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JUMPING THROUGH THE TRANSFINITE: THE MASTER CODE HIERARCHY OF TURING DEGREES[1]

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Abstract. Where $\underset{\sim}{a}$ is a Turing degree and ξ is an ordinal $< (\aleph_1)^{L^{\underset{\sim}{a}}}$, the result of performing ξ jumps on $\underset{\sim}{a}$, $\underset{\sim}{a}^{(\xi)}$, is defined set-theoretically, using Jensen's fine-structure results. This operation appears to be the natural extension through $(\aleph_1)^{L^{\underset{\sim}{a}}}$ of the ordinary jump operations. We describe this operation in more degree-theoretic terms, examine how much of it could be defined in degree-theoretic terms and compare it to the single jump operation.

§1. Basic definitions and results.

For $A \leq \omega$, let:

$$L_0[A] = M_0[A] = \{x \mid x \text{ is hereditarily finite}\};$$

$$L_{\alpha+1}[A] = \{x \mid x \text{ is first-order definable over } \langle L_\alpha[A]; \in \restriction L_\alpha[A], A; L_\alpha[A]\rangle\};$$

$$L_\lambda[A] = \bigcup_{\alpha<\lambda} L_\alpha[A];$$

$$M_{\omega\alpha+n}[A] = \Delta_n(\langle L_\alpha[A]; \in \restriction L_\alpha[A],\ A;\ L_\alpha[A]\rangle) \quad \text{for } n \geq 1;$$

$$M_{\omega\alpha}[A] = L_\alpha[A].$$

Clearly $M_{\omega(\alpha+1)}[A] - M_{\omega\alpha}[A] = L_{\alpha+1}[A] - L_\alpha[A]$. $\langle M_\alpha[A]\rangle_\alpha$ is introduced only for perspicacious statement of results. All proofs will use $\langle L_\alpha[A]\rangle_\alpha$. Note that if $A \equiv_T B$ then $M_\alpha[A] = M_\alpha[B]$. Thus for a Turing degree $\underset{\sim}{a}$, we may define $M_\alpha^{\underset{\sim}{a}} = M_\alpha[A]$ and $L_\alpha^{\underset{\sim}{a}} = L_\alpha[A]$, for $A \in \underset{\sim}{a}$. We let $M_\alpha[\varnothing] = M_\alpha^{\underset{\sim}{0}} = M_\alpha$ and $L_\alpha[\varnothing] = L_\alpha^{\underset{\sim}{0}} = L_\alpha$. All of the following definitions are given for $\underset{\sim}{a} = \underset{\sim}{0}$. They relativize to arbitrary $\underset{\sim}{a}$ in the obvious way. As usual, $L_\alpha^{\underset{\sim}{0}} = L_\alpha$, $M_\alpha^{\underset{\sim}{0}} = M_\alpha$. Unless otherwise indicated, lower case Greek letters range over $(\aleph_1)^L$; λ always ranges over limit ordinals.

$[\alpha, \beta) = \{\gamma \mid \alpha \leq \gamma < \beta\}$. $[\alpha, \beta)$ is an M-gap iff $(M_\beta - M_\alpha) \cap \omega^\omega = \varnothing$; $[\alpha, \beta)$ is an M-gap iff $(L_\beta - L_\alpha) \cap \omega^\omega = \varnothing$. α is an M-gap ordinal iff $[\alpha, \alpha + 1)$ is an M-gap; α is an L-gap ordinal iff $[\alpha, \alpha + 1)$ is an L-gap. α is an M-index iff α is not an M-gap ordinal; α is an L-index iff α is not an L-gap ordinal. α starts an M-gap iff α is an M-gap ordianl and is the supremum of M-indices; α starts an L-gap iff α is an L-gap ordinal and is the supremum of L-indices. Let $F(\alpha)$ be the maximum β such that $[\alpha, \alpha + \beta)$ is an M-gap. Thus α is an M-gap ordinal iff $F(\alpha) \neq 0$. If α starts an M-gap, $F(\alpha)$ is the length of that gap.

Let Ind: $(\aleph_1)^L \to (\aleph_1)^L$ enumerate the M-indices in increasing order. Clearly

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$\alpha \leq \mathrm{Ind}(\alpha)$. If $\alpha < \mathrm{Ind}(\alpha)$, it is because Ind was temporarily "thrown off" by an M-gap. $A \subseteq \omega$ is a master code for α iff $M_{\alpha+1} \cap 2^\omega = \{B \subseteq \omega \mid B \leq_T A\}$. Clearly this notion is invariant under Turing equivalence. Thus a Turing degree $\underline{b}$ is a master code for ξ iff $\underline{b}$ is the degree of a master code for ξ.

THE FUNDAMENTAL THEOREM. *ξ is an M-index iff there is a master code for ξ; furthermore, if $\underline{b}$ is the master code for ξ then $\underline{b}'$ is the master code for $\xi + 1$.*

We are now ready to extend the jump operation through $(\aleph_1)^L$. Let $\underline{0}^{(\xi)}$ = the master code for $\mathrm{Ind}(\xi)$. The previous definitions and the Fundamental Theorem relativize to an arbitrary degree $\underline{a}$. Thus we may define $\underline{a}^{(\xi)}$ = the $\underline{a}$-master code for $\mathrm{Ind}^{\underline{a}}(\xi)$, for $\xi < (\aleph_1)^{L\underline{a}}$.

The central results of this paper characterize the function $\xi \mapsto \underline{0}^{(\xi)}$ in more degree-theoretic terms. We now introduce the machinery needed to state these results.

$\mathrm{Ind}(\alpha) \neq \alpha$ iff $(\exists \beta)(\beta$ starts an M-gap and $\beta \leq \alpha < \beta + F(\beta) \cdot \omega)$. Let $J(\alpha)$ = the least strict upper-bound on $\{\mathrm{Ind}(\xi) \mid \xi < \alpha\}$. $\alpha \leq J(\alpha)$. In fact, $J(\alpha) > \alpha$ iff $\mathrm{Ind}(\alpha) > \alpha$ and α does not start an M-gap. $J(\alpha) \neq \mathrm{Ind}(\alpha)$ iff α starts an M-gap. $\mathrm{Ind}(\alpha) = J(\alpha) + F(J(\alpha))$.

We divide limit ordinals below $(\aleph_1)^L$ into three cases.

Case 1. $J(\lambda)$ is not a limit of M-gaps.

Case 2. $J(\lambda)$ is a limit of M-gaps and $F(J(\lambda)) < \omega$.

Case 3. Otherwise.

Notice that $F(J(\lambda)) \geq \omega$ iff λ falls under Case 3.

In subsequent proofs, further subdivision is needed.

Case 1.1. λ falls under Case 1 and $J(\lambda)$ is an M-index.

Case 1.2. λ falls under Case 1 but not Case 1.1.

$J(\lambda)$ is an M-gap ordinal iff $J(\lambda)$ is admissible iff λ is admissible and locally countable. Notice that if λ is not under Case 1.1, then $J(\lambda) = \lambda = \omega\lambda$. For λ under Case 1, λ falls under Case 1.1 iff $F(J(\lambda)) = 0$, and λ falls under Case 1.2 iff $F(J(\lambda)) = F(\lambda) = 1$. The least Case 1.1 ordinal is ω, and $\mathrm{Ind}(\omega) = \omega$. The least Case 1.2 ordinal is $\omega_1^{CK} = \omega_1$ and $\mathrm{Ind}(\omega_1) = \omega_1 + 1$. The least Case 2 ordinal is $\sup\{\omega_n^{CK} \mid n < \omega\} = \omega_\omega$, and $\mathrm{Ind}(\omega_\omega) = \omega_\omega$. The least Case 3 ordinal is β_0, and $\mathrm{Ind}(\beta_0) = \beta_0 + \omega$.

Between terms denoting Turing degrees, "$\leq$" represents Turing reducibility. A set I of Turing degrees is an ideal iff it is closed under join and downward-closed under $\leq$. If I is an ideal, the pair $(\underline{b}, \underline{c})$ is I-exact iff for any $\underline{a}$, $\underline{a} \in I$ iff $\underline{a} \leq \underline{b}$ and $\underline{a} \leq \underline{c}$. If $(\underline{b}, \underline{c})$ is I-exact we shall also call $(\underline{b} \vee \underline{c})$ I-exact. Let I_λ be the minimal ideal containing $\{\underline{0}^{(\xi)} \mid \xi < \lambda\}$. By definitions, $\bigcup I_\lambda = M_{J(\lambda)} \cap \omega^\omega = L_\gamma \cap \omega^\omega = L_\beta \cap \omega^\omega$ where $\omega \cdot \gamma = J(\lambda)$ and $\mathrm{Ind}(\lambda) = \omega \cdot \beta + n$ for $n < \omega$.

The following results extend the characterization of $(\lambda\xi.\ \underline{0}^{(\xi)}) \restriction \beta_0$ provided in [6]. Let μ_λ = the least μ such that $\{\underline{a}^{(\mu)} \mid \underline{a}$ is I_λ-exact$\}$ has a least member.

THEOREM 1. *μ_λ exists. In fact,*

$$\mu_\lambda = \begin{cases} 2 + F(J(\lambda)) & \text{for } \lambda \text{ under Case } 1, \\ 3 + F(J(\lambda)) & \text{for } \lambda \text{ under Case } 2 \text{ or Case } 3. \end{cases}$$

Thus $\mu_\lambda = 3 + F(\lambda)$ for λ under Cases 2 or 3, and $\mu_\lambda = F(\lambda)$ in Case 3.

THEOREM 2. *$\underset{\sim}{0}^{(\lambda)}$ is the least member of $\{\underset{\sim}{a}^{(\mu_\lambda)} \mid \underset{\sim}{a}$ is I_λ-exact$\}$. Moreover, it is the least member of $\{\underset{\sim}{a}^{(\mu_\lambda)} \mid \underset{\sim}{a}$ is I_λ-exact and* $\mathrm{Ind}(\lambda)$ *is recursive in $\underset{\sim}{a}\}$.*

Where U is a predicate of Turing degrees, $\underset{\sim}{a}$ is a ξ-low U iff $\underset{\sim}{a}$ is a U and for any $\underset{\sim}{b} \in U$, $\underset{\sim}{a} \leq \underset{\sim}{b}^{(\xi)}$.

THEOREM 3. *For $\xi < \mu_\lambda$, there is no ξ-low I_λ-exact degree.*

These theorems shall all be derived from the following lemmas. Let

$$G(\lambda) = \begin{cases} 2 + F(J(\lambda)) & \text{if } \lambda \text{ falls under Case 1,} \\ 3 + F(J(\lambda)) & \text{if } \lambda \text{ falls under Case 2 or Case 3.} \end{cases}$$

LEMMA 1. *There is an I_λ-exact pair $(\underset{\sim}{b}, \underset{\sim}{c})$ such that $(\underset{\sim}{b} \vee \underset{\sim}{c})^{(G(\lambda))} \leq \underset{\sim}{0}^{(\lambda)}$ and* $\mathrm{Ind}(\lambda)$ *is recursive in $\underset{\sim}{b}$ and in $\underset{\sim}{c}$.*

LEMMA 2. *For any $\underset{\sim}{d} \notin I_\lambda$ there is an I_λ-exact pair $(\underset{\sim}{b}, \underset{\sim}{c})$ such that for any $\xi < G(\lambda)$, $\underset{\sim}{d} \not\leq (\underset{\sim}{b} \vee \underset{\sim}{c})^{(\xi)}$ and* $\mathrm{Ind}(\lambda)$ *is recursive in $\underset{\sim}{b}$ and in $\underset{\sim}{c}$.*

LEMMA 3. *If $(\underset{\sim}{b}, \underset{\sim}{c})$ is I_λ-exact, then $\underset{\sim}{0}^{(\lambda)} \leq (\underset{\sim}{b} \vee \underset{\sim}{c})^{(G(\lambda))}$.*

By Lemmas 1 and 3, $\underset{\sim}{0}^{(\lambda)}$ is the least member of $\{\underset{\sim}{a}^{(G(\lambda))} \mid \underset{\sim}{a}$ is I_λ-exact$\}$ and of $\{\underset{\sim}{a}^{(G(\lambda))} \mid \underset{\sim}{a}$ is I_λ-exact and $\mathrm{Ind}(\lambda) < \omega_1^{(\underset{\sim}{b} \vee \underset{\sim}{c})}\}$. Thus μ_λ exists. By definition of μ_λ, $\{\underset{\sim}{a}^{(\mu_\lambda)} \mid \underset{\sim}{a}$ is I_λ-exact$\}$ has a least member $\underset{\sim}{d}$. Since $\underset{\sim}{d} \notin I_\lambda$, by Lemma 2 if $\mu_\lambda < G(\lambda)$, $\underset{\sim}{d}$ is not least. Thus $\mu_\lambda = G(\lambda)$. Lemma 2 easily proves Theorem 3.

COROLLARY. *$\underset{\sim}{0}^{(\lambda)}$ is the least member of $\{\underset{\sim}{a}^{(-1+\mu_\lambda)} \mid \underset{\sim}{a}$ is a u.u.b. on $I_\lambda\}$ and of $\{\underset{\sim}{a}^{(\mu_\lambda)} \mid \underset{\sim}{a}$ is a weak u.u.b. on $I_\lambda\}$. For $\xi < (-1 + \mu_\lambda)$ there is no ξ-low u.u.b. on I_λ; for $\xi < \mu_\lambda$ there is no ξ-low weak u.u.b. on I_λ.*

(See [4] for the definition of a u.u.b. and a weak u.u.b.) This corollary connects these results with the apparatus of [2].

We state, mostly without proof, some basic facts about gaps.

1. If α starts an M-gap or an L-gap, α is a limit ordinal and $\omega\alpha = \alpha$.
2. If α starts an L-gap, then α starts an M-gap.
3. If α is a supremum of L-indices, $L_\alpha \models V = HC$ (i.e. "everything is countable").
4. If α starts an M-gap, α is the supremum of L-indices.

A Δ_n comprehension axiom is a sentence of the form:

$$(\forall x \in \omega)\,(\phi x \leftrightarrow \psi x) \rightarrow (\exists y)\,(y \subseteq \omega \;\&\; (\forall x)\,(x \in y \leftrightarrow \phi\, x)).$$

where ϕ is Σ_n and ψ is Π_n. Δ_n CA is the set of Δ_n comprehension axioms.

5. If $F(\omega\alpha) \geq n$ then $L_\alpha \models \Delta_n$ CA.
6. If α starts an M-gap then α starts an L-gap iff $F(\alpha) \geq \omega$.
7. If α starts an M-gap and $F(\alpha) \geq n$ then α is Σ_n-admissible.

PROOF. Use Jensen's result on the Σ_n uniformizability of L_α.

8. α starts an L-gap iff $L_\alpha \models \mathrm{ZF}^- + V = HC$; if α starts an L-gap, $L_\alpha \cap \omega^\omega$ is a β-model of analysis. ($\mathrm{ZF}^- = \mathrm{ZF} - \{\text{Power Set}\}$.) See [8].
9. λ is a limit of M-gaps iff M_λ is closed under hyperjump iff $M_\lambda \cap \omega^\omega$ is Π_1^1 absolute.

We freely identify binary relations on ω with subsets of ω via the coding scheme $n = \langle (n)_0, (n_1) \rangle$. Thus for $X \subseteq \omega$, structures $\langle X, R, A \rangle$, $R \subseteq X^2$, $A \subseteq X$, may be identified with reals. An arithmetic copy of $\langle L_\alpha[A]; \in \restriction (L_\alpha[A^2]), A\rangle$ hereafter called an arithmetic copy of $L_\alpha[A]$, is a structure $\langle X, R, A\rangle$, $X \subseteq \omega$, isomorphic to $\langle L_\alpha[A]; \in \restriction (L_\alpha[a])^2, A\rangle$, coded as single real. Hereafter $E_\alpha[A]$ ranges over arithmetic copies of $L_\alpha[A]$. Let $\mathrm{Th}_n(\langle X; R, A\rangle)$ be $\mathrm{Th}(\langle X; R, A; X\rangle) \cap (\Sigma_n \cup \Pi_n)$, the n-quantifier theory of $\langle X; R, A; X\rangle$, with each member of X viewed as a name of itself. For $X \subseteq \omega$, $\mathrm{Th}_n(\langle X; R, A\rangle)$ may be viewed as a single real. The following standard facts about the arithmetic hierarchy provide motivation for this paper: if $E_0[A] \leq_{\mathrm{T}} A$, then $\mathrm{Th}_n(E_0[A]) \equiv_{\mathrm{T}} A^{(n)}$, and there is an $E_1[A]$ canonically constructed from $E_0[A]$ such that $\mathrm{Th}(E_0[A]) \equiv_{\mathrm{T}} \mathrm{Th}_0(E_1[A]) \equiv_{\mathrm{T}} A^{(\omega)}$.

The Fundamental Theorem is proved in [6]. The proof makes use of Jensen's Σ_n uniformization theorem, transferred from the J to the L hierarchy. The proof of that uses Jensen's notion of a Σ_n master code for an arbitrary L_α. It might seem more direct to imitate Jensen's proof, which proves Σ_n uniformization and the existence of Σ_n master codes simultaneously, with Δ_n uniformization and Δ_n master codes, thereby avoiding mention of Σ_n master codes. But this seems to be impossible.

The proof of the Fundamental Theorem proceeds by proving the following fact, which we shall misleadingly call a corollary.

If $L_\alpha \not\models \Delta_{n+1}$ CA then $\Delta_{n+1}(L_\alpha)$ contains a real of the form $\mathrm{Th}_n(E_\alpha)$. Thus the master code for $\omega\alpha + n$ is the least degree of the form $\deg(\mathrm{Th}_n(E_\alpha))$.

The ordinary jump on $\mathscr{D}$ corresponds to a canonical jump function * on $P\omega$: $\deg(A)' = \deg(A^*)$. Unfortunately, an arbitrary transfinite jump on $\mathscr{D}$ seems to be associated with no canonical such function on $P\omega$.

§2. Proofs of Lemmas 1, 2 and 3. Lemmas 1 and 2 for λ under Cases 1 or 2 are proved in [6]. For the sake of a complete presentation we sketch those proofs here.

Suppose λ falls under Cases 1 or 2. Let $n = F(J(\lambda))$; let $J(\lambda) = \omega\alpha$. Thus for some E_α, $\underset{\sim}{0}^{(\lambda)} = \deg(\mathrm{Th}_n(E_\alpha))$. To prove Lemma 1 it suffices to construct B and $C \in 2^\omega$ such that

(1) (B, C) is exact for $L_\alpha \cap \omega^\omega$;

(2.1) if λ falls under Case 1 then

$$(B \oplus C)^{(2+n)} \in \Delta_{n+1}(L_\alpha);$$

(2.2) if λ falls under Case 2 then

$$(B \oplus C)^{(3+n)} \in \Delta_{n+1}(L_\alpha).$$

To prove Lemma 2 it suffices, given $\underset{\sim}{d} \notin I_\lambda$ and $f \in \underset{\sim}{d}$, to construct B and $C \in 2^\omega$ such that (1) is true and

(3.1) if λ falls under Case 1 then $f \not\leq_{\mathrm{T}} (B \oplus C)^{(1+n)}$;

(3.2) if λ falls under Case 2 then $f \not\leq_{\mathrm{T}} (B \oplus C)^{(2+n)}$.

We now prove Lemmas 1 and 2, using forcing with uniformly recursively pointed perfect trees in an arithmetic setting. Fix a forcing language built from number variables, numerals, predicate constants for primitive recursive predicates on $2^\omega \times 2^\omega \times \omega$, and generic predicate constants $\underset{\sim}{B}$ and $\underset{\sim}{C}$. Build prenex sentences from $\exists$ and $\neg$, with the usual Π_i^0, Σ_i^0 classification. Conditions are as in [6]: pairs (P, Q) where P and Q are uniformly recursively pointed perfect trees from $L_\alpha \cap \omega^\omega$ and

$P \equiv_T Q$. (P, Q) extends (R, S) iff P and Q are subtrees of R and S respectively. $[P]$ is the set of characteristic functions, identified with members of $P\omega$, which lie along branches of P. $[P, Q] \doteq [P] \times [Q]$. $(P, Q) \Vdash \emptyset$ iff for any $(B, C) \in [P, Q]$, $(B, C) \models \emptyset$, where $\emptyset \in \Pi^0_{i \leq 2}$. The other clauses are standard:

$$(P, Q) \Vdash (\exists x)\, \emptyset \text{ iff for some } n < \omega\ (P, Q) \Vdash \emptyset\,(x/\underline{n});$$

$$(P, Q) \Vdash \neg \emptyset \text{ iff for every } (R, S) \text{ extending } (P, Q),$$
$$(R, S) \not\Vdash \emptyset, \text{ for } \emptyset \in \Sigma^0_{i>2}.$$

By Lemma 3.5 of [6], some condition extending (P, Q) decides $\emptyset$.

We must compute the definitional complexity over L_α of forcing restricted to $(\Sigma^0_{G(\lambda)} \cup \Pi^0_{G(\lambda)})$. The class of conditions and the extends relation are Σ^0_3 and Π^0_2 respectively. By Lemma 3.8 of [6], forcing restricted to $(\Sigma^2_0 \cup \Pi^0_2)$ is Σ^0_3. So if λ falls under Case 1.1, forcing restricted to $(\Sigma_{G(\lambda)} \cup \Pi_{G(\lambda)})$ is Δ_1 over L_α. Forcing restricted to $(\Sigma_3 \cup \Pi_3)$ is Π^1_1 over $L_\alpha \cap \omega^\omega$, so clearly Δ_2 over L_α. So for λ under Case 1, forcing for $(\Sigma_{G(\lambda)} \cup \Pi_{G(\lambda)})$ is $\Delta_{F(J(\lambda))+1}$ over L_α).

Suppose λ falls under Case 2. $L_\alpha \cap \omega^\omega$ is Π^1_1 absolute. So forcing restricted to $(\Sigma^0_3 \cup \Pi^0_3)$ is Π_1 over L_α. But by the Kleene basis theorem we can show that it is also Σ_1 over L_α; suppose $(P, Q) \Vdash \emptyset$ if and only if $(\forall f \in \omega^\omega) R(f, P, Q, \emptyset)$, where $R \in \Sigma^0_1$, W^X is the hyperjump of X. Then $(\forall f \in \omega^\omega) R(f, P, Q, \emptyset)$ iff $(\forall f \leq_T W^{(P \oplus Q)}) \cdot R(f, P, Q, \emptyset)$ iff $L_\alpha \Vdash (\exists \xi)$ (ξ admissible & $(P, Q) \in L_\xi$ & $(\forall f \in L_{\xi+1} \cap \omega^\omega) \cdot R(f, P, Q, \emptyset)$). So in Case 2, following up the definition of forcing, forcing for $(\Sigma^0_{i+3} \cup \Pi^0_{i+3})$ is Δ_{i+1} over L_α, thus for $(\Sigma^0_{G(\lambda)} \cup \Pi^0_{G(\lambda)})$ is $\Delta_{F(J(\lambda))+1}$ over L_α.

Let $\langle \emptyset_i \rangle_{i \in \omega}$ and $\langle A_i \rangle_{i \in \omega}$ be $\Delta_{n+1}(L_\alpha)$ enumerations of $(\Sigma_{G(\lambda)} \cup \Pi_{G(\lambda)})$ and $L_\alpha \cap 2^\omega$ respectively. The latter exists by the corollary to the Fundamental Theorem. Let

$$P_0 = Q_0 = \text{id}, \qquad (P_{2i+1}, Q_{2i+1}) = (P_{2i} * A_i, Q_{2i} * A_i);$$

$$(P_{2i+2}, Q_{2i+2}) = \text{the } <_L \text{ least extension of } (P_{2i+1}, Q_{2i+1}) \text{ deciding } \emptyset_i,$$

where $P * A$ is the canonical result of coding A into P; see [6, Lemma 3.3]. $\langle (P_i, Q_i) \rangle_{i \in \omega} \in \Delta_{n+1}(L_\alpha)$. Let $(B, C) = \bigcap_{i<\omega}(P_i, Q_i]$. The usual forcing = truth lemma states that $(B, C) \Vdash \emptyset_i$ iff $(P_{2i+1}, Q_{2i+1}) \Vdash \emptyset_i$. (1) follows easily from the odd steps and Lemma 3.7 of [6]. Since $(B \oplus C)^{(G(\lambda))}$ is defined by a $\Sigma_{G(\lambda)}$ formula, (2.1) and (2.2) are satisfied.

To prove Lemma 2 it shall be necessary to prove the following.

SUBLEMMA 1. *For any i and any condition (P, Q) there is an m and a condition (R, S) extending (P, Q) such that either*

$$(R, S) \Vdash \text{“}\neg \{\underline{i}\}^{(\underline{B} \oplus \underline{C})^{(G(\lambda)-1)}}(\underline{m}) \textit{ converges”}$$

or

$$(\exists k)(k \neq f(m) \ \&\ (R, S) \Vdash \text{“}\{\underline{i}\}^{(\underline{B} \oplus \underline{C})^{(G(\lambda)-1)}}(\underline{m}) = \underline{k}\text{”}).$$

Suppose (P, Q) and i are a counterexample, i.e. for any m and any (R, S) extending (P, Q):

(4) $(\exists(R^*, S^*)$ extending $(R, S))$ $(R^*, S^*) \Vdash$ "$\{i\}^{(\underline{B}\oplus\underline{C})^{(G(\lambda)-1)}}(\underline{m})$ converges";
and

(5) $(\forall k)$(if $(R, S) \Vdash$ "$\{i\}^{(\underline{B}\oplus\underline{C})^{(G(\lambda)-1)}}(\underline{m}) = k$" then $f(m) = \underline{k}$).
Thus $f(m) = k$ iff

$$L_\alpha \models (\exists(R, S))\ (R, S) \text{ extends } (P, Q)\ \&\ (R, S) \Vdash \text{“}\{\underline{i}\}^{(\underline{B}\oplus\underline{C})^{(G(\lambda-1))}}(\underline{m}) = \underline{k}\text{”}.$$

By the previous results on the definitional complexity of forcing, the above definition is Σ_n. Thus, because f is a function, $f \in \Delta_n(L_\alpha)$. But $L_\alpha \models \Delta_n$ CA. So $f \in L_\alpha$, contrary to choice of f.

We now finish the proof of Lemma 2. Let $\langle \varnothing_i \rangle_{i\in\omega}$ and $\langle A_i \rangle_{i\in\omega}$ be enumerations of $(\Sigma_{G(\lambda)-1} \cup \Pi_{G(\lambda)-1})$ and $L_\alpha \cap 2^\omega$ respectively; let

$$P_0 = Q_0 = \text{id};$$

$$(P_{3i+1}, Q_{3i+1}) = (P_{3i} * A_i, Q_{3i} * A_i);$$

$$(P_{3i+2}, Q_{3i+2}) = \text{an extension of } (P_{3i+1}, Q_{3i+1}) \text{ deciding } \varnothing;$$

$$(P_{3i+3}, Q_{3i+3}) = \text{an extension of } (P_{3i+2}, Q_{3i+2}) \text{ such that for some } m,$$

either

$(P_{3i+3}, Q_{3i+3}) \Vdash$ "$\neg\{i\}^{(\underline{B}\oplus\underline{C})^{(G(\lambda-1))}}(\underline{m})$ converges" or

for some $k \neq f(m)$,

$(P_{3i+3}, Q_{3i+3}) \Vdash$ "$\{i\}^{(\underline{B}\oplus\underline{C})^{(G(\lambda-1))}}(\underline{m}) = \underline{k}$".

Let $(B, C) = \bigcap_{i<\omega} [P_i, Q_i]$. (1) is immediate as in Lemma 1. Stages of the form $3i$ insure the truth of (3.1) and (3.2). Note that if $f \in \Delta_{n+1}(L_\alpha)$ the above construction can be made Δ_{n+1} over L_α. So for $\underline{d} \leq \underline{0}^{(\lambda)}$ we could choose $(\underline{b}, \underline{c})$ so that $\underline{d} \not\leq (\underline{b} \vee \underline{c})^{(G(\lambda)-1)}$ and $(\underline{b} \vee \underline{c})^{(G(\lambda))} \leq \underline{0}^{(\lambda)}$, so $(\underline{b} \vee \underline{c})^{(G(\lambda))} = \underline{0}^{(\lambda)}$ by Lemma 3.

Suppose that λ falls under Case 3. Recall that $J(\lambda) = \lambda = \omega\lambda$. Let $F(\lambda) = \omega\beta + n$. For some $E_{\lambda+\beta}$, $\underline{0}^{(\lambda)} = \deg(\text{Th}_n(E_{\lambda+\beta}))$. To prove Lemma 1 it suffices to construct B and $C \in 2^\omega$ such that (1) is true and

(8) For some $E_\beta[B \oplus C]$, $\text{Th}_n(E_\beta[B \oplus C]) \in \Delta_{n+1}(L_{\lambda+\beta})$;

(9) $\lambda + \beta < \omega_1^B$ and $\lambda + \beta < \omega_1^C$;

(9) implies that $\text{Ind}(\lambda) < \omega_1^{(B\oplus C)}$. Its purpose is more than decorative. Let $\underline{b} = \deg(B)$, $\underline{c} = \deg(C)$. $\text{Ind}^{(\underline{b}\vee\underline{c})}(F(\lambda)) = \text{Ind}^{(\underline{b}\vee\underline{c})}(\omega\beta) + n$. Suppose $\text{Ind}^{(\underline{b}\vee\underline{c})}(\omega\beta) = \omega\gamma + m$. Then for some $E_\gamma[B \oplus C]$, $(\underline{b} \vee \underline{c})^{(G(\lambda))} = \deg(\text{Th}_{m+n}(E_\gamma[B \oplus C]))$. If $\omega\beta + n < \omega\gamma + m$, we have no reason to expect that we can find B, C and $E_\gamma[B \oplus C]$ such that $\text{Th}_{m+n}(E_\gamma[B \oplus C]) \in \Delta_{n+1}(L_{\lambda+\beta})$. However, (9) insures that $\text{Ind}^{(\underline{b}\vee\underline{c})}(\omega\beta) = \omega\beta$. Thus (8) suffices for Lemma 1. For Lemma 2, suppose that $\underline{d} \notin I_\lambda$ and $f \in \underline{d}$. If λ falls under Case 3 it suffices to construct B and $C \in 2^\omega$ such that (1) and (9) are true and

(10) $f \notin \Sigma_n(L_\beta[B \oplus C])$.

As before, (9) insures that $(\underline{b} \vee \underline{c})^{(F(\lambda))} = \deg(\text{Th}_n(E_\beta[B \oplus C]))$ for some $E_\beta[B \oplus C]$. Furthermore, $\bigcup I_{\beta}^{(\underline{b}\vee\underline{c})} = L_\beta[B \oplus C] \cap \omega^\omega$.

For both Lemmas 1 and 2 we shall obtain $B = B_0 \oplus B_1$ and $C = C_0 \oplus C_1$, such that B_0 and C_0 are Turing upper-bounds on $L_\lambda \cap \omega^\omega$ and such that B_1 and C_1

are wellfounded trees of height high enough to insure that $\omega_1^{B_1}$ and $\omega_1^{C_1}$ are greater than $\lambda + \beta$.

In Case 3, the proofs of Lemmas 1 and 2 use forcing for a ramified language. Fix a set $\boldsymbol{P}$ of one-place predicate constants. Let the lexicon of $\boldsymbol{L}^*_\eta[\boldsymbol{P}]$ consist of members of $\boldsymbol{P}$, $\neg$, &, $\exists$, parentheses, countably many unranked variables, and for each $\xi < \eta$, countably many variables of rank ξ. Let the formation rules be as usual, except that $\ulcorner \underline{P}(v) \urcorner$ for $\underline{P} \in \boldsymbol{P}$ is well-formed iff v has rank 0. Call a formula with no bound unranked variables "ranked". Let $C_0[\boldsymbol{P}]$ be a set of standard names for members of L_0. Let $C_{\xi+1}[\boldsymbol{P}]$ be the set of terms $\hat{x}^\xi \phi(x_1/c_1, \ldots, x_k/c_k)$ such that ϕ is ranked with exactly the free variables $x^\xi, x_1, \ldots, x_k$, no bound variables of rank $>\xi, c_1, \ldots, c_k \in \bigcup_{\alpha \le \xi} C_\alpha[\boldsymbol{P}]$. If ξ is a limit, let $C_\xi[\boldsymbol{P}] = \bigcup_{\alpha<\xi} C_\alpha[\boldsymbol{P}]$. Let $\boldsymbol{L}_\eta[\boldsymbol{P}]$ be the language which results by supplementing $\boldsymbol{L}^*_\eta[\boldsymbol{P}]$ by the constants in $C_\eta[\boldsymbol{P}]$.

Identify terms and formulae of $\boldsymbol{L}_\eta[\boldsymbol{P}]$ with members of L_η in some fixed way. The rank of term c, $\rho(c)$, is the least ξ such that $c \in C_\xi[\boldsymbol{P}]$. A formula ϕ of $L_\eta[\boldsymbol{P}]$ is ranked iff it has no bound unranked variables. Its rank, $\rho(\phi)$, is the supremum of the ranks of its contained constants, predicate constants, and bound variables, where members of $\boldsymbol{P}$ have rank 1. Suppose $\boldsymbol{P} = \{\underline{P}_0, \ldots, \underline{P}_k\}$. For $i \le k$, suppose $\underline{P}_i$ is assigned to $P_i \subseteq \omega$. $L_\eta[P_0, \ldots, P_k]$ is defined in the obvious way, and obviously equals $L_\eta[P_0 \oplus \cdots \oplus P_k]$. $\langle L_\eta[P_0, \ldots, P_k]; \in | L_\eta[P_0, \ldots, P_k], P_0, \ldots, P_k; L_\eta[P_0, \ldots, P_k] \rangle$ is the intended structure for $\boldsymbol{L}_\eta[\boldsymbol{P}]$. Note that for $\eta > 0$, the intended structure contains each P_i both as an extension of $\underline{P}_i$ and as an individual denoted by "$\hat{x}^0(\underline{P}_i(x^0))$". Variables of rank $\xi < \eta$ range over $L_\xi[P_0, \ldots, P_k]$. $c \in C_\eta[\boldsymbol{P}]$ denotes a member of $L_{\rho(c)}[P_0, \ldots, P_k]$. Thus if ϕ is ranked, ϕ is interpretable over $L_{\rho(\phi)}[P_0, \ldots, P_k]$.

Let $\Pi_0 = \Sigma_0 = \{\phi \mid \phi$ is ranked formula of $\boldsymbol{L}_\eta[\boldsymbol{P}]\}$. Define Σ_n and Π_n as usual. For the proofs to follow, let $\boldsymbol{P} = \{\underline{B}_0, \underline{B}_1, \underline{C}_0, \underline{C}_1\}$ and let $L_\beta[\boldsymbol{P}] = L$; let $C_\xi[\boldsymbol{P}] = C_\xi$.

To insure the truth of (9) we need conditions more complicated than those used up to now. Let δ be the maximum ordinal $\le \lambda + \beta$ which is either admissible or a limit of admissibles.

A modified Steel condition is a finite function z into δ such that $\text{dom}(z) \subseteq \text{Seq} - \{\langle\ \rangle\}$, $\text{dom}(z)$ is closed under initial segments, and for $\sigma, \tau \in \text{dom}(z)$, if σ properly extends τ then $z(\sigma) < z(\tau)$. (Think of $\langle\ \rangle$ as belonging to $\text{dom}(z)$ and $z(\langle\ \rangle) = \delta$.) If y and z are such conditions, z extends y iff $z \upharpoonright \text{dom}(y) = y$.

Let a condition be a quadruple (P, Q, y, z), where P and Q are Turing equivalent uniformly recursively pointed perfect trees in L_λ and y and z are modified Steel conditions. Understand "extends" componentwise. Hereafter, "K" etc. shall range over conditions. Let the height of K, $\text{ht}(K)$, $=$ $\max(\text{range}(y) \cup \text{range}(z))$. Let $\boldsymbol{K}$ = the set of conditions; $\boldsymbol{K}_\xi = \{K \mid \text{ht}(K) < \xi\}$, where ξ is a limit ordinal.

Let $<^*$ be the wellfounded relation on sentences of L introduced by Cohen in his definition of forcing [3, p. 115]. Forcing for sentences in L is defined by induction on $<^*$. Let $K = (P, Q, y, z)$.

$K \Vdash \varnothing$ iff $\rho(\varnothing) = 0$ and $\varnothing$ is true;

$K \Vdash \underline{B}_0(\underline{k})$ iff for every $X \in [P]$, $k \in X$;

$K \Vdash \underline{C}_0(\underline{k})$ iff for every $X \in [Q]$, $k \in X$;

$K \Vdash \underset{\sim}{B}_1(\underset{\sim}{k})$	iff $k \in \mathrm{dom}(y)$ or $k = \langle\ \rangle$;
$K \Vdash \underset{\sim}{C}_1(\underset{\sim}{k})$	iff $k \in \mathrm{dom}(z)$ or $k = \langle\ \rangle$;
$K \Vdash -\varnothing$	iff for every K' extending K, $K' \nVdash \varnothing$, for $\rho(-\varnothing) > 0$;
$K \Vdash \varnothing \ \&\ \psi$	iff $K \Vdash \varnothing$ and $K \Vdash \psi$ for $\rho(\varnothing \ \&\ \psi) > 0$;
$K \Vdash (\exists x^\xi)\varnothing$	iff for some $c \in C_\xi$, $K \Vdash \varnothing(x^\xi/c)$, for $\rho((\exists x^\xi)\varnothing) > 0$;
$K \Vdash (\exists x)\varnothing$	iff for some $c \in C$, $K \Vdash \varnothing(x/c)$;
$K \Vdash c_1 \in c_2$	iff either (i) $\rho(c_1) < \rho(c_2)$ and c_2 is $\hat{x}^\xi\varnothing$ and $K \Vdash \varnothing(x^\xi/c_1)$, or (ii) for some c_3, $\rho(c_3) < \rho(c_2)$ and $K \Vdash ((\forall x^\xi)(x^\xi \in c_1 \leftrightarrow x^\xi \in c_3) \ \&\ c_3 \in c_2)$ where $\rho(c_1) = \xi + 1$, and in either case $\rho(c_1 \in c_2) > 0$.

Let $|\varnothing|$ be the ordinal for the position of $\varnothing$ in $<^* \restriction \{\psi \mid \psi \in \Pi_0 \text{ and } \rho(\psi) > 0\}$. Thus $\sup|\varnothing| \le \omega\cdot(\lambda+\beta)$. In order to refer to $|\varnothing|$ in $L_{\lambda+\beta}$, code $\omega\cdot(\lambda+\beta)$ into $\omega \times (\lambda+\beta)$ in the canonical way; we shall freely identify $|\varnothing|$ with the appropriate member of $\omega \times (\lambda+\beta)$. For $\eta < \lambda+\beta$ and $\varnothing$ such that $\rho(\varnothing) > 0$, $|\varnothing| < \omega\cdot\eta$ iff $\rho(\varnothing) < \eta$. From right to left this is clear; if $|\varnothing| < \omega\cdot\eta$, then for some $k < \omega$, $|\varnothing| = \omega\cdot\rho(\varnothing) + k$; so $\omega\cdot\rho(\varnothing) < \omega\eta$; so $\rho(\varnothing) < \eta$.

SUBLEMMA 2. *Forcing restricted to* $(\Sigma_n \cup \Pi_n)$ *sentences is* Δ_{n+1} *over* $L_{\lambda+\beta}$.

Let $\Vdash$ be the characteristic function for forcing. We shall prove that $\Vdash \restriction (\boldsymbol{K} \times \Pi_0) \in \Sigma_1(L_{\lambda+\beta})$. Because it is a function, it then belongs to $\Delta_1(L_{\lambda+\beta})$.

For $\omega\eta \le \delta$, let $\Vdash_{\omega\eta} = \Vdash \restriction (\boldsymbol{K}_{\omega\eta} \times \{\varnothing \mid \rho(\varnothing) = 0 \text{ or } |\varnothing| < \eta\})$. For $\delta \le \eta < \lambda+\beta$ and $k < \omega$, let $\Vdash^{\omega\eta+k} = \Vdash \restriction (\boldsymbol{K} \times \{\varnothing \mid |\varnothing| < \omega\eta + k \text{ or } \rho(\varnothing) = 0\})$. We shall find Σ_1 formulas $\Phi_1(f)$ and $\Phi_2(f, K, \varnothing, i)$ such that for $\omega\xi \le \delta$:

(12) if $\eta < \xi$, $\Vdash_{\omega\eta} \in L_{\omega\xi}$;

(13) $L_{\omega\xi} \models \Phi_1(f)$ iff for some $\eta < \xi$, $f = \Vdash_{\omega\eta}$;

(14) if ξ is a successor, $\Vdash_{\omega\xi}(K, \varnothing) = i$ iff

$$L_{\omega\xi} \models (\exists f)(\Phi_1(f) \ \&\ \Phi_2(f, K, \varnothing, i));$$

(15) if ξ is a limit, $\Vdash_{\omega\xi}(K, \varnothing) = i$ iff

$$L_{\omega\xi} \models (\exists f)(\Phi_1(f) \ \&\ f(K, \varnothing) = i).$$

We shall find a Σ_1 formula $\Phi_3(f)$ and for each $k < \omega$ a formula $\chi_k(f, K, \varnothing, i)$ such that:

(16) for $k < \omega$ $\Vdash^{\delta+k}(K, \varnothing) = i$ iff $L_\delta \models (\exists f)(\Phi_1(f) \ \&\ \chi_k(f, K, \varnothing, i))$; and for ξ such that $\delta < \xi \le \lambda + \beta$;

(17) for $\delta \le \eta < \xi$ and $k < \omega$, $\Vdash^{\omega\eta+k} \in L_\xi$;

(18) $L_\xi \models \Phi_3(f)$ iff for some η and k, $\delta \le \eta < \xi$ and $k < \omega$, $f = \Vdash^{\omega\eta+k}$;

(19) for ξ such that $\delta < \xi < \lambda + \beta$ and $k < \omega$,

$$\Vdash^{\omega\xi+k}(K, \varnothing) = i \quad \text{iff } L_\xi \models (\exists f)(\Phi_3(f) \ \&\ \chi_k(f, K, \varnothing, i));$$

and finally for $\varnothing \in \Pi_0$;

(20) $\Vdash (K, \varnothing) = i$ iff $L_{\lambda+\beta} \models (\exists f)(\Phi_3(f) \ \&\ \chi(f, K, \varnothing, i))$.

As a first approximation to $\Phi_1(f)$, consider

(21) f is a function into 2 & $(\exists\eta)$ ($\omega\eta$ exists & $\mathrm{dom}(f) = \boldsymbol{K}_{\omega\eta} \times \{\varnothing \mid |\varnothing| < \eta\}$ & $(\forall\langle K, \varnothing\rangle \in \mathrm{dom}\, f)$:

(1) $\rho(\varnothing) = 0 \Rightarrow (f(K, \varnothing) = 1$ iff $\varnothing$ is true),

(2) $(\forall k \in \omega)$ $(\varnothing = \ulcorner \underline{B}_0(\underline{k})\urcorner \Rightarrow (f(K, \varnothing) = 1$ iff ...)),

⋮

(6) $(\forall\phi \in \{\varnothing \mid 0 < |\varnothing| < \eta\})(\varnothing = \ulcorner\neg\phi\urcorner \Rightarrow (f(K, \varnothing) = 1$ iff $(\forall K')(K'$ extends $K \Rightarrow f(K', \phi) = 0)))$,

⋮

(8) $(\forall\xi)$ $(\forall\phi \in \{\varnothing \mid |\varnothing| < \eta\})(\varnothing = \ulcorner(\exists x^{\xi})\phi\urcorner \Rightarrow (f(K, \varnothing) = 1$ iff $(\exists c \in C_{\xi})$ $(f(K, \phi(X^{\xi}/c) = 1))))$

⋮).

What are the failings of (1)? The '$(\forall\xi)$' in clause (8) is unrestricted. But for $|\varnothing| < \eta$, if $\varnothing = (\exists^{\xi}x)\phi$, $\xi \le \rho(\varnothing) \le |\varnothing| < \eta$. Thus it may be replaced by '$(\forall\xi < \eta)$'. More seriously, the quantifier over conditions in clause (6) is not only unrestricted within $L_{\omega\xi}$ for $\omega\,\xi \le \delta$, but if $\omega\xi < \delta$, its intended range includes more than $L_{\omega\xi}$. We shall show that in fact it may be replaced by '$(\forall K' \in \boldsymbol{K}_{\omega\eta})$'. This shall require several facts about modified Steel conditions due, essentially, to Steel [10].

If x and y are modified Steel conditions and η is a limit ordinal, then x is an η-retag of y iff: $\mathrm{dom}(x) = \mathrm{dom}(y)$; if $x(\sigma) < \eta$ then $x(\sigma) = y(\sigma)$; if $x(\sigma) \ge \eta$ then $y(\sigma) \ge \eta$. Notice that 'is an η-retag of' is symmetric.

Retagging Lemma. *Suppose that x, x' and y are modified Steel conditions, x' extends x, and $\xi < \eta$ are two limit ordinals. If y is an η-retag of x then some modified Steel condition y' extends y, and is a ξ-retag of x'.*

Proof. Let

$$r(\sigma) = \begin{cases} 0 & \text{if } x'(\sigma) < \xi, \\ 1 + \max\{r(\sigma^\frown j) \mid \sigma^\frown j \in \mathrm{dom}\, x'\} & \text{otherwise.} \end{cases}$$

Clearly $\mathrm{dom}(r) = \mathrm{dom}(x')$, since $\mathrm{dom}(x')$ is finite and wellfounded under $\prec$, where $\sigma \prec \tau$ iff τ properly extends σ.
Let

$$y'(\sigma) = \begin{cases} x'(\sigma) & \text{if } r(\sigma) = 0, \\ 1 + \max\{y'(\sigma^\frown j) \mid \sigma^\frown j \in \mathrm{dom}\, x'\} & \text{if } r(\sigma) \ne 0 \text{ and } \sigma \notin \mathrm{dom}\, y, \\ y(\sigma) & \text{otherwise.} \end{cases}$$

Clearly $\mathrm{dom}(y') = \mathrm{dom}(x')$.

Claim. If $\sigma \prec \tau \in \mathrm{dom}(y')$ then $y'(\sigma) > y'(\tau)$.

This is straightforward unless $\sigma \in \mathrm{dom}(y)$ and $\tau \notin \mathrm{dom}(y)$. Then, by induction on $r(\tau)$, $y'(\tau) \le x'(\tau)$. But $x'(\tau) < x'(\sigma) = x(\sigma)$. If $x(\sigma) < \eta$, $x(\sigma) = y(\sigma) = y'(\sigma)$, yielding $y'(\tau) < y'(\sigma)$. If $x(\sigma) \ge \eta$, $y'(\sigma) \ge \eta$. But by induction on $r(\tau)$, $y'(\tau) < \xi + \omega \le \eta$. So again $y'(\tau) < y'(\sigma)$. Thus y' is a modified Steel condition. Clearly y' is a ξ-retag of x'. Suppose $\sigma \in \mathrm{dom}(y)$. If $r(\sigma) \ne 0$, $y'(\sigma) = y(\sigma)$. If $r(\sigma) = 0$, $y'(\sigma) = x'(\sigma) = x(\sigma) = y(\sigma)$ because $x(\sigma) < \xi < \eta$ and y is an η-retag of x. Thus y' extends y. Note that if $\eta \le \mathrm{ht}\, y$, $\mathrm{ht}(y') = \mathrm{ht}(y)$; if $\mathrm{ht}(y) < \eta$, $\mathrm{ht}(y') < \xi + \omega$.

COROLLARY 1. *If x' and x are modified Steel conditions, x' extending x, and ξ is a limit ordinal, then there is a modified Steel condition y extending x such that y is a ξ-retag of x and* $\mathrm{ht}(y) < \max(\mathrm{ht}(x) + \omega, \xi + \omega)$.

PROOF. Because x is a $(\xi + \omega)$-retag of itself, the desired y exists by the retagging lemma and the concluding remark in its proof.

Let condition $\langle P, Q, x, y\rangle$ be a ξ-retag of $\langle P, Q, x', y'\rangle$ iff x and y are ξ-retags of x and y' respectively. The previous lemma and corollary remain true when modified Steel conditions are replaced by conditions.

COROLLARY 2. *For $\varnothing \in \Pi_0$ such that $|\varnothing| \le \xi$ or $\rho(\varnothing) = 0$ and K' an $\omega \cdot \xi$-retag of K: $K \Vdash \varnothing$ iff $K' \Vdash \varnothing$.*

PROOF. If $\rho(\varnothing) = 0$, this is trivial. We now induce on $|\varnothing|$. Clearly $K \Vdash \underline{B}_0(\underline{k})$ iff $K' \Vdash \underline{B}_0(\underline{k})$.

Similarly for the other base clauses. All induction steps except the one for negation are trivial. Suppose $K \nVdash \neg\varnothing$. Let K_0 extend K such that $K_0 \Vdash \varnothing$. Since $|\neg\varnothing| \le \xi$, $|\varnothing| < \xi$. The retagging lemma provides K_0' extending K' which is an $\omega\cdot|\varnothing|$-retag of K. By induction hypothesis, $K_0 \Vdash \varnothing$ iff $K_0' \Vdash \varnothing$. Thus $K' \nVdash \neg\varnothing$. The converse follows symmetrically.

COROLLARY 3. *For $K \in \boldsymbol{K}_{\omega\xi}$ and $\varnothing$ such that $|\varnothing| < \xi$, $K \Vdash \neg\varnothing$ iff for any $K' \in \boldsymbol{K}_{\omega\xi}$, if K' extends K, $K' \nVdash \varnothing$.*

PROOF. ($\Rightarrow$) is clear. ($\Leftarrow$) Suppose K^* extends K, $K^* \Vdash \varnothing$. By Corollary 1 there is a K' extending K, K' an $\omega\cdot|\varnothing|$-retag of K^*, and $\mathrm{ht}(K^*) < \max(\mathrm{ht}(K) + \omega, \omega\cdot|\varnothing| + \omega) \le \omega\xi$. Thus $K^* \Vdash \varnothing$ by Corollary 2 and $K^* \in \boldsymbol{K}_{\omega\xi}$. So the quantifier restriction in (*) (6) may be introduced.

We now construct Φ_2. Let t be a Σ_1 term such that for any η, $L_{\omega(\eta+1)} \models t = \eta$. As a first approximation let $\Phi_2(f, K, \varnothing, i)$ be:

$(|\varnothing| < t \;\&\; f(K, \varnothing) = i) \vee (|\varnothing| = t \;\&\; \mathrm{dom}(f) = (\boldsymbol{K}_{\omega\cdot t} \times \{\varnothing \mid |\varnothing| < t\} \;\&$

(1) $\rho(\varnothing) = 0 \Rightarrow (i = 1 \text{ iff } \varnothing \text{ is true})$

(2) $(\forall k \in \omega)(\varnothing = \ulcorner \underline{B}_0(\underline{k}) \urcorner \Rightarrow (i = 1 \text{ iff } \ldots))$

⋮

(6) $(\forall \psi \in \{\varnothing \mid |\varnothing| < t\})(\varnothing = \ulcorner \neg\psi \urcorner \Rightarrow (i = 1 \text{ iff } (\forall K' \in \boldsymbol{K}_{\omega t})(f(K', \psi) = 0)))$

⋮).

The arguments used in revising (*) show that for $|\varnothing| = \eta$,

$$L_{\omega(\eta+1)} \models \Phi_1(f) \;\&\; \Phi_2(f, K, \varnothing, i) \quad \text{iff } f = \Vdash_{\omega\eta} \text{ and } \Vdash_{\omega(\eta+1)}(K, \varnothing) = i.$$

We now prove (12)–(15) by simultaneous induction. (12) is vacuously true for $\xi = 0$. For any ξ such that $\omega\xi \le \delta$, if (12) is true, so is (13). Then so are (14) and (15). Thus $\Vdash_{\omega\xi} \in L_{\omega\xi+1}$, implying (12) for $\xi + 1$. If ξ is a limit, (12) holds for ξ induction. Thus (12)–(15) are all true. Consequently $\Vdash_\delta \in \Sigma_1(L_\delta)$ and thus $\Vdash_\delta \in L_{\delta+1}$. If $\delta = \lambda + \beta$, this proves Sublemma 2.

If $\delta < \lambda + \beta$, notice that $\boldsymbol{K} \in L_{\delta+1} \subseteq L_{\lambda+\beta}$. $\Phi_3(f)$ may be taken to have the form:

$$(f = \Vdash_\delta) \vee (\exists\eta)(\exists k < \omega)(\delta \le \eta \;\&\; \mathrm{dom}(f) = \boldsymbol{K} \times \{\varnothing \mid |\varnothing| < \omega\eta + k\} \;\&\; \Phi'(f)),$$

where the construction of Φ' is easy. Notice that the quantifier $(\forall K)$ in clause (6) may be restricted to $\boldsymbol{K}$. Let $\chi_0(f, K, \varnothing, i)$ be $f(k, \varnothing) = i$. Suppose χ_k has been constructed. Let $\chi_{k+1}(f, K, \varnothing, i)$ be

⋮

(5) $(\forall\psi)(\varnothing = \ulcorner - \psi \urcorner \Rightarrow (i = 1 \text{ iff } (\forall K' \text{ extending } K)(X_k(F, K', \psi, 0))))$ &
(6) $(\forall\psi)(\forall\psi')(\varnothing = \ulcorner \psi \wedge \psi' \urcorner \Rightarrow (i = 1 \text{ iff } (X_k(F, K, \psi, 1) \;\&\; (X_k(F, K, \psi', 1))))$,

⋮

(16) is clear, using (13), yielding (17) for $\xi = \delta + 1$. Assume (17) for arbitrary ξ. By construction, (18) and (19) are true for ξ. Thus (17) is true for $\xi + 1$. If ξ is a limit and (17) is true for all $\xi' < \xi$, (17) is true for ξ. Thus (17)–(19) are true for all ξ such that $\delta < \xi \leq \lambda + \beta$. So (20) is also true, proving that $\Vdash \upharpoonright (\boldsymbol{K} \times \Pi_0) \in \Sigma_1(L_{\lambda+\beta})$. Sublemma 2 is proven.

Suppose $\langle K_i \rangle_{i\in\omega}$ is a Σ_n generic sequence, i.e. for each $i \in \omega$, K_{i+1} extends K_i and for every $\varnothing \in \Sigma_n$ there is an i such that K_i decides $\varnothing$. Where $K_i = \langle P_i, Q_i, x_i, y_i \rangle$, let $T_0 = \bigcup x_i$, $T_1 = \bigcup y_i$, $B_1 = \text{dom}(T_0)$ and $C_1 = \text{dom}(T_1)$. By the usual forcing = truth lemma, for any $k \in \omega$, $k \in B_1$ iff for some i, $K_i \Vdash \underset{\sim}{B}_1(\underset{\sim}{k})$, and similarly for C_1. By definition of modified Steel conditions, $T_0 \colon \subseteq \text{seq} - \{\langle\,\rangle\} \to \delta$ such that if $\sigma \prec \tau \in \text{dom}(T_0)$, $\sigma \in \text{dom}(T_0)$ and $T_0(\sigma) > T_0(\tau)$. Similarly for T_1. Thus $\langle B_1, \prec \rangle$ and $\langle C_1, \prec \rangle$ are wellfounded.

SUBLEMMA 3. *If $\sigma \in B_1$, for any ξ, $|\sigma|_{B_1} = \xi$ iff for some i, $x_i(\sigma) = \xi$.*

Furthermore, the order type of $\langle B_1, \prec \rangle = \delta$. *Similarly for C_1 and y_i.* Proof by induction on $|\sigma|_{B_1}$. Suppose $|\sigma|_{B_1} = \xi$. For some i, $\sigma \in \text{dom}(x_i)$ and $K_i \Vdash$ "$|\underset{\sim}{\sigma}|_{B_1} = \underset{\sim}{\xi}$". By the induction hypothesis ($\Leftarrow$), $|\sigma|_{B_1} \leq x_i(\sigma)$. So $x_i(\sigma) \geq \xi$. If $x(\sigma) > \xi$, let $K = \langle P_i, Q_i, x_i \cup \{\langle \sigma^\frown j, \xi \rangle\}, y_i \rangle$, for some j such that $\sigma^\frown j \notin \text{dom}(x_i)$, and form another generic sequence extending K and yielding $\hat{B}_1$ in place of B_1. By choice of i and forcing = truth, $|\sigma|_{\hat{B}_1} = \xi$. But by the induction hypothesis ($\Leftarrow$), $|\sigma^\frown j|_{\hat{B}_1} = \xi$. Contradiction. Thus $x_i(\sigma) = \xi$. Now suppose that $x_i(\sigma) = \xi$. If $|\sigma|_{\hat{B}_1} > \xi$, for some j, $\sigma^\frown j \in B_1$ and $|\sigma^\frown j|_{\hat{B}_1} \geq \xi$. By the induction hypothesis ($\Rightarrow$) there is an $i' \geq i$ such that $x_{i'}(\sigma^\frown j) \geq \xi$, which is impossible. If $\eta = |\sigma|_{B_1} < \xi$, select $i' \geq i$ such that $K_{i'} \Vdash$ "$|\underset{\sim}{\sigma}|_{B_1} = \underset{\sim}{\eta}$". Where $K = \langle P_{i'}, Q_{i'}, x_i \cup \{\langle \sigma^\frown j, \eta \rangle\}, y_{i'} \rangle$ for j such that $\sigma^\frown j \notin \text{dom } x_i$, form another generic sequence extending K and yielding $\hat{B}_1$ in place of B. By choice of i and forcing = truth, $|\sigma|_{\hat{B}_1} = \eta$. By the induction hypothesis ($\Leftarrow$), $|\sigma^\frown j|_{\hat{B}_1} = \eta$.Contradiction. Thus $|\sigma|_{B_1} = \xi$. Suppose $\sup\{|\sigma|_{B_1} | \sigma \in B_1\} = \xi < \delta$. For some i, $K_i \Vdash$ "$\sup\{|\sigma|_{B_1} | \underset{\sim}{B}_1(\sigma)\} = \underset{\sim}{\xi}$". Let $K = \langle P_i, Q_i, x_i \cup \{\langle \langle j \rangle, \xi + 1 \rangle\}, y_i \rangle$ for $\langle j \rangle \in \text{dom}(x_i)$. Form a generic sequence extending K and yielding $\hat{B}_1$ in place of B_1. By forcing = truth, $\sup\{|\sigma|_{\hat{B}_1} | \sigma \in \hat{B}_1\} = \xi$. But by previous parts of this sublemma, $|\langle j \rangle|_{\hat{B}_1} = \xi + 1$. Contradiction. A symmetric argument applies to C_1.

We now construct B and C. Let $\langle A_i \rangle_{i\in\omega}$ and $\langle \varnothing_i \rangle_{i\in\omega}$ be enumerations of $L_\lambda \cap 2^\omega$ and the Σ_n sentences of $\boldsymbol{L}$, both members of $\Delta_{n+1}(L_{\lambda+\beta})$. Such enumerations exist by the corollary to the Fundamental Theorem. Define $\langle K_i \rangle_{i\in\omega} \in \Delta_{n+1}(L_{\lambda+\beta})$ by:

$$K_0 = \langle \text{id}, \text{id}, \wedge, \wedge \rangle;$$

$$K_{2i+1} = \text{the } <_L\text{-least condition extending } K_{2i} \text{ and deciding } \varnothing_i;$$

$$K_{2i+2} = \langle P * A_i, Q * A_i, x, y \rangle \quad \text{where } K_{2i+1} = \langle P, Q, x, y \rangle.$$

Letting $K_i = \langle P_i, Q_i, x_i, y_i \rangle$, let $B_0 = \bigcap_{i<\omega}[P_i]$, $C_0 = \bigcap_{i<\omega}[Q_i]$ and B_1, C_1, T_0 and T_1 as above. Let $B = B_0 \oplus B_1$ and $C = C_0 \oplus C_1$. By the standard argument, if $f \leq_T B$ and $f \leq_T C$, $f \in L_{\lambda+\beta}$, and so $f \in L_\lambda$. By the even stages of the construction, for any $f \in L_\lambda$, $f \leq_T B_0$ and $f \leq_T C_0$. Thus (1) is true. Because $\delta < \omega_1^{B_1}$ and $\delta < \omega_1^{C_1}$, (9) is true. Because $\langle K_i \rangle_{i\in\omega} \in \Delta_{n+1}(L_{\lambda+\beta})$, there is an M, a term-model copy of $L_\beta[B, \oplus C]$, such that the $\mathrm{Th}_n(M) \in \Delta_{n+1}(L_{\lambda+\beta})$. Because $\Delta_{n+1}(L_{\lambda+\beta})$ contains a counting of the set of terms C, the preimage of ε_M under this counting is the desired $E_\beta[B \oplus C]$, verifying (8).

We now prove Lemma 2 for Case 3. Recall that $f \in \omega^\omega - L_\lambda$. The language $\boldsymbol{L}$ is as before. We now extend Sublemma 1 to this setting.

SUBLEMMA 4. *$\varnothing(x^0, y^0) \in \Sigma_n$, $K \in \boldsymbol{K}$. There are $k < \omega$ and K' extending K such that either $K' \Vdash \neg(\exists y^0)\varnothing(\underline{k}, y^0)$ or for some $m \neq f(k)$, $K' \models \varnothing(\underline{k}, \underline{m})$.*

PROOF. Suppose not. Then for any $k \in \omega$ and any K' extending K there is a K^* extending K' s.t. $K^* \Vdash (\exists y^0)\varnothing(\underline{k}, y^0)$, and for any such K', if $K' \Vdash \varnothing(\underline{k}, \underline{m})$ then $f(k) = m$. This gives us a Σ_n definition of f over $L_{\lambda+\beta}$. Thus $f \in \Delta_n(L_{\lambda+\beta})$. But $L_{\lambda+\beta} \models \Delta_n\mathrm{CA}$. So $f \in L_{\lambda+\beta}$. Contradiction. Select such a k and K' and call them $k(K, \varnothing)$ and $K'(K, \varnothing)$ respectively.

We may now construct B and C. Let $\langle A_i \rangle_{i\in\omega}$ and $\langle \varnothing_i \rangle_{i\in\omega}$ enumerate $L_\lambda \cap 2^\omega$ and the Σ_n sentences of $\boldsymbol{L}$. Let $\langle K_i \rangle_{i\in\omega}$ be such that:

$K_0 = \langle \mathrm{id}, \mathrm{id}, \wedge, \wedge \rangle$;

K_{3i+1} decides $\varnothing_i$;

$K_{3i+2}(P * A_i, Q * A_i, x, y)$ where the $K_{3i+1} = (P, Q, x, y)$;

$K_{3i+3} = K_{3i+2}$ if $\varnothing_i$ does not contain exactly the free variables x^0 and y^0, $K'(K_{3i+2}, \varnothing_i)$ otherwise.

Form B_0, B_1, C_0, C_1, B, C from $\langle K_i \rangle_{i\in\omega}$ as before. As with Lemma 1, (1) and (9) are true. Suppose $f \in \Sigma_n(L_\beta[B \oplus C])$. Then f is defined over $L_\beta[B \oplus C]$ by some $\varnothing_i(x^0, y^0)$. For some j, $K_j \Vdash \varnothing_i(\underline{k}, \underline{f}(\underline{k}))$ where $k = k(K_{3i+2}, \varnothing_i)$. But either $K_i \Vdash \neg(\exists y^0)\varnothing_i(\underline{k}, y^0)$ or $K_i \Vdash \varnothing_i(\underline{k}, \underline{m})$ for $m \neq f(k)$. Contradiction. Thus (10) is also true.

Again we note that if $\underline{d} \leq \underline{0}^{(\lambda)}$, $(\underline{b}, \underline{c})$ could be constructed so that $(\underline{b} \vee \underline{c})^{(G(\lambda))} \leq \underline{0}^{(\lambda)}$, thus $(\underline{b} \vee \underline{c})^{(G(\lambda))} = \underline{0}^{(\lambda)}$ by Lemma 3.

We now turn to Lemma 3. Suppose that $(\underline{b}, c)$ is I_λ-exact, $B \in \underline{b}$ and $C \in \underline{c}$. Let $J(\lambda) = \omega\gamma$. In Cases 1 and 2 we want to construct a real $\mathrm{Th}_{F(J(\lambda))}(E_\gamma)$ which is recursive in $(B \oplus C)^{(G(\lambda))}$. By an easy modification of Definition 8 from [1], there is an operator $*$ on $2^\omega \times \omega$ such that (i) $(X, y)^* \leq_T X^{(\omega)}$ uniformly in y, and (ii) for any E_ξ and $b \in \omega$ such that for no $x \in \mathrm{Fld}(E_\xi)$, $(x)_1 = b$, $(\mathrm{Th}_0(E_\xi), b)^* = \mathrm{Th}_0(E_{\xi+1})$ for an $E_{\xi+1}$ extending E_ξ. We also note that the relation $X = Y^{(\omega)}$ is Π_1^0 over $2^\omega \times 2^\omega$ (although it is only Π_2^0 over $P\omega \times P\omega$!).

Case 1.1. $F(\lambda) = 0$ and $G(\lambda) = 2$. If $\gamma = \gamma' + 1$, γ' is an L-index for otherwise λ falls under Case 3. Applying the corollary to the Fundamental Theorem to γ', there is a real $\mathrm{Th}_0(E_{\gamma'}) \in L_\gamma$. $E_{\gamma'}$ may be chosen so that for any $x \in \mathrm{Fld}(E_{\gamma'})$, $(x)_1 \neq 0$. Let $\mathrm{Th}_0(E_\gamma) = (\mathrm{Th}_0(E_{\gamma'}), 0)^* \leq_T \mathrm{Th}_0(E_{\gamma'})^{(\omega)}$. By results in [4], $(B \oplus C)^{(2)}$ can compute a nice parametrization of $L_\gamma \cap \omega^\omega$. Thus $\mathrm{Th}_0(E_{\gamma'})^{(\omega)} \leq_T (B \oplus C)^{(2)}$.

Now suppose that γ is a limit. Select a $\gamma' < \gamma$ such that $[\gamma', \gamma)$ contains no M-gaps. As before, select a real $\mathrm{Th}_0(E_{\gamma'}) \in L_{\gamma'+1}$. By choice of γ', there is a linear system of notation $R \in L_{\gamma'+1}$ of height $(\gamma - \gamma')$. Working over L_γ we construct a sequence $\langle \mathrm{Th}_0(E_x)\rangle_{x\in \mathrm{Fld}(R)}$, each E_x is an $E_{|x|_R}$, as follows.

$$\mathrm{Th}_0(E_x) = \begin{cases} \mathrm{Th}_0(E_{\gamma'}) & \text{if } |x|_R = 0; \\ (\mathrm{Th}_0(E_y), y)^* & \text{if } |x|_R = |y|_R + 1; \\ \bigcup_{y<R} \mathrm{Th}_0(E_y) & \text{if } |x|_R \text{ is a limit.} \end{cases}$$

$$\mathrm{Th}_0(E_\gamma) = \bigcup_{x\in \mathrm{Fld}(R)} \mathrm{Th}_0(E_x).$$

To show that $\mathrm{Th}_0(E_\gamma) \leq_{\mathrm{T}} (B \oplus C)^{(2)}$, we introduce another such sequence.

$$H_x(R) = \begin{cases} \mathrm{Th}_0(E_{\gamma'}) & \text{if } |x|_R = 0, \\ H_y(R)^{(\omega)} & \text{if } |x|_R = |y|_R + 1, \\ \{\langle y, z\rangle \mid z \in H_y(R) \;\&\; y <_R x\}. \end{cases}$$

$$H(R) = \{\langle x, z\rangle \mid z \in H_x(R) \;\&\; x \in \mathrm{Fld}(R)\}.$$

By induction along R, $\mathrm{Th}_0(E_x) \leq_{\mathrm{T}} H_x(R)$ uniformly in x and $H_x(R) \in L_\gamma$ for each $x \in \mathrm{Fld}(R)$. Thus $\mathrm{Th}_0(E_\gamma) \leq_{\mathrm{T}} H(R)$ and $H(R) \in \Delta^1_1(L_\gamma \cap \omega^\omega)$. Again because $(B \oplus C)^{(2)}$ computes a nice parametrization of $L_\gamma \cap \omega^\omega$, $H(R) \leq_{\mathrm{T}} (B \oplus C)^{(2)}$.

Case 1.2. $\lambda = \gamma$, $F(\lambda) = 1$ and $G(\lambda) = 3$. We use the previous argument with a twist. Let $\gamma' < \gamma$ be maximum such that γ' is admissible or a limit of admissibles. L_λ contains no system of notation for λ. But because $(B \oplus C)^{(2)}$ computes a nice parametrization of $L_\lambda \cap \omega^\omega$, there is a linear system of notation R of $\lambda = \lambda - \gamma'$ such that $H(R) \leq_{\mathrm{T}} (B \oplus C)^{(2)}$ and each initial segment of R belongs to L_λ. This follows from Theorem 2 of [4], replacing the ordinary jump by the ω-jump. Select $\mathrm{Th}_0(E_{\gamma'}) \in L_{\gamma'+1}$ and construct $\langle \mathrm{Th}_0(E_x)\rangle_{x\in \mathrm{Fld}(R)}$ and $\mathrm{Th}_0(E_\lambda)$ as before, with E_x an $E_{\gamma'+|x|_R}$. Again $\mathrm{Th}_0(E_\lambda) \leq_{\mathrm{T}} H(R) \leq_{\mathrm{T}} (B \oplus C)^{(2)}$. But then $\mathrm{Th}_1(E_\lambda) \leq_{\mathrm{T}} \mathrm{Th}_0(E_\lambda)' \leq_{\mathrm{T}} (B \oplus C)^{(3)}$.

Case 2. $\lambda = \gamma$. Let $F(\lambda) = n$, $G(\lambda) = n + 3$. Again, by a slight revision of Theorem 3 of [4], we may select a linear system of notation R for λ, such that $H(R) \leq_{\mathrm{T}} (B \oplus C)^{(3)}$. Select $\mathrm{Th}_0(E_0) \in L_1$. We construct $\langle \mathrm{Th}_0(E_x)\rangle$, with E_x an $E_{|x|_R}$, and $\mathrm{Th}_0(E_\lambda)$ as before. Again $\mathrm{Th}_0(E_\lambda) \leq_{\mathrm{T}} H(R)$. Thus $\mathrm{Th}_n(E_\lambda) \leq_{\mathrm{T}} \mathrm{Th}_0(E_\lambda)^{(n)} \leq_{\mathrm{T}} (B \oplus C)^{(G(\lambda))}$.

Case 3. $\lambda = \gamma$, $F(\lambda) = G(\lambda) = \omega\beta + n$. The argument divides into two subcases. Suppose $\beta < \lambda + \beta$. Thus $\beta < \omega\lambda$, $\mathrm{Ind}^{(b\vee c)}(G(\lambda)) = G(\lambda)$. We wish to find an $E_{\lambda+\beta}$ such that $\mathrm{Th}_n(E_{\lambda+\beta}) \in \Delta_{n+1}(L_\beta[B \oplus C])$. By the argument for Case 2, there is a real $\mathrm{Th}_0(E_\lambda) \leq_{\mathrm{T}} (B \oplus C)^{(3)}$, and so belonging to $L_1[B \oplus C]$. Let R be a linear system of notation for β such that $R \leq_{\mathrm{T}} (B \oplus C)^{(3)}$ and all initial segments of R belong to L_λ. Within $L_\beta[B \oplus C]$ we construct a sequence $\langle \mathrm{Th}_0(E_x)\rangle_{x\in \mathrm{Fld}(R)}$ starting with $\mathrm{Th}_0(E_x)$ and such that E_x is an $E_{\lambda+|x|_R}$. $\mathrm{Th}_0(E_{\lambda+\beta}) = \bigcup_{x\in \mathrm{Fld}(R)} \mathrm{Th}_0(E_x)$ as before; so $\mathrm{Th}_0(E_{\lambda+\beta}) \leq_{\mathrm{T}} H(R)$. $H(R) \in \Delta_1(L_\beta[B \oplus C])$; so $H(R)^{(n)} \in \Delta_{n+1}(L_\beta[B \oplus c])$ and $\mathrm{Th}_n(E_{\lambda+\beta}) \leq_{\mathrm{T}} \mathrm{Th}_0(E_{\lambda+\beta})^{(n)} \leq_{\mathrm{T}} H(R)^{(n)}$.

Now suppose that $\beta = \lambda + \beta$. Let $\mathrm{Ind}^{(b\vee c)}(G(\lambda))\, \omega \cdot \delta + m$. The strategy used up to now is no longer available, for we cannot count on there being a system of notation for β belonging to $L_1[B \oplus C]$. Let $\hat{L} = \{\langle \xi, x\rangle \mid x \in L_{\xi+1} - L_\xi \text{ for } \xi < \lambda + \beta\}$ and let $\langle \xi, x\rangle \hat{\in} \langle \eta, y\rangle$ iff $x \in y$. Thus $\langle \hat{L}, \hat{\in} \restriction (\hat{L} \times \hat{L})\rangle$ is isomorphic to $\langle L_{\lambda+\beta}, \in\rangle$ and $\hat{L} \in \Delta_1(L_\beta[B \oplus C])$. Select $\mathrm{Th}_m(E_\delta[B \oplus C]) \in (\underset{\sim}{b} \vee \underset{\sim}{c})^{(G(\lambda))}$. Let $E_{\lambda+\beta}$ be the copy within $E_\delta[B \oplus C]$ of $\langle \hat{L}, \hat{\in} \restriction (\hat{L} \times \hat{L})\rangle$. Because $\omega\beta + n \le \omega\delta + m$, $\mathrm{Th}_n(E_{\lambda+\beta}) \le_T \mathrm{Th}_m(E_\delta[B \oplus C])$.

§3. **Defining $\lambda\xi.\underset{\sim}{0}^{(\xi)}$ inductively.** $\lambda\xi.\underset{\sim}{0}^{(\xi)}$ has been defined set-theoretically. In [6] $(\lambda\xi.\underset{\sim}{0}^{(\xi)}) \restriction \beta_0$ is shown to have a sort of degree-theoretic inductive definition over $\langle \mathscr{D};\ \le, '\rangle$; viz. there is a sequence of formulae $\langle \psi_i(x)\rangle_{i<\omega}$ in the language of $\langle \mathscr{D};\ \le, ', I\rangle$ such that for any $\lambda < \beta_0$ there is an i such that $\langle \mathscr{D};\ \le, ', I_\lambda\rangle \models (\exists x)\psi_i(x)$ and for the least such i, $\langle \mathscr{D};\ \le, ', I_\lambda\rangle \models \underset{\sim}{0}^{(\lambda)} = (\imath x)\,\psi_i(x)$. Allowing the appearance of $\lambda\underset{\sim}{a}.\underset{\sim}{a}^{(\xi)}$ for $\xi < \beta_0$ in the structure, we can bootstrap up to a larger initial segment of $\lambda\xi.\underset{\sim}{0}^{(\xi)}$. How far may this be iterated? We define a sequence of suchinitial segments $\langle d_i\rangle_{i\le\omega}$ as follows.

Given any partial function d on ordinals, let $\mathrm{dom}^*(d)$ be the maximal initial segment of the ordinals on which d is defined.

$d_0(n) = \underset{\sim}{0}^{(n)}$ for $n \in \omega$.

$d_{i+1}(\lambda) =$ the least member of $\{d_i^{\underset{\sim}{a}}(\mu_\lambda) \mid \underset{\sim}{a} \text{ is } I_\lambda\text{-exact}\}$ if for every I_λ-exact $\underset{\sim}{a}$, $\mu_\lambda \in \mathrm{dom}^*(d_i^{\underset{\sim}{a}})$; undefined, otherwise;

$d_{i+1}(\lambda + n) = d_{i+1}(\lambda)^{(n)}$.

$d_\omega(\lambda) =$ the least member of $\{d_i^{\underset{\sim}{a}}(\mu_\lambda) \mid \underset{\sim}{a} \text{ is } I_\lambda\text{-exact},\ i \in \omega\}$ if there is an $i \in \omega$ such that for every I_λ-exact $\underset{\sim}{a}$, $\mu_\lambda \in \mathrm{dom}(d_i^{(\underset{\sim}{a})})$;

$d_\omega(\lambda + n) = d_\omega(\lambda)^{(n)}$.

Notice that $d_{i+1}(\lambda) = \max\{\underset{\sim}{d} \mid \text{for any } I_\lambda\text{-exact } \underset{\sim}{a},\ \underset{\sim}{d} \le d_i^{\underset{\sim}{a}}(\mu_\lambda)\}$, under the above conditions for definition. So the definition of d_1 in terms of d_0 coincides with the inductive degree-theoretic definition of $\lambda\xi < \beta_0.\underset{\sim}{0}^{(\xi)}$ provided in [6].

α is a local $\aleph_\xi$ iff $L_{\alpha+1} \models \alpha = \aleph_\xi$. Let δ_ξ be the least local $\aleph_\xi$. Let $\delta_{<\omega}$ be the least α such that $L_\alpha \models (\forall n \in \omega)$ ($\aleph_n$ exists). Note that $\sup\{\delta_n \mid n \in \omega\} < \delta_{<\omega} < \delta_\omega$. For $n \ge 1$, let λ_n be that λ such that $L_{\delta_n+1} \models \lambda = \aleph_1$. Let λ_ω be that λ such that $L_{\delta_{<\omega}} \models \lambda = \aleph_1$. Again, $\lambda_\omega < \lambda$ for that λ such that $L_{\delta_\omega} \models \lambda = \aleph_1$.

THEOREM 4. *For $\xi \le \omega$, $\mathrm{dom}^*(d_\xi) = \lambda_\xi$.*

This approach to defining $\underset{\sim}{0}^{(\lambda)}$ in terms of I_λ stops at λ_ω. This follows from the following.

THEOREM 5. *There is an I_{λ_ω}-exact $\underset{\sim}{a}$ such that $d_\omega^{\underset{\sim}{a}}(\mu_{\lambda_\omega})$ is undefined. Thus if $d_{\omega+1}$ is defined in terms of d_ω just as d_{i+1} is defined in terms of d_i, then* $\mathrm{dom}^*(d_{\omega+1}) = \lambda_\omega$.

We simultaneously prove Theorem 4 and the following lemma.

LEMMA 4. *Suppose that $[\lambda, \lambda + \alpha)$ is a maximal L-gap. $[\lambda, \lambda + \alpha)$ contains a local $\aleph_{i+1}$ iff for the some I_λ-exact $\underset{\sim}{a}$, $\mu_\lambda \notin \mathrm{dom}^*(d_i^{\underset{\sim}{a}})$.*

PROOF. $\lambda_0 = \omega$. Thus for $\xi = 0$, Theorem 4 is trivial. For $i = 0$, by results from §2, both sides of the biconditional in Lemma 4 are true. $\lambda_1 = \beta_0$. By results in §2, for $\xi = 1$, Theorem 4 follows. Suppose $1 < \xi < \omega$ and $\xi = i + 1$. We assume as our induction hypothesis that for any $\underset{\sim}{a}$, $\mathrm{dom}^*(d_i^{\underset{\sim}{a}}) = \lambda_i^{\underset{\sim}{a}}$. This is legitimate because this proof, though presented relative to $\underset{\sim}{0}$, may be relativized to any degree. In what follows, we write $L_\alpha^{\underset{\sim}{a}}$ as $(L^{\underset{\sim}{a}}, \alpha)$.

We first prove Lemma 4 for i as above. ($\Leftarrow$). Let $\underset{\sim}{a}$ be I_λ-exact, $\mu_\lambda \notin \mathrm{dom}^*(d_i^{\underset{\sim}{a}})$. By induction hypothesis relativized to $\underset{\sim}{a}$, $\mathrm{dom}^*(d_i^{\underset{\sim}{a}}) = \lambda_i^{\underset{\sim}{a}}$. So $\mu_\lambda \ge \lambda_i^{\underset{\sim}{a}}$.

Claim. $\delta_i^{\underline{a}} \in [\lambda, \lambda + \alpha)$.

First we show that $\lambda_i^{\underline{a}} \leq \lambda + \alpha$. If $\lambda + \alpha < \lambda_i^{\underline{a}}$, then $\alpha < \lambda_i^{\underline{a}}$; so $\omega\alpha < \omega\lambda_i^{\underline{a}} = \lambda_i^{\underline{a}}$. For some $n, \in \omega$, $\mu_\lambda = \omega\alpha + n$ and $\lambda_i^{\underline{a}}$ is a limit; so $\mu_\lambda < \lambda_i^{\underline{a}}$. Now we show that $\delta_i^{\underline{a}} < \lambda + \alpha$. Suppose $\lambda + \alpha \leq \delta_i^{\underline{a}}$. $\lambda + \alpha$ is an L-index. But $(L^{\underline{a}}, \delta_i^{\underline{a}}) \models \mathrm{ZF}^-$; thus $(L, \delta_i^{\underline{a}}) \models \mathrm{ZF}^-$. So $\delta_i^{\underline{a}}$ is not an L-index. Thus $\lambda_i^{\underline{a}} < \lambda + \beta < \delta_i^{\underline{a}}$. $L_{\lambda+\alpha+1}$ contains a wellordering of height $\lambda + \alpha$. So does $(L^{\underline{a}}, \lambda + \alpha + 1)$. But $[\lambda_i^{\underline{a}}, \delta_i^{\underline{a}} + 1)$ is an $L^{\underline{a}}$-gap. Contradiction. Therefore $\delta_i^{\underline{a}} < \lambda + \alpha$.

Claim. $[\lambda, \delta_i^{\underline{a}} + 1)$ contains a local $\aleph_{i+1}$. Let $\alpha_1, \ldots, \alpha_{i-1}$, be such that:

$$(L^{\underline{a}}, \delta_i^{\underline{a}} + 1) \models \alpha_1 = \aleph_1 \;\&\; \cdots \;\&\; \alpha_{i-1} = \aleph_{i-1} \;\&\; \delta_i^{\underline{a}} = \aleph_i.$$

Thus

$$(L, \delta_i^{\underline{a}} + 1) \models \lambda = \aleph_1 \;\&\; \alpha_1 \geq \aleph_2 \;\&\; \cdots \;\&\; \alpha_{i-1} \geq \aleph_i \;\&\; \delta_i^{\underline{a}} \geq \aleph_{i+1}.$$

Thus $[\lambda, \lambda + \alpha)$ contains a local $\aleph_{i+1}$.

($\Rightarrow$). Suppose that $[\lambda, \lambda + \alpha)$ contains a local $\aleph_{i+1}$. Thus there are $\alpha_1, \ldots, \alpha_i \in [\lambda, \lambda + \alpha)$ such that

$$L_{\alpha_i+1} \models \lambda = \aleph_1 \;\&\; \alpha_1 = \aleph_2 \;\&\; \cdots \;\&\; \alpha_i = \aleph_{i+1}.$$

Clearly $\lambda + \alpha_1 = \alpha_1$. We construct an I_λ-exact pair $(\underline{b}, \underline{c})$ such that $\lambda_i^{(\underline{b}\vee\underline{c})} \leq \mu_\lambda$. By the induction hypothesis, $\lambda_i^{(\underline{b}\vee\underline{c})} = \mathrm{dom}^*(d_i^{(\underline{b}\vee\underline{c})})$. So this suffices. We construct B and $C \in 2^\omega$ such that $\underline{b} = \deg(B)$ and $\underline{c} = \deg(C)$. Let conditions be as in the proof of Lemmas 1 and 2 under Cases 1 and 2. Let the forcing language $\boldsymbol{L}$ be $L_{\alpha_i+1}[\underline{B}, \underline{C}]$. Let $\langle (P_j, Q_j) \rangle_{j\in\omega}$ be a sequence of conditions such that

$P_0 = Q_0 = \mathrm{id}$;

(P_{2i+1}, Q_{2i+1}) extends (P_{2i}, Q_{2i}) and decides $\varnothing_i$;

$(P_{2i+2}, Q_{2i+2}) = (P_{2i+1} * A_i, Q_{2i+1} * A_i)$;

where $\langle \varnothing_i \rangle_{i\in\omega}$ and $\langle A_i \rangle_{i\in\omega}$ are enumerations of the sentences of $\boldsymbol{L}$ and of $\boldsymbol{L}_\lambda \cap 2^\omega$ respectively. Let $(B, C) = \bigcap_j [P_j, Q_j]$. $\lambda \leq \omega_1^{(B\oplus C)}$. By the standard argument, all cardinals of L_{α_i+1} except for $\aleph_1$ are preserved.

$$L_{\alpha_i+1}[B \oplus C] \models \alpha_1 = \aleph_1 \;\&\; \cdots \;\&\; \alpha_i = \aleph_i.$$

Thus $\alpha_i \geq \delta_i^{(\underline{b}\vee\underline{c})}$. So $\lambda_i^{(\underline{b}\vee\underline{c})} \leq \alpha_1 < \alpha \leq \omega\alpha \leq \mu_\lambda$. Thus Lemma 4 is proved for this choice of i.

We now prove that $\mathrm{dom}^*(d_{i+1}) = \lambda_{i+1}$. If $\lambda < \lambda_{i+1}$, λ does not start an L-gap containing a local $\aleph_{i+1}$. By Lemma 4, for any $\underline{a}$ which is I_λ-exact, $\mu_\lambda \in \mathrm{dom}^*(d_i^{\underline{a}})$. Thus $d_{i+1}(\lambda)$ is defined. λ_{i+1} starts an L-gap containing a local $\aleph_{i+1}$. By Lemma 4, $d_{i+1}(\lambda_{i+1})$ is undefined.

Finally, suppose $\xi = \omega$. If $\lambda < \lambda_\omega$, for some $i \in \omega$, λ does not start an L-gap containing a local $\aleph_{i+1}$. By Lemma 4, for any I_λ-exact $\underline{a}$, $\mu_\lambda \in \mathrm{dom}^*(d_i^{\underline{a}})$. Thus $d_\omega(\lambda)$ is defined. λ_ω starts an L-gap containing a local $\aleph_{i+1}$ for every $i \in \omega$. Thus for any $i \in \omega$ there is an I_{λ_ω}-exact $\underline{a}$ such that $\mu_{\lambda_\omega} \notin \mathrm{dom}^*(d_i^{\underline{a}})$. Thus $d_\omega(\lambda_\omega)$ is undefined.

PROOF OF THEOREM 5. Let $\langle \alpha_i \rangle_{i\in\omega}$ be such that for all $i \in \omega$, $(L, \delta_{<\omega}) \models \alpha_i = \aleph_{i+1}$. It suffices to construct $(\underline{b}, \underline{c})$ I_{λ_ω}-exact such that $\alpha_1 = \lambda_\omega^{(\underline{b}\vee\underline{c})}$. Because $\mu_{\lambda_\omega} > \omega\alpha_1 = \alpha_1$, by Theorem 4, $\mu_{\lambda_\omega} \notin \mathrm{dom}^*(d_\omega^{(\underline{b}\vee\underline{c})})$. Thus $d_{\omega+1}(\lambda_\omega)$ is undefined. As in the proof of Lemma 4 ($\Rightarrow$), we construct $(\underline{b}, \underline{c})$ such that for all $i \in \omega$, $(L^{(\underline{b}\vee\underline{c})}, \delta_{<\omega}) \models \alpha_i = \aleph_i$. Thus $\delta_{<\omega} = \delta_{<\omega}^{(\underline{b}\vee\underline{c})}$. So $\alpha_1 = \lambda_\omega^{(\underline{b}\vee\underline{c})}$.

Let A_n, for $n < \omega$, be nth-order number theory, i.e. Peano's axioms set in an nth order language, where variables of order i, $1 \leq i \leq n$, range over sets of type $i - 1$. Let $A_\omega = \bigcup_n A_n$, in the language with variables of all finite orders. A_2 is analysis. For $\xi \leq \omega$, we imitate the construction of the ramified analytical hierarchy. A structure for A_ξ has the form $\langle\langle U_i\rangle_{1\leq i<\xi}; +, \cdot, S; 0\rangle$ where $U_0 = \omega$ and $U_{j+1} \subseteq P(U_j)$; variables of order $j + 1$ range over U_j. Let M_0^ξ be the structure for A_ξ with all U_j's, for $j \neq 0$, empty. Form the transfinite sequence $\langle M_\eta^\xi\rangle_\eta$ by iterating closure under definability in the language of A_ξ. This hierarchy stops. Let the final structure be M^ξ with domains $\langle U_i^\xi\rangle_{1\leq i<n}$; let the closure ordinal be γ_ξ. Then $M^\xi \models A_\xi$. Let A_ξ^* be A_ξ translated into the language of set theory. $L_{\delta_n} \cap P^n(\omega)$ and $L_{\delta_{<\omega}} \cap P^n(\omega)$ are, respectively, the minimal models for A^*_{n+1} and A^*_ω in which wellfoundedness is absolute. Thus $U^n_{i+1} = (L, \delta_n) \cap P^i(\omega)$; $U_i^\omega = (L, \delta_{<\omega}) \cap P^i(\omega)$ where U_i^{n+1}, U_i^ω are from M^{n+1}, M^ω respectively. Thus for any $n < \omega$, $\gamma_n = \delta_{n-1}$; furthermore, $\gamma_\omega = \delta_{<\omega}$. Let $I^\xi = \lambda_\xi$ for $\xi \leq \omega$. By Theorem 4, $\bigcup I_\xi \cap 2^\omega = U_1^\xi$. Thus d_ξ classifies the degrees of reals in M^ξ.

§4. **Comparison with the single jump operation.** How similar is the single jump operation $\underset{\sim}{a} \mapsto \underset{\sim}{a}'$ to an arbitrary operation of the form $\underset{\sim}{a} \mapsto \underset{\sim}{a}^{(\xi)}$ for $\xi < (\aleph_1)^{L\underset{\sim}{a}}$? In this section we examine an analogy and a striking disanalogy. As usual, all results are stated for $\underset{\sim}{a} = 0$, but easily generalize to arbitrary $\underset{\sim}{a}$.

The analogy: Friedberg's completeness theorem $((\forall \underset{\sim}{a} \geq \underset{\sim}{0}')(\exists \underset{\sim}{c})(\underset{\sim}{a} = \underset{\sim}{c}'))$ generalizes to arbitrary transfinite jumps:

THEOREM 6. *For any* $\xi < (\aleph_1)^L$ *and any* $\underset{\sim}{a}$, *if* $\underset{\sim}{a} \geq \underset{\sim}{0}^{(\xi)}$ *then there is a* $\underset{\sim}{c}$ *such that* $\underset{\sim}{a} = \underset{\sim}{c}^{(\xi)}$.

The disanalogy: the trivial fact that $\underset{\sim}{0}' \leq \underset{\sim}{a}'$ does not generalize in the most straightforward way.

THEOREM 7. *For any* $\xi < (\aleph_1)^L$ *and any* $\underset{\sim}{a}$, $\underset{\sim}{0}^{(\xi)} \leq \underset{\sim}{a}^{(\mathrm{Ind}(\xi))}$.

THEOREM 8. *For any* $\xi < (\aleph_1)^L$ *there is a* $\underset{\sim}{b}$ *such that* $\underset{\sim}{0}^{(\xi)} = \underset{\sim}{b}^{(\mathrm{Ind}(\xi))}$.

Before we present proofs, notice that if $\xi + \eta < (\aleph_1)^L$, $\underset{\sim}{0}^{(\xi)(\eta)} = \underset{\sim}{0}^{(\xi+\eta)}$. This follows by an easy induction on η.

PROOF OF THEOREM 6. Suppose $\underset{\sim}{a} \geq \underset{\sim}{0}^{(\xi)}$. If $\xi = \alpha + 1$, by the relativization of Friedberg's theorem to $\underset{\sim}{0}^{(\alpha)}$, there is a $\underset{\sim}{d} \geq \underset{\sim}{0}^{(\alpha)}$, $\underset{\sim}{d}' = \underset{\sim}{a}$. By the induction hypothesis on α, there is a $\underset{\sim}{c}$ such that $\underset{\sim}{d} = \underset{\sim}{c}^{(\alpha)}$. Thus $\underset{\sim}{d}' = \underset{\sim}{c}^{(\alpha+1)} = \underset{\sim}{a}$.

Suppose that ξ is a limit ordinal. By analogy with Friedberg's argument, we construct a $\underset{\sim}{c}$ such that $\underset{\sim}{c}^{(\xi)} \leq \underset{\sim}{a} \leq \underset{\sim}{c} \vee \underset{\sim}{0}^{(\xi)} \leq \underset{\sim}{c}^{(\xi)}$. Clearly such a $\underset{\sim}{c}$ is as desired. Suppose $\mathrm{Ind}(\xi) = \omega\beta + n$ and $A \in \underset{\sim}{a}$. Let the forcing language $\boldsymbol{L}$ be $\boldsymbol{L}_\beta[\underset{\sim}{C}]$. Let $\langle\phi_i\rangle_{i\in\omega}$ be a $\Delta_{n+1}(L_\beta)$ enumeration of the $\Sigma_n \cup \Pi_n$ sentences of $\boldsymbol{L}$. We shall force with Cohen condition, viewed as finite strings of 0 and 1. There is a sequence of Cohen conditions $\langle\sigma_i\rangle_{i\in\omega} \in \Delta_{n+1}(L_\beta)$ such that for every i:

σ_{i+1} extends σ_i;

σ_{2i} decides ϕ_i;

$\sigma_{2i+1} = \sigma_{2i}\frown A(i)$.

Let $C = \lim_{i\to\omega}\sigma_i$; let $\underset{\sim}{c} = \deg(C)$. For any E_β, $\langle\sigma_i\rangle_{i\in\omega}$ is coded by a single real recursive in $\mathrm{Th}_n(E_\beta) \oplus A$. Select E_β such that $\mathrm{Th}_n(E_\beta) \in \underset{\sim}{0}^{(\xi)}$. Because $\mathrm{Th}_n(E_\beta) \leq_T A$, $C \leq_T A$. Because the even steps are determined only by $\mathrm{Th}_n(E_\beta)$ and A is coded into C at the odd steps, $A \leq_T C \oplus \mathrm{Th}_n(E_\beta)$. Thus $\underset{\sim}{c}^{(\xi)} \leq \underset{\sim}{a} \leq \underset{\sim}{c} \vee \underset{\sim}{0}^{(\xi)}$. Finally because all conditions belong to L_0, for any $\eta \leq \beta$ and any m, $L_\eta \models \Delta_m$ CA iff $L_\eta[C] \models \Delta_m$CA. Thus for any $\eta \leq \xi$, $\underset{\sim}{0}^{(\eta)} \leq \underset{\sim}{c}^{(\eta)}$. Therefore $\underset{\sim}{c} \vee \underset{\sim}{0}^{(\xi)} \leq \underset{\sim}{c}^{(\xi)}$. Q.E.D.

PROOF OF THEOREM 7. Suppose $\xi = \gamma + 1$. By induction hypothesis, $\underset{\sim}{0}^{(\gamma)} \leq \underset{\sim}{a}^{(\mathrm{Ind}(\gamma))}$. Thus $\underset{\sim}{0}^{(\gamma+1)} \leq \underset{\sim}{a}^{(\mathrm{Ind}(\gamma)+1)}$. But $\mathrm{Ind}(\xi) = \mathrm{Ind}(\gamma) + 1$. Suppose ξ is a limit. Let $\mathrm{Ind}(\xi) = \omega\beta + n$; let $\mathrm{Ind}^{\underset{\sim}{a}}(\mathrm{Ind}(\xi)) = \omega\alpha + m \geq \omega\beta + n$. Let $A \in \underset{\sim}{a}$. By the procedure used in the proof of Lemma 3, Case 3, in the final paragraph, there is a uniform way of obtaining an E_β from any $E_\alpha[A]$ such that $\mathrm{Th}_n(E_\beta) \leq^{\mathrm{T}} \mathrm{Th}_m(E_\alpha[A])$. This suffices.

PROOF OF THEOREM 8. If $\mathrm{Ind}(\xi) = \xi$, Theorem 8 is trivial. So suppose $\mathrm{Ind}(\xi) > \xi$. It suffices to construct a $\underset{\sim}{b}$ such that $\underset{\sim}{b}^{(\mathrm{Ind}(\xi))} \leq \underset{\sim}{0}^{(\xi)}$. Let $\mathrm{Ind}(\xi) = \omega\alpha + n$. Let $\boldsymbol{L}$ be $\boldsymbol{L}_\alpha[B]$. Let δ be the maximum ordinal $\leq \alpha$ which is admissible or a limit of admissibles. We force with modified Steel conditions, with ordinal labels $< \delta$. Let $\langle \phi_i \rangle_{i \in \omega} \in \Delta_{n+1}(L_\alpha)$ enumerate the $(\Sigma_n \cup \Pi_n)$ sentences of $\boldsymbol{L}$. Let $\langle z_i \rangle_{i \in \omega} \in \Delta_{n+1}(L_\alpha)$ be a sequence of modified Steel conditions such that for any i, z_{i+1} extends z_i and z_i decides ϕ_i. By the standard construction, such a sequence exists. Let B be the extension of $\underset{\sim}{B}$ determined by this sequence. Let $\underset{\sim}{b} = \deg(B)$. Because $\langle z_i \rangle_{i \in \omega} \in \Delta_{n+1}(L_\alpha)$, for any E_α there is an $E_\alpha[B]$ such that $\mathrm{Th}_n(E_\alpha[B]) \leq_{\mathrm{T}} \mathrm{Th}_n(E_\alpha)$. Select E_α such that $\mathrm{Th}_n(E_\alpha) \in \underset{\sim}{0}^{(\xi)}$; let $E_\alpha[B]$ be determined by E_α. As in the proof of Lemmas 1 and 2 under Case 3, B is a wellfounded tree of sequence numbers of height δ. Thus $\mathrm{Ind}(\xi) < \omega_1^B$. Thus $\underset{\sim}{b}^{(\omega\alpha+n)} \leq \deg(\mathrm{Th}_n(E_\alpha[B]))$.

We finish this section with an application of Theorem 6.

COROLLARY. *For any* λ, $\{\underset{\sim}{d} \mid \underset{\sim}{d} \geq \underset{\sim}{0}^{(\lambda)}\} = \{\underset{\sim}{a}^{(\mu_\lambda)} \mid \underset{\sim}{a}$ *is* I_λ*-exact*$\}$.

PROOF. Let $(\underset{\sim}{b}, \underset{\sim}{c})$ be an I_λ-exact pair such that $(\underset{\sim}{b} \vee \underset{\sim}{c})^{(\mu_\lambda)} \geq \underset{\sim}{0}^{(\lambda)}$. Suppose $\underset{\sim}{d} \geq \underset{\sim}{0}^{(\lambda)}$. By Theorem 6, for some $\underset{\sim}{d}^* \geq (\underset{\sim}{b} \vee \underset{\sim}{c})^{(2)}$, $(\underset{\sim}{d}^*)^{(-2+\mu_\lambda)} = \underset{\sim}{d}$. By (J) of [4, §2], for some I_λ-exact pair $(\underset{\sim}{b}^*, \underset{\sim}{c}^*)$, $(\underset{\sim}{b}^* \vee \underset{\sim}{c}^*)^{(2)} = d^*$. The other direction is just Lemma 3.

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