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FINITE LEVEL BOREL GAMES AND A PROBLEM CONCERNING THE JUMP HIERARCHY

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§1. Introduction. The jump hierarchy of Turing degrees assigns to each $\xi < (\aleph_1)^L$ the degree $\mathbf{0}^{(\xi)}$; we presuppose familiarity with its definition and with the basic terminology of [5]. Let λ be a limit ordinal, $\lambda < (\aleph_1)^L$. The central result of [5] concerns the relation between $\mathbf{0}^{(\lambda)}$ and exact pairs on $I_\lambda = \{\mathbf{0}^{(\xi)} \mid \xi < \lambda\}$. In [6] this question is raised: Where $\mathbf{a}$ is an upper bound on I_λ, how far apart are $\mathbf{a}$ and $\mathbf{0}^{(\lambda)}$? It is there shown that if λ is locally countable and admissible, they may be very far apart: $\mathbf{0}^{(\lambda)}$ = the least member of $\{\mathbf{a}^{(\mathrm{Ind}(\lambda))} \mid \mathbf{a}$ is an upper bound on $I_\lambda\}$; this is rather pathological, for Ind(λ) may be larger than λ. If λ is locally countable but neither admissible nor a limit of admissibles, we are essentially in the case of $\lambda < \omega_1^{CK}$; by results of Sacks [12] and Enderton and Putnam [2], $\mathbf{0}^{(\lambda)}$ = the least member of $\{\mathbf{a}^{(2)} \mid \mathbf{a}$ is an upper bound on $I_\lambda\}$. If λ is not locally countable, Ind(λ) is neither admissible nor a limit of admissibles, so we are again in a case like that of $\lambda < \omega_1^{CK}$. But what if λ is locally countable and nonadmissible, but is a limit of admissibles? For the rest of this paper let λ be such an ordinal. The central result of this paper answers this question for some such λ.

Let "Det(Σ_n^0, Y)" for a field of play Y be the statement: "Any two-player infinite game on Y is determined if the set of plays for which I wins is Σ_n^0 (relative to the Baire topology on $[Y]$)." (The definition of a field of play will be given in §2.) Let Det(Σ_n^0) = Det($\Sigma_n^0, \omega^{<\omega}$). The connection between our initial question and the determinacy of games was discussed in [4]; the following improves the results presented there.

THEOREM 1. (i) *If* $L_\lambda \models \neg\,\mathrm{Det}(\Sigma_3^0)$, *then* $\mathbf{0}^{(\lambda)}$ = *the least member of* $\{\mathbf{a}^{(3)} \mid \mathbf{a}$ *is an upper bound on* $I_\lambda\}$.

Recall that α is a local $\aleph_m$ iff $L_{\alpha+1} \models \alpha = \aleph_m$. Let λ be m-well-behaved iff there are $\beta, \gamma < \lambda$ so that for all α, if α is a local $\aleph_{m+1}$ and $\beta < \alpha < \lambda$ then $L_{\alpha+\gamma} \models \alpha \neq \aleph_{m+1}$.

(ii) *If* λ *is n-well-behaved,* $L_\lambda \models (\mathrm{Det}(\Sigma_{n+3}^0) \,\&\, \neg\,\mathrm{Det}(\Sigma_{n+4}^0))$, *then* $\mathbf{0}^{(\lambda)}$ = *the least member of* $\{\mathbf{a}^{(n+4)} \mid \mathbf{a}$ *is an upper bound on* $I_\lambda\}$.

CONJECTURE 1. The restriction to λ which are n-well-behaved may be eliminated from (ii).

Can the λ for which Theorem 1 answers our question be characterized in other terms? The following result goes some distance in that direction.

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THEOREM 2. (i) *If λ is not a limit of $\alpha < \lambda$ such that $L_\alpha \models \mathrm{Det}(\boldsymbol{\Sigma}_3^0)$, then $L_\lambda \not\models \mathrm{Det}(\boldsymbol{\Sigma}_3^0)$.*

(ii) *If λ is n-well-behaved and λ is not a limit of α such that $L_\alpha \models \mathrm{Det}(\boldsymbol{\Sigma}_{n+4}^0)$, then $L_\lambda \not\models \mathrm{Det}(\boldsymbol{\Sigma}_{n+4}^0)$.*

CONJECTURE 2. The restrictions to λ which are n-well-behaved can be eliminated from (ii).

On the positive side, we will show:

THEOREM 3. *If α is a local $\aleph_{n+1}$ (in fact if L_α is a model for the Σ_3^{n+1}-comprehension fragment of $(n + 2)$th order number theory and $L_\alpha \models \aleph_n$ exists), then $L_\alpha \models \mathrm{Det}(\boldsymbol{\Sigma}_{n+3}^0)$ (in fact if $n > 0$ and $L_\alpha \models \gamma = \aleph_1$, $L_\gamma \models \mathrm{Det}(\boldsymbol{\Sigma}_{n+3}^0)$).*

Applying Π_1^1 absoluteness twice, this yields the following.

THEOREM 4. *If λ is a limit of ordinals meeting the conditions on α in the antecedent of Theorem 3, then $L_\lambda \models \mathrm{Det}(\boldsymbol{\Sigma}_{n+3}^0)$.*

§2. $\boldsymbol{\Sigma}_3^0$ games in general. We begin with a careful look at $\boldsymbol{\Sigma}_3^0$ games on arbitrary fields of play. The key ideas (except for one small but important change) are implicit in Morton Davis' original proof of $\mathrm{Det}(\boldsymbol{\Sigma}_3^0)$.

A set Y with $p \in Y$ is a field of play starting at p iff Y is a set of finite sequences such that:

if $q \in Y$ and $p \subseteq r \subseteq q$, then $r \in Y$;
if $q \in Y$, then for some x, $q^\frown\langle x\rangle \in Y$;
length (p) is even.

Let $[Y] = \{f \mid f$ is a function on ω, $f \restriction n = p \restriction n$ for $n \le$ length (p), and for all $n \in \omega$, $f \restriction (n + 1) = (f \restriction n)^\frown\langle x\rangle \in Y$ for some $x\}$. Thus $[Y]$ is the set of plays on field Y. Where $B \subseteq [Y]$ and $Z \subseteq Y$ is a field of play starting at $p' \in Y$, $G(B, Z)$ is the two-player infinite game of perfect information played from p' as follows: I selects an x_0 so that $p'^\frown\langle x_0\rangle \in Z$; II selects an x_1 so that $p'^\frown\langle x_0, x_1\rangle \in Z$; etc.; where f is the play produced, I wins iff $f \in B$.

If Z does not start at $q \in Z$, by "$G(B, Z)$ from q" we mean the game $G(B, Z^{\supseteq q})$ where $Z^{\supseteq q} = \{r \in Z \mid q \subseteq r\}$. Z is a II-imposed subgame of Y iff Z is a field of play starting with p and for any $q \in Z$, if length(q) is even and $q^\frown\langle x\rangle \in Y$, then $q^\frown\langle x\rangle \in Z$; similarly, for "$Z$ is a I-imposed subgame of Y," except with "odd" replacing "even."

In the Baire topology on $[Y]$, a closed set is one of the forms $[S]$ where S is a tree in Y, i.e. $S \subseteq Y$, S is a field of play starting with p. Where S is a function carrying $(i, j) \in \omega^2$ to a tree $S_i(j)$, a set $B = \bigcap_{i\in\omega}\bigcup_{j\in\omega}[S_i(j)]$ is a $\boldsymbol{\Pi}_3^0$ set. We fix a $\boldsymbol{\Sigma}_3^0$ game $G = G([Y] - B, Y)$ for the next two sections. We suppose that Y starts at the empty sequence $\langle\ \rangle$. We will provide an inductive analysis of $\{p \in Y \mid \neg$ I has a winning strategy for G from $p\}$.

Suppose $Z \subseteq Y$, Z a field of play, $p \in Z$. For $i \in \omega$, $X \subseteq Y$, let $H_i(Z, X, p)$ be the game which is played as follows. First, *player II* selects $j \in \omega$; play continues in $Z^{\supseteq p}$. I picks an x so that $p^\frown\langle x\rangle \in Z$, etc. The play $\langle j\rangle^\frown f$, $f \in [Z^{\supseteq p}]$, is a win for I iff $f \notin [S_i(j) \cup X] \cap B$. $H_i(Z, X, p)$ is a Σ_3^0 game (on an appropriate field of play).

Let $\Phi_{i,Z}(X) = \{p \in Z \mid \neg$ I has a winning strategy in $H_i(Z, X, p)\}$. It is not hard to see that $\Phi_{i,Z}$ is a monotone (in fact positive) Π_2^1 inductive operator on $\mathscr{P}(Y)$, where the second-order quantifiers range (roughly) over $\mathscr{P}(Y)$.

The following fact is hidden in [1].

FUNDAMENTAL TECHNICAL LEMMA. *The following are equivalent.*

(1) $p \in \Phi^{\infty}_{i,Z}$.

(2) *There is a* II-*imposed subgame of* $Z^{\supseteq p}$, $(Z, i, p)^*$, *so that*

(a) $[(Z, i, p)^*] \subseteq \bigcup_{j \in \omega} [S_i(j)]$, *and*

(b) I *does not have a winning strategy in* $G([Y] - B, (Z, i, p)^*)$.

(3) $p \in Z$ *and* I *does not have a winning strategy in* $G([Y] - B, Z)$ *from* p.

LEMMA 1. *For* $p \in Y$, $p \in \Phi_{i,Z}(X)$ *iff* $p \in Z$ *and for some* $j \in \omega$:

(4) II *has a winning strategy in* $G([Y] - [S_i(j) \cup X], Z)$ *from* p; *and*

(5) *where* U *is the* II-*imposed subgame of* $Z^{\supseteq p}$ *produced by* II's *aforementioned strategy,* I *has no winning strategy in* $G([Y] - B, U)$.

PROOF. ($\Leftarrow$) For $p \in Z$, suppose $j \in \omega$ satisfies (4), but $p \notin \Phi_{i,Z}(X)$. Let s be I's winning strategy in $H_i(Z, X, p)$. Let II start a play of $H_i(Z, X, p)$ by choosing j; let I follow s. After her initial move, let II impose U. Where f is the play produced in Z, since $\langle j \rangle ^\frown f$ is a win for I in $H_i(Z, X, p)$ and $[U] \subseteq [S_i(j) \cup X]$, $f \notin B$; thus I has a winning strategy for $G([Y] - B, U)$ from p, contrary to (5).

($\Rightarrow$) Suppose no j satisfies (4) and (5) and $p \in Z$. We describe a winning strategy for I in $H_i(Z, X, p)$. Let II start a play of $H_i(Z, X, p)$ with j. If (4) fails for the chosen j, let I play to win $G([Y] - [S_i(j) \cup X], Z)$ from p; that game is open, so I may do this. Then I wins $H_i(Z, X, p)$. If (4) holds for j, then U exists and (5) fails. Let s be I's winning strategy for $G([Y] - B, U)$. As long as II stays inside U let I follow s; if II never leaves U, I wins $H_i(Z, X, p)$; if II leaves U at position q, I has a winning strategy for $G([Y] - [S_i(j) \cup X], Z)$ from q, since U was designed to keep the play in a closed set; let I then play to win that game, thereby also winning $H_i(Z, X, p)$. Thus $p \notin \Phi_{i,Z}(X)$. QED.

PROOF OF THE FUNDAMENTAL TECHNICAL LEMMA. (1) $\Rightarrow$ (2). Suppose $p \in \Phi^{\infty}_{i,Z}$. We describe how II imposes $(Z, i, p)^*$. Let $p_0 = p$, $|p_0|_\Phi = \xi_0$ for $\Phi = \Phi_{i,Z}$. Since $p \in \Phi(\Phi^{<\xi_0})$, by Lemma 1 there are a $j_0 \in \omega$ and a U_0, so that U_0 is a II-imposed subgame on $Z^{\supseteq p}$, $[U_0] \subseteq [S_i(j_0) \cup \Phi^{<\xi_0}]$ and I has no winning strategy in $G([Y] - B, U_0)$ from p_0. Let II keep the play in U_0 until the end of time or until a $p_1 \notin S_i(j_0)$ is reached. In the latter case, $p_1 \in \Phi^{<\xi_0}$; let $|p_1|_\Phi = \xi_1 < \xi_0$; since $p_1 \in \Phi(\Phi^{<\xi_1})$ we may fix j_1 and U_1, and iterate. Eventually we reach a final j_n and U_n and the play ends up in $S_i(j_n)$. At no position does I get a winning strategy in $G([Y] - B, U_k)$ for $k \leq n$. The resulting II-imposed game, hereafter denoted $(Z, i, p)^*$, is clearly as desired.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Suppose $p \in Z$, $p \notin \Phi^\infty$. We now show how I can win $G([Y] - B, Z)$ from p. Let s_0 be I's winning strategy in $H_i(Z, \Phi^\infty, p_0)$ for $p_0 = p$. I pretends that he is playing $H_i(Z, \Phi^\infty, p_0)$ and that II started that play with $j = 0$; I follows s_0. Either the play in Z produced is not in B or else a $p_1 \notin S_i(0) \cup \Phi^\infty$ is reached. In the latter case, since $p_1 \notin \Phi^\infty$, I has a winning strategy s_1 for $H_i(Z, \Phi^\infty, p_1)$; now I drops the previous pretense and instead pretends to be playing $H_i(Z, \Phi^\infty, p_1)$, and that II started that play with $j = 1$. I now follows s_1. And so on. If a final p_j is reached, the play in Z produced is not in B. Otherwise for all $j \in \omega$ a p_j is reached, $p_j \notin S_i(j)$; thus the play does not belong to $\bigcup_{j<\omega} [S_i(j)] \supseteq B$; so I wins $G([Y] - B, Z)$ from p_0. QED.

Supposed I has no winning strategy in $G([Y] - B, Y)$ from $\langle\ \rangle = p_0$. We describe a strategy for II. It will be important that this construction, unlike Davis' construction in [1], does not assume that $[Y]$ is compact. Let $Z(p_0) = (Y, 0, p_0)^*$, using the Fundamental Technical Lemma, (3) $\Rightarrow$ (2). I moves and II responds by selecting $p_1 \in Z(p_0)$; since $Z(p_0)$ is II-imposed this is possible. I has no winning strategy for $G([Y] - B, Z(p_0))$. Let $Z(p_1) = (Z(p_0), 1, p_1)^*$, using (3) $\Rightarrow$ (2) again. Continue in this manner. Since $[Z(p_i)] \subseteq \bigcup_{j<\omega}[S_i(j)]$, II wins $G([Y] - B, Y)$. Notice that the II-imposed subgame corresponding to this strategy is $\{p \in Y \mid Z(p)$ is defined$\}$.

Observation 1. $\mathrm{Det}(\Sigma^0_{n+3})$ is a theorem of $(n+2)$th order number theory, in fact of the Σ^{n+2}_3-comprehension fragment of $(n+2)$th order number theory.

PROOF. Suppose S is a function on ω^{n+2} whose values are trees in $Y_0 = \omega^{<\omega}$. Let

$$B = \begin{cases} \bigcap_i \bigcup_j \bigcap_{i_1} \cdots \bigcup_{i_n} [S(i, j, i_1, \ldots, i_n)] & \text{if } n \text{ is even;} \\ \bigcap_i \bigcup_j \bigcap_{i_1} \cdots \bigcap_{i_n} ([Y] - [S(i, j, i_1, \ldots, i_n)]) & \text{otherwise.} \end{cases}$$

Let $G^0 = G([Y_0] - B, Y_0)$; G^0 is a typical Σ^0_{n+3} game. For $i < n$ let G^{i+1} be the result of applying Martin's *-operation from [9] to G^i; G^{i+1} is a game on $Y_{i+1} = Y^*_i$, which may be viewed as a subfied of play of $\mathscr{P}^{i+1}(\omega)^{<\omega}$. Thus G^n is a Σ^0_3 game on $Y = Y_n$. $[Y]$ is not compact; hence the need to revise the Davis proof.

In $(n+2)$th order number theory, we can formalize the previous proof that G^n is determined. In fact, Σ^{n+1}_3-comprehension suffices to prove the existence of the fixed points for the Π^{n+1}_2 monotonic operators involved in that proof. Suppose s^n is a winning strategy for G^n; in [9] Martin describes a procedure which converts a strategy s^{i+1} for G^{i+1} into a strategy s^i for G^i. This procedure can be described and shown to work in $(n+2)$th order number theory. Thus in $(n+2)$th order number theory we can show that s^0, a winning strategy for G, exists.

If α is a local $\aleph_{n+1}$, then L_α is a model of $(n+2)$th order number theory; thus $L_\alpha \models \mathrm{Det}(\Sigma^0_{n+3})$. Theorems 3 and 4 follow immediately.

We will now relativize the previous discussion to models of $V = L$. Let T be the set consisting of these sentences:

Extensionality, Pairing, Union, Infinity, Foundations,
$(\forall\xi)(\exists x)(x = L_\xi)$, $\quad V = L$.

Let $\mathscr{M} = (M, \varepsilon^{\mathscr{M}})$ be an arbitrary ω-model of T; $M_a = \{x \in M \mid \mathscr{M} \models x \in L_a\}$ for $a \in \mathrm{On}(\mathscr{M})$; $\mathscr{M}_a = (M_a, \varepsilon^{\mathscr{M}} \upharpoonright M_a)$; $\alpha = o(\mathscr{M})$ = the least ordinal not represented in $\mathscr{M}$. Suppose $Y \in \mathscr{M}$, $\mathscr{M} \models Y$ is a field of play starting at $\langle\ \rangle$.

LEMMA 2. *If $\mathscr{M} \models (\forall\xi)(\exists\eta > \xi)(\eta$ is admissible), then there is a Π_1 formula defining $p \in \Phi_{i,Z}(X)$ over $\mathscr{M}$, where "Z" and "X" are regarded as first order variables.*

PROOF. Recall that $p \in \Phi_{i,Z}(X)$ is defined by the following Π_2 formula:

$(\forall s)$(if s is a strategy for I in the field of play for
$H_i(Z, X, p)$ then $(\exists f)(f$ is a play of $H_i(Z, X, p)$
in which I follows s and which II wins)).

Fix $s \in M$, $\mathscr{M} \models s$ is a strategy for I in the field of play for $H_i(Z, X, p)$. What follows the "$(\exists f)$" above may be rewritten in this form:

$$(\exists f)(\exists g)(f \text{ and } g \text{ are functions on } \omega \text{ and } (\forall n \in \omega)\psi(s, Y, Z, i, p, f \upharpoonright n, g \upharpoonright n, n)),$$

where ψ is Σ_0. The formula is equivalent to the statement that a certain tree $\hat{T}$ has an infinite branch, where $\hat{T}$ depends in a Σ_0 way on the parameters s, Y, Z, i, p. Where $\hat{T} \in M_a$, if $\mathscr{M} \models (a < b$ and b is admissible and $\hat{T}$ has an infinite branch), then $\mathscr{M} \models (\hat{T}$ has an infinite branch in $L_{b+1})$, by a relativized version of the Kleene basis theorem. Thus our original formula holds in $\mathscr{M}$ iff the following does:

$$(\forall s)(\text{if } s \text{ is a strategy for I in the field of play for } H_i(Z, X, p) \text{ then } (\forall \xi)(\text{if } \hat{T}(s, Y, Z, i, p) \in L_\xi \text{ and } \xi \text{ is admissible, then } (\exists f)(f \in L_{\xi+1} \text{ and } f \in [\hat{T}(s, Y, Z, i, p)]))).$$

The latter is clearly equivalent to a Π_1 formula. QED.

For the rest of this section, we will assume that $\mathscr{M} \models (\forall \xi)(\exists \eta > \xi)(\eta$ is admissible). For $a \in \mathrm{On}(\mathscr{M})$, a is $\mathscr{M}$-stable iff $\mathscr{M}_a \prec_1 \mathscr{M}$. Where $\mathscr{M} = (L_\alpha, \varepsilon \restriction L_\alpha)$, $\mathscr{M}$-stability coincides with α-stability. Let "$\boldsymbol{\Sigma}_n$ Projectibility" be the sentence: "There is a $\boldsymbol{\Sigma}_n$ function projecting the ordinals one-one into a set"; "Projectibility" is "$\boldsymbol{\Sigma}_1$ Projectibility." Clearly α is $\boldsymbol{\Sigma}_n$-projectible (i.e. $\rho^n_a < \alpha$) iff $L_\alpha \models \boldsymbol{\Sigma}_n$ Projectibility. The familiar Skolem argument, showing that α is not projectible iff α is a limit of α-stables, generalizes to $\mathscr{M}$: $\mathscr{M} \models (\neg$ Projectibility) iff the $\mathscr{M}$-stables are unbounded in $\mathrm{On}(\mathscr{M})$ under $<^{\mathscr{M}}$; $\mathscr{M} \models (a$ is not projectible) iff the $\mathscr{M}$-stables $<^{\mathscr{M}}$-below a are unbounded under $<^{\mathscr{M}}$. Let $\{a_b\}_{b \in B}$ for $B \subseteq \mathrm{On}(\mathscr{M})$ be the increasing enumeration of the $\mathscr{M}$-stables (under $<^{\mathscr{M}}$). This listing is continuous under $<^{\mathscr{M}}$.

Suppose $c, d \in M_{a_b}$ and $\mathscr{M} \models a_b + 1 = a'$. Where ψ is a Π_1 formula, there is an $e \in M$ such that $\mathscr{M} \models (e = \{x \in c \mid \psi(x, c, d)\}$ and $e \in L_{a'})$; this is because $\mathscr{M}_{a_b} \prec_1 \mathscr{M}$.

Suppose $S \in M, \mathscr{M} \models (S$ is a function on ω^2 such that for all $i, j \in \omega$, $S_i(j)$ is a tree in Y). In this case we will say that the game $G = G([Y] - B, Y)$, for $B = \bigcap_i \bigcup_j [S_i(j)]$, is defined in $\mathscr{M}$. Fix i, $Z \in M$, $\mathscr{M} \models (i \in \omega$ and Z is a subfield of play of Y). Let $\Phi = \Phi_{i,Z}$. For $a \in \mathrm{On}(\mathscr{M})$ fix $\theta^{<a} = \theta$ such that

$$\mathscr{M} \models \theta \text{ is a function on } a \text{ and } (\forall \xi < a)(\theta(\xi) = \Phi(\textstyle\bigcup \mathrm{Range}(\theta^{<\xi}))),$$

provided that for all $b <^{\mathscr{M}} a, \theta^{<b}$ is defined; let $\theta^a = \theta^{<a'}$ where $\mathscr{M} \models a' = a + 1$; let "$p \in \Phi^a$" abbreviate "$p \in \theta^a(a)$", "$p \in \Phi^{<a}$" abbreviate "$p \in \bigcup \mathrm{Range}\, \theta^{<a}$", "$|p|_\Phi = a$" abbreviate "$p \in \Phi^a - \Phi^{<a}$", and "$\Phi^\infty$ exists" abbreviate "$(\exists \xi)(\bigcup \mathrm{Range}(\theta^{<\xi}) = \Phi(\bigcup \mathrm{Range}(\theta^{<\xi})))$." Suppose that Y, $Z \in M_{a_c}$. For $b' = c +^{\mathscr{M}} b \in B, \mathscr{M} \models \theta^{<b}$ is definable over $L_{a_{b'}}$; so $\theta^{<b} \in M_{a'}$ where $\mathscr{M} \models a' = a_{b'} + 1$; this is proved by induction on b within $\mathscr{M}$, using Lemma 2. We are now ready to consider the Fundamental Technical Lemma within $\mathscr{M}$.

LEMMA 3. *If $\mathscr{M} \models p \in \Phi^a$, then $\mathscr{M}$ satisfies proposition* (2) *of the Fundamental Technical Lemma.*

PROOF. Within $\mathscr{M}$ we carry out the construction of $(Z, i, p)^*$ using $\theta^{<a} \in M$; notice that $(Z, i, p)^*$ is actually a member of M, since it is Δ_1 in $\theta^{<a}$ and relevant parameters; it clearly meets conditions (2a) and (2b). QED.

LEMMA 4. *Suppose that $\mathscr{M} \models (\Phi^\infty$ exists) and $Y, Z, \Phi^\infty \in L_{a_b}$. If $\mathscr{M} \models (p \notin \Phi^\infty)$, then $\mathscr{M} \models ($I has a winning strategy in G from p which belongs to $L_{a'})$, where $\mathscr{M} \models a_b + 1 = a'$.*

PROOF. Carry out the construction used in proving the Fundamental Technical Lemma, (3) $\Rightarrow$ (1), within $\mathscr{M}$. Notice that all of I's subsidiary strategies, the s_j's of that proof, belong to M_{a_b}; thus definably over M_{a_b} we may assemble them into a strategy for I in G from p. QED.

Suppose $\mathscr{M} \models \neg$ I has a winning strategy in G from p. We now construct a system of notation within $\mathscr{M}$. Let $Y \in M_{a_c}$, where c is the $<^{\mathscr{M}}$-least such ordinal. We will define a partial two-place function $g_{\mathscr{M}} = g$. Let $g(\langle\ \rangle^{\mathscr{M}}, p) = a$ iff $\mathscr{M} \models a = c + |p|_\Phi$, where $\Phi = \Phi_{0,Y}$. If $g(\langle\ \rangle^{\mathscr{M}}, \langle\ \rangle^{\mathscr{M}})$ is defined, $\mathscr{M} \models (\langle\ \rangle \in \Phi^b)$ for some $b \in \mathrm{On}(\mathscr{M})$; by Lemma 3 we may fix $Z(\langle\ \rangle^{\mathscr{M}})$ by $\mathscr{M} \models (Y, 0, \langle\ \rangle)^* = Z(\langle\ \rangle^{\mathscr{M}})$. Now suppose that $g(q,q)$ and $Z(q)$ are defined, $\mathscr{M} \models (q \in Y$ and length$(q) = 2i)$ and $\mathscr{M} \models \neg$ I has a winning strategy in $G([Y] - B, Z(q))$ from q. Suppose $\mathscr{M} \models q' = q^\frown\langle x, y\rangle \in Z(q)$. Let $g(q', p) = a$ iff $\mathscr{M} \models a = g(q,q) + |p|_\Phi$, where $\Phi = \Phi_{i+1,Z(q)}$. If $g(q', q')$ is defined, for some b, $\mathscr{M} \models q' \in \Phi^b$; fix $Z(q')$ by $\mathscr{M} \models (Z(q), i+1, q')^* = Z(q')$. Thus $\mathscr{M} \models \neg$ I has a winning strategy for $G([Y] - B, Z(q'))$ from q'; so the induction hypothesis is preserved.

LEMMA 5. *Suppose that $b \in B$ and for all q, p: if $g(q,p)$ is defined, then $g(q,p) <^{\mathscr{M}} b$. Then $\mathscr{M} \models$ II has a winning strategy in G.*

PROOF. For $\Phi = \Phi_{0,Y}$, $\mathscr{M} \models$ (if $|p|_\Phi$ exists, then $|p|_\Phi < b$); so $\mathscr{M} \models \Phi^\infty$ exists. Using Lemma 4, $\mathscr{M} \models \langle\,\rangle \in \Phi^\infty$; so $\mathscr{M} \models (|\langle\,\rangle|_\Phi = b')$, for some $b' \in \mathrm{On}(\mathscr{M})$. $\mathscr{M} \models c + b'$ exists; otherwise fix b'' so that $\mathscr{M} \models c + b'' = b$; since $b'' <^{\mathscr{M}} b'$, for some r, $\mathscr{M} \models |r|_\Phi = b''$; so $g(\langle\ \rangle^{\mathscr{M}}, r) = b$, a contradiction. Thus $g(\langle\ \rangle^{\mathscr{M}}, \langle\ \rangle^{\mathscr{M}})$ is defined, and so is $Z(\langle\ \rangle^{\mathscr{M}})$. In fact $Z(\langle\ \rangle^{\mathscr{M}})$ is Δ_1 in $\theta^{<b'}$, and so belongs to M_{a_b}. Now suppose that $Z(q)$ and $g(q,q)$ are defined, $Z(q) \in M_{a_b}$, $\mathscr{M} \models (q \in Y$ and length$(q) = 2i)$, $\mathscr{M} \models \neg$ I has a winning strategy in $G([Y] - B, Z(q))$ from q. Let $\mathscr{M} \models q' = q^\frown\langle x, y\rangle \in Z(q)$. For $\Phi = \Phi_{i+1,Z(q)}$, as above we have $\mathscr{M} \models \Phi^\infty$ exists. By Lemma 4, $\mathscr{M} \models q' \in \Phi^\infty$; thus there is a $b' \in \mathrm{On}(\mathscr{M})$ so that $\mathscr{M} \models |q'|_\Phi = b'$. As before, $\mathscr{M} \models c + b'$ exists; so $g(q', q')$ is defined, as is $Z(q')$; again $Z(q') \in M_{a_b}$. Since for all q so that $Z(q)$ is defined, $Z(q) \in M_{a_b}$, we can define $\{q \mid Z(q)$ is defined$\}$ over M_{a_b}; it is a winning strategy for II in G which belongs to M. QED.

We will use $g_{\mathscr{M}}$ later. For now we note the following fact.

LEMMA 6. *Suppose α is a limit of admissibles, $L_\alpha \models (\aleph_n$ exists and Y is a subfield of play of $\mathscr{P}^n(\omega)^{<\omega})$. If $L_\alpha \models \mathrm{Det}(\mathbf{\Sigma}^0_3, Y)$, then $L_\alpha \models \mathrm{Det}(\mathbf{\Sigma}^0_{n+3})$.*

PROOF. We use the notation of Observation 1, where G is defined in L_α, i.e. $S \in L_\alpha$. We define the sequence G^i and Y_i using the Martin *-operation within L_α, i.e. $L_\alpha \models Y_{i+1} = Y_i^*$, where Y_i is a subfield of play of $\mathscr{P}^i(\omega)^{<\omega} \cap L_\alpha$. Suppose for $s^n \in L_\alpha$, $L_\alpha \models s^n$ is a winning strategy for G^n. It suffices to note that s^i may be defined from s^{i+1} within L_α. If s^{i+1} is a winning strategy for I, this is straightforward. If s^{i+1} is a winning strategy for II and $s^{i+1} \in L_\beta$, where $L_\alpha \models \beta > \aleph_{i+1}$, then $s_i \in L_{\beta'}$ where $\beta' = \beta^+ + 1$. ($\beta^+ =$ the least admissible $< \beta$.) To see this, recall the closed games of the form G' from [9, p. 367]. The set of winning positions for I in G' belongs to L_β. By Theorem 7B.2 of [11], a winning strategy for G' belongs to $L_{\beta'}$. By finding such strategies and using them as detailed in [9], s^i is defined in L. Thus $L_\alpha \models s^i$ is a winning strategy; so s^0 is as required. QED.

§3. Computing infinite descending chains. Suppose $\mathscr{M}$ is a nonstandard ω-model of T, $\mathscr{M} = (M, \varepsilon^{\mathscr{M}})$, $M \subseteq \omega$; let $\alpha = o(\mathscr{M})$ and suppose that $L_\alpha \models \gamma = \aleph_m$ is the greatest cardinal, and $\mathscr{M} \models \gamma^{\mathscr{M}} = \aleph_m$ is the greatest cardinal, where $m \in \omega$. Let $\mathscr{E}$ be an arithmetic copy of $L_{\hat{\alpha}}$, i.e. $\mathscr{E} = (E, \varepsilon^{\mathscr{E}})$ for $E \subseteq \omega$, $\mathscr{E}$ isomorphic to $L_{\hat{\alpha}}$, for $\alpha \leq \hat{\alpha}$. We will investigate various cases of this question: how hard is it to compute an infinite descending $\varepsilon^{\mathscr{M}}$ chain given an oracle for $A = \mathrm{Sat}(\mathscr{M}) \oplus \mathrm{Sat}(\mathscr{E})$?

For $a \in \mathrm{On}(\mathscr{M})$ let $M_a = \{b \mid \mathscr{M} \models b \in L_a\}$, $\mathscr{M}_a = (M_a, \varepsilon^{\mathscr{M}} \restriction M_a)$; for $a \in \mathrm{On}(\mathscr{E})$ define $\mathscr{E}_a$ analogously. Let $M' = \bigcup\{M_a \mid a$ is a standard ordinal of $\mathscr{M}\}$, $\mathscr{M}' =$

$(M', \varepsilon^{\mathcal{M}} \upharpoonright M')$. Let E_α = the domain of $\mathcal{E}_a$ for $a = \alpha^{\mathcal{E}}$ if $\hat{\alpha} > \alpha$; let E_α be E if $\alpha = \hat{\alpha}$. Let $F: E_\alpha \to M$ be the unique isomorphic embedding of $(E_\alpha, \varepsilon^{\mathcal{E}} \upharpoonright E_\alpha)$ onto $\mathcal{M}'$. Let $F_i = F \upharpoonright \{x^{\mathcal{E}} \mid x \in \mathcal{P}^i(\omega)\}$ for $i \le m + 1$. Where confusion is unlikely, we will identify $\mathcal{E}$ and $L_{\hat{\alpha}}$.

LEMMA 7. *F_i is recursive in $A^{(i)}$.*

PROOF. For $i = 0$, this is clear. $F_{i+1}(a) = b$ iff $L_\alpha \models a \subseteq \mathcal{P}^i(\omega)$, $\mathcal{M} \models b \subseteq \mathcal{P}^i(\omega)$, and for all c such that $L_\alpha \models c \in \mathcal{P}^i(\omega)$: $L_\alpha \models c \in a$ iff $\mathcal{M} \models F_i(c) \in b$; so F_{i+1} is Π^0_1 in F_i, so by induction is recursive in $A^{(i+1)}$.

COROLLARY 1. *If $L_\alpha \models \mathrm{card}(\delta) = \aleph_i$, then $F \upharpoonright \delta$ is recursive in $A^{(i)}$ for $i \le m$.*

PROOF. Fix $W, g \in L_\alpha$, W a well-ordering of height δ, $\mathrm{Fld}(W) \subseteq \mathcal{P}^i(\omega) \cap L_\alpha$ and g the order-preserving map of $\mathrm{Fld}(W)$ onto δ. For $\xi < \delta$, $F(\xi) = b$ iff for some a, $L_\alpha \models g(a) = \xi$ and $\mathcal{M} \models g^{\mathcal{M}}(F_i(a)) = b$. So $F \upharpoonright \delta$ is recursive in $A \oplus F_i$, and thus in $A^{(i)}$. QED.

LEMMA 8. *$F \upharpoonright \alpha$ and F are recursive in $A^{(m+1)}$.*

PROOF. Using any reasonable way of coding constructible sets as ordinals, it suffices to prove this for $F \upharpoonright \alpha$. Let $\varphi(x, y)$ be the Σ_1 formula which defines the enumeration of $\mathcal{P}^{m+1}(\omega) \cap L$ in increasing order under $<_L$, i.e. $L \models \varphi(x, \xi)$ iff x is the ξth member of $\mathcal{P}^{m+1}(\omega)$ under $<_L$. This remains true within L_α. Let $x_b = a$ iff $\mathcal{M} \models \varphi(a, b)$ for $b \in \mathrm{On}(\mathcal{M})$, $y_\xi = a$ iff $L_\alpha \models \varphi(a, \xi)$ for $\xi < \alpha$. Then $F(\xi) = b$ if $F_{m+1}(y_\xi) = x_b$; so $F \upharpoonright \alpha$ is recursive in $A \oplus F_{m+1}$ and so in $A^{(m+1)}$. QED.

COROLLARY 2. *There is an infinite descending $\varepsilon^{\mathcal{M}}$-chain recursive in $A^{(m+2)}$.*

PROOF. By Lemma 7, $\mathrm{On}(\mathcal{M}) - \mathrm{On}(\mathcal{M}') = \mathrm{On}(\mathcal{M}) - F''\alpha$, which is co-r.e. in $A^{(m+1)}$, and so is recursive in $A^{(m+2)}$; an infinite descending $\varepsilon^{\mathcal{M}}$-chain may now be easily constructed. QED.

The rest of this section concerns improvements of Corollary 2. We recall the generalization of projectibility from Σ_1 to Σ_n: α is Σ_n-projectible iff there is an f mapping α one-one into some $\delta < \alpha$, where f is Σ_n over L_α.

LEMMA 9. *Suppose that α is Σ_{n+1}-projectible. If there is a nonstandard a such that $\mathcal{M}' \prec_n \mathcal{M}_a$, then there is an infinite descending $\varepsilon^{\mathcal{M}}$-chain recursive in $A^{(m)}$.*

PROOF. Let f be a Σ_{n+1} over L_α projection of α into γ, where $L_\alpha \models \aleph_m = \gamma$. Suppose f is defined over L_α by $(\exists z)\varphi(p, x, y, z)$, $p \in L_\alpha$, φ a Π_n formula. Suppose $\mathcal{M}' \prec_n \mathcal{M}$; otherwise replace $\mathcal{M}$ by an appropriate $\mathcal{M}_a$. Then $\mathcal{M} \models (\exists z)\varphi(p^{\mathcal{M}}, \xi^{\mathcal{M}}, f(\xi)^{\mathcal{M}}, z)$ for all $\xi < \alpha$.

Claim. For $b \in \mathrm{On}(\mathcal{M})$, b is nonstandard iff one of the following conditions obtains:

(1) $\mathcal{M} \models \neg(\exists \eta < \gamma^{\mathcal{M}})(\exists z)\varphi(p^{\mathcal{M}}, b, \eta, z)$;

(2) for some $\eta < \gamma$, $\eta \notin \mathrm{Range}(f)$ and $\mathcal{M} \models (\exists z)\varphi(p^{\mathcal{M}}, b, \eta^{\mathcal{M}}, a)$;

(3) $\mathcal{M} \models (\exists \xi)(\exists \xi')(\exists \eta < \gamma^{\mathcal{M}})(\exists z)(\exists z')(\xi \ne \xi' \mathbin{\&} \langle \xi', z' \rangle <_L \langle \xi, z \rangle \mathbin{\&} \langle \xi, z \rangle \in L_b \mathbin{\&} \varphi(p^{\mathcal{M}}, \xi, \eta, z) \mathbin{\&} \varphi(p^{\mathcal{M}}, \xi', \eta, z'))$.

Thus the set of nonstandard $b \in \mathrm{On}(\mathcal{M})$ is RE in $A \oplus (F \upharpoonright \gamma)$, which by Corollary 1, is recursive in $A^{(m)}$; this suffices to compute a descending $\varepsilon^{\mathcal{M}}$-chain as in the proof of Lemma 8. QED.

LEMMA 10. *Suppose that for all nonstandard $a \in \mathrm{On}(\mathcal{M})$, $\mathcal{M}' \not\prec_1 \mathcal{M}_a$. If α is not Σ_1-projectible and the order-type of the α-stable ordinals $= \beta < \alpha$, then there is an infinite descending $\varepsilon^{\mathcal{M}}$-chain recursive in $A^{(m)}$.*

PROOF. Let $\{\alpha_\xi\}_{\xi < B}$ be the increasing enumeration of the α-stable ordinals. Let $\{a_b\}_{b \in B}$ for $B \subseteq \mathrm{On}(\mathcal{M})$ be the $<^{\mathcal{M}}$-increasing enumeration of the $\mathcal{M}$-stable ordinals. We describe a procedure effective in $A^{(m)}$ for selecting a nonstandard $a \in \mathrm{On}(\mathscr{M})$; it is sufficiently independent of $\mathscr{M}$ to be repeatable with $\mathscr{M}_a$ in the place of $\mathscr{M}$; iterating this procedure, we will obtain our desired $\varepsilon^{\mathscr{M}}$-chain. Assume without loss of generality that $\mathscr{M}' \not\prec_1 \mathscr{M}$.

If B is nonempty, there is a b_0 which is the $<^{\mathscr{M}}$-maximal b such that a_b is standard; otherwise $\mathscr{M}' \prec_1 \mathscr{M}$. Clearly b_0 is standard; let $b_0 = (\xi_0)^{\mathscr{M}}$ and $a^* = a_{b_0}$. Where $\gamma^{\mathscr{M}} = a_{b_0}$, γ is α-stable: for suppose $e \in L_\gamma$, φ is Π_0 and $L_\alpha \models (\exists x)\varphi(x,e)$; then $\mathscr{M} \models (\exists x)\varphi(x,e^{\mathscr{M}})$ and $e^{\mathscr{M}} \in \mathscr{M}_{a*}$; thus $\mathscr{M}_{a*} \models (\exists x)\varphi(x,e^{\mathscr{M}})$ and so $L_\gamma \models (\exists x)\varphi(x,e)$. We may prove more along these lines: for $\delta < \gamma$, δ is α-stable iff $\delta^{\mathscr{M}}$ is $\mathscr{M}$-stable. Let $e \in L_\delta$ and let φ be Π_0. Suppose that δ is α-stable; if $\mathscr{M} \models (\exists x)\varphi(x,e^{\mathscr{M}})$, then $\mathscr{M}_{a*} \models (\exists x)\varphi(x,e^{\mathscr{M}})$; so $L_\gamma \models (\exists x)\varphi(x,e)$; so $L_\delta \models (\exists x)\varphi(x,e)$; so $\mathscr{M}_{\delta^{\mathscr{M}}} \models (\exists x)\varphi(x,e^{\mathscr{M}})$; thus $\delta^{\mathscr{M}}$ is $\mathscr{M}$-stable. Suppose that $\delta^{\mathscr{M}}$ is $\mathscr{M}$-stable; if $L_\alpha \models (\exists x)\varphi(x,e)$ then $\mathscr{M} \models (\exists x)\varphi(x,e^{\mathscr{M}})$; so $\mathscr{M}_{\delta^{\mathscr{M}}} \models (\exists x)\varphi(x,e^{\mathscr{M}})$; so $L_\delta \models (\exists x)\varphi(x,e)$, showing δ to be α-stable. This implies the following important fact: For $\xi \le \xi_0$ and $b = \xi^{\mathscr{M}}$, $a_b = (a_\xi)^{\mathscr{M}}$.

We now describe three search procedures; we will engage in Search 1 if B is nonempty, in Search 2 if B is nonempty and has a $<^{\mathscr{M}}$-maximum member, and in Search 3 if B is empty. All of these searches can be carried out effectively in $A \oplus (F \restriction \beta)$.

Search 1. Search for $\xi < \beta$ so that for $b = \xi^{\mathscr{M}} = F(\xi) \in B$, $a_b \ne (\alpha_\xi)^{\mathscr{M}}$. We try to determine whether $a_b \ne (\alpha_\xi)^{\mathscr{M}}$ as follows: search for $r \in L_\alpha$ and $s \in M$ so that

$$L_\alpha \models r \text{ is the } \alpha_\xi\text{th subset of } \mathscr{P}^m(\omega) \text{ under } <_L;$$
$$\mathscr{M} \models s \text{ is the } a_b\text{th subset of } \mathscr{P}^m(\omega) \text{ under } <_L;$$

then search for an $e \in L_\alpha \cap \mathscr{P}^m(\omega)$ so that $L_\alpha \models e \in r$ iff $\mathscr{M} \not\models e^{\mathscr{M}} \in s$, using F_m. Such an e exists, and so will be found, iff $a_b \ne (\alpha_\xi)^{\mathscr{M}}$. We output the first a_b found in this manner. Such an a_b is nonstandard; otherwise $b \le^{\mathscr{M}} b_0$, in which case $a_b = (\alpha_\xi)^{\mathscr{M}}$.

Search 2. Let c_0 be the $<^{\mathscr{M}}$-maximum member of B. Find η_0 so that $\eta_0^{\mathscr{M}} = c_0$. Then search for $\eta_1, \ldots, \eta_k < \eta_0$ and $c_1, \ldots, c_k \in B$, $a \in \mathrm{On}(\mathscr{M})$ and φ a Π_0 formula so that $(F \restriction \beta)(\eta_i) = c$; for $1 \le i \le k$, $a_{c_0} <^{\mathscr{M}} a$ and:

$$L_\alpha \not\models (\exists x)\varphi(x, \alpha_{\eta_0}, \alpha_{\eta_1}, \ldots, \alpha_{\eta_k}),$$
$$\mathscr{M} \models (\exists x \in L_a)\varphi(x, a_{c_0}, a_{c_1}, \ldots, a_{c_k}).$$

Output a.

Claim. If this search succeeds, a is nonstandard. If a_{c_0} is nonstandard, so is a; otherwise $c_0 = b_0$, $\eta_0 = \xi_0$; thus $a_{c_i} = (\alpha_{\eta_i})^{\mathscr{M}}$ for $i \le k$. If $\mathscr{M} \models \varphi(e, a_{c_0}, a_{c_i}, \ldots, a_{c_k})$ and $e \in \mathscr{M}'$, then $e = d^{\mathscr{M}}$ for $d \in L_\alpha$ and $L_\alpha \models \varphi(d, \alpha_{\eta_0}, \alpha_{\eta_1}, \ldots, \alpha_{\eta_k})$, contrary to what holds in L_α; thus for a witness $e \in \mathscr{M}_a$, a must be nonstandard.

If B has no $<^{\mathscr{M}}$-maximum member, then $b = b_0 +^{\mathscr{M}} 1 \in B$ and $\xi_0 + 1 < \beta$ meet the conditions of Search 1; so eventually Search 1 succeeds. If B has a $<^{\mathscr{M}}$-maximum member c_0 and $c_0 >^{\mathscr{M}} b_0$, Search 1 will succeed. Now suppose that $c_0 = b_0$. Also suppose that for all Π_0 formulae φ and all $\eta_1, \ldots, \eta_k < \eta_0 = \xi_0 < \beta$ and $c_i = (\eta_i)^{\mathscr{M}}$ for $i \le k$: if

$$\mathscr{M} \models (\exists x)\varphi(x, a_{c_0}, a_{c_1}, \ldots, a_{c_k}),$$

then

$$L_\delta \models (\exists x)\varphi(x, \alpha_{\eta_0}, \alpha_{\eta_1}, \ldots, \alpha_{\eta_k}),$$

for $\delta = \alpha_{\xi_0+1}$. Then for $b = \delta^{\mathcal{M}}$, $\mathcal{M}_b \models (\exists x)\varphi(x, a_{c_0}, a_{c_1}, \ldots, a_{c_k})$. Thus b is $\mathcal{M}$-stable, standard, and $b >^{\mathcal{M}} a_{b_0}$, contrary to the choice of b_0. So our supposition is false and Search 2 will succeed.

Search 3. We suppose that B is empty. Search for a Π_0 formula φ without parameters, and an $a \in \mathrm{On}(\mathcal{M})$ so that $L_\alpha \not\models (\exists x)\varphi$ and $\mathcal{M} \models (\exists x \in L_a)\varphi$; output a. Clearly a is nonstandard. If no such ϕ and a exist, $(\alpha_0)^{\mathcal{M}}$ would be $\mathcal{M}$-stable, and so $0^{\mathcal{M}} \in B$.

To construct an infinite descending $\varepsilon^{\mathcal{M}}$-chain, proceed as follows. If $\mathcal{M} \models$ (a is the greatest ordinal), output a; otherwise run the appropriate searches, outputting the first appropriate a we find. Replace $\mathcal{M}$ by $\mathcal{M}_a$ and do it again; etc. QED.

We are now ready to use the apparatus of §2 to obtain another improvement of Corollary 2.

LEMMA 11. *If* $L_\alpha \not\models \mathrm{Det}(\Sigma^0_{m+3})$, *then there is an infinite descending* $\varepsilon^{\mathcal{M}}$*-chain recursive in* $A^{(m)}$.

First we show that without loss of generality we may suppose that α is Σ_2 projectible. Fix Y so that $L_\alpha \models Y = Y^m$, for $Y^i, i \leq m$, as defined in the observation from §2.

LEMMA 12. *If α is not Σ_2 projectible, then* $L_\alpha \models \mathrm{Det}(\Sigma^0_{m+3})$.

PROOF. Suppose α is not Σ_2 projectible. By Lemma 6 it suffices to show that $L_\alpha \models$ all Σ^0_3 games on Y are determined. By the analysis of such games in §2, it suffices to show that if Φ is a monotone inductive operator on $\mathcal{P}(Y) \cap L_\alpha$ with a Π_1 definition over L_α, then $L_\alpha \models \Phi^\infty$ exists. Observe that "$|p|_\Phi = \xi$" is expressible over L_α as:

$$(\exists f)(f \text{ is a function on } \xi + 1 \text{ and } p \in f(\xi) - \bigcup f''\xi \text{ and}$$
$$(\forall \eta \leq \xi)(\forall q \in Y)(q \in f(\eta) \text{ iff } q \in \Phi(\bigcup f''\eta))).$$

Since $L_\alpha \models \Sigma_2$ Bounding, this formula may be put into Σ_2 form. By the Σ_2 uniformization of L_α (see [8]) there is a function h uniformizing $\{(\xi, p) \mid L_\alpha \models |p|_\Phi = \xi\}$; h is Σ_2 over L_α. Familiar arguments show that α is the limit of α many α-stable ordinals; so by results of §2, $L_\alpha \models (\forall \xi)(\Phi^\xi$ exists). If $L_\alpha \not\models \Phi^\infty$ exists, for any $\xi < \alpha$ there is a p so that $L_\alpha \models |p|_\Phi = \xi$; thus $\mathrm{dom}(h) = \alpha$. Clearly h is one-one and projects α into $Y^n \in L_\alpha$; this violates the fact that α is not Σ_2 projectible. QED.

Therefore we may as well suppose that $\mathcal{M}' \not\prec_1 \mathcal{M}$ and $\mathcal{M}' \not\prec_1 \mathcal{M}_a$ for all $a \in \mathrm{On}(\mathcal{M})$, a nonstandard. By Lemma 10 we also may as well suppose that the order-type of the α-stable ordinals is α. Let $\{\alpha_\xi\}_{\xi<\alpha}$ be the increasing enumeration of the α-stables.

LEMMA 13. *If $a \in \mathrm{On}(\mathcal{M})$ is nonstandard, then there is a nonstandard $b <^{\mathcal{M}} a$ such that $\mathcal{M} \models b$ is nonprojectible.*

PROOF. Since the order-type of the α-stables is α, α is a limit of limits, and thus a limit of nonprojectibles. If $\xi < \alpha$ is nonprojectible, $\mathcal{M} \models \xi^{\mathcal{M}}$ is nonprojectible; so the standard nonprojectible ordinals in $\mathcal{M}$ are unbounded. If this lemma fails for a, then $\{b \mid \mathcal{M} \models b < a$ and $(\forall\eta)$(if $b \leq \eta < a$, then η is nonprojectible)$\}$ is represented in $\mathcal{M}$ but has no $<^{\mathcal{M}}$-least member. QED.

Without loss of generality, suppose that $\mathcal{M} \models \neg$ Projectibility; otherwise select a nonstandard $b \in \mathrm{On}(\mathcal{M})$ so that $\mathcal{M} \models$ (b is nonprojectible), and replace $\mathcal{M}$ by $\mathcal{M}_b$. Trivially $\mathcal{M} \models (\forall\xi)(\exists\eta \geq \xi)$ (η is admissible). Let $\{a_b\}_{b \in B}$ be as in the proof of

Lemma 10. Since $\mathcal{M} \models \neg$ Projectibility, the $\mathcal{M}$-stables are unbounded in $<^{\mathcal{M}}$ and B has no maximum member. We now use the apparatus of §2. Let $G^0 = G([Y_0] - B, Y_0)$ be our typical Σ^0_{n+3} game defined in L_α. Form $G = G^n$ on $Y = Y^n$ as in the proof of Lemma 6. Suppose that $L_\alpha \not\models$ (there is a winning strategy in G^0). By Lemma 6, $L_\alpha \not\models$ (there is a winning strategy in G). Let $g = g_{L_\alpha}$ and $\hat{g} = g_{\mathcal{M}}$. Fix $\eta =$ the least ordinal so that $Y \in L_{\alpha_\eta}$, $c =$ the $<^{\mathcal{M}}$-least member of B so that $Y^{\mathcal{M}} \in M_{a_c}$. Let ξ_0 and b_0 be as in the proof of Lemma 10.

LEMMA 14. *For* $g(q, p) = \xi, \hat{g}(q^{\mathcal{M}}, p^{\mathcal{M}}) = b$: *if* $\xi < \xi_0$ *or* $b <^{\mathcal{M}} b_0$ *then* $b = \dot{\xi}^{\mathcal{M}}$; *and if* $\hat{g}(q^{\mathcal{M}}, p^{\mathcal{M}}) = b_0$ *then* $g(q, p) = \xi_0$.

PROOF. Suppose $q = \langle\,\rangle$. Since $\eta \leq \xi$ and $c \leq^{\mathcal{M}} b$, if $\xi < \xi_0$ or $b <^{\mathcal{M}} b_0$, $\eta^{\mathcal{M}} = c$. So suppose $\eta^{\mathcal{M}} = c$. By induction on ξ' such that $\eta + \xi' < \xi_0$: for all $r \in Y$, $L_\alpha \models r \in (\Phi_{o,Y})^{\xi'}$ iff $\mathcal{M} \models r^{\mathcal{M}} \in (\Phi_{0^{\mathcal{M}},Y^{\mathcal{M}}})^{\hat{b}}$ for $\hat{b} = (\xi')^{\mathcal{M}}$; we use the facts that $a_{c+^{\mathcal{M}}\hat{b}} = (\alpha_{\eta+\xi'})^{\mathcal{M}}$, $\theta^{\xi'}_{0,Y} \in L_{\alpha_{\eta+\xi'}+1}$ and $\mathcal{M} \models (\theta^{b}_{0^{\mathcal{M}},Y^{\mathcal{M}}} \in L_{a_{c+\hat{b}}+1})$. Where $\xi = \eta + \xi'$ and $b = c +^{\mathcal{M}} b'$, $L_\alpha \models |p|_{\Phi_{0,Y}} = \xi'$ and $\mathcal{M} \models |p^{\mathcal{M}}|_{\Phi_{0^{\mathcal{M}},Y^{\mathcal{M}}}} = b'$, which is to say:

$$p \in \theta^{\xi'}_{0,Y}(\xi') - \bigcup \text{Range}\, \theta^{\xi'}_{0,Y} \quad \text{and} \quad \mathcal{M} \models (p^{\mathcal{M}} \in \theta^{b'}_{0^{\mathcal{M}},Y^{\mathcal{M}}}(b') - \bigcup \text{Range}\, \theta^{b'}_{0^{\mathcal{M}},Y^{\mathcal{M}}}).$$

If $\xi < \xi_0$, $\mathcal{M} \models ((\theta^{\xi'}_{0,Y}) = \theta^{\hat{b}}_{0^{\mathcal{M}},Y^{\mathcal{M}}})$, and thus $b' = \hat{b}$ and $b = \xi^{\mathcal{M}}$; similarly if $b <^{\mathcal{M}} b_0$.

Now suppose that $q = q' {}^\wedge \langle x, y\rangle$, length $(q') = 2i$; assume as an induction hypothesis that if $g(q', q') < \xi_0$ or $\hat{g}(q'^{\mathcal{M}}, q'^{\mathcal{M}}) <^{\mathcal{M}} b_0$, then $\hat{g}(q'^{\mathcal{M}}, q'^{\mathcal{M}}) = g(q', q')^{\mathcal{M}}$ and $\mathcal{M} \models Z(q')^{\mathcal{M}} = Z(q'^{\mathcal{M}})$. If $\xi < \xi_0$ or $b <^{\mathcal{M}} b_0$ the antecedent of the induction hypothesis obtains. Suppose it does. By induction on ξ' so that $g(q', q') + \xi' < \xi_0$, we show that for all $r \in Y$ and $\hat{b} = (\xi')^{\mathcal{M}}$, $L_\alpha \models r \in (\Phi_{i+1,Z(q')})^{\xi'}$ iff $\mathcal{M} \models r^{\mathcal{M}} \in (\Phi_{i+1^{\mathcal{M}},Z(q')^{\mathcal{M}}})^{\hat{b}}$. Where $\xi = g(q', q') + \xi'$ and $b = \hat{g}(q'^{\mathcal{M}}, q'^{\mathcal{M}}) +^{\mathcal{M}} b'$, we have

$$p \in \theta^{\xi'}_{i+1,Z(q')}(\xi') - \bigcup \text{Range}(\theta^{\xi'}_{i+1,Z(q')})$$

and

$$\mathcal{M} \models p^{\mathcal{M}} \in \theta^{b'}_{i+1^{\mathcal{M}},Z(q'^{\mathcal{M}})}(b') - \bigcup \text{Range}(\theta^{b'}_{i+1^{\mathcal{M}},Z(q'^{\mathcal{M}})});$$

if $\xi < \xi_0$, then $\mathcal{M} \models (\theta^{\xi'}_{i+1,Z(q')})^{\mathcal{M}} = \theta^{\hat{b}}_{i+1^{\mathcal{M}},Z(q'^{\mathcal{M}})}$; so $b' = \hat{b}$ and $b = \xi^{\mathcal{M}}$. A similar argument applies if $b <^{\mathcal{M}} b_0$. Furthermore,

$$\mathcal{M} \models (Z(q'), i + 1, p)^{*\mathcal{M}} = (Z(q'^{\mathcal{M}}), i + 1, p^{\mathcal{M}})^*,$$

preserving our induction hypothesis.

Now suppose $\hat{g}(q^{\mathcal{M}}, p^{\mathcal{M}}) = b_0$. If $g(q, p)$ is defined, $g(q, p) \geq \xi_0$. Suppose $q = \langle\,\rangle$. If $g(\langle\,\rangle, p)$ is undefined or defined and $\neq \xi_0$, for ξ' so that $\eta + \xi' = \xi_0$ and for $\Phi = \Phi_{0,Y}$, $p \notin \Phi^{\xi'}$. Suppose d is a witness to the Σ_1 fact that $p \notin \Phi(\Phi^{<\xi'})$. By the preceding part of the lemma,

$$\mathcal{M} \models ((\theta^{<\xi'}_{o,Y})^{\mathcal{M}} = \theta^{<b'}_{0^{\mathcal{M}},Y^{\mathcal{M}}}), \text{ for } b' = (\xi')^{\mathcal{M}}.$$

Thus $d^{\mathcal{M}}$ witnesses in $\mathcal{M}$ the fact that

$$\mathcal{M} \models (p^{\mathcal{M}} \notin \Phi_{0^{\mathcal{M}},Y^{\mathcal{M}}}(\bigcup \text{Range}\, \theta^{<b'}_{0^{\mathcal{M}},Y^{\mathcal{M}}})).$$

This contradicts our supposition that $\hat{g}(\langle\,\rangle, p) = b_0$, since $\mathcal{M} \models c + b' = b_0$. For $q = q' {}^\wedge \langle x, y\rangle, q'$ of length $2i$, the argument is similar.

At last we are prepared for the construction which proves Lemma 11. As in our proof of Lemma 10, we describe a procedure for selecting a nonstandard

$a \in \mathrm{On}(\mathscr{M})$; we require that $\mathscr{M} \models a$ is nonprojectible. This will enable us to iterate the process with $\mathscr{M}_a$ in place of $\mathscr{M}$.

If $\mathscr{M} \models$ (there is a greatest nonprojectible ordinal), output that $a \in \mathrm{On}(\mathscr{M})$ such that $\mathscr{M} \models a$ is the greatest nonprojectible ordinal. By Lemma 13, a is nonstandard. Now assume that $\mathscr{M} \models (\forall \xi)(\exists \eta \leq \xi)(\eta$ is nonprojectible). If (and only if) $\mathscr{M} \not\models$ (there is a winning strategy in G), we engage in a variant of Search 1 from Lemma 10.

Search 1$'$. Search for $q, p \in Y$, $\xi < \alpha$ and $b \in B$ so that $L_\alpha \models g(q, p) = \xi$, $\mathscr{M} \models \hat{g}(q^{\mathscr{M}}, p^{\mathscr{M}}) = b$ and $a_{b +_{\mathscr{M}} 1} \neq (\alpha_{\xi+1})^{\mathscr{M}}$. The last clause is "checked" as in the proof of Lemma 10. If this search succeeds, output an $a \in \mathrm{On}(\mathscr{M})$ such that $\mathscr{M} \models (a_{b+1} \leq a$ and a is nonprojectible).

If $\mathscr{M} \models$ (there is a winning strategy in G), find an $a \in \mathrm{On}(\mathscr{M})$ so that $\mathscr{M} \models (a$ is nonprojectible and there is a winning strategy in G belonging to L_a). Such an a must be nonstandard, for if $s \in L_\alpha$ and $\mathscr{M} \models (s^{\mathscr{M}}$ is a winning strategy in G), then $L_\alpha \models (s$ is a winning strategy in G), contrary to our assumptions.

We now show that if we engage in Search 1$'$, we succeed. It suffices that there be $q, p \in Y$ so that $\hat{g}(q^{\mathscr{M}}, p^{\mathscr{M}}) = b_0$; for then $g(q, p) = \xi_0$ and b_0 and ξ_0 are as required. Suppose not. Then $\mathscr{M} \models$ (if $\hat{g}(q^{\mathscr{M}}, p^{\mathscr{M}})$ is defined, then $\hat{g}(q^{\mathscr{M}}, p^{\mathscr{M}}) < b_0$). By Lemma 5, $\mathscr{M} \models$ (II has a winning strategy in G), contrary to our case assumption.

Since our output a is such that $\mathscr{M}_a \models \neg$ Projectibility, this process may be iterated. This construction is effective in $A \oplus F_m$, and thus in $A^{(m)}$. QED.

Lemmas 9, 10 and 11 permitted us to shave two jumps off of Corollary 2. We now consider ways to shave a single jump off of Corollary 2. Generalize the notion of $\mathscr{M}$-stability from Σ_1 to Σ_k as follows: a is Σ_k-$\mathscr{M}$-stable iff $\mathscr{M}_a \prec_k \mathscr{M}$. So α-stability is just Σ_1-L_α-stability. As usual, $\mathscr{M} \models (\neg\, \Sigma_k$-Projectibility) iff the Σ_k-$\mathscr{M}$-stables are unbounded in $\mathscr{M}$.

LEMMA 15. *If for some k, α is Σ_k-projectible, then there is an infinite descending $\varepsilon^{\mathscr{M}}$-chain recursive in $A^{(m+1)}$.*

PROOF. By Lemma 9 we may assume that α is not Σ_1-projectible. Let k be least so that α is Σ_{k+1}-projectible; again by Lemma 9 we may assume that for no nonstandard a is $\mathscr{M} \prec_k \mathscr{M}_a$; thus $k \geq 1$. Let k' be least such that either for all nonstandard b, $\mathscr{M}' \not\prec_{k'+1} \mathscr{M}_b$, or such that there is a nonstandard $a \in \mathrm{On}(\mathscr{M})$ so that for all nonstandard $b <^{\mathscr{M}} a$, $\mathscr{M}' \not\prec_{k'+1} \mathscr{M}_b$. Then $k' + 1 \leq k$. Without loss of generality we may suppose that for all nonstandard $b \in \mathrm{On}(\mathscr{M})$, $\mathscr{M}' \not\prec_{k'+1} \mathscr{M}_b$; otherwise replace $\mathscr{M}$ by an appropriate $\mathscr{M}_a$.

Suppose $a \in \mathrm{On}(\mathscr{M})$ is nonstandard, $\mathscr{M}' \prec_{k'} \mathscr{M}_a$. We describe a procedure, which is sufficiently independent of a to be iterated recursively in $A^{(m+1)}$, for choosing a $b <^{\mathscr{M}} a$, b nonstandard. If $\mathscr{M} \models$ (there is a maximum L_a-$\Sigma_{k'}$-stable ordinal), find b so that $\mathscr{M} \models$ (b is the maximum L_a-$\Sigma_{k'}$-stable ordinal). Since α is not $\Sigma_{k'}$-projectible, α is a limit of L_α-Σ_k-stables; furthermore $\mathscr{M}' \prec_{k'} \mathscr{M}_a$; thus b is nonstandard. Since $\mathscr{M}_b \prec_{k'} \mathscr{M}_a$, $\mathscr{M}' \prec_{k'} \mathscr{M}_b$, and we may iterate with b in place of a.

Suppose $\mathscr{M} \not\models$ (there is a maximum L_α-$\Sigma_{k'}$-stable ordinal).

Claim. There are arbitrarily low (in $<^{\mathscr{M}}$) nonstandard $b \in \mathrm{On}(\mathscr{M}_a)$ that are $\mathscr{M}_a$-$\Sigma_{k'}$-stable. Since α is not Σ_k-projectible and $\mathscr{M}' \prec_{k'} \mathscr{M}_a$, the standard $\mathscr{M}_a$-$\Sigma_{k'}$-stables are unbounded in $<^{\mathscr{M}}$. If for $c <^{\mathscr{M}} a$, c nonstandard, there are no nonstandard $\mathscr{M}_a$-$\Sigma_{k'}$-stables below c, then $\{d \mid d <^{\mathscr{M}} c$ and $(\forall \eta)$ (if $d \leq \eta < c$ then η is not $\mathscr{M}_a$-$\Sigma_{k'}$-stable$\}$ is represented in $\mathscr{M}$ but has no $\varepsilon^{\mathscr{M}}$-least member; contradiction. Thus the $\mathscr{M}_a$-$\Sigma_{k'}$-

stables are unbounded below a under $<^{\mathcal{M}}$. We will apply a technique hereafter called "$\Sigma_{k'+1}$-witnessing." For some $\Pi_{k'}$ formula φ and some $p \in L_\alpha$, $\mathcal{M}_a \models (\exists x)\varphi(x, F(p))$ and $L_\alpha \not\models (\exists x)\varphi(x, p)$; the search for φ and p is recursive in $A^{(m+1)}$. Then we find $b <^{\mathcal{M}} a$ so that $\mathcal{M}_a \models (\exists x \in L_b)\varphi(x, F(p))$ and b is $\mathcal{M}_a$-$\Sigma_{k'}$-stable; output b. This b must be nonstandard; since $\mathcal{M}_b \prec_{k'} \mathcal{M}_a$, $\mathcal{M}' \prec_{k'} \mathcal{M}_b$; thus this process may be iterated with b in place of a. QED.

CONJECTURE 3. Even if for all k α is not Σ_k-projectible (i.e. α is a local $\aleph_{m+1}$), the consequent of Lemma 15 is true.

A proof of Conjecture 3 would yield proofs of Conjectures 1 and 2. Unfortunately, the technique used in Lemma 15 does not generalize to a proof of Conjecture 3 in any straightforward way. Suppose that for any $k \in \omega$ there are arbitrarily low (in $<^{\mathcal{M}}$) nonstandard b so that $\mathcal{M}' <_k \mathcal{M}_b$. For example, suppose $\mathcal{M}' \prec_2 \mathcal{M}_a$ and $\mathcal{M}_a \models$ (a is not Σ_2-projectible). If the Σ_3, or even the Σ_4, witnessing technique yields an output, that will yield a nonstandard $b <^{\mathcal{M}} a$. But if $\mathcal{M}' \prec_3 \mathcal{M}_a$ (and $\mathcal{M}' \prec_4 \mathcal{M}_a$), then they will not yield an output; there seems to be no way effective in $A^{(m+1)}$ to decide this; if we also apply Σ_5 witnessing and it yields an output before Σ_3 or Σ_4 witnessing does so, that output is nonstandard if $\mathcal{M}' \prec_3 \mathcal{M}_a$; but otherwise it might be standard.

What follows is a case of making the best of a bad situation.

Let $\hat{\alpha}$ be the least ordinal such that $L_{\hat{\alpha}} \models \alpha \neq \aleph_{m+1}$. Then $\hat{\alpha} = \alpha' + 1$ for α' of the form $\alpha + \hat{\xi}$. Suppose that $\hat{\xi} < \alpha$. Recall that $\mathscr{E}$ is an arithmetic copy of $L_{\hat{\alpha}}$. Where $a \in \mathrm{On}(\mathcal{M})$ and $\xi < \alpha$ we define $F^{a,\xi}: L_{\alpha+\xi} \to M_b$ for $b = a +^{\mathcal{M}} \xi^{\mathcal{M}}$. Recall that for any $\xi' < \xi$, $p \in L_{\alpha+\xi'+1} - L_{\alpha+\xi'}$ may be defined over $L_{\alpha+\xi'}$ by a formula in which all parameters are ordinals. $F^{a,\xi} \restriction L_\alpha = F$; $F^{a,\xi}(\xi') = a +^{\mathcal{M}} \xi'^{\mathcal{M}}$; where $p \in L_{\alpha+\xi'+1} - L_{\alpha+\xi'}$ and $\varphi(x, \bar{q})$ is the $<_L$-least formula defining p over $L_{\alpha+\xi'}$ so that $\bar{q}$ consists of ordinals, let $F^{a,\xi}(p) = p' \in M_b$, $b = a +^{\mathcal{M}} (\xi' + 1)^{\mathcal{M}}$ so that $\mathcal{M} \models (\forall x)(x \in p'$ iff $L_{a+\xi'} \models \varphi(x, F^{a,\xi'}(\bar{q})))$. We call $a \in \mathrm{On}(\mathcal{M})$ *(ξ, n)-reflecting* iff for every Σ_n formula φ and every $\bar{p}$ from $L_{\alpha+\xi}$

$$L_{\alpha+\xi} \models \varphi(\bar{p}) \quad \text{iff} \quad \mathcal{M}_b \models \varphi(F^{a,\xi}(\bar{p}))$$

for $b = a +^{\mathcal{M}} \xi^{\mathcal{M}}$. Note: where $\bar{p} = (\ldots, p_i, \ldots)$, $F^{a,\xi}(\bar{p}) = (\ldots, F^{a,\xi}(p_i), \ldots)$. Since $F^{a,\xi} \restriction \mathrm{On}(L_{\alpha+\xi})$ is recursive in $F \restriction \xi$, it is recursive in $A^{(m)}$.

Suppose $\hat{n}$ is least so that a projection f of α' into $\gamma < \alpha$ (where $L_\alpha \models \gamma = \aleph_m$) is $\Sigma_{\hat{n}}$ over α'. Clearly such $\hat{n}$ and f exist.

LEMMA 16. *If there is a $(\hat{\xi}, \hat{n} + 1)$-reflecting $a \in \mathrm{On}(\mathcal{M})$, then there is an infinite descending $\varepsilon^{\mathcal{M}}$-chain recursive in $A^{(m+1)}$.*

PROOF. Suppose a is $(\hat{\xi}, \hat{n} + 1)$ reflecting, $b = a +^{\mathcal{M}} \hat{\xi}^{\mathcal{M}}$, and f is defined over $L_{\alpha+\hat{\xi}}$ by $\varphi(x, y, p)$. Then

$$\mathcal{M}_b \models (\forall x)(\varphi(x, F^{a,\hat{\xi}}(f(\eta)), F^{a,\hat{\xi}}(p)) \text{ iff } x = F^{a,\hat{\xi}}(\eta))$$

for $\eta < \alpha'$. For $c \in \mathrm{On}(\mathcal{M})$, $c \in \mathcal{M}'$ iff for some $\eta < \alpha$ and $\xi < \gamma$, $\mathcal{M}_b \models \varphi(c, \eta^{\mathcal{M}}, F^{a,\hat{\xi}}(p))$ and $L_{\alpha'} \models \varphi(\xi, \eta, p)$. Since $F \restriction \gamma$ and $F^{a,\hat{\xi}}$ are recursive in $A^{(m)}$, $\mathrm{On}(\mathcal{M}')$ is RE in $A^{(m)}$; the lemma follows easily. QED.

We now assume that no $a \in \mathrm{On}(\mathcal{M})$ is $(\hat{\xi}, \hat{n} + 1)$-reflecting. Where W is any well-ordering, f and h are functions on ω^2, $\mathrm{Range}(h) \subseteq \mathrm{Fld}(W)$, we will say that h and W bound the convergence of f iff for all x, $t_0 < \cdots < t_{l-1}$:

for all $i < l$, if $f(x, t_i) \neq f(x, t_i + 1)$, then
$h(x, t_i + 1) <^W h(x, t_i)$.

Let $(\xi', n') < (\xi, n)$ iff $\xi' < \xi$ or ($\xi' = \xi$ and $n' = n$). Fix

$$W = \{(a,b) \mid a = (\xi', n')^{\mathscr{E}} \text{ and } b = (\xi, n)^{\mathscr{E}} \text{ for } (\xi', n') < (\xi, n)\}.$$

The following lemma is the source of the restrictions in Theorems 1 and 2 to λ which are n-well-behaved. The natural strategy for proving Conjectures 1 and 2, short of proving Conjecture 3, would be to improve Lemma 17, e.g. by replacing W with a well-ordering of type ω.

LEMMA 17. *There are functions f and h recursive in $A^{(m+1)}$ such that h and W bound the convergence of f and f converges to an infinite descending $\varepsilon^{\mathscr{M}}$-chain.*

PROOF. We describe a procedure which, given a nonstandard $a \in \mathrm{On}(\mathscr{M})$, guesses at a nonstandard $c <^{\mathscr{M}} a$; to each guess we associate a pair $(\xi, n) < (\hat{\xi}, \hat{n} + 1)$; each time we change our guess we pick a new pair below the previous one. Note: if (η, m) is least such that a is (η, m) reflecting, φ is Π_m and $\bar{p} \in L_{\alpha+n}$: if $L_{\alpha+n} \models (\exists x)\varphi(x, \bar{p})$ then $\mathscr{M}_b \models (\exists x)\varphi(x, F^{a,\eta}(\bar{p}))$ for $b = \alpha +^{\mathscr{M}} \eta^{\mathscr{M}}$. By assumption there is such an $(\eta, m) < (\hat{\xi}, \hat{n} + 1)$. We search for a Π_n formula $\varphi, \xi \leq \hat{\xi}, \bar{p} \in L_{\alpha+\xi}$ and $c \in \mathrm{On}(\mathscr{M})$ so that:

$$\mathscr{M}_b \models (\exists x \in L_c)\varphi(x, F^{a,\xi}(\bar{p})) \quad \text{and} \quad L_{\alpha+\xi} \models \neg(\exists x)\varphi(x, F^{a,\xi}(\bar{p}))$$

for $b = a +^{\mathscr{M}} \xi^{\mathscr{M}}$. By the remark about (η, m), eventually we find these. We output guess c associated with the pair (ξ, n). If we later find a $\Pi_{n'}$ formula $\varphi', \xi' \leq \hat{\xi}$ with $(\xi', n') < (\xi, n), \bar{p}' \in L_{\alpha+\xi'}$ and $c' \in \mathrm{On}(\mathscr{M})$ so that

$$\mathscr{M}_{b'} \models (\exists x \in L_{c'})\varphi'(x, F^{a,\xi'}(\bar{p}')) \quad \text{and} \quad L_{\alpha+\xi'} \models \neg(\exists x)\varphi'(x, \bar{p}')$$

for $b' = a +^{\mathscr{M}} \xi'^{\mathscr{M}}$, we change our guess to c' and associate it with (ξ', n')—for we know that a was not (ξ, n)-reflecting. Eventually we reach a guess c associated with (η, m); this c must be nonstandard. We iterate guessing in the usual manner to define the desired f and h. QED.

§4. Proof of Theorem 1.

LEMMA 18. *Consider any $n \in \omega$. If $n > 0$, suppose λ is $(n-1)$-well-behaved; suppose that $L_\lambda \not\models \mathrm{Det}(\Sigma^0_{n+3})$ and $A \subseteq \omega$ is a Turing upper bound on $L_\lambda \cap \mathscr{P}(\omega)$. Then there is an arithmetic copy $\mathscr{E}_\lambda$ of L_λ so that $\mathrm{Sat}_0(\mathscr{E}_\lambda)$ is recursive in $A^{(n+3)}$. ($\mathrm{Sat}_0(\mathscr{E}_\lambda)$ is the Σ_0 satisfaction relation for $\mathscr{E}_\lambda$.)*

PROOF. If $n = 0$, let $\beta_0 < \lambda$ bound $\{\xi \mid \xi \text{ is a local } \aleph_1\}$ below λ. (If no such β_0 existed, by Theorem 4 and the Π^1_1 absoluteness of λ and of any local $\aleph_1$, $L_\lambda \models \mathrm{Det}(\Sigma^0_3)$, contrary to our supposition.) If $n > 0$, using the fact that λ is $(n-1)$-well-behaved, fix β_0 and γ_0 so that: for any α which is a local $\aleph_n$, if $\beta_0 < \alpha < \lambda$, then $L_{\alpha+\gamma_0} \models \alpha \neq \aleph_n$. We might as well take $\gamma_0 < \beta_0$. Thus for $\beta_0 < \alpha < \lambda$, $L_\alpha \not\models \aleph_{n+1}$ exists. Fix $\beta_1 < \lambda$ so that for any limit of admissibles α, if $\beta_1 < \alpha < \lambda$, then $L_\alpha \not\models \mathrm{Det}(\Sigma^0_{n+3})$. If no such β_1 exists, where G is a Σ^0_{n+3} game on $\omega^{<\omega}$ defined in L_λ, select $\alpha < \lambda$, α a limit of admissibles sufficiently large for G to be defined in L_α, so that $L_\alpha \models \mathrm{Det}(\Sigma^0_{n+3})$; by the Π^1_1 absoluteness of α and λ, $L_\lambda \models G$ is determined; thus $L_\lambda \models \mathrm{Det}(\Sigma^0_{n+3})$, contrary to our supposition. Let β be admissible and locally countable, where $\max\{\beta_0, \beta_1\} \leq \beta < \lambda$. Fix an arithmetic copy $\mathscr{E}$ of L_β, $\mathrm{Sat}(\mathscr{E}) \in L_{\beta^+}$; fix $e_0 \in \omega$ so that $\mathrm{Sat}(\mathscr{E}) = \{e_0\}^A$. Let $W^* = \{(a,b) \mid a = (\xi^{\mathscr{E}}, n^{\mathscr{E}}), b = (\xi'^{\mathscr{E}}, n'^{\mathscr{E}}) \text{ for } (\xi, n) < (\xi', n') \text{ and } \xi < \gamma_0\}$.

Let $C_0 = \{e \in \omega \mid \{e\}^A$ is total and codes some $\mathrm{Sat}(\mathscr{M})$ where $\mathscr{M}$ is an ω-model for $T\}$. C_0 is Π^0_2 in A. For $e \in C_0$, let $\mathscr{M}(e)$ be such that $\{e\}^A$ codes $\mathrm{Sat}(\mathscr{M}(e))$. If for some $e \in C_0$, $o(\mathscr{M}(e)) > \lambda$, Lemma 18 follows immediately. Since $o(\mathscr{M}(e))$ is admissible and λ is not, $o(\mathscr{M}(e)) \neq \lambda$. We assume that for all $e \in C_0$, $o(\mathscr{M}(e)) < \lambda$. Let $C_1 = \{e \in C_0 \mid$ for some $a \in M(e)$, a codes $\{e_0\}^A$ in $\mathscr{M}(e)\}$; since "a codes $\{e_0\}^A$ in $\mathscr{M}(e)$" is Π^0_1 in A, C_1 is Δ^0_3 in A. For $e \in C_1$, $o(\mathscr{M}(e)) < \beta$.

If $n = 0$, let $C_2 = \{e \in C_1 \mid$ for every $x \in \omega$, if $\{x\}^A$ is total then $\{x\}^A$ is not an infinite descending $\varepsilon^{\mathscr{M}(e)}$-chain$\}$. If $n > 0$, let $C_2 = \{e \in C_1 \mid$ for every x and $y \in \omega$, if $\{x\}^{A^{(n)}}$ and $\{y\}^{A^{(n)}}$ are total and $\{y\}^{A^{(n)}}$ and W^* bound the convergence of $\{x\}^{A^{(n)}}$ then $\{x\}^{A^{(n)}}$ does not converge to an infinite descending $\varepsilon^{\mathscr{M}(e)}$-chain$\}$. C_2 is Π^0_{n+3} in A. We now use the results of §3.

LEMMA 19. *If $e \in C_2$, then $\mathscr{M}(e)$ is well-founded.*

PROOF. Suppose $e \in C_2$, $\mathscr{M} = \mathscr{M}(e)$ is nonstandard, $\alpha = o(\mathscr{M})$. Let $L_\alpha \models (\aleph_m$ is the greatest cardinal). Since $\beta < \alpha, m \leq n$. Select $\mathscr{E}_\alpha$, an arithmetic copy of L_α, where $\mathrm{Sat}(\mathscr{E}_\alpha) \in L_\lambda$; $\mathrm{Sat}(\mathscr{M}) \oplus \mathrm{Sat}(\mathscr{E}_\alpha)$ is recursive in A.

Case 1. $m = n$. If α is Σ_1-projectible, by Lemma 9, there is an infinite descending $\varepsilon^{\mathscr{M}}$-chain recursive in $A^{(n)}$. If α is not Σ_1-projectible, α is a limit of admissibles; since $\beta < \alpha$, $L_\alpha \not\models \mathrm{Det}(\Sigma^0_{n+3})$; so by Lemmas 10 and 11 there is an infinite descending $\varepsilon^{\mathscr{M}}$-chain recursive in $A^{(n)}$. All this contradicts $e \in C_2$. If $n = 0$, we are done. Suppose $n > 0$.

Case 2. $m = n - 1$. If α is not a local $\aleph_n$, by Lemma 15 there is an infinite descending $\varepsilon^{\mathscr{M}}$-chain recursive in $A^{(n)}$. If α is a local $\aleph_n$, we cannot be so straightforward. Fix $\mathscr{E}_{\hat\alpha}$, an arithmetic copy of $L_{\hat\alpha}$ so that $\mathrm{Sat}(\mathscr{E}_{\hat\alpha}) \in L_\lambda$, for $\hat\alpha$, $\hat\xi$ and $\hat n$ as in Lemma 16. By choice of β_0 and γ_0, $\hat\xi < \gamma_0$ and so $\hat\xi < \beta < \alpha$. By Lemma 16, if some $a \in \mathrm{On}(\mathscr{M})$ is $(\hat\xi, \hat n + 1)$-reflecting, then there is an infinite descending chain recursive in $A^{(n)}$. Otherwise, fix W as in Lemma 17. Let f and h be the functions recursive in $(\mathrm{Sat}(\mathscr{M}) \oplus \mathrm{Sat}(\mathscr{E}_{\hat\alpha}))^{(n)}$ delivered by Lemma 17; they are also recursive in $A^{(n)}$. The function $\hat F$ such that for $\xi \leq \hat\xi$, $\hat F(\xi^{\mathscr{E}_{\hat\alpha}}) = \xi^{\mathscr{E}}$ is recursive in finitely many jumps of $(\mathrm{Sat}(\mathscr{M}) \oplus \mathrm{Sat}(\mathscr{E}_{\hat\alpha})) \in L_\lambda$, and so in A; thus via $\hat F$, h may be "translated" to an $\hat h$ into W^* so that $\hat h$ and W^* bound the convergence of f; this contradicts $e \in C_2$. If $n = 1$, we are done. Suppose $n > 1$.

Case 3. $m \leq n - 2$. Use Corollary 2 for a contradiction with $e \in C_2$. QED.

Let $C_3 = \{e \in C_2 \mid \mathscr{M}(e) \models KP$ and $(\forall x)(x$ is countable$)\}$. C_3 is Π^0_{n+3} in A. Since λ is locally countable, for every $\alpha < \lambda$ there is $e \in C_3$ with $o(\mathscr{M}(e)) > \alpha$. For $e, e' \in C_3$ and $o(\mathscr{M}(e)) \leq o(\mathscr{M}(e'))$, let $h_{e,e'}\colon M(e) \to M(e')$ be the isomorphic embedding of $\mathscr{M}(e)$ onto an initial segment of $\mathscr{M}(e')$. Recall the coding of hereditarily countable sets by trees on ω; see [8] for details. For $x \in L$, x hereditarily countable, let $c(x)$ be the $<_L$-least tree on ω coding x; if α is admissible and locally countable, for $x \in L_\alpha$, $c(x) \in L_\alpha$. Thus $h_{e,e'}(x) = y$ iff for all $n \in \omega$, $\mathscr{M}(e) \models n^{\mathscr{M}(e)} \in c(x)$ iff $\mathscr{M}(e') \models n^{\mathscr{M}(e')} \in c(y)$; so $h_{e,e'}$ is Π^0_1 is A, uniformly in e and e'. Furthermore, $o(\mathscr{M}(e)) < o(\mathscr{M}(e'))$ iff for some $y \in M(e')$ there is no $x \in M(e)$ so that $h_{e,e'}(x) = y$. This question is Σ^0_3 in A. We define a sequence $\{e_i\}_{i<\omega}$ for $e_i \in C_3$. Fix $e_0 \in C_3$. Let e_{i+1} be the least $e \in C_3$ so that $e > e_i$ and $o(\mathscr{M}(e)) > o(\mathscr{M}(e_i))$. Since C_3 is recursive in $A^{(n+3)}$, so is $\{e_i\}_{i<\omega}$. We now construct our desired $\mathscr{E}_\lambda$ recursively in $A^{(n+3)}$.

Let $E = \{\langle i, x\rangle \mid x \in M(e_i)$ and if $i > 0$ then $x \notin \mathrm{Range}(h_{e_{i-1},e_i})\}$. Let

$$\varepsilon^{\mathscr{E}_\lambda} = \{\langle\langle i_1, x_1\rangle, \langle i_2, x_2\rangle\rangle \mid \mathscr{M}(e_i) \models y_1 \in y_2, \text{ where } i = \max\{i_1, i_2\} \text{ and } y_j = h_{e_{i_j},e_i}(x_j) \text{ for } j = 1, 2\}.$$

For a Σ_0 formula $\varphi(v_1,\dots,v_k)$ and values $\langle i_1,x_1\rangle,\dots,\langle i_k,x_k\rangle\in E$, let $i=\max\{i_1,\dots,i_k\}$, $y_j=h_{e_{i_j},e_i}(x_j)$ for $1\le j\le k$, and let

$$\varphi(\langle i_1,x_1\rangle,\dots,\langle i_k,x_k\rangle)\in \mathrm{Sat}_0(\mathscr{E}_\lambda)\quad\text{iff}\quad \mathscr{M}(e_i)\models\varphi(y_1,\dots,y_k).$$

For $\mathscr{E}_\lambda=(E,\varepsilon^{\mathscr{E}_\lambda})$, $\mathrm{Sat}_0(\mathscr{E}_\lambda)$ is recursive in $A^{(n)}$. QED.

The following argument, when combined with Lemma 18, proves Theorem 1.

LEMMA 20. *Let $\mathscr{E}$ be an arithmetic copy of L_λ.*

(i) *There is an $A\subseteq\omega$, a Turing upper bound on $L_\lambda\cap\mathscr{P}(\omega)$, with $A^{(3)}$ recursive in* $\mathrm{Sat}_0(\mathscr{E})$.

(ii) *If $L_\lambda\models\mathrm{Det}(\Sigma^0_{n+3})$, then there is an $A\subseteq\omega$, a Turing upper bound on $L_\lambda\cap\mathscr{P}(\omega)$, with $A^{(n+4)}$ recursive in* $\mathrm{Sat}_0(\mathscr{E})$.

PROOF. We force with uniformly recursive pointed perfect trees in L_λ. A *perfect tree* is a function P: $2^{<\omega}\to2^{<\omega}$ such that $P(\sigma^\frown\langle0\rangle)$ and $P(\sigma^\frown\langle1\rangle)$ are incompatible extensions of $P(\sigma)$ for $\sigma\in2^{<\omega}$. P is *uniformly recursively pointed* iff for some $c\in\omega$ for all $A\in[P]$, $P=\{e\}^A$; *Q extends P* iff for all $\sigma\in2^{<\omega}$, $P(\sigma)\subseteq Q(\sigma)$. We refer to such trees in L_λ as *conditions*. Let $\mathscr{L}$ be an arithmetic forcing language with primitives '$\exists$', '$\neg$', '&', '=', a predicate for each primitive recursive relation, and 'A', an uninterpreted one-place predicate. We suppose that all sentences are prenex. If φ is a $\Pi^0_2\cup\Sigma^0_2$ sentence of $\mathscr{L}$, let $P\Vdash\varphi$ iff for all $A\in[P]$, $A\models\varphi$. For a proof of the density lemma, that for every such φ and every condition P there is a condition Q which extends P and either $Q\Vdash\varphi$ or $Q\Vdash\neg\varphi$, see [12]. To prove (i), we extend the definition of forcing to $\Pi^0_3\cup\Sigma^0_3$ sentences as follows:

$$P\Vdash(\exists x)\varphi(x)\quad\text{iff}\quad\text{for some }k\in\omega,\ P\Vdash\varphi(k);$$

$$P\Vdash\neg(\exists x)\varphi\quad\text{iff}\quad\text{for every condition }Q\text{ extending }P, Q\Vdash(\exists x)\varphi,$$

where φ is Π^0_2. Density under this definition is trivial; forcing for sentences in $\Sigma^0_3\cup\Pi^0_3$ is Π^1_1 and so Δ_1 over L_λ.

To prove (ii) we extend the defining of forcing for $\Sigma^0_2\cup\Pi^0_2$ sentences to $\Sigma^0_{n+4}\cup\Pi^0_{n+4}$ sentences in the simplest possible way:

$$P\Vdash\varphi\quad\text{iff}\quad\text{for every }A\in[P],\ A\models\varphi.$$

Again forcing is Δ_1 over L_λ. I owe the key idea in the following lemma to Leo Harrington.

LEMMA 21. *Suppose $L_\lambda\models\mathrm{Det}(\Sigma^0_{n+3})$, $\varphi(x)$ is a Π^0_{n+3} formula of $\mathscr{L}$ with only x free, and P is a condition. There is a condition Q extending P such that either $Q\Vdash(\exists x)\varphi(x)$ or $Q\Vdash\neg(\exists x)\varphi(x)$.*

PROOF. Let $G(P,\varphi)$ be the following game. I selects $k\in\omega$; hereafter both players proceed in $2^{<\omega}$. Where $\langle k\rangle^\frown f_1$ is I's play and f_2 is II's play, I wins iff $f_1\in[P]$, $f_1\models\varphi(k)$ and $f_2=\{e_1\}^{\langle k\rangle^\frown f_1}$, where e_1 is a specific number in ω; we will postpone specifying it for a moment. $G(P,\varphi)$ is clearly a Π_{n+3} game which is defined over L_λ. Thus L_λ contains a winning strategy s for $G(P,\varphi)$. Let $\hat{s}=$ the characteristic function of $\{\langle x,s(x)\rangle\mid x\in2^{<\omega}\}$.

Case I. s is a winning strategy for I. We construct a condition Q so that $Q\Vdash(\exists x)\varphi(x)$. Suppose s tells I to first select k. Consider the tree T_0 of initial segments of plays by II which encode $\hat{s}$ at even places, i.e. $T_0=\{\langle\hat{s}(0),i_0,\dots,s(x),i_x\rangle\mid x\in\omega\}$. Let

T_1 be the set of I's responses under s to II's moves in T_0, with I's initial move deleted. Since s wins for I, $T_1 \subseteq \text{Range}(P)$. Claim: $[T_1]$ is a perfect set. If not, then for some $\sigma \in T_1$ there is a unique $f_1 \in [T_1]$, $f_1 \restriction \text{length}(\sigma) = \sigma$. Suppose σ is I's response to $\tau \in T_0$. For any $f_2 \in [T_0]$ such that $f_2 \restriction \text{length}(\tau) = \tau$, $\langle k \rangle {}^\frown f_1$ is I's play against II's play of f_1. But we may choose f_2 so that f_2 is not recursive in $\langle k \rangle {}^\frown f_1$, contrary to $f_2 = \{e_1\}^{\langle k \rangle {}^\frown f_1}$; this establishes the claim. Therefore there is a perfect tree Q extending P so that $[Q] = [T_1]$. Claim: for any $A \in [Q]$, $A \models \varphi(k)$. It suffices to show that for any $A = f_1 \in [Q]$ there is an $f_2 \in [T_0]$ so that $\langle k \rangle {}^\frown f_1$ is I's response to II's play of f_2 under s. Suppose $Q(\sigma) = f_1 \restriction z$ and $D_\sigma = \{\tau \in T_0 \mid \langle k \rangle {}^\frown (f_1 \restriction z)$ is I's response to τ under $s\}$. D_σ is nonempty and if $Q(\sigma {}^\frown \langle i \rangle) = f_1 \restriction z'$, then $z' > z$, $D_{\sigma {}^\frown \langle i \rangle}$ is nonempty, and any $\tau' \in D_{\sigma {}^\frown \langle i \rangle}$ extends some $\tau \in D_\sigma$; by König's lemma the desired f_2 exists.

We finally show that Q is uniformly recursively pointed. For $f_1 \in [Q]$, $\{e_1\}^{\langle k \rangle {}^\frown f_1} \in [T_2]$ and so encodes $\hat{s}$ and thus s; so f_1 computes s by a single procedure independent of f_1; but Q is recursive in $P \oplus s$; since $f_1 \in [P]$, P is recursive in f_1 by a procedure independent of f_1; putting these together, Q is recursive in f_1 by a procedure independent of f_1.

Case II. s is a winning strategy for II. Let Q be the result of coding $\hat{s}$ into P at the odd places, i.e. $Q(\sigma) = P(\langle (\sigma)_0, \hat{s}(0), \ldots, (\sigma)_z, \hat{s}(z) \rangle)$ where $z = \text{length}(\sigma) - 1$. By a familiar argument (see e.g. [12]), Q is uniformly recursively pointed; since $s \in L_\lambda$, $Q \in L_\lambda$. For $A = f_1 \in [Q]$ we show that $A \models \neg(\exists x)\varphi(x)$. We first complete our description of $G(P, \varphi)$ by specifying e_1: let e_2 be a procedure which, given a play $\langle k \rangle {}^\frown f_1$ by I, computes the real encoded at the odd places in f_1; let e_3 be the procedure which, given a strategy for II and a play by I, computes the play of II under that strategy in response to that play by I; e_1 is the procedure which first applies e_2 to $\langle k \rangle {}^\frown f_1$, regards the result as the characteristic function of a strategy for II, and applies e_3 to that strategy and $\langle k \rangle {}^\frown f_1$. Now suppose I plays $\langle k \rangle {}^\frown f_1$, $f_1 \in [Q]$; let f_2 be II's response under s. Since $\{e_2\}^{\langle k \rangle {}^\frown f_1} = \hat{s}$ and $\{e_3\}^{s, \langle k \rangle {}^\frown f_1} = f_2$,

$$f_2 = \{e_1\}^{\langle k \rangle {}^\frown f_1}.$$

But $f_1 \in [P]$ and I loses this play of $G(P, \varphi)$; so $A \models \neg\varphi(k)$. Since k was arbitrary, $A \models \neg(\exists x)\varphi x$. QED.

The rest of the construction for Lemma 20 is routine. We fix a listing $\langle \varphi_i \rangle_{i < \omega}$ of all Σ^0_{n+4} sentences of $\mathscr{L}$, and a Δ_1 over L_λ listing $\{A_i\}_{i < \omega}$ of $L_\lambda \cap \mathscr{P}(\omega)$. We form a Δ_1 (over L_λ) sequence $\langle P_i \rangle_{i < \omega}$ of conditions, P_{i+1} extending P_i, so that:

$$\text{either } P_{2i} \Vdash \varphi_i \text{ or } P_{2i} \Vdash \neg\varphi_i;$$

P_{2i+1} is the result of coding A_i into P_{2i} at the odd places. Then $\bigcap_{i < \omega} [P_i] = \{A\}$ for some $A \subseteq \omega$. The odd steps ensure that A computes A_i for all $i < \omega$; the even steps ensure that $A \models \varphi_i$ iff $P_{2i} \Vdash \varphi_i$; since $\langle P_i \rangle_{i < \omega}$ and the forcing relation are Δ_1 over L_λ, $A^{(n+4)}$ is recursive in $\text{Sat}_0(\mathscr{E}_\lambda)$. QED.

§5. Failure of determinacy. The results of §3, together with techniques developed by H. Friedman [3] and Martin [10], enable us to show that certain initial segments of L do not satisfy certain determinacy conditions. Clearly $L_\alpha \models \text{Det}(\Sigma^0_{n+3})$ iff $L_\gamma \models \text{Det}(\Sigma^0_{n+3})$, where $\gamma = (\aleph_1)^{L_\alpha}$. Thus we confine our attention to locally countable initial segments of L. We now suspend the assumption that λ is not admissible; the following theorem clearly implies Theorem 2 if λ is not admissible.

THEOREM 5. *Let λ be a locally countable limit of admissibles. Suppose that λ is not a limit of $\alpha < \lambda$ such that $L_\alpha \models \mathrm{Det}(\Sigma^0_{n+3})$. If $n > 0$ suppose that λ is $(n-1)$-well-behaved. Furthermore, suppose that if λ is not projectible, then the order-type of the λ-stable ordinals is less than λ. Then $L_\lambda \not\models \mathrm{Det}(\Sigma^0_{n+3})$.*

PROOF. Suppose not; let α be the least counterexample. Fix $\beta_0, \gamma_0, \beta_1, \beta, \mathscr{E} = \mathscr{E}_\beta$ and W^* as in the proof of Lemma 18. Let T' be the result of adding to T these further sentences:

$(\forall x)$ x is countable;

if Projectibility fails then the stable ordinals have the order-type of some ordinal;

$(\forall \alpha)$ α satisfies Theorem 5.

By our choice of α, $L_\alpha \models T'$.

We associate with each formula φ in which $\mathrm{Sat}(\mathscr{E})$ is the sole parameter a game $G(\varphi)$ on $2^{<\omega}$. Where f_1 and f_2 are the plays produced by I and II, respectively, I wins $G(\varphi)$ iff:

(i) f_1 encodes $\mathrm{Sat}(\mathscr{M})$, where $\mathscr{M}$ is an ω-model of $T' \cup \{\varphi\}$ in which $\mathrm{Sat}(\mathscr{E})$ is represented;

(ii.1) if $n = 0$, for every $x \in \omega$ such that $\{x\}^{f_1 \oplus f_2}$ is total, $\{x\}^{f_1 \oplus f_2}$ is not an infinite descending $\varepsilon^{\mathscr{M}}$-chain; and

(ii.2) if $n > 0$, for every $x, y \in \omega$ such that $\{x\}^{(f_1 \oplus f_2)^{(n)}}$ and $\{y\}^{(f_1 \oplus f_2)^{(n)}}$ are total and $\{y\}^{(f_1 \oplus f_2)^{(n)}}$ and W^* bound the convergence of $\{x\}^{(f_1 \oplus f_2)^{(n)}}$, then $\{x\}^{(f_1 \oplus f_2)^{(n)}}$ does not converge to an infinite descending $\varepsilon^{\mathscr{M}}$-chain.

We note that $G(\varphi)$ is a Π^0_{n+3} (in $\mathrm{Sat}(\mathscr{E})$) game. By hypothesis there is an $s \in L_\lambda$ so that $L_\lambda \models$ (s is a winning strategy for $G(\varphi)$). Since λ is a limit of admissibles, s is a winning strategy for $G(\varphi)$. We show: s is winning strategy for I iff $L_\lambda \models \varphi$. This implies that truth in the structure $\langle L_\lambda; \varepsilon \restriction L_\lambda; \mathrm{Sat}(\mathscr{E})\rangle$ is definable over that structure, contrary to Tarski's well-known result.

If $L_\lambda \models \varphi$, then I has this winning strategy: encode $\mathrm{Sat}(\mathscr{E}_\lambda)$ for $\mathscr{E}_\lambda$ an arithmetic copy of L_λ.

Claim. If $L_\lambda \models \neg\varphi$, then II has this winning strategy: encode $\mathrm{Sat}(\mathscr{E}_\lambda)$ for $\mathscr{E}_\lambda$ an arithmetic copy of L_λ. Suppose II plays f_2 encoding $\mathrm{Sat}(\mathscr{E}_\lambda)$ and I plays f_1, encoding an $\mathscr{M}$ which satisfies condition (i). Clearly $\mathscr{M}$ is nonstandard. We show that condition (ii) fails. Where $\alpha = o(\mathscr{M})$, $\beta < \alpha$ since $\mathrm{Sat}(\mathscr{E})$ is represented in $\mathscr{M}$. We cannot have $\lambda < \alpha$, by the third new sentence of T'. Let $L_\alpha \models$ ($\aleph_m$ is the greatest cardinal). If $\lambda = \alpha$, then $m = 0$; by the assumptions on λ either Lemma 9 or Lemma 10 provides an infinite descending $\varepsilon^{\mathscr{M}}$-chain recursive in $f_1 \oplus f_2$ and violating (ii). Suppose $\alpha < \lambda$. If $n = 0$, since $\beta_0 \leq \beta < \alpha$ we have $m = 0$; since $\beta_1 \leq \beta < \alpha$, $L_\alpha \not\models \mathrm{Det}(\Sigma^0_3)$; by Lemma 9 or 10 or 11 there is an infinite descending $\varepsilon^{\mathscr{M}}$-chain recursive in $f_1 \oplus f_2$, violating (ii). If $n > 0$, then $m \leq n$, since $\gamma_0 \leq \beta_0 < \beta < \alpha$ and if γ were a local $\aleph_{n+1}$ and $\beta_0 < (\aleph_n)^{L_\gamma} + \delta = \gamma$ we would have $\delta \leq \gamma_0$, which is impossible. We now argue by cases as in the proof of Lemma 19. If $m = n$, Lemmas 9, 10 or 11 apply; if $m = n - 1$, Lemmas 15, 16 or 17 apply; if $n \geq 2$ and $m \leq n - 2$, Corollary 2 applies; so in all cases (ii) fails and II wins $G(\varphi)$. QED.

REFERENCES

[1] MORTON DAVIS, *Infinite games with perfect information*, ***Advances in game theory***, Annals of Mathematics Studies, no. 52, Princeton University Press, Princeton, New Jersey, 1964, pp. 85–101.

[2] H. B. ENDERTON and HILARY PUTNAM, *A note on the hyperarithmetical hierarchy*, this JOURNAL, vol. 35 (1970), pp. 429–430.

[3] HARVEY FRIEDMAN, *Higher set theory and mathematical practice*, ***Annals of Mathematical Logic***, vol. 2 (1971), pp. 325–357.

[4] HAROLD T. HODES, ***Jumping through the transfinite***, Ph.D. Thesis, Harvard University, Cambridge, Massachusetts, 1977.

[5] ———, *Jumping through the transfinite*, this JOURNAL, vol. 45 (1980), pp. 204–220.

[6] ———, *Upper bounds on locally countable admissible initial segments of a Turing degree hierarchy*, this JOURNAL, vol. 46 (1981), pp. 753–760.

[7] RONALD JENSEN, *The fine structure of the constructible hierarchy*, ***Annals of Mathematical Logic***, vol. 4 (1972), pp. 229–308.

[8] CARL JOCKUSCH, JR., and STEPHEN SIMPSON, *A degree-theoretic definition of the ramified analytical hierarchy*, ***Annals of Mathematical Logic***, vol. 10 (1976), pp. 1–32.

[9] DONALD MARTIN, *Borel determinacy*, ***Annals of Mathematics***, ser. 22, vol. 102 (1975), pp. 363–371.

[10] ———, *Analysis and* Σ_4^0 *games* (unpublished manuscript).

[11] YIANNIS MOSCHOVAKIS, ***Elementary induction on abstract structures***, North-Holland, Amsterdam, 1974.

[12] GERALD SACKS, *Forcing with perfect closed sets*, ***Axiomatic set theory***, Proceedings of Symposia in Pure Mathematics, vol. 13, part 1, American Mathematical Society, Providence, Rhode Island, 1971, pp. 331–355.

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