

$$(\underline{\text{exactly } 0}\, v)\varphi: \qquad \neg(\exists v)\varphi;$$

$$(\underline{\text{exactly } k+1}\, v)\varphi: \quad (\exists v')(\varphi(v/v')\,\&\,(\underline{\text{exactly } k}\, v)(\varphi \,\&\, v \neq v')).$$

For $\mu \in Var(2)$ and distinct $v, v_0 \in Var(0)$:

$$\underline{0 =} \mu: \qquad (\underline{\text{exactly}}\, \mu\, v) \perp;$$

$$\underline{k+1 =} \mu: \quad (\forall v_0) \cdots (\forall v_k)\Big(\Big(\bigwedge_{i<j\leqslant k} v_i \neq v_j\Big) \supset (\underline{\text{exactly}}\, \mu\, v)\Big(\bigvee_{i\leqslant k} v = v_i\Big)\Big).$$

For $\mu \in Var(2j+2)$, $\rho \in Var(2j)$ with $0 < j < \omega$:

$$\underline{0 =} \mu: \qquad (\underline{\text{exactly}}\, \mu\, \rho) \perp;$$

$$\underline{k+1 =} \mu: \quad (\underline{\text{exactly}}\, \mu\, \rho)\Big(\bigvee_{i\leqslant k} \underline{i =} \rho\Big).$$

Notice: if φ is a formula of $\mathscr{L}^{i,2j}(\underline{\text{exactly}}, \leqslant)$, then (1) so is $(\underline{\text{exactly } k}\, v)\varphi$; but as just defined, (2) it uses '=' between variables of type $\geqslant 2$ when v is of type $\geqslant 2$. Using these definitions, for $\mu \in Var(2j)$ with $j > 0$, and $1 \leqslant k < \omega$, adopt this abbreviation:

$$\boldsymbol{E}_k(\mu): \quad (\exists \eta)(\boldsymbol{G}_0(\eta)\,\&\,(\underline{\text{exactly } k}\, \rho)(\eta \leqslant \rho \,\&\, \rho < \mu)).$$

Thus for $\kappa \geqslant \aleph_0$ and $n < \kappa$: $\mathscr{A} \vDash_\kappa \boldsymbol{E}_l(\mathbf{n})$ iff $n = \aleph_l$.

Feature (2) of our definition of $(\underline{\text{exactly } k}\, v)\varphi$ may be avoided, provided we consider only infinite κ, by adopting this abbreviation:

$$\underline{(\text{exactly } k\ \nu)}\varphi: \quad (\exists\mu)(\underline{k = }\,\mu\ \&\ \underline{(\text{exactly}}\ \mu\ \nu)\varphi).$$

But for $\nu \in Var(2j)$, the right-hand side either requires $\mu \in Var(2j+2)$, making $\underline{(\text{exactly } k\ \nu)}\varphi \notin \text{Fml}(\mathscr{L}^{i,2j}(\underline{\text{exactly}}, \leqslant))$, or else it requires $\mu \in Var(2j)$, making $\underline{(\text{exactly } k\ \mu)}\varphi \notin \text{Fml}(\mathscr{L}^{i,\omega}(\underline{\text{exactly}}, \leqslant))$. The second kind of abbreviation, taking $\mu \in Var(2j+2)$ [$var(2j)$] will be used in §4.1 [§4.2].

2.5. Observation. *If κ is finite, then $\mathscr{L}^{i,2*}(\underline{\text{exactly}}, \leqslant) \overset{\kappa}{\preceq} \mathscr{L}^i$.*

Proof is left to the reader; the important thing to see is that if ρ is a variable of type $\geqslant 2$ and $k < \kappa$, then $\underline{(\text{exactly } k}\ \rho)\psi$ is replaced by $\bigvee \{\theta(b): b \subseteq \bar{\kappa}, \text{card}(b) = k\}$, where $\theta(b)$ is:

$$\bigwedge \{\varphi(\rho/\mathbf{k}): k \in b\}\ \&\ \bigwedge \{\neg\varphi(\rho/\mathbf{k}): \kappa \in \bar{\kappa} - b\}.$$

Hereafter κ shall always be an infinite cardinal.

2.6. This section concerns assertion of identity across types. For $0 < i < \omega$, $\rho \in Var(2i)$, $\mu \in Var(2i+2)$ let:

$$\rho =_{2i} \mu: \quad \underline{(\text{exactly}}\ \mu\ \nu)\ \nu < \rho,$$

where $\nu \in Var(2i)$ is distinct from ρ. If $\text{ncb}^{i-1}(\kappa) = \aleph_0$ and either n or m is finite:

$$\mathscr{A} \vDash_k \mathbf{n} =_{2i} \mathbf{m} \quad \text{iff} \quad n = m.$$

This idea will now be pushed a little further. For $k < \omega$ let $\rho =_{2i,k} \mu$ abbreviate:

$$((\boldsymbol{Fin}(\rho) \vee \boldsymbol{Fin}(\mu)) \supset \rho =_{2i} \mu)$$

$$\&\ (\neg(\boldsymbol{Fin}(\rho) \vee \boldsymbol{Fin}(\mu)) \supset \bigvee_{l<k} (\boldsymbol{E}_l(\rho)\ \&\ \boldsymbol{E}_l(\mu)).$$

Thus: if $\text{ncb}^{i-1}(\kappa) = \aleph_k$ and either n or $m < \aleph_k$:

$$\mathscr{A} \vDash_\kappa \mathbf{n} =_{2i,k} \mathbf{m} \quad \text{iff} \quad n = m.$$

Note that $\rho =_{2i,0} \mu$ is just $\rho =_{2i} \mu$.

Where $\alpha < \omega^\omega$, let the Cantor-coefficient sequence for $\aleph_\alpha$ be $\langle n_0, \ldots, n_q \rangle$, where $\alpha = \omega^q \cdot n_q + \cdots + \omega \cdot n_1 + n_0$. For $0 < j < \omega$, $\rho \in Var(2j)$ and $\mu_0, \ldots, \mu_{q-1} \in Var(2j+2)$, there is a $\boldsymbol{Cc}^q(\rho, \mu_0, \ldots, \mu_{q-1}) \in \text{Fml}(\mathscr{L}^{0,2j+2}$ $(\underline{\text{exactly}}, \leqslant))$ so that for any $n < \aleph_{\omega^q} \leqslant \kappa$ and $n_0, \ldots, n_{q-1} < \aleph_0$:

$$\mathscr{A} \vDash_\kappa \boldsymbol{Cc}^q(\mathbf{n}, \mathbf{n}_0, \ldots, \mathbf{n}_{q-1}) \quad \text{iff} \quad \langle n_0, \ldots, n_{q-1} \rangle \text{ is the Cantor-coefficient sequence for } n.$$

For $\rho, \rho' \in Var(2j)$ and $0 < k < \omega$ let $\boldsymbol{M}^k(\rho, \rho')$ say "ρ' is the maximum cardinal of the form $\aleph_{\omega^k\cdot\beta}$ that is $\leqslant\rho$". Let $\boldsymbol{Cc}^q(\rho, \mu_0, \ldots, \mu_{q-1})$ be:

$$(\exists\rho_1)\cdots(\exists\rho_{q-1})\Big(\bigwedge_{0<k<q} \boldsymbol{M}^k(\rho, \rho_k)$$
$$\&\ (\underline{\text{exactly}}\ \mu_{q-1}\ \nu)(\boldsymbol{Lim}^{q-1}(\nu)\ \&\ 0<\nu\ \&\ \nu\leqslant\rho_{q-1})$$
$$\&\ (\underline{\text{exactly}}\ \mu_{q-2}\ \nu)(\boldsymbol{Lim}^{q-2}(\nu)\ \&\ \rho_{q-1}<\nu\ \&\ \nu\leqslant\rho_{q-2})$$
$$\&\cdots\&\ (\underline{\text{exactly}}\ \mu_1\ \nu)(\boldsymbol{Lim}^{1}(\nu)\ \&\ \rho_2<\nu\ \&\ \nu\leqslant\rho_1)$$
$$\&\ (\underline{\text{exactly}}\ \mu_0\ \nu)(\rho_1<\nu\ \&\ \nu\leqslant\rho)\Big),$$

where $\nu, \rho_0, \ldots, \rho_{q-1} \in Var(2j)$. It is easy to see that $\boldsymbol{Cc}^q$ is as required. For $\rho \in Var(2j)$ and $\mu \in Var(2j+2)$, let $\rho =_{2j,q}' \mu$ abbreviate:

$$(\exists\rho_0)\cdots(\exists\rho_{q-1})(\exists\mu_0)\cdots(\exists\mu_{q-1})$$
$$\Big(\bigwedge_{i<q} \boldsymbol{Fin}(\rho_i)\ \&\ \bigwedge_{i<q} \rho_i =_{2j} \mu_i\ \&\ \boldsymbol{Cc}^q(\rho, \vec{\rho})\ \&\ \boldsymbol{Cc}^q(\mu, \vec{\mu})\Big),$$

for $\rho_0, \ldots, \rho_{q-1} \in Var(2j+2)$, $\mu_0, \ldots, \mu_{q-1} \in Var(2j+4)$. Then for any κ, any $n, m < \text{ncb}^{j-1}(\kappa)$ and any model $\mathscr{A}$, if either n or $m < \aleph_{\omega^q}$:

$$\mathscr{A} \vDash_\kappa \mathbf{n} =_{2j,q}' \mathbf{m} \quad \text{iff} \quad n = m.$$

2.7. This section describes cases in which final segments of the inclusion-hierarchy described in §2.1 collapse (with respect to expressive power) to a lower language.

Collapsing Theorem. *Suppose* $\kappa \in \text{Card}$ *is an aleph,* $i \in 2$ *and* $1 \leqslant k < \omega$.

(i) *If* $\text{ncb}^k(\kappa) < \aleph_\omega$, *then* $\mathscr{L}^{i,2k*}(\underline{\text{exactly}}, \leqslant) \overset{\kappa}{\preceq} \mathscr{L}^{i,2k+2}(\underline{\text{exactly}}, \leqslant)$.

(ii) *If* $\text{ncb}^k(\kappa) < \aleph_{\omega^\omega}$, *then* $\mathscr{L}^{i,2k*}(\underline{\text{exactly}}, \leqslant) \overset{\kappa}{\preceq} \mathscr{L}^{i,2k+4}(\underline{\text{exactly}}, \leqslant)$.

So, for example, if $\kappa < \aleph_\omega$, then hierarchy of type $\omega + 1 + \omega^*$ going from $\mathscr{L}^{1,6}(\underline{\text{exactly}}, \leqslant)$ through $\mathscr{L}^{1,\omega}(\underline{\text{exactly}}, \leqslant)$ up to $\mathscr{L}^{1,2*}(\underline{\text{exactly}}, \leqslant)$ collapses down to $\mathscr{L}^{i,4}(\underline{\text{exactly}}, \leqslant)$: under $\vDash_\kappa$ these languages have equal expressive power.

Proof. We will consider the case of $k = 1$. Suppose $\text{ncb}(\kappa) = \aleph_q$ for $q < \omega$. Given $\varphi \in \text{Sent}(\mathscr{L}^{i,2*}(\underline{\text{exactly}}, \leqslant))$ form φ' by replacing each subformula of φ of the form $(\underline{\text{exactly}}\ \mu\ \rho)\psi$ for $\mu, \rho \in Var(2)$ by:

$$(\exists\mu')((\underline{\text{exactly}}\ \mu'\ \rho)\psi'\ \&\ \mu =_{2,q+1} \mu'),$$

where $\mu' \in Var(4)$ does not occur free in ψ'. To see that φ' is as required, note the following. For $(\underline{\text{exactly}}\ \mu\ \rho)\psi \in \text{Fml}(\mathscr{L}^{i,2*}(\underline{\text{exactly}}, \leqslant))$ with μ the only free variable, and for any $\mathscr{A}$ and $n < \kappa$:

$$\mathscr{A} \vDash_\kappa (\underline{\text{exactly}}\ \mathbf{n}\ \rho)\psi \quad \text{iff} \quad \mathscr{A} \vDash_\kappa (\exists\mu')((\underline{\text{exactly}}\ \mu'\ \rho)\psi\ \&\ \mathbf{n} =_{2,q+1} \mu'));$$

this is because the left-hand side implies that $n \leqslant \text{ncb}(\kappa)$, since $\hat{\rho}\psi \subseteq \bar{\kappa}$, and so $\mathscr{A} \models_\kappa \mathbf{n} =_{2i,q+1} \mathbf{n}$, yielding the right-hand side; to go from right to left, notice that for any $n, n' < \kappa$: if $\mathscr{A} \models_\kappa \mathbf{n} =_{2i,q+1} \mathbf{n}'$, then $n = n' \leqslant \aleph_q$.

Now suppose $\text{ncb}(\kappa) = \aleph_\alpha$ for $\alpha < \omega^\omega$; let $q < \omega$ be least so that $\alpha < \omega^q$. Given $\varphi \in \text{Sent}(\mathscr{L}^{i,2*}(\underline{\text{exactly}}, \leqslant))$, form φ' by replacing each subformula of φ of the form $(\underline{\text{exactly}}\, \mu\, \rho)\psi$ for $\mu, \rho \in \mathit{Var}(2)$ by:

$$(\exists \mu')((\underline{\text{exactly}}\, \mu'\, \rho)\psi' \;\&\; \mu =_{2,q}' \mu'),$$

where $\mu' \in \mathit{Var}(4)$ and does not occur free in ψ'. The reason why φ' works is as above, using the fact that for any $n, n' \in \bar{\kappa}$: if $\mathscr{A} \models_\kappa \mathbf{n} =_{2,q}' \mathbf{n}'$, then $n = n' < \aleph_{\omega^q}$.

For $1 < k < \omega$, replace types 2 and 4 by types $2k$ and $2k+2$ respectively in the preceding argument. □

3. The hierarchy problem

3.1. A proof of the following conjecture would be the best possible complement to the Collapsing Theorem of §2.7.

Conjecture. For any $i \in 2$ and $1 \leqslant k < \omega$ there are *Pred* and *Funct* so that:

(1a) For every infinite $\kappa \in \text{Card}$,

$$\mathscr{L}^{i,4}(\underline{\text{exactly}}) \not\leqslant_\kappa \mathscr{L}^{i,2}(\underline{\text{exactly}}, \leqslant).$$

(1b) If $k > 1$, for every κ with $\aleph_\omega \leqslant \text{ncb}^{k-1}(\kappa)$,

$$\mathscr{L}^{i,2k+2}(\underline{\text{exactly}}) \not\leqslant_\kappa \mathscr{L}^{i,2k}(\underline{\text{exactly}}, \leqslant).$$

(2) For every κ with $\aleph_{\omega^\omega} \leqslant \text{ncb}^k(\kappa)$,

$$\mathscr{L}^{i,2k*}(\underline{\text{exactly}}) \not\leqslant_\kappa \mathscr{L}^{i,2k+2*}(\underline{\text{exactly}}, \leqslant).$$

We will prove (1a) for $i = 0$. Fix $\mathit{Pred} = \{\mathbf{R}_0, \mathbf{R}_1\}$, with $\mathbf{R}_0$ and $\mathbf{R}_1$ both 2-place, and let *Funct* be empty. Let $\varphi_{0,2}$ be:

$$(\forall \mu)((\underline{\text{exactly}}\, \mu\, \rho)(\exists v_0)(\underline{\text{exactly}}\, \rho\, v)\mathbf{R}_0(v_0, v)$$
$$\equiv (\underline{\text{exactly}}\, \mu\, \rho)(\exists v_0)(\underline{\text{exactly}}\, \rho\, v)\mathbf{R}_1(v_0, v)).$$

Clearly $\varphi_{0,2} \in \text{Sent}(\mathscr{L}^{0,4}(\underline{\text{exactly}}))$.

Theorem. *For any infinite* $\kappa \in \text{Card}$, $\varphi_{0,2}$ *is equivalent*$_\kappa$ *to no sentence of* $\mathscr{L}^{0,2}(\underline{\text{exactly}}, \leqslant)$.

Before the proof, some conjectures deserve mention.

Fix $Pred = (\mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3\}$, all 2-place. Let $\varphi_{0,4}$ be:

$$(\forall\mu)[(\underline{\text{exactly}}\ \mu\ \rho)(\exists v_1)(\underline{\text{exactly}}\ \rho\ \eta)(\exists v_0)(\mathbf{R}_1(v_1, v_0)\ \&\ (\underline{\text{exactly}}\ \eta\ v)\mathbf{R}_0(v_0, v)) \equiv (\underline{\text{exactly}}\ \mu\ \rho)(\exists v_1)(\underline{\text{exactly}}\ \rho\ \eta)(\exists v_0)(\mathbf{R}_3(v_1, v_0)\ \&\ (\underline{\text{exactly}}\ \eta\ v)\mathbf{R}_2(v_0, v))].$$

(Here $\mu \in Var(6)$, $\rho \in Var(4)$, $\eta \in Var(2)$, v, v_0, $v_1 \in Var(0)$.)

Conjecture (A). This choice of $\varphi_{0,4}$ meets the conditions required in (1b) for $i = 0$, $k = 2$.

The sequence $\varphi_{0,2}$, $\varphi_{0,4}$, extends, following the obvious pattern, to include likely candidates for (1b) when $i = 0$ and $k > 2$.

Where $Pred = \{\mathbf{P}, \mathbf{R}\}$, $\mathbf{P}$ 1-place and $\mathbf{R}$ 2-place, let $\varphi^*_{0,2}$ be:

$$(\forall\mu)((\underline{\text{exactly}}\ \mu\ \rho)(\exists v_0)(\underline{\text{exactly}}\ \rho\ v)\mathbf{R}(v_0, v) \equiv (\underline{\text{exactly}}\ \mu\ v)\mathbf{P}v),$$

for μ, $\rho \in Var(2)$, $v_0, v \in Var(0)$ and distinct.

Conjecture (B). This choice of $\varphi^*_{0,2}$ meets the conditions required by (2) for $i = 0$, $k = 1$.

Similarly, let $\varphi^*_{0,4}$ be:

$$(\forall\mu)[(\underline{\text{exactly}}\mu\ \rho)(\exists v_1)(\underline{\text{exactly}}\ \rho\ \eta)(\mathbf{R}_1(v, v_0)\ \&\ (\underline{\text{exactly}}\ \eta\ v)\mathbf{R}_0(v_0, v)) \equiv (\underline{\text{exactly}}\ \mu\ \rho)(\exists v_0)(\underline{\text{exactly}}\ \rho\ v)\mathbf{R}_2(v_0, v)];$$

it seems likely that this is as required by (2) for $i = 0$, $k = 2$. This pattern also extends to yield likely candidates for (2) when $i = 0$ and $k > 2$.

The $\varphi_{0,2}$ above is expressible in $\mathscr{L}^{1,2}(\underline{\text{exactly}})$. To see this, let $\psi_i(Y)$ be:

$$(\forall\rho)((\underline{\text{exactly}\ 1}\ v_0)(Yv_0\ \&\ (\underline{\text{exactly}}\ \rho\ v)\mathbf{R}_i(v_0, v)) \equiv (\exists v_0)(\underline{\text{exactly}}\ \rho\ v)\mathbf{R}_i(v_0, v)).$$

where $Y \in Var(1)$: then for any infinite κ, $n < \kappa$ and any model $\mathscr{A}$ for $\{\mathbf{R}_0, \mathbf{R}_1\}$, $\mathscr{A} \models_\kappa (\exists Y)\psi_i(Y)$; furthermore, if

$$\mathscr{A} \models_\kappa (\underline{\text{exactly}}\ \mathbf{n}\ \rho)(\exists v_0)(\underline{\text{exactly}}\ \rho\ v)\mathbf{R}_i(v_0, v),$$

then for any $B \subseteq |\mathscr{A}|$: $\mathscr{A} \models_\kappa \psi_i[B]$ iff $\text{card}(B) = n$. Thus

$$(\exists Y_0)(\exists Y_1)(\psi_0(Y_0)\ \&\ \psi_1(Y_1)\ \&\ (\forall\mu)((\underline{\text{exactly}}\ \mu\ v)Y_0v \equiv (\underline{\text{exactly}}\ \mu\ v)Y_1v))$$

is super-equivalent to $\varphi_{0,2}$. But the idea behind the construction of $\varphi_{0,2}$ suggests the following. Fix $(\mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3\}$, all 2-place; let $\varphi_{1,2}$ be:

$$(\forall\mu)[(\underline{\text{exactly}}\ \mu\ \rho)(\exists v_0)(\exists v_1)(\underline{\text{exactly}}\ \rho\ v)(\mathbf{R}_0(v_0, v)\ \&\ \mathbf{R}_1(v_1, v)) \equiv (\underline{\text{exactly}}\ \mu\ \rho)(\exists v_0)(\exists v_1)(\underline{\text{exactly}}\ \rho\ v)(\mathbf{R}_2(v_0, v)\ \&\ \mathbf{R}_3(v_1, v))].$$

Conjecture (C). This $\varphi_{1,2}$ meet the requirements of (1a) for $i=1$.

The idea behind this suggestion extends to yield likely candidates to meet the requirements of (1) and (2) when $i>1$.

3.2. To prove Theorem 3.1, we will use Ehrenfeucht-games for languages of the form $\mathscr{L}^{0,2}(\underline{\text{exactly}}, \leqslant, Pred, Funct)$. Given models $\mathscr{A}_0$, $\mathscr{A}_1$ for *Pred, Funct*, $q<\omega$ and $\kappa \in \text{Card}$, we consider the game $G=G_\kappa^{0,2}(\underline{\text{exactly}}, \leqslant, Pred, Funct, \mathscr{A}_0, \mathscr{A}_1, q)$. A position p in G is a finite sequence of 'exchanges, between players I and II with $|p|\leqslant q$; each such p is associated with a 'situation' $h(p)=\langle h_0(p), h_1(p)\rangle$, where for $i\in 2$:

$$h_i(p)=\langle \vec{a}_i, \vec{n}_i\rangle \quad \text{for } \vec{a}_i \in |\mathscr{A}|^{l(0)}, \qquad \vec{n}_i \in \bar{\kappa}^{l(2)}, \qquad |p|=l(0)+l(2).$$

Play of G begins at $\langle\ \rangle$, with $h_i(\langle\ \rangle)=\langle\langle\ \rangle, \langle\ \rangle\rangle$. Suppose that play of G has reached p, with $|p|=q'\leqslant q$. Fix $v_0, \ldots, v_{l(0)-1} \in Var(0)$, $\mu_0, \ldots, \mu_{l(2)-1} \in Var(2)$. If $q'=q$, play is over; II wins iff for every atomic formula φ of $\mathscr{L}^{0,2}(\underline{\text{exactly}}, \leqslant)$ with free variables among those listed:

$$\mathscr{A}_0 \vDash_\kappa \varphi[h_0(p)] \quad \text{iff} \quad \mathscr{A}_1 \vDash_\kappa \varphi[h_1(p)].$$

Now suppose $q'<q$. I initiates an exchange of one of three sorts. In what follows, p' shall be the position reached at the end of the exchange.

(1) I selects $i\in 2$ and $a_{i,l(0)} \in |\mathscr{A}_i|$; II must select an $a_{1-i,l(0)} \in |\mathscr{A}_{1-i}|$; then for $j\in 2$, $h_j(p')=\langle \vec{a}_j * a_{j,l(0)}, \vec{n}_j\rangle$.

(2) I selects $i\in 2$ and $n_{i,l(2)} \in \bar{\kappa}$; II must select an $n_{1-i,l(2)} \in \bar{\kappa}$; then for $j\in 2$, $h_j(p')=\langle \vec{a}_j, \vec{n}_j * n_{j,l(2)}\rangle$.

(3) I selects $i\in 2$, $w<l(2)$ and $B_i \subseteq |\mathscr{A}_i|$ with $\text{card}(B_i)=n_{i,w}$; II must select $B_{1-i} \subseteq |\mathscr{A}_{1-i}|$ with $\text{card}(B_{1-i})=n_{1-i,w}$. I then selects an $a_{1-i,l(0)} \in |\mathscr{A}_{1-i}|$; II selects $a_{i,l(0)} \in |\mathscr{A}_i|$ so that $a_{0,l(0)} \in B_0$ iff $a_{1,l(0)} \in B_1$; if II can't do this, she loses; for $j\in 2$, $h_j(p')=\langle \vec{a}_j * a_{j,l(0)}, \vec{n}_j\rangle$.

Lemma 1. *If* II *has a winning strategy for* G, *then for every* $\varphi \in \text{Sent}(\mathscr{L}^{0,2}(\underline{\text{exactly}}, \leqslant))$, *if* quantifier-depth$(\varphi)\leqslant q$, $\mathscr{A}_0 \vDash_\kappa \varphi$ *iff* $\mathscr{A}_1 \vDash_\kappa \varphi$.

Before proving this, we will consider another sort of game.

Let $\mathscr{A}$ be a model for *Pred, Funct* and $\varphi \in \text{Fml}(\mathscr{L}^{0,2}(\underline{\text{exactly}}, \leqslant))$ with free variables among $v_0, \ldots, v_{l(0)-1} \in Var(0)$, $\mu_0, \ldots, \mu_{l(2)-1} \in Var(2)$. For convenience, suppose that each variable ρ occurring in φ is bound at at most one occurrence, i.e. occurs at most once in a prefex of the form $(\exists\rho)$ or $(\underline{\text{exactly}}\ \mu\ \rho)$. Fix $\vec{a} \in |\mathscr{A}|^{l(0)}$, $\vec{n} \in \bar{\kappa}^{l(2)}$. We describe the game $\text{SAT}_\kappa(\varphi, \mathscr{A}, \vec{a}, \vec{n})$ inductively. There are two players, I and II, and two hats, TRUE and FALSE. At any position, each player wears one hat, and the other the other; at a position, the players shall be referred to by the hats they wear.

If φ is atomic, play is over; TRUE wins iff $\mathscr{A} \vDash_\kappa \varphi[\vec{a}, \vec{n}]$. If φ is $(\varphi_0 \supset \varphi_1)$, TRUE picks $i < 2$; if $i = 1$ they go on to play $\mathrm{SAT}_\kappa(\varphi_1, \mathscr{A}, \vec{a}, \vec{n})$ with hats as they are; if $i = 0$ they switch hats and go on to play $\mathrm{SAT}_\kappa(\varphi_0, \mathscr{A}, \vec{a}, \vec{n})$. If φ is $(\exists v_{l(0)})\varphi_0$, TRUE picks $a_{l(0)} \in |\mathscr{A}|$; with hats as they are, they go on to play $\mathrm{SAT}_\kappa(\varphi_0, \mathscr{A}, \vec{a} * a_{l(0)}, \vec{n})$. If φ is $(\exists \mu_{l(2)})\varphi_0$ TRUE picks $n_{l(2)} \in \bar{\kappa}$; they go on to play $\mathrm{SAT}_\kappa(\varphi_0, \mathscr{A}, \vec{a}, \vec{n} * n_{l(2)})$. If φ is $(\underline{\text{exactly}}\ \mu_w\ v_{l(0)})\varphi_0$, TRUE selects $B \subseteq |\mathscr{A}|$ with $\mathrm{card}(B) = n_w$; FALSE selects $a \in |\mathscr{A}|$; they exchange hats iff $a \notin B$, and go on to play $\mathrm{SAT}_\kappa(\varphi_0, \mathscr{A}, \vec{a} * a_{l(0)}, \vec{n})$. Since this game is finite, it is determined.

Lemma 2. *$\mathscr{A} \models_\kappa \varphi[\vec{a}, \vec{n}]$ iff* TRUE *has a winning strategy for* $\mathrm{SAT}_\kappa(\varphi_0, \mathscr{A}, \vec{a}, \vec{n})$.

Proof is straightforward. Where φ is $(\underline{\text{exactly}}\ \mu_w\ v_{l(0)})\varphi_0$, think of TRUE's choice of B as a claim that $B = \hat{v}_{l(0)}\varphi_0[\vec{a}, \vec{n}]$; so if FALSE takes $a_{l(0)} \in B$, TRUE must defend the claim that $a_{l(0)} \in \hat{v}_{l(0)}\varphi_0[\vec{a}, \vec{n}]$; otherwise TRUE must refute that claim—and so must put on the FALSE hat for $\mathrm{SAT}_\kappa(\varphi_0, \mathscr{A}, \vec{a}, \vec{n})$.

We now describe I's strategy for G. Until I wins, I associates each position p in G with a formula φ_p, depth $(\varphi_p) \leq q - |p|$, so that for $h(p)$ as above:

$$\mathscr{A}_0 \models_\kappa \varphi_p[\vec{a}_0, \vec{n}_0] \quad \text{iff} \quad \mathscr{A}_1 \not\models_\kappa \varphi_p[\vec{a}_1, \vec{n}_1].$$

Let $\varphi_{\langle\rangle} = \varphi$. Suppose p has been reached, $|p| = q'$. If φ_p is truth-functionally compound I first finds a non-truth-functionally compound truth-functional component of φ_p, φ'_p, so that:

$$\mathscr{A}_0 \models_\kappa \varphi'_p[\vec{a}_0, \vec{n}_0] \quad \text{iff} \quad \mathscr{A} \not\models_\kappa \varphi'_p[\vec{a}_1, \vec{n}_1].$$

Otherwise let $\varphi'_p = \varphi_p$.

Suppose that no p^* with φ_{p^*} of the form $(\underline{\text{exactly}}\ \mu\ v)\psi$ has yet been reached; I selects $i < 2$ so that $\mathscr{A}_i \models_\kappa \varphi'_p[\vec{a}_i, \vec{n}_i]$. If φ'_p is $(\exists v_{l(0)})\psi$ I selects $a_{i,l(0)}$ so that $\mathscr{A}_i \models_\kappa \psi[\vec{a}_i * a_{i,l(0)}, \vec{n}_i]$ and sets $\varphi_{p'} = \psi$. No matter what $a_{1-i,l(0)}$ II takes, $\mathscr{A}_{1-i} \not\models_\kappa \psi[\vec{a}_{1-i} * a_{1-i,l(0)}, \vec{n}_i]$. If φ'_p is $(\exists \mu_{l(2)})\psi$, I again plays a witnessing $n_{i,l(2)}$; no matter what $n_{1-i,l(2)}$ II takes, $\mathscr{A}_{1-i} \not\models_\kappa \psi[\vec{a}_{1-i}, \vec{n}_{1-i} * n_{1-i,l(2)}]$. If φ'_p is $(\underline{\text{exactly}}\ \mu_w\ v_{l(0)})\psi$, I plays $B_i = \hat{v}_{l(0)}\psi[\vec{a}_i, \vec{n}_i]^{\mathscr{A}_i}$ and w; $\mathrm{card}(B_i) = n_{i,w}$; no matter what B_{1-i} II picks, if $\mathrm{card}(B_{1-i}) = n_{1-i,w}$ then $B_{1-i} \neq \hat{v}_{l(0)}\psi[\vec{a}_{1-i}, \vec{n}_{1-i}]^{\mathscr{A}_{1-i}}$; so I may select $a_{1-i,l(0)}$ in their symmetric difference. No matter what $a_{i,l(0)}$ II now takes, since II must have $a_{0,l(0)} \in B_0$ iff $a_{1,l(1)} \in B_1$: $\mathscr{A}_0 \models_\kappa \psi[\vec{a}_0 * a_{0,l(0)}, \vec{n}_0]$ iff $\mathscr{A}_1 \not\models_\kappa \psi[\vec{a}_1 * a_{1,l(0)}, \vec{n}_1]$. Let $\varphi_{p'} = \psi$.

As soon as the above sort of exchange takes place, I changes his approach: he pretends to be playing both:

$$\mathrm{SAT}_0 = \mathrm{SAT}_\kappa(\psi, \mathscr{A}_0, \vec{a}_0 * a_{0,l(0)}, \vec{n}_0);$$
$$\mathrm{SAT}_1 = \mathrm{SAT}_\kappa(\psi, \mathscr{A}_1, \vec{a}_1 * a_{1,l(0)}, \vec{n}_1).$$

For $j \in 2$, let TRUE_j and FALSE_j be the hats from SAT_j. Fix $j \in 2$ so that $\mathscr{A}_j \models_\kappa \psi[\vec{a}_j * a_{j,l(0)}, \vec{n}_j]$. I begins SAT_j wearing TRUE_j and SAT_{1-j} wearing the FALSE_{1-j}. By Lemma 2, I has winning strategies for SAT_0 and SAT_1. At all subsequent positions, I wears TRUE_0 iff I wears FALSE_1. Suppose play has reached p, where for some p^* an initial segment of p, p^* initiated I's change of approach. Suppose that I is wearing TRUE_{k_0} and FALSE_{1-k_0}. If φ_p is not a conditional, in G I plays k_0 and lets $\varphi'_p = \varphi_p$. If φ_p is $\psi_0 \supset \psi_1$, in his pretend play of SAT_{k_0}, I chooses $i_0 \in 2$ according to his strategy for SAT_{k_0}. In the pretended play of SAT_{1-k_0}, I pretends that TRUE_{1-k_0} also plays i_0. Let $\theta_1 = \psi_{i_0}$. Let k_1 be such that after these moves, I wears TRUE_{k_1} and FALSE_{1-k_1}. Iterate this until a θ_z is reached which is not a conditional; I is, of course, wearing TRUE_{k_z} and FALSE_{1-k_z}. I plays $i = k_z$ in G and lets $\varphi'_p = \theta_z$. Thus $\mathcal{A}_i \models_\kappa \varphi'_p[\vec{a}_i, \vec{n}_i]$ and $\mathcal{A}_{1-i} \not\models_\kappa [\vec{a}_{1-i}, \vec{n}_{1-i}]$. I now moves in the pretended play of SAT_i as dictated by his strategy in that game; he makes the same move in G. II responds in G; then I pretends that this response is TRUE_{1-i}'s move in SAT_{1-i}. If φ'_p was of the form ($\underline{\text{exactly}}\ \mu_w\ \nu)\psi$, we are not yet done; I responds in SAT_{1-i} according to his strategy there, and makes that move in G; II responds in G; I regards this as FALSE_i's response in SAT_i. Thus we preserve the following at each such p:

$$\text{I wears } \mathrm{TRUE}_i \quad \text{and} \quad \mathcal{A}_i \models_\kappa \varphi_p[\vec{a}_i, \vec{n}_i];$$
$$\text{I wears } \mathrm{FALSE}_{1-i} \quad \text{and} \quad \mathcal{A}_{1-i} \not\models_\kappa \varphi_p[\vec{a}_{1-i}, \vec{n}_{1-i}].$$

The last φ_p to be defined is atomic and witnesses I's victory in the play of G.

It is important to notice that in the proof of Lemma 2 nothing would be lost by requiring that in exchanges of the third sort, I select B_i of the form $\hat{v}_{l(0)}\varphi[\vec{a}_i, \vec{n}_i]^{\mathcal{A}_i}$ for some φ with $\operatorname{depth}(\varphi) \leqslant q - q'$. Hereafter we take $G^{0,2}_\kappa(\mathcal{A}_0, \mathcal{A}_1, q)$ to involve this constraint on I's moves.

One other sort of game needs to be mentioned. Where $\vec{\alpha}_0, \vec{\alpha}_1 \in \omega^l$, $\vec{\alpha}_i = \langle \alpha_{i,0}, \ldots, \alpha_{i,l-1} \rangle$, let $\langle \alpha_0, \alpha_1 \rangle$ be 0-congruent iff for all $w, u < l$: $\alpha_{0,w} < \alpha_{0,u}$ iff $\alpha_{1,w} < \alpha_{1,u}$. Let $M(\vec{\alpha}_0, \vec{\alpha}_1, q)$ be the Ehrenfeucht game on $\langle \omega, <\restriction\omega \rangle$ with 'situation' function g, played as follows. Play starts at $\langle\ \rangle$, with $g(\langle\ \rangle) = \langle \vec{\alpha}_0, \vec{\alpha}_1 \rangle$. Let p be a position with $g(p) = \langle \vec{\beta}_0, \vec{\beta}_1 \rangle$. If $|\vec{\beta}_0| = l + q$, play is over; II wins if $g(p)$ is 0-congruent. If $m = |\vec{\beta}_0| < l + q$, I chooses $i \in 2$ and n with $0 < n < q - (l + m)$, and $\beta_{i,m}, \ldots, \beta_{i,m+n-1} \in \omega$; II selects $\beta_{1-i,m}, \ldots, \beta_{1-i,m+n-1} \in \omega$; where p' is the resulting position, let:

$$g(p') = \langle \vec{\beta} * \langle \beta_{0,m}, \ldots, \beta_{0,m+n-1} \rangle, \vec{\beta}_1 * \langle \beta_{1,m}, \ldots, \beta_{1,m+n-1} \rangle \rangle.$$

For $\alpha, \beta, n \in \omega$, let:

$$\alpha \sim_n \beta \quad \text{iff} \quad \text{either} \quad \alpha = \beta < n \quad \text{or} \quad \alpha, \beta \geqslant n.$$

Let $\langle \vec{\alpha}_0, \vec{\alpha}_1 \rangle$ be n-congruent iff it is 0-congruent and for any $w, u < l$: $|\alpha_{0,u} - \alpha_{0,w}| \sim_{2^n} |\alpha_{1,u} - \alpha_{1,w}|$. The following is easy to prove:

II has a winning strategy for $M(\vec{\alpha}_0, \vec{\alpha}_1, q)$ iff $\langle \vec{\alpha}_0, \vec{\alpha}_1 \rangle$ is q-congruent.

3.3. Proof of Theorem 3.1. Fix a set I of cardinality κ. Consider $t_0, t_1 < \omega$, $0 < t_i$ for $i \in 2$, and an increasing sequence $\langle y_j \rangle_{j < t_0 + t_1}$ in ω with $0 < y_0$. Letting

$W = \langle t_0, t_1, \langle y_j \rangle_{j<t_0+t_1} \rangle$, an array for W has the form:

$$\langle \langle Y_{\alpha,j} \rangle_{\alpha \in I, j<t_0+t_1}, \langle Z_{\alpha,j} \rangle_{\alpha \in I, j<t_0+t_1} \rangle$$

where any two sets in these sequences are disjoint in every possible way, i.e.:

- for all α, $\alpha' \in I$ and $j, j' < t_0 + t_1$: $Y_{\alpha,j} \cap Z_{\alpha',j}$ is empty;
- for all distinct α, $\alpha' \in I$ and distinct j, $j' < t_0 + t_1$: $Y_{\alpha,j} \cap Y_{\alpha,j'}$, $Y_{\alpha,j} \cap Y_{\alpha',j}$, $Y_{\alpha,j} \cap Y_{\alpha',j'}$, $Z_{\alpha,j} \cap Z_{\alpha,j'}$, $Z_{\alpha,j} \cap Z_{\alpha',j}$, $Z_{\alpha,j} \cap Z_{\alpha',j'}$ are empty; and
- for all $\alpha \in I$ and $j < t_0 + t_1$: $\text{card}(Y_{\alpha,j}) = \kappa$, $\text{card}(Z_{\alpha,j}) = y_j$.

Such an array determines the sequence $\langle A, R_0, R_1, E_0, E_1, F_0, F_1, f \rangle$ where:

$$A = \bigcup \{Y_{\alpha,j} \cup Z_{\alpha,j}: \alpha \in I, j < t_0 + t_1\},$$
$$R_0 = \bigcup \{Y_{\alpha,j} \times Z_{\alpha,j}: \alpha \in I, j < t_0\},$$
$$R_1 = \bigcup \{Y_{\alpha,j} \times Z_{\alpha,j}: \alpha \in I, t_0 \leq j < t_0 + t_1\},$$
$$E_0 = \bigcup \{Y_{\alpha,j} \times Y_{\alpha,j}: \alpha \in I, j < t_0\},$$
$$E_1 = \bigcup \{Y_{\alpha,j} \times Y_{\alpha,j}: \alpha \in I, t_0 \leq j < t_0 + t_1\},$$
$$F_0 = \bigcup \{Z_{\alpha,j} \times Z_{\alpha,j}: \alpha \in I, j < t_0\},$$
$$F_1 = \bigcup \{Z_{\alpha,j} \times Z_{\alpha,j}: \alpha \in I, t_0 \leq j < t_0 + t_1\},$$
$$f(a) = j \quad \text{for } a \in Y_{\alpha,j} \cup Z_{\alpha,j} \text{ for any } \alpha \in A.$$

$\mathscr{A}$ is a W-model if $\mathscr{A}$ is a model for $\{\mathbf{R}_0, \mathbf{R}_1, \mathbf{E}_0, \mathbf{E}_1, \mathbf{F}_0, \mathbf{F}_1\}$, all 2-place, where an array for W determines the sequence:

$$\langle |\mathscr{A}|, \mathbf{R}_0^{\mathscr{A}}, \mathbf{R}_1^{\mathscr{A}}, \mathbf{E}_0^{\mathscr{A}}, \mathbf{E}_1^{\mathscr{A}}, \mathbf{F}_0^{\mathscr{A}}, \mathbf{F}_1^{\mathscr{A}}, f_{\mathscr{A}} \rangle.$$

Clearly for $l \in 2$:

$$\mathscr{A} \models_\kappa (\underline{\text{exactly}}\ \mathbf{n}\ \rho)(\exists v_1)(\underline{\text{exactly}}\ \rho\ v_0)\mathbf{R}_l(v_1, v_0) \quad \text{iff} \quad n = t_l.$$

Thus $\mathscr{A} \models_\kappa \varphi_{0,2}$ iff $t_0 = t_1$. Our approach to Theorem 3.1 will be: given $q < \omega$, find W_0 and W_1 where $W_1 = \langle t_0, t_0 + 1, \langle y_j \rangle_{j<2t_0+1} \rangle$ and $W_0 = \langle t_0, t_0, \langle y_j \rangle_{j<2t_0} \rangle$ so that where $\mathscr{A}_l$ is a W_l-model for $l \in 2$, II has a winning strategy for $G = G_\kappa^{0,2}(\underline{\text{exactly}}, \leq, \mathscr{A}_0, \mathscr{A}_1, q)$. Then $\mathscr{A}_0 \models_\kappa \varphi_{0,2}$, $\mathscr{A}_1 \not\models_\kappa \varphi_{0,2}$, and for every $\varphi \in \text{Sent}(\mathscr{L}^{0,2}(\underline{\text{exactly}}, \leq))$ with $\text{depth}(\varphi) \leq q$ $\mathscr{A}_0 \models_\kappa \varphi$ iff $\mathscr{A}_1 \models_\kappa \varphi$. Thus $\varphi_{0,2}$ cannot be equivalent$_\kappa$ to any such φ.

Lemma 1. *Let $\mathscr{A}$ be a W-model for $W = \langle t_0, t_1, \langle y_j \rangle_{j<t_0+t_1} \rangle$. Let $\varphi \in \text{Fml}(\mathscr{L}^{0,2}(\underline{\text{exactly}}, \leq))$ with free variables among $v, v_0, \ldots, v_{l(0)-1} \in Var(0)$, $\mu_0, \ldots, \mu_{l(2)-1} \in Var(2)$, and $\vec{a} \in |\mathscr{A}|^{l(0)}$, $\vec{n} \in \bar{\kappa}^{l(2)}$, and $B = \hat{v}\varphi[\vec{a}, \vec{n}]^{\mathscr{A}}$. Let $B' = B - \{a_0, \ldots, a_{l(0)-1}\}$. For $j < t_0 + t_1$ let:*

$$V_j = \{\alpha: \textit{for some } w < l(0),\ a_w \in Y_{\alpha,j} \cup Z_{\alpha,j}\}.$$

For any $\alpha \in I$ and $j < \omega$:

- *if $B' \cap Y_{\alpha,j} \neq \{\ \}$, then $Y_{\alpha,j} - \{a_0, \ldots, a_{l(0)-1}\} \subseteq B$;*
- *if $\alpha \notin V_j$ and $B' \cap Z_{\alpha,j} \neq \{\ \}$, then for any $\beta \in I - V_j$,*

  $$Z_{\beta,j} - \{a_0, \ldots, a_{l(0)-1}\} \subseteq B.$$

Proof. Consider permutations: in the first case permute members of $Y_{\alpha,j} - \{a_0, \ldots, a_{l(0)-1}\}$; in the second case switch $Z_{\alpha,j} - \{a_0, \ldots, a_{l(0)-1}\}$ with $Z_{\beta,j} - \{a_0, \ldots, a_{l(0)-1}\}$.

Fix $q < \omega$. Let:

$$S_W = \left\{ v \in \omega^{t_0+t_1} : \sum_{j<t_0+t_1} v(j) \leq q \right\};$$

$$Q_W = \left\{ \left(\sum_{j<t_0+t_1} v(j) \cdot y_j \right) + e : v \in S_W,\ -q \leq e \leq q \right\}.$$

Lemma 2. *For $\mathcal{A}$, φ, B, etc. as in Lemma* 1 *and* $l(0) \leq q$: *if* $\operatorname{card}(B) < \kappa$, *then* $\operatorname{card}(B) \in Q_W$.

Proof. For $j < t_0 + t_1$, let:

$$v_{\vec{a}}(j) = \operatorname{card}\{\alpha \in V_j : B' \cap Z_{\alpha,j} \text{ is non-empty}\}.$$

Since $\sum_{j<t_0+t_1} v_{\vec{a}}(j) \leq l(0) \leq q$, $v_{\vec{a}} \in S_W$. Suppose that $\operatorname{card}(B) < \kappa$. By Lemma 1, for all $\alpha \in I$ and $j < t_0 + t_1$, $B' \cap Y_{\alpha,j}$ is empty; furthermore, if $\alpha \notin V_j$, then $B' \cap Z_{\alpha,j}$ is empty. If $B' \cap Z_{\alpha,j}$ is non-empty, then $\alpha \in V_j$ and $Z_{\alpha,j} - \{a_0, \ldots, a_{l(0)-1}\} \subseteq B$; this follows by a permutation argument in which members of $Z_{\alpha,j} - \{a_0, \ldots, a_{l(0)-1}\}$ are permuted. There are $\leq l(0)$ many $\langle \alpha, j \rangle$'s with $\alpha \in V_j$; thus:

$$\left[\sum_{j<t_0+t_1} v_{\vec{a}}(j) \cdot y_j \right] - l(0) \leq \operatorname{card}(B') \leq \sum_{j<t_0+t_1} v_{\vec{a}}(j) \cdot y_j;$$

so:

$$\left[\sum_{j<t_0+t_1} v_{\vec{a}}(j) \cdot y_j \right] - l(0) \leq \operatorname{card}(B) \leq \sum_{j<t_0+t_1} v_{\vec{a}}(j) \cdot y_j + l(0);$$

since $l(0) \leq q$, $\operatorname{card}(B) \in Q_W$.

Suppose that $y_j = (2^{3q} + q^2 + 2q)^{j+1}$ for all $j < t_0 + t_1$; then given $[\sum_{j<t_0+t_1} v(j) \cdot y_j] + e \in Q_W$, we may uniquely recover v and e. This will make the cardinality of B when $\operatorname{card}(B) < \kappa$ carry information about membership in B.

Let $t_0 = 2^{(q^2)} + 1$, $t_1 = t_0 + 1$, $W_0 = \langle t_0, t_0, \langle y_j \rangle_{j<2t_0} \rangle$, $W_1 = \langle t_0, t_1, \langle y_j \rangle_{j<2t_0+1} \rangle$; for $i < 2$ let $f_i = f_{\mathcal{A}_i}$, $S_i = S_{W_i}$, $Q_i = Q_{W_i}$. A gap in Q_i is an interval (n_i, n_i') with $n_i, n_i' \in Q_i$ and $(n_i, n_i') \cap Q_i$ empty. $0 \in Q_i$; so if $n \notin Q_i$ and n belongs to no gap in Q_i, for all $n' \in Q_i$: $n' < n$. We have chosen $\langle y_j \rangle_{j<2t_0+1}$ so that if (n_i, n_i') is a gap in Q_i, $n_i' - n_i \geq 2^{3q}$. If $[n, m] \subseteq Q_i$ with $n - 1$, $m + 1 \notin Q_i$, call $[n, m]$ a block in Q_i; we have made sure that if $[n, m]$ is a block in Q_i, then $n - m = 2q$. As II plays G, she will 'match up' blocks of Q_0 with blocks of Q_1, and gaps in Q_0 with gaps in Q_1. As II plays G, she will pretend to also be playing:

$$M_2 = M(\langle 0, \ldots, t_0 - 1, 2t_0 - 1 \rangle, \langle 0, \ldots, t_0 - 1, 2t_0 \rangle, q^2).$$

II has a winning strategy for M_2, by choice of t_0. The playing of members of $\bar{\kappa}$

in G shall be controlled by the pretended play of M_2, and may be viewed as involving play of:

$$M_2' = M(\langle y_0, \ldots, y_{t_0-1}, y_{2t_0-1}\rangle, \langle y_0, \ldots, y_{t_0-1}, y_{2t_0}\rangle, 3q)$$

within which the 'matching up' of blocks and gaps occurs. Of course II also has a winning strategy for M_2'.

Where $i \in 2$, $\vec{a}_i \in |\mathscr{A}|^{l(0)}$, let $\langle \vec{a}_0, \vec{a}_1\rangle$ be a matched pair iff for every φ belonging to:

$$\{\mathbf{R}_j(v_w, v_u), \mathbf{E}_j(v_w, v_u), \mathbf{F}_j(v_w, v_u), v_w = v_u : j \in 2,\ w, u < l(0)\}$$

$\mathscr{A}_0 \models \varphi[\vec{a}_0]$ iff $\mathscr{A}_1 \models \varphi[\vec{a}_1]$.

Suppose $n \in Q_i$; fix $v \in S_i$ and e so that $-q \leq e \leq q$ and $n = [\sum_{j<t_0+t_1} v(j) \cdot y_j] + e$; let $n^* = \langle k_0, \ldots, k_{z-1}\rangle$ where $k_0 < \cdots < k_{z-1}$ is a list of exactly those $j < t_0 + t_i$ with $v(j) > 0$. Since $v \in S_i$, $z \leq q$. Given $\vec{n}_i \in \bar{\kappa}^{l(2)}$ for $i \in 2$, let $w_0 < \cdots < w_{c-1}$ be a list of those $w < l(2)$ so that $n_{i,w} \in Q_i$; let $(\vec{n}_i)^* = n^*_{i,w_0} * \cdots * n^*_{i,w_{c-1}}$.

We may now describe II's strategy for G. each position p of G shall be associated with a position p_2 of M_2. Suppose play of G has reached p with $|p| = q' \leq q$, $h_i(p) = \langle \vec{a}_i, \vec{n}_i\rangle$ for $i \in 2$, $\langle \vec{a}_0, \vec{a}_1\rangle$ is a matched pair, and $\langle (\vec{n})^* * f_0(\vec{a}_n), (\vec{n}_1)^* * f_1(\vec{a}_1)\rangle$ is a situation in II's winning subgame for M_2. (Here $f_i(\vec{a}_1) = \langle f_i(a_0), \ldots, f_i(a_{l(0)-1})\rangle$.) Suppose that for $w < l(2)$, $n_{0,w} \in Q_0$ iff $n_{1,w} \in Q_1$. If $n_{i,w} \in (n_i, n_i')$ where (n_i, n_i') is a gap in Q_i, then $n_{1-i,w} \in (n_{1-i}, n_{1-i}')$, where that is a gap in Q_{1-i}; we will say that as of p the gaps (n_0, n_0') and (n_1, n_1') have been matched. Similarly if $n_{i,w} \in [n_i, n_i']$, where $[n_i, n_i']$ is a block in Q_i, we will have $n_{1-i,w} \in [n_{1-i}, n_{1-i}']$, a block in Q_{1-i} with which as of p, $[n_i, n_i']$ has been matched. Suppose that $\langle \vec{n}_0, \vec{n}_1\rangle$ is a situation in II's winning subgame for M_2' (i.e. it is q'-congruent).

Suppose that $q' < q$, and I picks $i \in 2$. If I now selects $a_{i,l(0)} \in |\mathscr{A}_i|$, II pretends that I plays i and $f_i(a_{i,l(0)})$ in M_2; in the pretend-play of M_2, II follows her strategy and plays n; since $n < t_0 + t_{1-i}$, II must find $a_{1-i,l(0)} \in |\mathscr{A}_{1-i}|$ so that $\langle \vec{a}_0 * a_{0,l(0)}, \vec{a}_1 * a_{1,l(0)}\rangle$ is a matched-pair and $f_{1-i}(a_{1-i,l(0)}) = n$. This is easy to do.

Suppose I selects $n_{i,l(2)} \in \bar{\kappa}$. Letting $n_{i'} = \max(Q_{i'})$ for $i' \in 2$, suppose that $n_{i,l(2)} > n_i$; II plays $n_{1-i,l(2)} = n_{1-i} + (n_{i,l(2)} - n_i)$. Suppose $n_{i,l(2)} \in Q_i$, $n_{i,l(2)} = [\sum_{j<t_0+t_1} v_i(j) \cdot y_j] + e$. II pretends that I plays i and $n^*_{i,l(2)} = \langle k_{i,0}, \ldots, k_{i,z-1}\rangle$ in M_2; II follows her strategy for M_2, playing $\langle k_{1-i,0}, \ldots, k_{1-i,z-1}\rangle$; clearly for all $w < z$, $k_{1-i,w} < t_0 + t_{1-i}$. Let:

$$v_{1-i}(k_{1-i,w}) = v_i(k_{i,w}) \quad \text{for } w < z;$$

$$v_{1-i}(j) = 0 \quad \text{for } j \in (t_0 + t_i) - \{k_{1-i,0}, \ldots, k_{1-i,z-1}\};$$

$$n_{1-i,l(2)} = \left[\sum_{j<t_0+t_{1-i}} v_{1-i}(j) \cdot y_j\right] + e;$$

II plays $n_{1-i,l(2)}$. Notice that $\langle \vec{n}_0 * n_{0,l(2)}, \vec{n}_1 * n_{1,l(2)}\rangle$ is now $q'-1$-congruent. Suppose that $n_{i,l(2)} \in (m_i, m_i')$, a gap in Q_i. Where $m_i' = [\sum_{j<t_0+t_i} v_i(j) \cdot y_j] - q$, II

computes $\langle k_{i,0}, \ldots, k_{i,z-1}\rangle$ as above, pretends that I plays i and it in M_2, and obtains $\langle k_{1-i,0}, \ldots, k_{1-i,z-1}\rangle$ and then v_{1-i} as above; let $m'_{1-i} = [\Sigma_{j<t_0+t_{1-i}} v_{1-i}(j)\cdot y_j] - q$; fix m_{1-i} so that (m_{1-i}, m'_{1-i}) is a gap in Q_{1-i}. II matches (m_0, m'_0) with (m_1, m'_1) by playing $n_{1-i,l(2)} \in (m_{1-i}, m'_{1-i})$ so as to keep $\langle \vec{n}_0 * \langle m_0, n_{0,l(2)}, m'_0\rangle, \vec{n}_1 * \langle m_1, n_{1,l(2)}, m'_1\rangle\rangle$ $q'-1$-congruent. Since $m'_i - m_i > 2^{3q}$, this may be done.

Suppose I initiates the third sort of exchange, selecting $w < l(2)$ and $B_i = \hat{v}\varphi[\vec{a}_i, \vec{n}_i]^{\mathscr{A}_i}$ with $\text{card}(B_i) = n_{i,w}$. Letting $k_{i'} = f_{i'}(a_{i',u})$ for $i' \in 2$, $u < l(0)$, and $a_{i,u} \in Y_{i,\alpha,k_i} \cup Z_{i,\alpha,k_i}$, let $U_{i,u} = (B_i - \vec{a}_i) \cap Z_{i,\alpha,k_i}$. By Lemma 1 either $U_{i,u}$ is empty or $Z_{i,u,k_1} - \vec{a}_i \subseteq B_i$. Let:

$$U_{1-i,u} = \begin{cases} Z_{1-i,\alpha,k_{1-i}} - \vec{a}_{1-i} & \text{if } U_{i,w} \neq \{\ \}, \\ \{\ \} & \text{otherwise.} \end{cases}$$

II plays:

$$B_{1-i} = \{a_{1-i,u}\colon a_{i,u} \in B_i,\ u < l(0)\} \cup \bigcup \{U_{1-i,u}\colon u < l(0)\}.$$

By Lemma 2, $n_{i,w} \in Q_i$. II has played so that $n_{1-i,w} \in Q_{1-i}$ and $\text{card}(B_{1-i}) = n_{1-i,w}$. Whatever $a_{1-i,l(0)}$ II now picks, I can find $a_{i,l(0)}$ so that $\langle \vec{a}_0 * a_{0,l(0)}, \vec{a}_1 * a_{1,l(0)}\rangle$ is a matched-pair, and $a_{0,l(0)} \in B_0$ iff $a_{1,l(0)} \in B_1$. Clearly when p with $|p| = q$ is reached, II wins G.

3.4. Where $\mathbf{P}$ is a one-place predicate and $Pred = \{\mathbf{P}\}$, there is a $\varphi_0 \in \text{Sent}(\mathscr{L}^{0,4}(\underline{\text{exactly}}, \leqslant))$ so that for any model $\mathscr{A}$ for $\{\mathbf{P}\}$:

$$\mathscr{A} \models_\kappa \varphi_0 \quad \text{iff} \quad \text{card}(\mathbf{P}^{\mathscr{A}}) \text{ is finite and even.}$$

Theorem. *For any infinite* $\kappa \in \text{Card}$, φ_0 *is not equivalent*$_\kappa$ *to any sentence* $\mathscr{L}^{0,2}(\underline{\text{exactly}}, \leqslant)$.

This is weaker than the previous result, since φ_0 contains '$\leqslant$'; but its proof is much easier and is left to the reader.

We can also construct a $\varphi_2 \in \text{Sent}(\mathscr{L}^{0,6}(\underline{\text{exactly}}, \leqslant))$ so that for any $\mathscr{A}$ as above:

$$\mathscr{A} \models_\kappa \varphi_2 \quad \text{iff} \quad \text{for some even } q < \omega \quad \text{ncb}(\text{card}(\mathbf{P}^{\mathscr{A}})) = \aleph_q.$$

Let $\mathbf{P}^2(\mu)$ be $(\forall\rho)((\underline{\text{exactly}}\,\rho\, v)\mathbf{P}v \supset \mu < \rho)$, for $\mu, \rho \in \text{Var}(2)$; $\mathbf{P}^2(\mu)$ pins the value of μ to $\text{ncb}(\text{card}(\mathbf{P}^{\mathscr{A}}))$; construction of φ_2, using $\mathbf{P}^2(\mu)$, is left to the reader.

Let $\kappa = \aleph_\alpha$ for $\alpha = \aleph_\omega$. The previous result suggests that φ_2 is not equivalent$_\kappa$ to a sentence of $\mathscr{L}^{0,4}(\underline{\text{exactly}}, \leqslant)$. This turns out to be false! Since this shows something of the expressive power of $\mathscr{L}^{0,4}(\underline{\text{exactly}}, \leqslant)$, I will give details.

For $n < \kappa$ let:

$$\text{code}(n) = \{q < \omega\colon \text{for some } m < n,\ \aleph_q = \text{card}[m, n)\}.$$

For any finite $A \subset \omega$ there is an $n < \kappa$ with $A = \text{code}(n)$. Clearly, if $\text{ncb}(n) = \aleph_q$, then $q = \max(\text{code}(n))$. The key to expressing φ_2 is that $\mathcal{A} \models_\kappa \varphi$ iff for some $n < \kappa$:

(i) $\text{ncb}(\text{card}(P^{\mathcal{A}})) = \text{ncb}(n)$;
(ii) $0 \in \text{code}(n)$;
(iii) if $\text{ncb}(n) = \aleph_q$ and $r + 1 \leqslant q$, then:

$$r \in \text{code}(n) \quad \text{iff} \quad r + 1 \notin \text{code}(n).$$

Let φ' be:

$$\begin{aligned}
&(\underline{\text{exactly}}\ \rho_0\ \rho')\mathbf{P}^2\rho' \ \&\ (\underline{\text{exactly}}\ \rho_0\ \rho')\ \rho' < \mu \\
&\&\ (\exists\mu')(\exists\rho)(E_0(\rho)\ \&\ (\underline{\text{exactly}}\ \rho\ \nu)(\mu' \leqslant \nu\ \&\ \nu < \mu) \\
&\&\ (\forall\rho)(\forall\rho')([\rho < \rho'\ \&\ \neg(\exists\rho'')(\rho < \rho''\ \&\ \rho'' < \rho')\ \&\ G_0(\rho)\ \&\ \rho' \leqslant \rho_0] \\
&\quad \supset [(\exists\mu')(\underline{\text{exactly}}\ \rho\ \nu)(\mu' \leqslant \nu\ \&\ \nu < \mu) \\
&\qquad\quad \equiv \neg(\exists\mu')(\underline{\text{exactly}}\ \rho'\ \nu)(\mu' \leqslant \nu\ \&\ \nu < \mu)]).
\end{aligned}$$

Then $(\exists\mu)(\exists\rho_0)\varphi'$ expresses φ_1. (Help: the value of μ will be the above-mentioned n.)

Proving the following may be easier than proving (A).

Conjecture (D). For **P**, **Q** 1-place, and $\kappa = \aleph_{\aleph_\omega}$, no sentence of $\mathcal{L}^{0,4}(\underline{\text{exactly}}, \leqslant)$ is equivalent$_\kappa$ to the easily constructed sentence of $\mathcal{L}^{0,6}(\underline{\text{exactly}}, \leqslant)$ expressing the following:

$$\text{for some } q < \omega, \quad \text{ncb}(\text{card}(\mathbf{P}^{\mathcal{A}})) = \aleph_q \quad \text{and} \quad \text{ncb}(\text{card}(\mathbf{Q}^{\mathcal{A}})) = \aleph_{q\cdot 2}.$$

3.5. Let a weak language be one introduced in §1 without type-1 variables and without '$\leqslant$' in its logical lexicon. We will now show that such languages really are weak, i.e. cannot express '$\leqslant$'. Let **P**, **Q** be 1-place, *Pred* = {**P**, **Q**} and *Funct* be empty.

Observation. *For* $0 < k < \omega$ *and any infinite* $\kappa \in$ Card:

(i) $\mathcal{L}^{0,2}(\underline{\text{exactly}}, \leqslant) \not\leqslant_\kappa \mathcal{L}^{0,2k}(\underline{\text{exactly}}, =)$,

(ii) $\mathcal{L}^{0,2}(\underline{\text{exactly}}, \leqslant) \not\leqslant_\kappa \mathcal{L}^{0,2k*}(\underline{\text{exactly}}, =)$.

Indeed, the following sentence witnesses both (i) and (ii) for all choices of k and κ:

$$(\exists\mu)(\exists\mu')(\mu \leqslant \mu'\ \&\ (\underline{\text{exactly}}\ \mu\ \nu)\mathbf{P}\nu\ \&\ (\underline{\text{exactly}}\ \mu'\ \nu)\mathbf{Q}\nu).$$

To prove (ii) it suffices to show that for every $q < \omega$ there are models $\mathcal{A}_0$ and $\mathcal{A}_1$ of cardinality κ with $\text{card}(\mathbf{P}^{\mathcal{A}_0}) < \text{card}(\mathbf{Q}^{\mathcal{A}_0})$, $\text{card}(\mathbf{Q}^{\mathcal{A}_1}) < \text{card}(\mathbf{P}^{\mathcal{A}_1})$, and such that for all $\psi \in \mathrm{Sent}(\mathscr{L}^{0,2k*}(\underline{\text{exactly}}))$:

$$\text{if depth}(\psi) \leq q, \quad \text{then} \quad \mathscr{A}_0 \vDash_\kappa \psi \quad \text{iff} \quad \mathscr{A}_1 \vDash_\kappa \psi.$$

A similar sufficient condition applies to (i).

We will discuss the case in which $k = 1$; generalizing the argument to $k > 1$ is straightforward. For $v_0, \ldots, v_{l(0)-1} \in Var(0)$, $\mu_0, \ldots, \mu_{l(2)-1} \in Var(2)$, let Φ be a 0-profile for $\vec{v}$, $\vec{\mu}$ iff Φ is a minimal consistent set so that:

- for any $j < j' < l(0)$: $v_j = v_{j'} \in \Phi$ or $v_j \neq v_{j'} \in \Phi$;
- for any $j < l(0)$: $\mathbf{P}v_j \in \Phi$ or $\neg\mathbf{P}v_j \in \Phi$; and similarly for $\mathbf{Q}v_j$;
- for any $j < j' < l(2)$: $\mu_j = \mu_{j'} \in \Phi$ or $\mu_j \neq \mu_{j'} \in \Phi$;

if $\mathrm{ncb}(\kappa) < \kappa$ we also require:

- for any $j < l(2)$: $(\underline{\mathrm{ncb} =} \mu_j) \in \Phi$ or $(\underline{\mathrm{ncb} \neq} \mu_j) \in \Phi$.

For $\rho_0, \ldots, \rho_{l-1}, v_0, \ldots, v_{n-1} \in Var(0)$ and $n < \omega$ let $\theta(\mathbf{P}, \mathbf{n}, \rho_0, \ldots, \rho_{l-1})$ abbreviate:

$$(\exists v_0 \cdots \exists v_{n-1})\Big(\bigwedge_{j<j'<n} v_j \neq v_{j'} \;\&\; \bigwedge_{\substack{j<n\\ j'<l}} v_j \neq \rho_{j'} \;\&\; \bigwedge_{j<n} \mathbf{P}v_j\Big).$$

We will also use these abbreviations, for $\mu \in Var(2)$:

$$\underline{\mathrm{card}(\mathbf{P}) + n =} \mu\colon \quad (\exists v_0 \cdots \exists v_{n-1})\Big(\bigwedge_{j<j'<n} v_j \neq v_{j'} \;\&\; \bigwedge_{j<n} \neg\mathbf{P}v_j$$
$$\&\; (\underline{\text{exactly}}\; \mu\; v)\Big(\mathbf{P}v \vee \bigvee_{j<n} v = v_j\Big)\Big),$$

$$\underline{\mathrm{card}(\mathbf{P}) - n =} \mu\colon \quad (\exists v_0 \cdots \exists v_{n-1})\Big(\bigwedge_{j<j'<n} v_j \neq v_{j'} \;\&\; \bigwedge_{j<n} \mathbf{P}v_j$$
$$\&\; (\underline{\text{exactly}}\; \mu\; v)\Big(\mathbf{P}v \;\&\; \bigwedge_{j<n} v \neq v_j\Big)\Big);$$

$$\underline{\mathrm{card}(\mathbf{P}) + n = \mathrm{card}(\mathbf{Q})}\colon \quad (\exists\mu)(\underline{\mathrm{card}(\mathbf{P}) + n =} \mu \;\&\; (\underline{\text{exactly}}\; \mu\; v)\mathbf{Q}v).$$

Similar abbreviations are in force with '**Q**' and '**P**' switched.

Where $B \subseteq \aleph_0$ is finite, let Φ be a 1-profile for $\vec{v}, \vec{\mu}$ relative to B iff Φ is a minimal set so that for any $n \in B$ and $j < l(2)$:

either $\underline{n =} \mu_j \in \Phi$ or $\underline{n \neq} \mu_j \in \Phi$;

either $\theta(\mathbf{P}, \mathbf{n}, v_0, \ldots, v_{l(0)-1}) \in \Phi$ or $\neg\theta(\mathbf{P}, \mathbf{n}, v_0, \ldots, v_{l(0)-1}) \in \Phi$;

either $\underline{\mathrm{card}(\mathbf{P}) + n =} \mu_j \in \Phi$ or $\underline{\mathrm{card}(\mathbf{P}) + n \neq} \mu_j \in \Phi$;

either $\underline{\mathrm{card}(\mathbf{P}) - n =} \mu_j \in \Phi$ or $\underline{\mathrm{card}(\mathbf{P}) - n \neq} \mu_j \in \Phi$;

either $\underline{\mathrm{card}(\mathbf{P}) + n = \mathrm{card}(\mathbf{Q})} \in \Phi$ or $\underline{\mathrm{card}(\mathbf{P}) + n \neq \mathrm{card}(\mathbf{Q})} \in \Phi$;

and similarly with '**Q**' and '**P**' switched. Let $\mathscr{A}$ be a nice model iff $\mathbf{P}^{\mathscr{A}} \cap \mathbf{Q}^{\mathscr{A}}$ is empty and card($\mathbf{P}^{\mathscr{A}}$), card($\mathbf{Q}^{\mathscr{A}}$) $<\kappa$. Let a B-profile for $\vec{v}$, $\vec{\mu}$ be a union of a 0-profile and a 1-profile for $\vec{v}$, $\vec{\mu}$ relative to B which is κ-satisfiable in a nice model. Let ψ be κ-equivalent* to ψ' iff they are κ-equivalent restricted to nice models.

Lemma. *For any formula ψ of $\mathscr{L}^{0,2*}(\underline{\text{exactly}}, =)$ with free variables among $\vec{v}$, $\vec{\mu}$ there is a finite $B_\psi \subseteq \aleph_0$ so that ψ is κ-equivalent* to a disjunction of B_ψ-profiles for $\vec{v}$, $\vec{\mu}$.*

Proof is by induction on the construction of ψ. Details are left to the reader.

Let $B_q = \bigcup \{B_\psi\colon \text{depth}(\psi) \leq q\}$; B_q is finite; suppose $n = \max(B_q)$. Let $\mathscr{A}_0$ and $\mathscr{A}_1$ be nice models with:

$$n < \text{card}(\mathbf{P}^{\mathscr{A}_0}), \qquad \text{card}(\mathbf{P}^{\mathscr{A}_0}) + n < \text{card}(\mathbf{Q}^{\mathscr{A}_0}),$$
$$n < \text{card}(\mathbf{Q}^{\mathscr{A}_1}), \qquad \text{card}(\mathbf{Q}^{\mathscr{A}_1}) + n < \text{card}(\mathbf{P}^{\mathscr{A}_1}).$$

$\mathscr{A}_0$ and $\mathscr{A}_1$ are then as required; details are left to the reader.

4. Inclusions between weak languages

Even in weak languages we may define a prefex $(\underline{\infty}\rho)$ so that for any model $\mathscr{A}$, $\mathscr{A} \models_\kappa (\underline{\infty}\rho)\varphi$ iff $\hat{\rho}\varphi^{\mathscr{A}}$ is Dedekind-infinite. Let $(\underline{\infty}\rho)\varphi$ be:

if $\rho, \rho' \in Var(0)$, ρ' not occurring in φ: $(\exists\rho')(\varphi(\rho/\rho')$
$\quad \& (\forall\mu)[(\underline{\text{exactly}}\,\mu\,\rho)\varphi \equiv (\underline{\text{exactly}}\,\mu\,\rho)(\varphi \,\&\, \rho \neq \rho')])$;

if $\rho \notin Var(0)$:
$\quad (\varphi(\rho/\mathbf{0})^* \supset (\forall\mu)[(\underline{\text{exactly}}\,\mu\,\rho)\varphi \equiv (\underline{\text{exactly}}\,\mu\,\rho)(\varphi \,\&\, \neg \underline{0 =} \rho)])$
$\quad \& (\neg\varphi(\rho/\mathbf{0})^* \supset (\forall\mu)[(\underline{\text{exactly}}\,\mu\,\rho)\varphi \equiv (\underline{\text{exactly}}\,\mu\,\rho)(\varphi \vee \underline{0 =} \rho)])$,

where $\varphi(\rho/\mathbf{0})^*$ is formed from $\varphi(\rho/\mathbf{0})$ by replacing all subformulae of the forms $(\underline{\text{exactly}}\,\mathbf{0}\,v)\psi$, $\mathbf{0} = v$, $v = \mathbf{0}$ and $\mathbf{0} = \mathbf{0}$ by $\neg(\exists v)\psi$, $\underline{0 =} v$, $\underline{0 =} v$ and '$\neg\bot$' respectively. In this section we assume that all Dedekind-finite sets are finite.

4.1. Observation. $\mathscr{L}^{0,2}(\underline{\text{exactly}}, =) \overset{s}{\prec} \mathscr{L}^{0,4}(\underline{\text{exactly}})$.

Proof. A profile for $\mu_0, \ldots, \mu_{p-1} \in Var(2)$ is a consistent formula of the form $\bigwedge_{j'<j<r} (\mu_{j'} = \mu_j)^{a(j',j)}$, for $a(j', j) \in 2$. (Notation: for any formula θ, θ^0 is θ; θ^1 is $\neg\theta$.) Suppose $\varphi \in \text{Fml}(\mathscr{L}^{0,2}(\underline{\text{exactly}}, =))$ has free variables among $v_0, \ldots, v_{l-1} \in Var(0)$ and $\mu_0, \ldots, \mu_{p-1} \in Var(2)$. Call these the 'distinguished' variables. For each profile Φ for $\mu_0, \ldots, \mu_{p-1}$ we will construct $\varphi_\Phi \in \text{Fml}(\mathscr{L}^{0,4}(\underline{\text{exactly}}))$, so that for any $\kappa \in \text{Card}$, any model $\mathscr{A}$, $\vec{a} \in |\mathscr{A}|^l$, and $\vec{n} \in \bar{\kappa}$:

$(*)$ if $\mathscr{A} \models \Phi[\vec{n}]$, $\mathscr{A} \models_\kappa \varphi_\Phi[\vec{a}, \vec{n}]$ iff $\mathscr{A} \models_\kappa \varphi[\vec{a}, \vec{n}]$.

For $p=0$ this yields our observation. Without loss of generality, suppose that no distinguished variable occurs bound in φ, and that all conjuncts of Φ are inequalities; if the latter is not the case, substitute one equated variable for another, decreasing the number of type-2 variables free in φ until it is the case. We construct φ_Φ by induction on the construction of φ. If φ is $\mu_j = \mu_{j'}$, let:

$$\varphi_\Phi = \begin{cases} \neg\bot & \text{if } \varphi \text{ is a conjunct of } \Phi, \\ \bot & \text{if } \neg\varphi \text{ is a conjunct of } \Phi. \end{cases}$$

The only other case worth discussing is where φ is $(\exists\mu)\varphi_0$, for μ of type 2, and Φ is $\bigwedge_{j'<j<p}\mu_{j'} \neq \mu_j$. For $i<p$, let Φ_i be:

$$\Phi \,\&\, \mu = \mu_i \,\&\, \bigwedge \{\mu \neq \mu_{i'}\colon i' < l,\ i' \neq i\}.$$

Let $\varphi_{0,i}$ be $\varphi_{0,\Phi_i}(\mu/\mu_i)$. Clearly for κ, $\mathscr{A}$, $\vec{a}$, $\vec{n}$ as above and $n<\kappa$, if $\mathscr{A} \models_\kappa \Phi_i[\vec{n}, n]$ then:

$$\mathscr{A} \models_\kappa \varphi_{0,\Phi_i}[\vec{a}, \vec{n}, n] \quad \text{iff} \quad \mathscr{A} \models_\kappa \varphi_{0,i}[\vec{a}, \vec{n}].$$

Let Φ' be $\Phi \,\&\, \bigwedge_{i<p} \mu \neq \mu_i$. For each $b \subseteq p$, let ψ_b be:

$$\bigwedge_{i\in b} \varphi_{0,\Phi'}(\mu/\mu_i) \,\&\, \bigwedge_{i\in p-b} \neg\varphi_{0,\Phi'}(\mu/\mu_i);$$

let φ_b be $\psi_b \,\&\, \neg(\underline{\text{exactly}\, m}\ \rho)\varphi_{0,\Phi'}$, where $m = \text{card}(b)$ and where the second conjunct is expressed in $\mathscr{L}^{0,4}(\underline{\text{exactly}}, \leqslant)$ without use of '=' between variables of type 2, as described at the end of §2.4; this clause introduces the variables of type 4. Note that if $\mathscr{A} \models_\kappa \psi_b[\vec{a}, \vec{n}]$ then:

$$\hat{\mu}\varphi_0[\vec{a}, \vec{n}]^{\mathscr{A}} - \{n_i\colon i<p\} \neq \{\ \} \quad \text{iff} \quad \text{card}(\hat{\mu}\varphi_{0,\Phi'}[\vec{a}, \vec{n}]^{\mathscr{A}}) \neq m.$$

Furthermore for a unique $b \subseteq p$, $\mathscr{A} \models_\kappa \psi_b[\vec{a}, \vec{n}]$. Let φ_Φ be $\bigvee_{i<p} \varphi_{0,i} \vee \bigvee_{b\subseteq p} \varphi_b$. This proves (i).

4.2. Observation. $\mathscr{L}^{0,2*}(\underline{\text{exactly}}, =) \overset{s}{\rightarrowtail\!\!\!\prec} \mathscr{L}^{0,2*}(\underline{\text{exactly}})$.

Proof. Given $\varphi \in \text{Fml}(\mathscr{L}^{0,2*}(\underline{\text{exactly}}, =))$, with free variables among $v_0, \ldots, v_{l-1}, \mu_0, \ldots, \mu_{p-1}$ as in §4.1, and given a profile Φ for $\mu_0, \ldots, \mu_{p-1}$ we will construct $\varphi_\Phi \in \text{Fml}(L^{0,2*}(\underline{\text{exactly}}))$ so that for any $\mathscr{A}$, $\vec{a}$, $\vec{n}$, (*) of §4.1 holds. We use induction on the structure of φ as in the proof from §4.1; the notation and assumptions of §4.1 are in force. The cases worth discussing are where φ is $(\exists\mu)\varphi_0$ or $(\underline{\text{exactly}\, \mu_i}\ \mu)\varphi_0$ for μ of type 2, with Φ as above. The former case is handled as in §4.1, except that in forming φ_b, $(\underline{\text{exactly}\, m}\ \rho)\varphi_{0,\Phi'}$ is expressed in $\mathscr{L}^{0,2*}(\underline{\text{exactly}})$, as described at the end of §2.4.

Suppose φ is $(\underline{\text{exactly}\, \mu_i}\ \mu)\varphi_0$. Given a model $\mathscr{A}$, $\vec{a} \in |\mathscr{A}|^l$, $\vec{n} \in \bar{\kappa}^p$, let

$$A = \hat{\mu}\varphi_0[\vec{a}, \vec{n}]^{\mathscr{A}}, \qquad B = \{n_j\colon n_j \in A \text{ and } j<p\},$$

$$A' = \hat{\mu}\varphi_{0,\Phi'}[\vec{a}, \vec{n}]^{\mathscr{A}}, \qquad B' = \{n_j\colon n_j \in A' \text{ and } j<p\},$$

$$r = \text{card}(B - B') - \text{card}(B' - B).$$

By our induction hypothesis, for any $n \in \bar{\kappa} - \{n_0, \ldots, n_{p-1}\}$: $n \in A$ iff $n \in A'$. φ_Φ must say that the value of μ_i is card(A), using card(A') and r. In two cases card(A) = card(A'), making this easy:

Case 1: $r = 0$.

Case 2: *A' is infinite.*

If A' is finite and $r \neq 0$, we will want a formula that looks at sets D of cardinality r so that:

- if $r < 0$, then $D \subseteq A'$; so card(A) = card($A' - D$);
- if $r > 0$, then $D \subseteq \bar{\kappa} - A'$; so card($A$) = card($A' \cap D$).

For $j \in 2$ let

$$X_{2j} = \{\langle \theta, \rho \rangle : \ \underline{\text{(exactly}}\ \eta\ \rho)\theta \text{ is a subformula of } \varphi_{0,\Phi'} \text{ and } \rho \in Var(2j),\ \mu \in Var(2)\} \cup \{\langle \perp, \rho \rangle\},$$

where ρ is a selected new variable of type $2j$. Let:

$$N_{2j}(\mathscr{A}, \vec{a}, \vec{n}) = \{n < \kappa : \mathscr{A} \vDash_\kappa \vec{\exists}(\underline{\text{exactly}}\ \mathbf{n}\ \rho)\theta[\vec{a}, \vec{n}] \text{ for some } \langle \theta, \rho \rangle \in X_{2j}\},$$

where '$\vec{\exists}$' binds all non-distinguished variables free in its scope. Let:

$$N_{2j} = N_{2j}(\mathscr{A}, \vec{a}, \vec{n}), \qquad X = X_0 \cup X_2, \qquad N = N_0 \cup N_2,$$

$$\bar{N}_2 = \{n < \kappa : \ \text{for some } \langle \theta, \rho \rangle \in X_1 \text{ and assignments } \vec{a}^0, \vec{n}^0 \text{ for non-distinguished variables other than } \rho \text{ free in } \theta,\ \hat{\rho}\theta[\vec{a}, \vec{a}^0, \vec{n}, \vec{n}^0]^{\mathscr{A}} \subseteq N \text{ and has cardinality } n\}.$$

N is important because its members can be defined by formulae from a finite set; so members of $\bar{\kappa}$ can be distinguished from members of N without use of '='. Notice that $0 \in N$. Further facts: (1) For any $n, n' \in \bar{\kappa} - N$: $n \in A'$ iff $n' \in A'$. This follows by induction on the construction of $\varphi_{0,\Phi'}$. (2) For any $n \in N_2$ either $n \in \bar{N}_2$ or card($\bar{\kappa} - N$) $\leq n$. For $n \in N_2$, fix $\langle \theta, \rho \rangle \in X$ and $\vec{a}^0, \vec{n}^0$ so that

$$\mathscr{A} \vDash_\kappa \underline{\text{(exactly}}\ \mathbf{n}\ \rho)\theta[\vec{a}, \vec{a}^0, \vec{n}, \vec{n}^0];$$

by fact (1) either $\hat{\rho}\theta[\vec{a}, \vec{a}^0, \vec{n}, \vec{n}^0]^{\mathscr{A}} \subseteq N$, putting n into $\bar{N}_2$, or else $\bar{\kappa} - N \subseteq \hat{\rho}\theta[\vec{a}, \vec{a}^0, \vec{n}, \vec{n}^0]^{\mathscr{A}}$, yielding card($\bar{\kappa} - N$) $\leq n$.

If $r \neq 0$ and A' is finite, we consider these cases:

Case 3: $A' \subseteq N$, $r < 0$.

Case 4: $A' \subseteq N$, $r > 0$, *N in finite.*

Case 5: $A' \subseteq N$, $r > 0$, *N is infinite.*

Case 6: *not* $A' \subseteq N$, $r > 0$.

Case 7: *not* $A' \subseteq N$, $r < 0$.

Some further facts: (3) In case 6 and 7, N is infinite, since by fact (1) $\bar{\kappa} - N \subseteq A'$, making $\bar{\kappa} - N$ finite. (4) In case 4, $\aleph_0 \cap N_2 \subseteq \bar{N}_2$; for if $n \in N_2 - \bar{N}_2$, then by fact (2) card($\bar{\kappa} - N$) $\leq n$; since in this case N is finite, $\bar{\kappa} - N$ is infinite.

For any $b, b' \subseteq p$ let $\varphi_{b,b'}$ be:

$$\bigwedge_{j\in b} \varphi_{0,\Phi_j}(\mu/\mu_j) \mathbin{\&} \bigwedge_{j\in p-b} \neg\varphi_{0,\Phi_j}(\mu/\mu_j) \mathbin{\&} \bigwedge_{j\in b'} \varphi_{0,\Phi'}(\mu/\mu_j) \mathbin{\&} \bigwedge_{j\in p-b'} \neg\varphi_{0,\Phi'}(\mu/\mu_j).$$

Thus: $\mathscr{A} \vDash_\kappa \varphi_{b,b'}[\vec{a}, \vec{n}]$ iff

$$B = \{n_j: j \in b\} \quad \text{and} \quad B' = \{n_j: j \in b'\}.$$

We will construct $\varphi^*_{b,b'} \in \mathrm{Fml}(\mathscr{L}^{0,2*}(\underline{\text{exactly}}))$ so that if $\mathscr{A} \vDash_\kappa \varphi_{b,b'}[\vec{a}, \vec{n}]$ then:

$$\mathscr{A} \vDash_\kappa \varphi^*_{b,b'}[\vec{a}, \vec{n}] \quad \text{iff} \quad n_i = \mathrm{card}(A).$$

We will then let φ_Φ be

$$\bigvee \{\varphi^*_{b,b'}: b \cup b' = p,\ b \cap b' \text{ is empty}\}.$$

Let $r = \mathrm{card}(b - b') - \mathrm{card}(b' - b)$. If $r = 0$, then we are in case 1; so let $\varphi^*_{b,b'}$ be $(\underline{\text{exactly}}\ \mu_i\ \mu)\varphi_{0,\Phi'}$. Suppose $r \neq 0$; we will let $\varphi^*_{b,b'}$ be:

$$\bigvee \{\alpha_j \mathbin{\&} \varphi^*_j: j \in \{2, 3, 7\}\} \qquad \text{if } r < 0,$$
$$\bigvee \{\alpha_j \mathbin{\&} \varphi^*_j: j \in \{2, 4, 5, 6\}\} \quad \text{if } r > 0,$$

where each α_j says that case j holds and φ^*_j fixes the value of μ_i in case j.

Let α_2 be $(\underline{\infty}\mu)\varphi_{0,\Phi'}$ and φ^*_2 be $(\underline{\text{exactly}}\ \mu_i\ \mu)\varphi_{0,\Phi'}$. For $\eta \in Var(2)$ let $\mathrm{Def}(\eta)$ be $\bigvee \{\vec{\exists}(\underline{\text{exactly}}\ \eta\ \bar{\rho})\bar{\theta}: \langle \theta, \rho \rangle \in X\}$, where $\bar{\theta}$, $\bar{\rho}$ are formed from θ, ρ by replacing any free occurrences of η by some new variable. Since X is finite, this is well-defined. Clearly for any $n < \kappa$, $\mathscr{A} \vDash_\kappa \mathrm{Def}(\mathbf{n})[\vec{a}, \vec{n}]$ iff $n \in N$. Thus the construction of α_j for $j \in \{3, 4, 5, 6, 7\}$ is easy. For example, let α_4 be:

$$\neg\alpha_2 \mathbin{\&} (\forall\mu)(\varphi_{0,\Phi'} \supset \mathrm{Def}(\eta)) \mathbin{\&} \neg(\underline{\infty}\eta)\mathrm{Def}(\eta).$$

Cases 3, 5 and 6 are easy, since we may then take $D \subseteq N$. Fix $\eta_0, \ldots, \eta_{s-1} \in Var(2)$ and not occurring free in left-components of members of X. Suppose $Y \in X^s$, $Y = \langle\langle \theta_0, \rho_0\rangle, \ldots, \langle \theta_{s-1}, \rho_{s-1}\rangle\rangle$. Form $\bar{Y} = \langle\langle \bar{\theta}_0, \bar{\rho}_0\rangle, \ldots, \langle \bar{\theta}_{s-1}, \bar{\rho}_{s-1}\rangle\rangle$ by replacing the free variables in the θ_j's by new free variables as needed to insure that for all $j' < j < s$ all non-distinguished free variables in $\bar{\theta}_j$ do not occur free in $\bar{\theta}_{j'}$, and vice-versa, are not μ, and where if ρ_j is replaced in θ_j it is replaced by $\bar{\rho}_j$ which is not among $\eta_0, \ldots, \eta_{s-1}, \mu$. Let $\mathrm{Distinct}_Y(\eta_0, \ldots, \eta_{s-1})$ be

$$\bigwedge_{j<s} (\underline{\text{exactly}}\ \eta_j\ \bar{\rho}_j)\bar{\theta}_j \mathbin{\&} \bigwedge_{j'<j<s} \neg(\underline{\text{exactly}}\ \eta_{j'}\ \bar{\rho}_j)\bar{\theta}_j.$$

Let ψ^+_Y be

$$\vec{\exists}(\mathrm{Distinct}_Y(\eta_0, \ldots, \eta_{s-1}) \mathbin{\&} \bigwedge_{j<r} \neg\varphi_{0,\Phi'}(\mu/\eta_j)$$

$$\mathbin{\&} (\underline{\text{exactly}}\ \mu_i\ \mu)(\varphi_{0,\Phi'} \vee \bigvee_{j<s} (\underline{\text{exactly}}\ \mu\ \bar{\rho}_j)\bar{\theta}_j)),$$

where '$\vec{\exists}$' binds all non-distinguished variables free in its scope. Suppose $r>0$. Let φ_5^* and φ_6^* be $\bigvee\{\psi_Y^+: Y \in X^r\}$. This formula looks for $D \subseteq \bar{\kappa} - A'$ and says that the value of μ_i is card$(A' \cup D)$. Using fact (3), in cases 5 and 6 $N - A'$ is infinite; so such a D exists.

Similarly, for $Y \in X^s$ let ψ_Y^- be:

$$\vec{\exists}\Big(\text{Distinct}_Y(\eta_0, \ldots, \eta_{s-1}) \mathbin{\&} \bigwedge_{j<s} \varphi_{0,\Phi'}$$

$$\mathbin{\&} (\underline{\text{exactly}}\ \mu_i\, \mu)\Big(\varphi_{0,\Phi'} \mathbin{\&} \bigwedge_{j<s} \neg(\underline{\text{exactly}}\ \mu\, \bar{\rho}_j)\bar{\theta}_j\Big)\Big).$$

Where $r<0$ and $s=|r|$ let φ_3^* be $\bigvee\{\psi_Y^-: Y \in X^s\}$. This formula looks for $D \subseteq A' \cap N$ and says that the value of μ_i is card$(A' - D)$. Cases 4 and 7 require more work.

Let a filling be a set $\{\langle p_0, q_0\rangle, \ldots, \langle p_{k-1}, q_{k-1}\rangle\}$ such that for any $j<k$, $q_j \in N \cap \bar{\aleph}_0$ and $\{\ \} \neq (p_j, q_j) \subseteq \bar{\aleph}_0 - N$. Where F is a filling, let:

$$\bigcup F = \bigcup_{j \in k} (p_j, q_j), \qquad \bar{F} = \langle q_0 - p_0 - 1, \ldots, q_{k-1} - p_{k-1} - 1\rangle;$$

the ordering does not matter, but we do not just want a set because we want to have card$(\bigcup F) = \sum \bar{F}$. Let $t = \max(\bar{\aleph}_0 \cap N)$; since $0 \in N$, t exists. F is the maximum filling iff $\bigcup F = \bar{t} - N$.

Suppose case 4 holds. Consider $S = \langle s_0, \ldots, s_{k-1}\rangle \in (\bar{\aleph}_0 - \{0\})^k$ and $r_0 < \aleph_0$. If $r \geq r_0 + \sum S$, let subcase $\langle r_0, S\rangle$ hold iff:

$$r_0 = \min\{r, \text{card}(N - A')\}, \qquad \sum S = \min\{r - r_0, \text{card}(\bar{t} - N)\}.$$

Let $r_1 = \sum S$; $r_2 = r - (r_0 + r_1)$. For each such $\langle r_0, S\rangle$ we will construct $\varphi_{r_0,S}$ so that if $\mathscr{A}$, $\vec{a}$, $\vec{n}$ fall under case 4 then:

$$\mathscr{A} \vDash_\kappa \varphi_{r_0,S}[\vec{a}, \vec{n}] \quad \text{iff} \quad \langle r_0, S\rangle \text{ holds and } n_i = \text{card}(A') + r.$$

This formula will look for $D = D_0 \cup D_1 \cup D_2 \subseteq \bar{\kappa} - A'$, with D_0, D_1, D_2 pairwise disjoint and card$(D_j) = r_j$ for $j \in 3$, and say that the value of μ_i is card$(A' \cup D)$. More precisely, we will have:

$D_0 \subseteq N - A'$; thus $D_0 = N - A'$ iff $r \geq r_0$;
if $r_i \neq 0$, then $D_1 = \bigcup F$ for some filling F with $\bar{F} = S$;
so F is the maximum filling iff $r > r_0 + \text{card}(\bar{t} - N) > 0$;
$D_2 = (t, t + r_2 + 1)$.

This fact will permit us to describe D_2 if it is non-empty: (5) if F is the maximum filling, then $(\aleph_0 \cap N) \cup \bigcup F = \overline{t+1}$. Thus:

$$\text{card}((\bar{\aleph}_0 \cap N) \cup \bigcup F) = t + 1 \notin N \cup \bigcup F.$$

$\varphi_{r,\{\}}$ is easy to construct. For $Y \in X^s$ form $\bar{Y}$ as before and let $\gamma_Y(\vec{\eta})$ be:

$$\mathrm{Distinct}_Y(\eta_0, \ldots, \eta_{s-1}) \,\&\, \bigwedge_{j<s} \neg\varphi_{0,\Phi'}(\mu/\eta_j).$$

For $s = r$, let β_Y be:

$$\vec{\exists}\Big(\gamma_Y \,\&\, (\underline{\text{exactly}}\; \mu_i\, \mu)\Big(\varphi_{0,\Phi'} \vee \bigvee_{j<r} (\underline{\text{exactly}}\; \mu\, \bar{\rho}_j)\bar{\theta}_j\Big)\Big)$$

where '$\vec{\exists}$' binds all non-distinguished variables free in its scope. Let $\varphi_{r,\{\}}$ be $\bigvee\{\beta_Y : Y \in X^r\}$.

Now suppose that $r_0 \neq r$ but $r = r_0 + r_1$. We must 'pin down' $N - A'$ and $\bigcup F$ for a filling F. We must first describe how to 'pin down' a block $(p, q) \subseteq \aleph_0 - N$ for $q \in N$. For $\langle \theta, \rho \rangle \in X$ and variables $\mu^*, \eta_0, \ldots, \eta_s \in Var(2)$ and $v'_0, \ldots, v'_s$ of the same type as ρ, we will construct a formula $\mathrm{Block}_{\theta,\rho}(\mu^*, \vec{\eta}, \vec{v}')$ that will describe such a block. Change θ, ρ to $\bar{\theta}, \bar{\rho}$ to make sure that none of the non-distinguished variables in $\bar{\theta}$ are among those we have fixed. Suppose that $\rho \in Var(0)$. For $j \leq s$ let $\theta_j^\wedge$ be $\bar{\theta} \,\&\, \bigwedge_{j' \leq j} \rho \neq v'_{j'}$. Let $\mathrm{Block}_{\theta,\rho,\langle\,\rangle}(\mu^*, \vec{\eta}, \vec{v}')$ be:

$$\neg(\underline{\infty}\bar{\rho})\bar{\theta} \,\&\, \bigwedge_{j<s} \theta(\bar{\rho}/v'_j) \,\&\, \bigwedge_{j'<j<s} v'_{j'} \neq v'_j \,\&\, \bigwedge_{j<s} \neg\mathrm{Def}(\eta_j)$$

$$\&\, (\underline{\text{exactly}}\; \mu^*\, \bar{\rho})\bar{\theta} \,\&\, \bigwedge_{j\leq s} (\underline{\text{exactly}}\; \eta_j\, \bar{\rho})\theta_j^\wedge.$$

For $\rho \in Var(2)$ we would like to do the same thing; but there is a problem: '=' was used in the above formula. Here we rely on fact (4). For $Z \in X^s$, form $\bar{Z}$ as before; let $\theta_j^\wedge$ be $\bar{\theta} \,\&\, \bigwedge_{j' \leq j} \neg(\underline{\text{exactly}}\; \bar{\rho}\, \bar{\rho}_{j'})\bar{\theta}_{j'}$. Let $\mathrm{Block}_{\theta,\rho,Z}(\mu^*, \vec{\eta}, \vec{v}')$ be:

$$\neg(\underline{\infty}\bar{\rho})\bar{\theta} \,\&\, \bigwedge_{j<s} \theta(\bar{\rho}/v'_j) \,\&\, \mathrm{Distinct}_Z(\vec{v}') \,\&\, \bigwedge_{j<s} \neg\mathrm{Def}(\eta_j)$$

$$\&\, (\underline{\text{exactly}}\; \mu^*\, \bar{\rho})\bar{\theta} \,\&\, \bigwedge_{j\leq s} (\underline{\text{exactly}}\; \eta_j\, \bar{\rho})\theta_j^\wedge.$$

Now fix variables $\eta_0, \ldots, \mu_{r_0-1}, \mu_0^*, \ldots, \mu_{k-1}^* \in Var(2)$, $Y = \langle\langle \theta_0^*, \rho_0^* \rangle, \ldots, \langle \theta_{r_0-1}^*, \rho_{r_0-1}^* \rangle\rangle \in X^{r_0}$, $U = \langle\langle \theta_0, \rho_0 \rangle, \ldots, \langle \theta_{k-1}, \rho_{k-1} \rangle\rangle \in X^k$, and for each $j \in k$ fix $\eta_{j,0}, \ldots, \eta_{j,s_j} \in Var(2)$ and:

- if $\rho_j \in Var(0)$, then $Z_j = \langle\;\rangle$ and $v_{j,0}, \ldots, v_{j,s_j} \in Var(0)$;
- if $\rho_j \in Var(2)$, then $Z_j = \langle\langle \theta_{j,0}, \rho_{j,0} \rangle, \ldots, \langle \theta_{j,s_j}, \rho_{j,s_j} \rangle\rangle \in X^{s_j+1}$ and $v_{j,0}, \ldots, v_{j,s_j} \in Var(2)$.

Transform $Y, U, Z_0, \ldots, Z_{k-1}$ to $\bar{Y}, \bar{U}, \bar{Z}_0, \ldots, \bar{Z}_{k-1}$ so that no two formulae in any of the latter sequences have a non-distinguished variable in common, and so that all variables in such formulae are distinct from those fixed so far. We will use $\vec{\mu}^*$ to 'pin down' RFld(F) and then variables of the form $\eta_{j,j'}$ for $j < k$ and $j' < s_j$ will 'pin down' the elements of $\bigcup F$ for a filling F. Let $\mathrm{Filling}_{U,\vec{Z}}(\vec{\mu}^*, \vec{\eta}_0, \ldots, \vec{\eta}_{k-1}, \vec{v}_o, \ldots, \vec{v}_{k-1})$ be:

$$\mathrm{Distinct}_U(\vec{\mu}^*) \,\&\, \bigwedge_{j<k} \mathrm{Block}_{\theta_j,\rho_j,Z_j}(\mu_j^*, \eta_{j,0}, \ldots, \eta_{j,s_j}, v_{j,0}, \ldots, v_{j,s_j}).$$

Let $\beta_{Y,U,\vec{Z}}$ be:

$$\vec{\exists}\Big(\gamma_Y \,\&\, (\forall\mu)\Big(\varphi_{0,\Phi'} \supset \bigvee_{j<r_0} (\underline{\text{exactly}}\ \mu\ \rho_j^*)\theta_j^*\Big)$$

$$\&\, \text{Filling}_{U,\vec{Z}}(\vec{\eta}^*, \vec{\eta}_0, \ldots, \vec{\eta}_{k-1}, \vec{\nu}_0, \ldots, \vec{\nu}_{k-1})$$

$$\&\, (\underline{\text{exactly}}\ \mu_i\ \mu)\Big[\varphi_{0,\Phi'} \vee \bigvee_{j<r_0} (\underline{\text{exactly}}\ \mu\ \bar{\rho}_j^*)\bar{\theta}_j^*$$

$$\vee \bigvee \{(\underline{\text{exactly}}\ \mu\ \rho_j)\bar{\theta}^{\wedge}_{j,j'}\colon j<k \text{ and } j'<s_j\}\Big]\Big),$$

where '$\vec{\exists}$' binds as usual. This formula looks at $D_0 = N - A'$ and $D_1 = \bigcup F$ for a filling F with $\tilde{F} = S$ and then says that the value of μ_i is card$(A' \cup D_0 \cup D_1)$. Let $\varphi_{r_0,S}$ be $\bigvee \{\beta_{Y,U,\vec{Z}}\colon Y, U, \vec{Z}$ as above$\}$.

Now suppose that $r > r_0$ and $r_2 \neq 0$. We must 'pin down' $t+1$. First we must pin down the maximum filling. For U, $\vec{Z}$ as above, a new μ^*, $\vec{\eta}_k = \eta_{k,0}, \eta_{k,1}$ and $\vec{\nu}_k = \nu_{k,0}, \nu_{k,1}$, let MaxFilling$_{U,\vec{Z}}(\vec{\mu}^*, \vec{\eta}_0, \ldots, \vec{\eta}_{k-1}, \vec{\nu}_0, \ldots, \vec{\nu}_{k-1})$ be:

$$\text{Filling}_{U,\vec{Z}}(\vec{\mu}^*, \vec{\eta}_0, \ldots, \vec{\eta}_{k-1}, \vec{\nu}_0, \ldots, \vec{\nu}_{k-1}) \,\&\, \bigwedge_{j<k} \text{Def}(\eta_{j,s_j})$$

$$\&\, \bigwedge \{\neg(\exists\mu_k^*)(\exists\vec{\eta}_k)(\exists\vec{\nu}_k)\text{Filling}_{U*\langle\theta_k,\rho_k\rangle,\vec{Z}*Z_k}(\vec{\mu}^*, \mu_k^*, \vec{\eta}_0, \ldots, \vec{\eta}_{k-1},$$
$$\vec{\eta}_k, \vec{\nu}_0, \ldots, \vec{\nu}_{k-1}, \vec{\nu}_k)\colon \langle\theta_k, \rho_k\rangle \in X,\ Z_k \in X^2\}.$$

The second conjunct shows the reason for including the variables η_{j,s_j} for $j<k$: that clause 'stretches' each j-th block down to p_j+1 for $p_j \in N$; such blocks exist, since $0 \in N$; $\eta_{k,1}$ and $\nu_{k,1}$ do no work in the third conjunct, but are included for the notational convenience of the second. Let FinDef(η) be:

$$\bigvee \{\vec{\exists}((\underline{\text{exactly}}\ \eta\ \rho)\theta \,\&\, \neg(\underline{\infty}\rho)\theta)\colon \langle\theta, \nu\rangle \in X\},$$

where '$\vec{\exists}$' binds as usual and η is as in our definition of Def(η); this formula says that the value of η is in $\bar{\aleph}_0 \cap N$. Let $\delta_{U,\vec{Z},0}(\ldots, \mu)$ be:

$$(\underline{\text{exactly}}\ \mu\ \eta)(\text{FinDef}(\eta) \vee \bigvee \{(\underline{\text{exactly}}\ \eta\ \bar{\rho}_{j,j'})\theta^{\wedge}_{j,j'}\colon j<k, j'<s_j\});$$

for $q < \omega$ let $\delta_{U,\vec{Z},q+1}(\ldots, \mu)$ be:

$$(\underline{\text{exactly}}\ \mu\ \eta)(\text{FinDef}(\eta) \vee \bigvee \{(\underline{\text{exactly}}\ \eta\ \bar{\rho}_{j,j'})\bar{\theta}_{j,j'}\colon j \leqslant k, j'<s_j\}$$
$$\vee\, \delta_{U,\vec{Z},0} \vee \cdots \delta_{U,\vec{Z},q}).$$

Then $\delta_{U,\vec{Z},q}(\ldots, \mu)$ 'pins down' the value of μ to be $t+q+1$, using fact (5) for $q=0$ and iterating. Let $\beta_{Y,U,\vec{Z}}$ be:

$$\vec{\exists}\Big(\gamma_Y \,\&\, (\forall\mu)\Big(\varphi_{0,\Phi'} \supset \bigvee_{j<r_0} (\underline{\text{exactly}}\ \mu\ \rho_j^*)\theta_j^*\Big)$$

$$\&\, \text{MaxFilling}_{U,\vec{Z}}(\vec{\mu}^*, \vec{\eta}_0, \ldots, \vec{\eta}_{k-1}, \vec{\nu}_0, \ldots, \vec{\nu}_{k-1})$$

$$\&\, (\underline{\text{exactly}}\ \mu_i\ \mu)\Big[\varphi_{0,\Phi'} \vee \bigvee_{j<r_0} (\underline{\text{exactly}}\ \mu\ \bar{\rho}_j^*)\bar{\theta}_j^*$$

$$\vee \bigvee_{j<r_2} \delta_{U,\vec{Z},j} \vee \bigvee \{(\underline{\text{exactly}}\ \mu\ \rho_j)\bar{\theta}^{\wedge}_{j,j'}\colon j<k \text{ and } j'<s_j\}\Big]\Big).$$

This formula looks at $D_0 = N - A'$, $D_1 = \bigcup F$ where F is the maximum filling, and at $D_2 = (t, t + r_2 + 1)$, and says that the value of μ_i is card$(A' \cup D_0 \cup D_1 \cup D_2)$. Let $\varphi_{r_0,S}$ be $\bigvee \{\beta_{Y,U,\vec{Z}}$: Y, U, $\vec{Z}$ as above$\}$. We let φ_4 be:

$$\bigvee \{\varphi_{r_0,S}\text{: } \langle r_0, S\rangle \text{ is a subcase of case 4}\}.$$

We now tackle case 7. Let $t = \text{card}(\bar{\kappa} - N)$. Since $A' - N = \bar{\kappa} - N$ and $r < 0$, $t > 0$. Let $t' < t$ be the greatest such that $t' \in N$. By fact (2), $t' \in N_0 \cup \bar{N}_2$. For $r_0, u < \aleph_0$ let subcase $\langle r_0, u\rangle$ hold iff $r_0 = \min\{|r|, \text{card}(A' \cap N)\}$ and $u = \min\{|r| - r_0, t - t' - 1\}$. We will construct $\varphi_{r_0,u}$ so that if $\mathscr{A}$, $\vec{a}$, $\vec{n}$ fall under case 7, then:

$$\mathscr{A} \vDash_\kappa \varphi_{r_0,u}[\vec{a}, \vec{n}] \quad \text{iff} \quad \langle r_0, u\rangle \text{ holds and } n_i = \text{card}(A') - |r|.$$

If subcase $\langle |r|, 0\rangle$ holds, then we can look at $D \subseteq A'$ with card$(D) = |r|$ and pin μ_i to card$(A' - D)$. For $Y = \langle\langle\theta_0, \rho_0\rangle, \ldots, \langle\theta_{|r|-1}, \rho_{|r|-1}\rangle\rangle \in X^{|r|}$ let β_Y be:

$$\text{Distinct}_Y(\eta_0, \ldots, \eta_{|r|-1}) \mathbin{\&} \bigwedge_{j<|r|} \varphi_{0,\Phi'}(\mu/\eta_j)$$

$$\mathbin{\&} (\underline{\text{exactly}}\; \mu_i \mu)\Big(\varphi_{0,\Phi'} \mathbin{\&} \bigwedge_{j<|r|} \neg(\underline{\text{exactly}}\; \mu\bar{\rho}_j)\bar{\theta}_j\Big);$$

let $\varphi_{|r|,0}$ be $\bigvee \{\vec{\exists}\beta_Y\text{: } Y \in X^{|r|}\}$.

Suppose that subcase $\langle r_0, u\rangle$ holds for $r_0 = \text{card}(A' \cap N) < |r|$ and $u = |r| - r_0$. Then $\bar{\aleph}_0 - N$ contains an interval with at least u members; since N is cofinite, there then is a filling $F = \{\langle p_0, q_0\rangle\}$ with $u = q_0 - p_0 - 1$; $\varphi_{\langle r_0,u\rangle}$ will say that the value of μ_i is card$(A' - (N \cup \bigcup F))$ for such an F. For $\langle\theta, \rho\rangle \in X$, $Y = \langle\langle\theta_0, \rho_0\rangle, \ldots, \langle\theta_{r_0-1}, \rho_{r_0-1}\rangle\rangle \in X^{r_0}$ and $\eta_0, \ldots, \eta_{r_0-1}$, $\mu^*, \eta'_0, \ldots, \eta'_{u-1} \in$ *Var*(2) and $v'_0, \ldots, v'_{u-1}$ of the same type as ρ, let $\beta_{Y,\beta,\rho}$ be:

$$\text{Distinct}_Y(\vec{\eta}) \mathbin{\&} \text{Block}_{\theta,\rho}(\mu^*, \vec{\eta}\,', \vec{v}\,') \mathbin{\&} (\underline{\text{exactly}}\; \mu_i\, \mu)$$
$$\Big(\varphi_{0,\Phi'} \mathbin{\&} \bigwedge_{j<r_0} \neg(\underline{\text{exactly}}\; \mu\, \bar{\rho}_j)\bar{\theta}_j \mathbin{\&} \bigwedge_{j<u} \neg(\underline{\text{exactly}}\; \mu\, \rho_j)\theta_j^{\wedge}\Big);$$

this looks for $D = (A' \cap N) \cup \bigcup F$ for a filling F as described above, and pins μ_i to card$(A' - D)$. Let $\varphi_{r_0,u}$ be:

$$\bigvee \{\vec{\exists}\beta_{Y,\theta,\rho}\text{: } \langle\theta, \rho\rangle \in X \quad \text{and} \quad Y \in X^{r_0}\}.$$

Now suppose that $r_0 < |r|$ and $u < |r| - r_0$; so $t = t' + u + 1$. Set $r_1 = |r| - r_0$. Fixing $\vec{\eta} = \eta_0, \ldots, \eta_{r_1-1}$ and $\vec{v}\,' = v'_0, \ldots, v'_{r_1-1}$, let α be:

$$\neg\bigvee \{\vec{\exists}\text{Block}_{\theta,\rho}(\mu^*, \vec{\eta}, \vec{v}\,')\text{: } \langle\theta, \rho\rangle \in X\};$$

α entails that $t - t' \leq r_1$. We will construct a $\varphi_{\langle r_1, u\rangle}$ to look for sets C and $D \subseteq C$, with card$(C) = t'$ and card$(D) = r_1 - u - 1$, and to say that the value of μ_i is card$(C - D)$. Let $s = r_1 - u - 1$. Fix $\mu^*, \eta_0, \ldots, \eta_u \in$ *Var*(2). For $\langle\theta, \rho\rangle \in X$:

– if $\rho \in$ *Var*(0) fix $v^*_0, \ldots, v^*_u$, $v'_0, \ldots, v'_{s-1} \in$ *Var*(0) and $Z = U = \langle\ \rangle$;

– if $\rho \in Var(2)$ fix $v_0^*, \ldots, v_u^*$, $v_0', \ldots, v_{s-1}' \in Var(2)$ and $Z = \langle\langle\theta_0, \rho_0\rangle, \ldots, \langle\theta_u, \rho_u\rangle\rangle \in X^{u+1}$, $U = \langle\langle\theta_0^*, \rho_0^*\rangle, \ldots, \langle\theta_{s-1}^*, \rho_{s-1}^*\rangle\rangle \in X^s$.

Form θ, ρ, $\bar{Z}$, $\bar{U}$ as usual to avoid collisions of non-distinguished variables. Say $\rho \in Var(0)$. For $j \leqslant u$ let $\theta_j^\vee$ be $\theta \vee \bigvee_{j' \leqslant j} \rho = v_{j'}^*$; let $\xi_{\theta,\rho,Z}(\mu^*, \vec{\eta}, \vec{v}^*)$ be:

$$\neg(\underline{\infty}\rho)\theta \,\&\, (\underline{\text{exactly}}\; \mu^*\; \rho)\theta \,\&\, \bigwedge_{j \leqslant u} (\underline{\text{exactly}}\; \eta_j)\theta_j^\vee$$

$$\& \bigwedge_{j<u} \neg\mathrm{Def}(\eta_j) \,\&\, (\underline{\text{exactly}}\; \eta_u\; \eta)\neg\mathrm{Def}(\eta).$$

If $\rho \in Var(2)$ for $j \leqslant u$ let $\theta_j^\vee$ be $\theta \vee \bigvee_{j' \leqslant j} (\underline{\text{exactly}}\; \rho\; \bar{\rho}_{j'})\bar{\theta}_{j'}^\vee$; let $\xi_{\theta,\rho,Z}(\mu^*, \vec{\eta}, \vec{v}^*)$ be:

$$\neg(\underline{\infty}\rho)\theta \,\&\, (\underline{\text{exactly}}\; \mu^*\; \rho)\theta \,\&\, \mathrm{Distinct}_Z(\vec{v}^*)$$

$$\& \bigwedge_{j \leqslant u} (\underline{\text{exactly}}\; \eta_j)\theta_j^\vee \,\&\, \bigwedge_{j<u} \neg\mathrm{Def}(\eta_j) \,\&\, (\underline{\text{exactly}}\; \eta_u\; \eta)\neg\mathrm{Def}(\eta).$$

In both situations, $\xi_{\xi,\rho,Z}(\mu^*, \vec{\eta}, \vec{v}^*)$ pins η_u to t and η^* to t'. If $\rho \in Var(0)$ let $\beta_{\theta,\rho,\langle\ \rangle,\langle\ \rangle}$ be:

$$\xi_{\theta,\rho,Z}(\mu^*, \vec{\eta}, \vec{v}^*) \,\&\, \bigwedge_{j<s} \theta(\rho/v_j') \,\&\, \bigwedge_{j'<j<s} v_{j'}' \neq v_j'$$

$$\& (\underline{\text{exactly}}\; \mu_i\; \rho)\Big(\theta \,\&\, \bigwedge_{j<s} \rho \neq v_j'\Big).$$

If $\rho \in Var(2)$ let $\beta_{\theta,\rho,Z,U}$ be:

$$\xi_{\theta,\rho,Z}(\mu^*, \vec{\eta}, \vec{v}^*) \,\&\, \bigwedge_{j<s} \theta(\rho/v_j') \,\&\, \mathrm{Distinct}_U(\vec{v}')$$

$$\& (\underline{\text{exactly}}\; \mu_i\; \rho)\Big(\theta \,\&\, \bigwedge_{j<s} \neg(\underline{\text{exactly}}\; \rho\; \rho_j^*)\theta_j^*\Big);$$

this formula handles the case of $t' \in N_2$; since then $t' \in \bar{N}_2$, the desired values for the v_j''s exist in N as required. Let $\varphi_{r_0,u}$ be:

$$\alpha \,\&\, \bigvee \{\vec{\exists}\beta_{\theta,\rho,Z,U}\colon \langle\theta, \rho\rangle, Z, U \text{ as described above}\}.$$

Let φ_7 be:

$$\bigvee \{\varphi_{r_0,u}\colon \langle r_0, u\rangle \text{ is a subcase of case 7}\}.$$

4.3. Theorems 4.1 and 4.2 suggest the following:

Conjecture (E). For any $\kappa \in$ Card:

(1) $\mathscr{L}^{0,4}(\underline{\text{exactly}}, =) \overset{\kappa}{\preccurlyeq} \mathscr{L}^{0,6}(\underline{\text{exactly}})$,

(2) $\mathscr{L}^{0,4*}(\underline{\text{exactly}}, =) \overset{\kappa}{\preccurlyeq} \mathscr{L}^{0,4*}(\underline{\text{exactly}})$.

Given $\varphi \in \text{Sent}(\mathscr{L}^{0,4}((\underline{\text{exactly}}, =))$ $[\text{Sent}(\mathscr{L}^{0,4*}(\underline{\text{exactly}}, =))]$, we may apply the procedures used in §4.1 [§4.2] to eliminate all equations between variables of type 4; but I can't see how to eliminate equations between variables of type 2 the scope of a prefex of the form $(\underline{\text{exactly}}\,\mu_i\,\mu)$ for $\mu \in Var(2)$ and $\mu_i \in Var(4)$. However for $\kappa < \aleph_\omega$ this obstacle can be avoided, even improving on (E.1).

Theorem. *For* $\kappa < \aleph_\omega$ *and* $1 \leq k < \omega$:

(i) $\mathscr{L}^{0,2k}(\underline{\text{exactly}}, =) \overset{\kappa}{\rightarrowtail\!\!\prec} \mathscr{L}^{0,2k}(\underline{\text{exactly}})$,

(ii) $\mathscr{L}^{0,2k*}(\underline{\text{exactly}}, =) \overset{\kappa}{\rightarrowtail\!\!\prec} \mathscr{L}^{0,2k*}(\underline{\text{exactly}})$.

In this section we will prove (i) for $k = 1$; in the next section we will consider (i) and (ii) with $k > 1$.

Let $\varphi \in \text{Fml}(\mathscr{L}^{0,2}(\underline{\text{exactly}}, =))$ with free variables among $v_0, \ldots, v_{l-1} \in Var(0)$, $\mu_0, \ldots, \mu_{p-1} \in Var(2)$, the 'distinguished' variables. Let Φ be a profile for $\mu_0, \ldots, \mu_{p-1}$. We will construct φ_Φ meeting the conditions met in §4.1 and 4.2. Only the case in which φ is $(\exists\mu)\varphi_0$, $\mu \in Var(2)$, needs discussion. As in §4.1 we may suppose that Φ is $\bigwedge_{j'<j<p} \mu_{j'} \neq \mu_j$, and that no distinguished variable occurs bound in φ or in $\varphi_{0,\Phi'}$. Let φ_j be $\varphi_{0,\Phi'}(\mu/\mu_j)$ for $j < p$. We will construct $\varphi' \in \text{Fml}(\mathscr{L}^{0,2}(\underline{\text{exactly}}))$ so that for any model $\mathscr{A}$, $\vec{a} \in |\mathscr{A}|^l$ and $\vec{n} \in \bar{\kappa}^p$ with $\mathscr{A} \vDash_\kappa \Phi[\vec{n}]$:

$$\mathscr{A} \vDash_\kappa (\exists\mu)\Big(\bigwedge_{i<p} \mu \neq \mu_i \,\&\, \varphi_{0,\Phi'}\Big)[\vec{a}, \vec{n}] \quad \text{iff} \quad \mathscr{A} \vDash \varphi'[\vec{a}, \vec{n}].$$

Then we will take φ_Φ to be $\bigvee_{i<p} \varphi_i \vee \varphi'$.

Let X_0 and $N_0 = N_0(\mathscr{A}, \vec{a}, \vec{n})$ be as in §4.2, $C = \{n_0, \ldots, n_{p-1}\} - N_0$. Let $\text{Def}_0(\eta)$ be $\bigvee \{\vec{\exists}(\underline{\text{exactly}}\,\eta\,\bar{v})\bar{\theta}\colon \langle\theta, v\rangle \in X_0\}$, for $\eta \in \text{Var}(2)$ as in §4.2 and '$\vec{\exists}$' binding all non-distinguished variables other than η free in its scope: clearly Def_0 defines N_0. Let φ^* be:

$$\bigvee\Big\{\vec{\exists}\Big((\underline{\text{exactly}}\,\mu\,\bar{v})\bar{\theta} \,\&\, \mathop{\&}_{j<p} \neg(\underline{\text{exactly}}\,\mu_j\,\bar{v})\bar{\theta} \,\&\, \varphi_{0,\Phi'}\Big)\colon \langle\theta, v\rangle \in X_0\Big\},$$

where θ, v are transformed into $\bar{\theta}$, $\bar{v}$ as usual to avoid collisions of variables, and where '$\vec{\exists}$' binds all non-distinguished variables in its scope, including μ. Thus for $\mathscr{A}, \vec{a}, \vec{n}$ as above:

$$\mathscr{A} \vDash_\kappa \varphi^*[\vec{a}, \vec{n}] \quad \text{iff} \quad \text{for some } n \in N_0 - \{n_0, \ldots, n_{p-1}\} \quad \mathscr{A} \vDash_\kappa \varphi_{0,\Phi'}[\vec{a}, \vec{n}].$$

For $\mathscr{A}$, $\vec{a}$, $\vec{n}$ and N_0 as above and $n, n' \in \bar{\kappa} - N_0$:

$$\mathscr{A} \vDash_\kappa \varphi_{0,\Phi'}[\vec{a}, \vec{n}, n] \quad \text{iff} \quad \mathscr{A} \vDash_\kappa \varphi_{0,\Phi'}[\vec{a}, \vec{n}, n'].$$

This follows by induction on the construction of $\varphi_{0,\Phi'}$. Thus for some $n \in \bar{\kappa} -$

$(N_0 \cup C)$:

$$\mathscr{A} \vDash_\kappa \varphi_{0,\Phi'}[\vec{a}, \vec{n}, n] \quad \text{iff}$$
$$\mathscr{A} \vDash_\kappa (\exists\mu)(\neg \mathrm{Def}_0(\mu) \,\&\, \varphi_{0,\Phi'})[\vec{a}, \vec{n}] \text{ and } \mathrm{card}(\bar{\kappa} - N) > \mathrm{card}(C).$$

For $c \subseteq p$ let Def_c be

$$\bigwedge_{j \in c} \neg \mathrm{Def}(\mu_j) \,\&\, \bigwedge_{j \in p-c} \mathrm{Def}(\mu_j).$$

Thus $\mathscr{A} \vDash_\kappa \mathrm{Def}_c[\vec{a}, \vec{n}]$ iff $C = \{n_j : j \in c\}$. We will construct a formula ψ_c saying that $\mathrm{card}(\bar{\kappa} - N_0) > \mathrm{card}(c)$. Letting φ^{**} be:

$$(\exists\mu)(\neg \mathrm{Def}_0(\mu) \,\&\, \varphi_{0,\Phi'}) \,\&\, \Big(\bigvee_{c \subseteq p} (\mathrm{Def}_c \,\&\, \psi_c)\Big),$$

we may then let φ' be $\varphi^* \vee \varphi^{**}$.

Suppose $\kappa = \aleph_z$, $z < \omega$. If $y = \mathrm{card}(N_0 \cap (\bar{\kappa} - \bar{\aleph}_0))$, then $\mathrm{card}(\bar{\kappa} - N_0) > \mathrm{card}(c)$ iff $\mathrm{card}(\bar{\aleph}_0 - N_0) > \mathrm{card}(c) - (z - y)$. For each $y \leqslant z$ we will construct δ_y and γ_y so that:

$$\mathscr{A} \vDash_\kappa \delta_y[\vec{a}, \vec{n}] \quad \text{iff} \quad y = \mathrm{card}(N_0 \cap (\bar{\kappa} - \bar{\aleph}_0)),$$
$$\mathscr{A} \vDash_\kappa \gamma_y[\vec{a}, \vec{n}] \quad \text{iff} \quad \mathrm{card}(\bar{\aleph}_0 - N_0) > \mathrm{card}(c) - (z - y).$$

Then we may let ψ_c be $\bigvee_{y \leqslant z} (\delta_y \,\&\, \gamma_y)$. The construction of δ_y relies on ideas used in §4.2, and so is left to the reader. The construction of γ_y uses a modified notion of a filling. Let F be an upward-filling iff $F = \{\langle p_0, q_0\rangle, \ldots, \langle p_{k-1}, q_{k-1}\rangle\}$ where $p_j \in N_0 \cap \bar{\aleph}_0$, $p_j < q_j$ and $(p_j, q_j + 1) \subseteq \bar{\aleph}_0 - N_0$ for all $j < k$. Let:

$$\bigcup F = \bigcup \{(p_j, q_j + 1) : j < k\}, \qquad \bar{F} = \langle q_0 - p_0, \ldots, q_{k-1} - p_{k-1}\rangle;$$

again order is unimportant; we only need that $\mathrm{card}(\bigcup F) = \sum \bar{F}$. For each $S \in (\bar{\aleph}_0 - \{0\})^k$ with $\sum S = \mathrm{card}(c) + y + 1 - z$ we construct γ_S saying that there is an upward-filling F with $\bar{F} = S$. We then take γ_y to be $\bigvee \{\gamma_S : S \text{ as above}\}$. Construction of γ_S resembles constructions in §4.2 and is left to the reader.

But the following deserves mention. In this construction we could not use fillings; fillings would be formed by counting downward from elements of $N_0 \cap \bar{\aleph}_0$; but if N_0 is finite, there might not be enough elements of $N_0 \cap \bar{\aleph}_0$ to yield an F with $\bigcup F$ sufficiently large. On the other hand, in case 4 of §4.2 we could not use upward-fillings; for in counting upward from an element of $N_2 \cap \bar{\aleph}_0$ we must 'count with' members of N; since in case 4 N is finite, we might not be able to count high enough.

4.4. We will now prove Theorem 4.3 for $k = 2$. Suppose that $\varphi \in \mathrm{Fml}(\mathscr{L}^{0,4}(\underline{\text{exactly}}, =))$ $[\mathrm{Fml}(\mathscr{L}^{0,4^*}(\underline{\text{exactly}}, =))]$ with free variables among $v_0, \ldots, v_{l-1} \in \mathit{Var}(0)$, $\mu_0, \ldots, \mu_{p-1} \in \mathrm{Var}(2)$, $\zeta_0, \ldots, \zeta_{q-1} \in \mathit{Var}(4)$; these are the distinguished variables. As indicated at the start of §4.3, it suffices to transform φ to a κ-equivalent $\hat{\varphi} \in \mathrm{Sent}(\mathcal{L}^{0,4}(\underline{\text{exactly}}, =))$ $[\mathrm{Sent}(\mathcal{L}^{0,4*}(\underline{\text{exactly}}, =))]$ with the same free variables such that $\hat{\varphi}$ contains no equations between variables of type 2. As usual, let distinctly bound variables be distinct from each other and from the distinguished variables. Let Φ be a profile for $\mu_0, \ldots, \mu_{p-1}$. We will construct $\varphi_\Phi \in \mathrm{Sent}\ (\mathcal{L}^{0,4}(\underline{\text{exactly}}))$ $[\mathrm{Sent}(\mathcal{L}^{0,4*}(\underline{\text{exactly}}))]$ so that for any model $\mathcal{A}$, $\vec{a} \in |\mathcal{A}|^l$, $\vec{n} \in \bar{\kappa}^p$, $\vec{m} \in \bar{\aleph}_0^q$, if $\mathcal{A} \models \Phi[\vec{n}]$ then:

$$\mathcal{A} \models_\kappa \varphi_\Phi[\vec{a}, \vec{n}, \vec{m}] \quad \text{iff} \quad \mathcal{A} \models_\kappa \varphi[\vec{a}, \vec{n}, \vec{m}].$$

The only cases worth discussing are where φ is $(\exists\mu)\varphi_0$ or $(\underline{\text{exactly}}\ \zeta_i\ \mu)\varphi_0$ for $\mu \in Var(2)$. The first case in handled as in §4.1; thus the assumption that $\kappa < \aleph_\omega$ is not used. Suppose that φ has the second form. We will try to mimic the construction from §4.2, with X_0 and N_0 playing the role that X and N played in §4.2. In cases 1 through 6 the construction is straightforward, not requiring use of the assumption that $\kappa < \aleph_\omega$. But case 7 poses a problem. Suppose subcase $\langle r_0, u\rangle$ obtains for $r_0 < |r|$ and $u < |r| - r_0 = r_1$, and for $\langle \theta, v\rangle \in X_0$ and appropriate $\vec{a}^0$, $\vec{n}^0$, $\vec{m}^0$ we have $C = \hat{v}\theta[\vec{a}, \vec{a}^0, \vec{n}, \vec{n}^0, \vec{m}, \vec{m}^0]^{\mathcal{A}} \subseteq |\mathcal{A}|$ with $\mathrm{card}(C) = t'$. For any $\mu' \in Var(2)$ we can produce a formula that pins the value of μ' to $t' - (r - u - 1) = \mathrm{card}(C - D)$ for any $D \subseteq C$ with $\mathrm{card}(D) = r_1 - u - 1$. But this will not enable us to produce a formula pinning ζ_i to $t' - (r_1 - u - 1)$, since $\zeta_i \in Var(4)$! This is the obstacle to the naive approach to proving conjecture (E).

The hypothesis that $\kappa = \aleph_z$ for $z < \omega$ makes possible a different approach to case 7. Under case 7 one of the following subcases holds:

(1) $\mathrm{card}(\bar{\aleph}_0 - N_0) \geq r$,

(2) $\mathrm{card}(A' \cap N_0) \geq r$,

(3) $\mathrm{card}(A') = \mathrm{card}(A' \cap N_0) + \mathrm{card}(\bar{\kappa} - N_0) < z + 2r$.

For each S such that $S \in (\aleph_0 - \{0\})^k$ for some k and $\sum S = r$, we may construct a formula α_S asserting the existence of a filling F with $\tilde{F} = S$ and such that the value of μ_i is $\mathrm{card}(A' - \bigcup F)$. In subcase (1) there is such an S and F. It is easy to construct a γ that 'looks for' $D \subseteq A' \cap N_0$ with $\mathrm{card}(D) = r$ and says that the value of μ_i is $\mathrm{card}(A' - D)$; in subcase (2) such a D exists. For each u with $r \leq u < z + 2r$ it is easy to construct a formula γ_u saying that $\mathrm{card}(A') = u$ and the value of μ_i is $u - r$. Let the disjunction of all of these formulae be φ_7; details are left to the reader.

This construction easily generalizes for $k > 2$.

4.5. We now show Theorem 4.3(i) is best-possible for $k = 1$. Let $\mathbf{R}$ be 2-place, *Pred* $= \{\mathbf{R}\}$, *Funct* $= \{\ \}$. For $\mu \in \mathrm{Var}(2)$ let $\theta(\mu)$ be $(\exists v_0)(\underline{\text{exactly}}\ \mu\ v)\mathbf{R}(v_0, v)$; let φ be:

$$(\exists\mu_0)(\exists\mu_1)(\neg\theta(\mu_0) \,\&\, \neg\theta(\mu_1) \,\&\, \mu_0 \neq \mu_1).$$

Observation. *For $\kappa \geq \aleph_\omega$, φ is not κ-equivalent to any sentence of $\mathcal{L}^{0,2}(\underline{\text{exactly}})$.*

Proof. We construct models $\mathscr{A}_0$ and $\mathscr{A}_1$ as follows. For each $n < \mathrm{ncb}(\kappa)$ fix sets X_n and Y_n with $\mathrm{card}(X_n) = \kappa$ and $\mathrm{card}(Y_n) = n$, all these sets pairwise disjoint. Let:

$$\mathbf{R}^{\mathscr{A}_0} = \bigcup \{X_n \times Y_n : n \neq \aleph_0\},$$
$$\mathbf{R}^{\mathscr{A}_1} = \bigcup \{X_n \times Y_n : n \notin \{\aleph_0, \aleph_1\}\}.$$

The members of $\bigcup X_n$ are X-objects, and the members of $\bigcup Y_n$ are Y-objects. For $a \in X_n \cup Y_n$ let $f(a) = n$. For $n, n' < \kappa$ let n match n' iff:

$$\text{if } n \text{ or } n' \text{ is finite or } \geq \aleph_\omega, \text{ then } \quad n = n';$$
$$n = \aleph_{t+1} \quad \text{iff} \quad n' = \aleph_{t+2} \quad \text{for all } t < \omega;$$
$$n = \aleph_0 \quad \text{iff} \quad n' \in \{\aleph_0, \aleph_1\}.$$

For $\vec{a}_i \in |\mathscr{A}_i|^l, \vec{n} \in \bar{\kappa}^p$ let $\langle \vec{a}_0, \vec{n}_0 \rangle$ match $\langle \vec{a}_1, \vec{n}_1 \rangle$ iff:

$$\text{for all } j < j' < l: \quad a_{0,j} = a_{0,j'} \quad \text{iff} \quad a_{1,j} = a_{1,j'};$$
$$f(a_{0,j}) = f(a_{0,j'}) \quad \text{iff} \quad f(a_{1,j}) = f(a_{1,j'});$$
$$\text{for all } j < l: \quad a_{0,j} \text{ is an } X\text{-object iff } a_{1,j} \text{ is an } X\text{-object};$$
$$a_{0,j} \text{ is a } Y\text{-object} \quad \text{iff} \quad a_{1,j} \text{ is a } Y\text{-object};$$
$$f(a_{0,j}) \text{ matches } f(a_{1,j});$$
$$\text{for all } j < p: \quad n_{0,j} \text{ matches } n_{1,j}.$$

Then for any formula ψ of $\mathscr{L}^{0,2}(\underline{\text{exactly}})$ with free variables among $v_0, \ldots, v_{l-1} \in Var(0)$, $\mu_0, \ldots, \mu_{p-1} \in Var(2)$: if $\langle \vec{a}_0, \vec{n}_0 \rangle$ matches $\langle \vec{a}_1, \vec{n}_1 \rangle$, then:

$$\mathscr{A}_0 \models_\kappa \psi[\vec{a}_0, \vec{n}_0] \quad \text{iff} \quad \mathscr{A}_1 \models_\kappa \psi[\vec{a}_1, \vec{n}_1].$$

This is easy to show. So for any $\psi \in \mathrm{Sent}(\mathscr{L}^{0,2}(\underline{\text{exactly}}))$, $\mathscr{A}_0 \models_\kappa \psi$ iff $\mathscr{A}_1 \models_\kappa \psi$, proving the observation.

4.6. We will now slightly improve the last remarks of §2.1.

Observation. *For* $1 \leq k < \omega$ *and* κ *an aleph, if either* $\mathrm{ncb}^k(\kappa) < \aleph_{\omega^\omega}$ *or* $\mathrm{ncb}^k(\kappa)$ *is a limit cardinal, then*:

(i) $\mathscr{L}^{0,2k+2}(\underline{\text{exactly}}, =) \overset{\kappa}{\preceq} \mathscr{L}^{0,2k*}(\underline{\text{exactly}}, =)$,

(ii) $\mathscr{L}^{0,2k+2}(\underline{\text{exactly}}) \overset{\kappa}{\preceq} \mathscr{L}^{0,2k*}(\underline{\text{exactly}})$.

To prove this, we will introduce another satisfaction relation. For a model $\mathscr{A}$ and $\kappa \in \mathrm{Card}$, we define $\mathscr{A} \models_\kappa^{2k} \varphi$ so that variables of type $\geq 2k+2$ range over $\overline{ncb^{k-1}(\kappa)}$ rather than over $\overline{ncb^k(\kappa)}$. That is, let $\mathrm{Sent}^{2k}(\mathscr{L}^{0,2k+2*}_{\mathscr{A},\kappa}(\underline{\text{exactly}}, =))$ be the set of sentences formed from formulae of $\mathscr{L}^{0,2k+2*}_{\mathscr{A},\kappa}(\underline{\text{exactly}}, =)$ by replacing variables of type-$2j$ by terms of the form $\mathbf{n}$, where:

$$\text{if } 1 \leq j \leq k, \text{ then } \quad n < \mathrm{ncb}^{j-1}(\kappa);$$
$$\text{if } j = k+1, \text{ then } \quad n < \mathrm{ncb}^{k-1}(\kappa).$$

For $\varphi \in \mathrm{Sent}^{2k}(\mathscr{L}^{0,2k+2*}_{\mathscr{A},\kappa}(\underline{\text{exactly}}, =))$ define $\mathscr{A} \vDash^{2k}_{\kappa} \varphi$ as in §1.1 except that for $\mu \in Var(2k+2)$:

$$\mathscr{A} \vDash^{2k}_{\kappa} (\exists \mu)\psi \quad \text{iff} \quad \text{for some } n < \mathrm{ncb}^{k-1}(\kappa) \quad \mathscr{A} \vDash^{2k}_{\kappa} \psi(\mu/\mathbf{n});$$
$$\mathscr{A} \vDash^{2k}_{\kappa} (\underline{\text{exactly}}\, \mathbf{m}\, \mu)\psi \quad \text{iff} \quad \mathrm{card}(\{n < \mathrm{ncb}^{k-1}(\kappa) \colon \mathscr{A} \vDash^{2k}_{\kappa} \psi(\mu/\mathbf{n})\}) = m.$$

(The reader might wonder why this paper investigates $\vDash_{\kappa}$ rather than $\vDash^{2}_{\kappa}$. The remarks of §1.2 only apply to the latter satisfaction relation if $\mathrm{ncb}(\kappa) = \kappa$ (when the relations coincide); also Theorem 2.7 fails for the latter relation.)

Given κ and $\varphi \in \mathrm{Sent}(\mathscr{L}^{0,2k+2*}(\underline{\text{exactly}}, =))$, we will construct $\varphi' \in \mathrm{Sent}(\mathscr{L}^{0,2k+2*}(\underline{\text{exactly}}, =))$ so that for any model $\mathscr{A}$: $\mathscr{A} \vDash_{\kappa} \varphi$ iff $\mathscr{A} \vDash^{2k}_{\kappa} \varphi'$. Form $\varphi'' \in \mathrm{Sent}(\mathscr{L}^{0,2k*}(\underline{\text{exactly}}, =))$ from φ' by replacing all variables of type $2k+2$ by new variables of type $2k$; for any model $\mathscr{A}$, $\mathscr{A} \vDash^{2k}_{\kappa} \varphi'$ iff $\mathscr{A} \vDash^{2k}_{\kappa} \varphi''$; but clearly $\mathscr{A} \vDash^{2k}_{\kappa} \varphi''$ iff $\mathscr{A} \vDash_{\kappa} \varphi''$; so φ'' is as required by (i). If $\varphi \in \mathrm{Sent}(\mathscr{L}^{0,2k+2*}(\underline{\text{exactly}}))$, we will make sure that $\varphi' \in \mathrm{Sent}(\mathscr{L}^{0,2k+2*}(\underline{\text{exactly}}))$; φ'' will be as required by (ii).

Suppose that $k = 1$. Suppose $\varphi \in \mathrm{Fml}(\mathscr{L}^{0,4}(\underline{\text{exactly}}))$ with free variables among $v_0, \ldots, v_{l(0)-1} \in Var(0)$, $\rho_0, \ldots, \rho_{l(2)-1} \in Var(2)$, $\mu_0, \ldots, \mu_{l(4)-1} \in Var(4)$, these to be called 'distinguished'. We will construct $\varphi' \in \mathrm{Fml}(\mathscr{L}^{0,4}(\underline{\text{exactly}}))$ with free variables among the distinguished ones, and so that for any model $\mathscr{A}$, $\vec{a} \in |\mathscr{A}|^{l(0)}$, $\vec{m} \in \bar{\kappa}^{l(2)}$, $\vec{n} \in \overline{\mathrm{ncb}(\kappa)}^{l(4)}$:

$$\mathscr{A} \vDash_{\kappa} \varphi[\vec{a}, \vec{m}, \vec{n}] \quad \text{iff} \quad \mathscr{A} \vDash^{2}_{\kappa} \varphi'[\vec{a}, \vec{m}, \vec{n}].$$

φ' is constructed by induction on the contraction of φ; the only case worth discussing is where φ is $(\exists \mu)\varphi_0$ for μ a non-distinguished type-4 variable. Suppose that φ'_0 has been constructed as desired.

We will transform the apparatus of §4.2 to use with $\vDash^{2}_{\kappa}$. For $i \in 2$ and a fixed $\rho' \in Var(2i)$ let

$$X_{2i} = \{\langle \theta, \rho \rangle \colon \text{for some } \eta \in Var(2i+2),\ (\underline{\text{exactly}}\, \eta\, \rho)\theta \text{ is a subformula of } \varphi'_0\} \cup \{\langle \bot, \rho' \rangle\};$$

$$N_{2i}(\mathscr{A}, \vec{a}, \vec{m}, \vec{n}) = \{n \colon \mathscr{A} \vDash^{2}_{\kappa} \vec{\exists}(\underline{\text{exactly}}\, \mathbf{n}\, \rho)\theta \text{ for some } \langle \theta, \rho \rangle \in X_{2i}\},$$

'$\vec{\exists}$' binding all non-distinguished variables in its scope. Where $\mathscr{A}$, $\vec{a}$, $\vec{m}$, and $\vec{n}$ are fixed, let $N_{2i} = N_{2i}(\mathscr{A}, \vec{a}, \vec{m}, \vec{n})$. Notice these facts. (1) If $n, n' \in \bar{\kappa} - N_2$ then:

$$\mathscr{A} \vDash^{2}_{\kappa} \varphi'_0[\mathscr{A}, \vec{a}, \vec{m}, \vec{n}, n] \quad \text{iff} \quad \mathscr{A} \vDash^{2}_{\kappa} \varphi'_0[\mathscr{A}, \vec{a}, \vec{m}, \vec{n}, n'].$$

(2) For any $n \in N_2$ either $n \leq \mathrm{card}(N_0)$ or $\mathrm{card}(\bar{\kappa} - N_0) \leq n$. For suppose that $\langle \theta, \rho \rangle \in X_2$ and $\mathscr{A} \vDash^{2}_{\kappa} (\underline{\text{exactly}}\, \mathbf{n}\, \rho)\theta[\vec{a}, \vec{a}^0, \vec{m}, \vec{m}^0, \vec{n}, \vec{n}^0]$, $\vec{a}^0$, $\vec{m}^0$ and $\vec{n}^0$ assigning values to the non-distinguished variables other than ρ free in θ; then for any m,

$m' \in \bar{\kappa} - N_0$:

$$\mathscr{A} \models^2_\kappa \theta[\vec{a}, \vec{a}^0, \vec{m}, \vec{m}^0, m, \vec{n}, \vec{n}^0] \quad \text{iff} \quad \mathscr{A} \models^2_\kappa \theta[\vec{a}, \vec{a}^0, \vec{m}, \vec{m}^0, m', \vec{n}, \vec{n}^0];$$

so either ${}^2\hat{\rho}\theta[\cdots]^{\mathscr{A}} \subseteq N_0$ or $\bar{\kappa} - N_0 \subseteq {}^2\hat{\rho}\theta[\cdots]^{\mathscr{A}}$. (Here ${}^2\hat{\rho}\theta[\cdots] = \{m < \kappa: \mathscr{A} \models^2_\kappa \theta[\ldots, m, \ldots]\}$.) (3) If $n \in N_2$, then $n \leq \text{ncb}(\kappa)$.

It is easy to construct a formula $\text{Def}_{2i}(\rho)$ for $\rho \in Var(2i+2)$ so that for any $\mathscr{A}, \vec{a}, \vec{m}, \vec{n}$ as above and $n < \kappa$:

$$\mathscr{A} \models^2_\kappa \text{Def}_{2i}(\mathbf{n})[\vec{a}, \vec{m}, \vec{n}] \quad \text{iff} \quad n \in N_{2i}.$$

Suppose we can construct a formula Φ of $\mathscr{L}^{0,4*}(\underline{\text{exactly}})$ so that for any $\mathscr{A}, \vec{a}, \vec{m}, \vec{n}$ as usual:

$$\mathscr{A} \models^2_\kappa \Phi[\vec{a}, \vec{m}, \vec{n}] \quad \text{iff} \quad \overline{\text{ncb}(\kappa)} - N_2 \text{ is non-empty.}$$

Then we may take φ' to be:

$$(\exists\mu)(\varphi_0' \,\&\, \neg\underline{\text{ncb} =}\, \mu \,\&\, (\text{Def}_2(\mu) \vee \Phi)).$$

Clearly if $\mathscr{A} \models_\kappa \varphi_0[\vec{a}, \vec{m}, \vec{n}, n]$ for $n < \text{ncb}(\kappa)$, $\mathscr{A} \models^2_\kappa \varphi'[\cdots]$. Suppose that

$$\mathscr{A} \models^2_\kappa (\varphi_0'(\mu/\mathbf{n}) \,\&\, \neg\underline{\text{ncb} =}\, \mathbf{n} \,\&\, (\text{Def}_2(\mathbf{n}) \vee \Phi))[\cdots].$$

If $n \in N_2$, then by fact (3) $n < \text{ncb}(\kappa)$, yielding $\mathscr{A} \models_\kappa \varphi[\cdots]$. Otherwise there is an $n' \in \overline{\text{ncb}(\kappa)} - N_2$; by fact (1) $\mathscr{A} \models^2_\kappa \varphi_0'(\mu/\mathbf{n}')[\cdots]$, again yielding $\mathscr{A} \models_\kappa \varphi[\cdots]$. So it suffices to construct Φ.

First we construct Φ_0 saying that $\bar{\aleph}_0 - N_2 \neq \{\ \}$. For $\langle\theta, \rho\rangle \in X_2$ and $\langle\theta', \nu'\rangle \in X_0$ form $\bar{\theta}, \bar{\rho}, \bar{\theta}', \bar{\nu}'$ as usual to avoid collisions of non-distinguished free variables; let $\beta_{\theta,\rho,\theta',\nu'}$ be:

$$\begin{aligned}&(\underline{\text{exactly}}\,\mu\,\bar{\rho})\bar{\theta} \,\&\, \neg\bar{\theta}(\bar{\rho}/\rho^*) \,\&\, (\underline{\text{exactly}}\,\rho^*\,\bar{\nu}')\bar{\theta}'\\ &\&\, (\underline{\text{exactly}}\,\eta^*\,\bar{\rho})(\bar{\theta} \vee (\underline{\text{exactly}}\,\rho\,\bar{\nu}')\bar{\theta}')\\ &\&\, \neg(\underline{\text{exactly}}\,\eta^*\,\bar{\rho})\bar{\theta} \,\&\, \neg\text{Def}_2(\eta^*),\end{aligned}$$

where $\eta, \eta^* \in Var(4)$, $\rho^* \in Var(2)$, all new. This will fix the values of η and η^* to be an n and $n+1$ with $n \in \bar{\aleph}_0 \cap N_2$ and $n+1 \notin N_2$. Let Φ_0 be:

$$\bigvee \{\vec{\exists}\beta_{\theta,\rho,\theta',\nu'}: \langle\theta, \rho\rangle \in X_2, \langle\theta', \nu'\rangle \in X_0\} \vee \neg(\underline{\underline{\infty}}\rho)\text{Def}_0(\rho),$$

where '$\vec{\exists}$' binds all non-distinguished variables in its scope. If N_0 is infinite, then $\bar{\aleph}_0 - N_2 \neq \{\ \}$ iff the first disjunct holds. The second disjunct says that N_0 is finite, in which case by fact (2) $\text{card}(N_0) + 1 \in \bar{\aleph}_0 - N_2$ and Φ_0 is satisfied.

Case 1: $\text{ncb}(\kappa) = \aleph_0$. Let Φ be Φ_0.

Case 2: $\text{ncb}(\kappa) = \aleph_1$. If N_2 is finite, $\bar{\aleph}_0 - N_2 \neq \{\ \}$;
otherwise $\text{card}(N_2) = \aleph_0 \in N_2$ iff $\bar{\aleph}_1 - N_2 \neq \{\ \}$. Let Φ be:

$$\Phi_0 \vee (\exists\mu)((\underline{\text{exactly}}\,\mu\,\eta)\text{Def}_2(\eta) \,\&\, \neg\text{Def}_2(\mu)).$$

This says "Either $\bar{\aleph}_0 - N_2 \neq \{\ \}$ or $\text{card}(N_2) \notin N_2$".

Case 3: $\text{ncb}(\kappa) = \aleph_{\delta+1}$ *for* $1 \leq \delta < \omega^\omega$. Suppose we can construct ψ_0, ψ_1, ψ_2 so that:

$$\mathscr{A} \models^2_\kappa \psi_0[\cdots] \quad \text{iff} \quad \mathrm{card}(N_0 - \bar{\aleph}_0) < \aleph_\delta;$$

$$\mathscr{A} \models^2_\kappa \psi_1[\cdots] \quad \text{iff} \quad \text{for some } n \notin N_2,\ n \leqslant \aleph_\delta \text{ and } n < \mathrm{card}(N_0 - \bar{\aleph}_0);$$

$$\mathscr{A} \models^2_\kappa \psi_2[\cdots] \quad \text{iff} \quad \mathrm{card}(N_0 - \bar{\aleph}) \notin N_2 \text{ and } \mathrm{card}(N_0 - \bar{\aleph}_0) \neq \aleph_{\delta+1}.$$

Thus:

$$\mathscr{A} \models^2_\kappa (\psi_1 \vee \psi_2)[\cdots] \quad \text{iff} \quad \text{for some } n \in N_2, \quad n \leqslant \aleph_\delta \quad \text{and} \quad n \leqslant \mathrm{card}(N_0 - \bar{\aleph}_0).$$

We may let $\psi_0 \vee \psi_1 \vee \psi_2$ be Φ. For, if $\mathscr{A} \not\models^2_\kappa \psi_0[\cdots]$, then $\mathrm{card}(N_0 - \bar{\aleph}_0) \geqslant \aleph_\delta$; and if $\mathscr{A} \not\models^2_\kappa (\psi_1 \vee \psi_2)[\cdots]$, then for any $n \leqslant \aleph_\delta$ $n \in N_2$; so $\bar{\aleph}_{\delta+1} - N_2 = \{\ \}$. Clearly, if $\mathscr{A} \models^2_\kappa (\psi_1 \vee \psi_2)[\cdots]$, then $\bar{\aleph}_{\delta+1} - N_2 \neq \{\ \}$. If $\mathscr{A} \models^2_\kappa \psi_0[\cdots]$, then $\mathrm{card}(N_0 - \bar{\aleph}_0) < \aleph_\delta$; since $\delta > 0$, $\mathrm{card}(N_0) < \aleph_\delta$; so $\aleph_\delta \notin N_2$, since otherwise $\aleph_{\delta+1} = \mathrm{card}(\bar{\kappa} - N_0) \leqslant \aleph_\delta$ by fact (2).

Since κ is assumed to be an aleph, it is convenient to identify cardinals with initial ordinals let $\langle \alpha_\xi \rangle_{\xi < \xi_0}$ be the listing of $N_0 - \bar{\aleph}_0$ in increasing order; clearly $\mathrm{card}(N_0 - \bar{\aleph}_0) \leqslant \xi_0$. Let:

$$M = \{\alpha_\xi : \xi < \mathrm{card}(N_0 - \bar{\aleph}_0)\}, \qquad \hat{M} = \{\mathrm{ncb}(\alpha) : \alpha \in M\}.$$

We will use these facts to construct ψ_0 and ψ_1:

$$\mathrm{card}(N_0 - \bar{\aleph}_0) < \aleph_\delta \quad \text{iff} \quad \text{order-type}(\hat{M}) = \bigcup \hat{M} < \omega + \delta;$$
$$\text{for some } n \notin N_2\text{:} \quad n \leqslant \aleph_\delta \quad \text{and} \quad n < \mathrm{card}(N_0 - \bar{\aleph}_0) \quad \text{iff}$$
$$\text{for some } \alpha \in M - N_2 \quad \mathrm{ncb}(\alpha) \leqslant \omega + \delta.$$

For non-distinguished $\rho, \rho' \in Var(2)$ and $\langle \theta, v \rangle$, $\langle \theta', v' \rangle \in X_0$, form $\bar{\theta}, \bar{v}, \bar{\theta}', \bar{v}'$ as usual to avoid collisions of non-distinguished free variables. Let $\rho \leqslant_{\theta, v, \theta', v'} \rho'$ abbreviate:

$$(\underline{\text{exactly}}\, \rho\ \bar{v})\bar{\theta}\ \&\ (\underline{\text{exactly}}\, \rho'\ \bar{v})\bar{\theta}'\ \&\ (\underline{\infty} \bar{v})\bar{\theta}\ \&\ (\underline{\infty} \bar{v}')\bar{\theta}'$$
$$\&\ (\underline{\text{exactly}}\, \rho'\ \bar{v}')(\bar{\theta}' \vee \bar{\theta}(\bar{v}/\bar{v}')).$$

Let $\rho \leqslant^* \rho'$ be:

$$\bigvee \{\vec{\exists}(\rho \leqslant_{\theta, v, \theta', v'} \rho') : \langle \theta, v \rangle, \langle \theta', v' \rangle \in X_0\},$$

where '$\vec{\exists}$' binds non-distinguished free variables other than ρ and ρ'. Then for any $m, m' < \kappa$:

$$\mathscr{A} \models^2_\kappa \mathbf{m} \leqslant^* \mathbf{m}'[\vec{a}, \vec{m}, \vec{n}] \quad \text{iff} \quad m, m' \in N_0 - \bar{\aleph}_0 \quad \text{and} \quad m \leqslant m'.$$

Let $\boldsymbol{M}(\rho)$ be:

$$\rho \leqslant^* \rho\ \&\ \neg(\exists \mu)((\underline{\text{exactly}}\, \mu\ \rho')(\rho' \leqslant^* \rho)\ \&\ (\underline{\text{exactly}}\, \mu\ \rho')(\rho' \leqslant^* \rho'));$$

Thus $\mathscr{A} \models^2_\kappa \boldsymbol{M}(\mathbf{m})[\cdots]$ iff $m \in M$. For $\mu_0, \mu_1 \in Var(4)$ let $\mu_0 \leqslant^{**} \mu_1$ abbreviate:

$$(\exists \rho_0)(\exists \rho_1)(\rho_0 \leqslant^* \rho_1\ \&\ \boldsymbol{M}(\rho_0)\ \&\ \boldsymbol{M}(\rho_1)$$
$$\&\ (\underline{\text{exactly}}\, \mu_0\ \rho)(\rho <^* \rho_0)\ \&\ (\underline{\text{exactly}}\, \mu_1\ \rho)(\rho <^* \rho_1)),$$

where $\rho <^* \rho_i$ is $\rho <^* \rho_i \,\&\, \neg(\rho_i \leqslant^* \rho)$. For $n, n' < \kappa$:

$$\mathscr{A} \models^2_\kappa \mathbf{n} \leqslant^* \mathbf{n}'[\cdots] \quad \text{iff} \quad n \leqslant n' \quad \text{and} \quad n, n' \in \hat{M}.$$

Using cardinality coefficients and the apparatus of §2.6 with '$\leqslant^{**}$' replacing '$\leqslant$' we can construct ψ_0 saying "the order-type of $\hat{M} < \omega + \delta$". Similarly we can construct ψ_1 saying "for some $\alpha \in M - N_2$, $\text{ncb}(\alpha) \leqslant \omega + \delta$". Details are left to the reader. Let ψ_2 be:

$$\neg(\exists\mu)((\underline{\text{exactly}}\ \mu\ \rho)(\rho \leqslant^* \rho) \,\&\, (\text{Def}_2(\mu) \vee \underline{\text{ncb}=}\ \mu)).$$

Case 4: ncb(κ) *is an uncountable limit cardinal.* If $\text{card}(N_0) \neq \text{ncb}(\kappa)$, then $\overline{\text{ncb}(\kappa)} - N_2$ is non-empty. For suppose that $\text{card}(N_0) \neq \text{ncb}(\kappa)$; by the case assumption and fact (3) fix an n with $\text{card}(N_0) < n < \text{ncb}(\kappa)$; by fact (2) if $n \in N_2$, then $\text{card}(\bar{\kappa} - N_0) \leqslant n$; but $\text{card}(\bar{\kappa} - N_0) = \text{ncb}(\kappa)$, a contradiction; so $n \notin N_2$. Let ψ_3 be:

$$\neg(\exists\mu)(\underline{\text{ncb}=}\ \mu \,\&\, (\underline{\text{exactly}}\ \mu\ \rho)\text{Def}_0(\rho)).$$

On the other-hand, if $\text{card}(N_0) = \text{ncb}(\kappa)$, then $\text{ncb}(\kappa) = \text{card}(N_0 - \bar{\aleph}_0)$. Let $\psi(\mu)$ be:

$$(\exists\rho)(\boldsymbol{M}(\rho) \,\&\, (\underline{\text{exactly}}\ \mu\ \rho')(\rho' <^* \rho));$$

then $\mathscr{A} \models^2_\kappa \psi(\mathbf{n})[\cdots]$ iff $n < \text{ncb}(\kappa)$. Let Φ be:

$$\psi_3 \vee (\exists\mu)(\psi(\mu) \,\&\, \neg\text{Def}_2(\mu)).$$

By the preceding remarks, this works.

It is easy to modify this construction to handle $\varphi \in \text{Sent}(\mathscr{L}^{0,4}(\underline{\text{exactly}}, =))$. For $k > 1$ simply replace types 2 and 4 by types $2k+2$ and $2k$.

For κ as above, part (i) of this Theorem with Theorem 4.2 yields the surprising inclusion:

$$\mathscr{L}^{0,4}(\underline{\text{exactly}}, =) \overset{\kappa}{\preccurlyeq} \mathscr{L}^{0,2*}(\underline{\text{exactly}}).$$

4.7. Here is another slight improvement on the concluding remarks of §2.1.

Observation. *For* $1 \leqslant k < \omega$, *if* $\text{ncb}^{k-1}(\kappa) < \aleph_\omega$, *then*:

$$\mathscr{L}^{0,2k+2*}(\underline{\text{exactly}}) \overset{\kappa}{\preccurlyeq} \mathscr{L}^{0,2k*}(\underline{\text{exactly}}).$$

Suppose $k = 1$. Let $\kappa = \aleph_z$ for $z < \omega$. If $z = 0$, then $\kappa = \text{ncb}(\kappa)$, and the above inclusion holds trivially. Suppose that $z \geqslant 1$. Let φ be a formula of $\mathscr{L}^{0,4*}(\underline{\text{exactly}})$ with free variables among $v_0, \ldots, v_{l(0)-1} \in Var(0)$, $\rho_0, \ldots, \rho_{l(2)-1} \in Var(2)$,

$\mu_0, \ldots, \mu_{l(4)-1} \in Var(4)$, the 'distinguished' variables. We will construct a formula φ' of $\mathscr{L}^{0,4^*}(\underline{\text{exactly}})$ meeting the conditions on φ' from §4.6; as there, this suffices to prove the observation. φ' is constructed by induction on the construction of φ. If φ is $(\exists\mu)\varphi_0$ for $\mu \in Var(4)$, φ' is constructed as in §4.6. Let φ be $(\underline{\text{exactly}}\,\mu_i\,\mu)\varphi_0$, for $i \leqslant l(4)$, $\mu \in Var(4)$. Suppose φ_0' has been constructed, and no distinguished variables occur bound in φ or φ_0'. Define X_2 and $N(\mathscr{A}, \vec{a}, \vec{m}, \vec{n})$ as in §4.6. Let $\mathrm{Def}_2(\mu)$ and $\mathrm{FinDef}_2(\mu)$ be the natural analogues of $\mathrm{Def}(\mu)$ and $\mathrm{FinDef}(\mu)$ from §4.2. For $\mathscr{A}$, $\vec{a}$, $\vec{m}$, $\vec{n}$, let $A = \{n < \kappa\colon \mathscr{A} \vDash^2_\kappa \varphi_0'[\vec{a}, \vec{m}, \vec{n}, n]\}$; we will pin μ_i to $\mathrm{card}(A \cap \bar{\aleph}_0)$. Let ψ be $(\forall\mu)(\varphi_0' \supset \mathrm{Def}_2(\mu))$ and ψ' be $(\exists\mu)(\underline{\mathrm{ncb}^1 =}\,\mu\ \&\ \varphi_0')$; ψ' says "$\aleph_0 \in A$". We will take φ' to be:

$$(\psi\ \&\ \varphi_1) \vee (\neg\psi\ \&\ \psi'\ \&\ \varphi_2) \vee (\neg\psi\ \&\ \neg\psi'\ \&\ \varphi_3).$$

To handle the case in which $A \subseteq N_2 \subseteq \bar{\aleph}_0$ let φ_1 be:

$$(\forall\mu)(\varphi_0' \supset \mathrm{Def}_2(\mu))\ \&\ (\underline{\text{exactly}}\,\mu_{i_0}\,\mu)(\varphi_0'\ \&\ \mathrm{FinDef}_2(\mu)).$$

As usual, if n, $n' \in \bar{\kappa} - N_2$: $n \in A$ iff $n' \in A$. So if $A - N_2$ is non-empty, then $\bar{\kappa} - N_2 \subseteq A$; since $N_2 \subseteq \bar{\aleph}_1$, $\aleph_1, \ldots, \aleph_{z-1} \in A$. For $j \in 2$, if $\mathrm{card}(A \cap \bar{\aleph}_0) \geqslant z - j$, then we want a formula $\varphi_{2+j,z-j}$ that looks for $D \subseteq A \cap \bar{\aleph}_0$ such that $\mathrm{card}(D) = z - j$ and 'pins' the value of μ_i to $\mathrm{card}(A - D)$. For each $u < z - j$ we construct $\varphi_{2+j,u}$ saying that $\mathrm{card}(A \cap \bar{\aleph}_0) = u$ and the value of μ_i is u. These constructions use easy ideas from §4.2; details are left to the reader. We let φ_{2+j} be $\bigvee \{\varphi_{2+j,u}\colon u \leqslant z - j\}$.

4.8. Here are some further questions, stated as conjectures in order of decreasing confidence.

Conjecture (F). For $0 < k < \omega$, if $\mathrm{ncb}^k(\kappa) \geqslant \aleph_\omega$, then:

$$\mathscr{L}^{0,2k+2}(\underline{\text{exactly}}, =) \not\leq_\kappa \mathscr{L}^{0,2k+2}(\underline{\text{exactly}}).$$

Conjecture (G). If $\kappa = \aleph_{\omega^\omega+1}$, then $\mathscr{L}^{0,4}(\underline{\text{exactly}}) \not\leq_\kappa \mathscr{L}^{0,2^*}(\underline{\text{exactly}})$.

The following sentence is a possible witness:

$$(\exists\mu)\neg(\exists v_1)(\underline{\text{exactly}}\,\mu\,\rho)(\exists v_0)(\mathbf{S}(v_1, v_0)\ \&\ (\underline{\text{exactly}}\,\rho\,v)\mathbf{R}(v_0, v)),$$

with $\mu \in Var(4)$, $\rho \in Var(2)$, and v, v_0, $v_1 \in Var(0)$.

Conjecture (H). For $\kappa = \aleph_\omega$, $\mathscr{L}^{0,6}(\underline{\text{exactly}}) \not\leq_\kappa \mathscr{L}^{0,2^*}(\underline{\text{exactly}})$.

The following sentence is a possible witness:

$$(\exists\eta)((\underline{\text{exactly}}\ \eta\ \mu)(\exists v_1)(\underline{\text{exactly}}\ \mu\ \rho)(\exists v_0)$$
$$(\mathbf{S}_0(v_1, v_0)\ \&\ (\underline{\text{exactly}}\ \rho\ v)\mathbf{R}_0(v_0, v))$$
$$\&\ (\underline{\text{exactly}}\ \eta\ \mu)(\exists v_1)(\underline{\text{exactly}}\ \mu\ \rho)(\exists v_0)$$
$$(\mathbf{S}_1(v_1, v_0)\ \&\ (\underline{\text{exactly}}\ \rho\ v)\mathbf{R}_1(v_0, v))),$$

where $\eta \in Var(6)$, $\mu \in Var(4)$, $\rho \in Var(2)$, v, v_0, $v_1 \in Var(0)$.

References

[1] D. Bostock, Logic and Arithmetic, Vol. I (Oxford University Press, 1974).
[2] H. Hodes, Logicism and the ontological commitments of arithmetic, J. Philosophy 81 (3) (1984) 123–149.
[3] H. Hodes, Cardinality Logics, part II: Definability, J. Symbolic Logic, to appear.
[4] H. Hodes, Where do the Natural Numbers come from?, Synthèse, to appear.
[5] A. Levy, Basic Set Theory (Springer, Berlin, 1979).