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11

# Ontological commitment
## Thick and thin

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Mathematical discourse is filled with existential assertions, assertions of the form "There is a number (or function, or set, or space or structure or whatever) such that . . . ." Some philosophers find such statements puzzling, or even unbelievable. This response is both healthy and misguided: misguided because some such statements are true, and not merely in some non-literal way; healthy because it indicates sensitivity to differences between the basis of the truth or falsity of such statements and that of existential statements living in other corners of our languages. The answers to questions like "Are there numbers?" and "Do sets exist?" are, trivially, "Yes." To not see these answers as trivialities bespeaks a misunderstanding of mathematical discourse. But to go on and say that there is a *realm* of mathematical objects is to engage in obscurantist hyperbole. Mathematical objects are second-rate; they are not among "the furniture of the universe." For a philosophically adequate understanding of mathematics, we must distinguish between what I'll call thick and thin ontological commitment.[1] And if ontological commitment is our subject, where should we begin but with Quine?

### 1

Quine's doctrine on this matter is, as he himself insists, a truism: ontological commitment is expressed by existential quantification. A chunk of discourse is ontologically committed to whatever it (or rather the assertive statements it contains or implies) says that there is.

> When I inquire into the ontological commitments of a given doctrine or body of theory, I am merely asking what, according to that theory, there is. [15, p. 203]

What is being said in a chunk of discourse might not be evident at a glance. Regimentation into formal languages can make this more evident. Regimentation clarifies logical aspects of syntactic, including quantificational, structure. Thus it can help us assess what I'll call the thin commitments of the chunk of discourse, a matter of *what is said to be*.

In his own characterizations of ontological commitment, Quine studiously avoids use of expressions like 'says that', or even 'implies that'. For example:

> A theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order that the affirmations made in the theory be true. [14, pp. 13–14]

> The entities to which a discourse commits us are the entities over which our variables of quantification have to range in order that the statements affirmed in that theory be true. [15, p. 205]

The question of *what there would have to be in order for certain statements to be true* is a question of what I'll call thick ontological commitment. To answer it we must assess the alethic underpinnings for the statements in question: the semantic properties of their basic constituents and the recursive "process" that determines their truth-conditions. These underpinnings are a matter of semantic[2] form.

Two tangential points deserve mention. To specify what is said is, at best, to give the sentences used (assuming that the language in which they were used is understood and known to be the language in which they were used). At worst, it's to give a paraphrase or translation of the sentences used. Thus such a specification can be tainted with intensionality. Perhaps this is why Quine has tended to avoid the "what is said to be" formulation. But it is important to realize that Quine's preferred explications of his phrase 'ontological commitment'[3] also involves intensional idioms. As Quine himself teaches, 'must' and 'have to' are also intensional, as is the subjunctive "in order to" formulation of thick commitment given above.[4] Suffice it to say that thin commitments are tied to logical syntax; so when we assess thin commitments, tightness of paraphrase is an important virtue.

Second, the sort of regimentation that Quine would have us use when assessing ontological commitments is regimentation into a first-order language. But this is unnecessarily restrictive. Some discourse is most naturally construed as involving higher-order quantification, though its order need not be syntactically explicit. Of course the syntax of mathematical discourse is virtually always first-order. Numbers, sets, and the like are values of first-order variables, that is, are objects (in Frege's sense of 'object'). So our initial focus will be on the commitments carried by first-order quantification, commitments to objects.[5]

In the cases most central to our languages, there is no difference between what is said to be and what there would have to be for what is said to be true. Semantic and logical form, and with them thick and thin ontological commitments, come apart only in peripheral sorts of discourse.

Consider singular terms. Singular termhood is a logico-syntactic matter. To classify an expression as a singular term is to assign it a certain role within a recursive description of the totality of sentences and valid inferences in a language. Although the second sort of description concerns what Quine called "the interanimation of sentences," it is as syntactic a project as that of characterizing sentencehood. In the cases most central to learning and mature use of language, all closed singular terms do the same sort of semantic work: designating objects.[6]

This fact greases the slide from logico-syntactic role to semantic role for all closed singular terms. Crispin Wright, for example, takes it as self-evident that

> what we say metalinguistically by "a' has a reference' is just the object-language, '$(\exists x)x = $ a'... [16, p. 83]

To make this slide when discussing a species of mathematical discourse is to adopt what I call "the Mathematical-Object theory" of that discourse. Frege's famous slogan "Numbers are objects" expressed his adherence to the Mathematical-Object theory of finite arithmetic. Frege took the semantic job of numerals and closed singular terms of the form 'the number of *F*s' to be designation. Correspondingly, he took the semantic job of arithmetic predicates to be applying or failing to apply to (tuples of) objects; and he took phrases like "for all natural numbers" to express quantification over objects of a special sort. He also accepted the analogous theory about set-theoretic discourse.

In this essay I'll present an alternative to the Mathematical-Object theory, to be called the Alternative theory.[7] I contend that some closed singular terms, including those that are properly mathematical, do a sort of semantic work that is not designation. Nor are mathematical predicates built for applying or failing to apply to (tuples of) objects. Rather the linguistic apparatus of a branch of mathematics is a package built to allow certain higher-order statements to be encoded "down" into a more familiar and tractable first-order form. When a singular term, represent it by '**a**', is part of such a package, '**a** exists' is still correctly parsed as '$(\exists x)x = \mathbf{a}$'; but it will not entail, let alone be or express, the metalinguistic statement '"**a**' designates something' or '"**a**' has a reference'. When a predicate, say a one-place predicate represented by '*P*', is part of such a package, 'There is a *P*' is still to be parsed as '$(\exists x)Px$'; but it will not entail that '*P*' applies to something. I'll introduce the Alternative theory obliquely, mixing some model-theory with philosophical claims about what this model-theory models.

## 2

Models, in the logician's sense, are sets.[8] Let *S* be a set of "uninterpreted" non-logical expressions; for our purposes each member is a predicate- or individual-constant (or, if you wish, a function-constant). A model $\mathscr{A}$ for *S* may then be taken to be an ordered pair, construed set-theoretically, whose left-component is a non-empty set $|\mathscr{A}|$ (the model's universe) and whose right-component is a function on *S* assigning each predicate-constant to a subset of an appropriate cartesian power of $|\mathscr{A}|$, and each individual-constant to a member of $|\mathscr{A}|$. (If *S* contains function-constants, it assigns to each function-constant a function from an appropriate cartesian power of $|\mathscr{A}|$ into $|\mathscr{A}|$.) All functions may be taken to be sets of ordered pairs. In keeping with usual notation, $\zeta^{\mathscr{A}}$ is the value the model assigns to any $\zeta \in S$.

Tarski first introduced models, in "On the Concept of Logical Consequence" (1935), to give a set-theoretic definition of logical consequence. They are appropriate for that project because they model in the engineering sense (that is, they mirror, reflect, represent) the relationship between possible sense-bearing languages and reality that would underlie the distribution of truth-values among the statements in such languages.[9] A model itself does not assign senses to the vocabulary items, or even grant them references. Genuine reference arises only with sense; it is a facet of the life words take on within a sense-bearing language.[10] Models are interesting sets because they model, in a set-theoretic way, the basic alethic underpinnings of possible sense-bearing languages (or fragments thereof), their basic semantic facts (whose specification would serve as base-clauses in a definition of truth). The elements of a model's universe represent objects (perhaps all objects, perhaps only special ones); its assignment of individual-constants to elements of its universe represents designation, each individual-constant and its value playing the parts of designator and designatum respectively; its assignment of sets to predicate-constants represents the "falling under" relation between (tuples of) objects and predicates of level-one.

Tarski showed how to set-theoretically define a binary relation, usually expressed by '$\models$', between models and sentences in their languages. Why is this relation of any interest? Taking a model $\mathscr{A}$ to represent the basic alethic underpinnings for a sense-bearing language, bearing the converse of this relation to $\mathscr{A}$ then represents being true for statements in that language. Truth-in-a-model is a model of truth.[11]

Although it's not essential to the project at hand, let's slightly modify the usual notion of modelhood. Sense-bearing languages can contain empty designators, a phenomenon that was idealized away by the notion of modelhood just introduced. To remedy this we broaden our notation of modelhood by weakening the requirements on naming-functions for models; we now say merely that a model $\mathscr{A}$ may assign an individual-constant in $S$ to any element of $|\mathscr{A}|$; we'll allow that it also may fail to assign such a constant to anything. (Similarly function-constants may be assigned to partial functions defined on subsets of appropriate cartesian powers of $|\mathscr{A}|$.) In the special case in which $\mathscr{A}$ assigns something to each individual-constant in $S$, call $\mathscr{A}$ total.[12]

Once we have non-total as well as total models, there are several relations that might reasonably be called "truth-in-a-model." Since our final model-theoretic semantics will be three-valued, we will adopt a three-valued approach from the start. This is not essential at this stage; but it has the advantage of making truth and falsity symmetric truth-values. There are several kinds of three-valued semantics. I prefer the so-called strong Kleene semantics with the "strong" semantics for '$=$'.

Fix an infinite set of variables. Let $L = L(S)$ be the uninterpreted language based on $S$, determined as follows. The terms of $L$ are the variables and individual-constants from $S$, and whatever is generated from these using function-constants in $S$. The formulae of $L$ are constructed as usual using the logical lexicon '$\supset$', '$\perp$', '$\exists$' and '$=$'. Let $\mathscr{A}$ be a model for the vocabulary set $S$.

To handle quantifiers with minimal clutter, expand $L$ to $L_{\mathscr{A}}$ by introducing for each $a \in |\mathscr{A}|$ an individual-constant $\mathbf{a}$ and letting $\mathbf{a}^{\mathscr{A}} = a$. We then define $\tau^{\mathscr{A}}$ for all closed terms $\tau$ in the usual way. Since $\mathscr{A}$ may be non-total, the assignment of $\tau$ to $\tau^{\mathscr{A}}$ in $|\mathscr{A}|$ may be non-total; "$\tau^{\mathscr{A}}\downarrow$" means that $\tau^{\mathscr{A}}$ is defined; i.e., for some $a$, $\tau^{\mathscr{A}} = a$; "$\tau^{\mathscr{A}}\uparrow$" means that $\tau^{\mathscr{A}}$ is undefined. We define the relations $\models$ ("makes true") and $\dashv$ ("makes false"), between $\mathscr{A}$ and sentences of $L_{\mathscr{A}}$, by a simultaneous recursion. Suppose $\gamma$ is an $n$-place predicate-constant and the $\tau$'s are closed terms; let:

$$\mathscr{A} \models \gamma(\tau_0, \ldots, \tau_{n-1}) \text{ iff } \tau_i^{\mathscr{A}}\downarrow \text{ for all } i < n \text{ and } \langle \tau_0^{\mathscr{A}}, \ldots, \tau_{n-1}^{\mathscr{A}} \rangle \in \gamma^{\mathscr{A}};$$

$$\mathscr{A} \dashv \gamma(\tau_0, \ldots, \tau_{n-1}) \text{ iff } \tau_i^{\mathscr{A}}\downarrow \text{ for all } i < n \text{ and } \langle \tau_0^{\mathscr{A}}, \ldots, \tau_{n-1}^{\mathscr{A}} \rangle \notin \gamma^{\mathscr{A}};$$

$$\mathscr{A} \models \tau_0 = \tau_1 \text{ iff } \tau_0^{\mathscr{A}}\downarrow \text{ and } \tau_0^{\mathscr{A}} = \tau_1^{\mathscr{A}};$$

$$\mathscr{A} \dashv \tau_0 = \tau_1 \text{ iff either } \tau_0^{\mathscr{A}}\downarrow, \tau_1^{\mathscr{A}}\downarrow \text{ and } \tau_0^{\mathscr{A}} \neq \tau_1^{\mathscr{A}}, \text{ or } \tau_0^{\mathscr{A}}\downarrow \text{ and } \tau_1^{\mathscr{A}}\uparrow, \text{ or } \tau_0^{\mathscr{A}}\uparrow \text{ and } \tau_1^{\mathscr{A}}\downarrow;$$

$$\mathscr{A} \dashv \perp.$$

For any sentences $\varphi$, $\psi$, and $(\exists \nu)\theta$ of $L_{\mathscr{A}}$, let:

$$\mathscr{A} \models (\varphi \supset \psi) \text{ iff either } \mathscr{A} \dashv \varphi \text{ or } \mathscr{A} \models \psi;$$
$$\mathscr{A} \dashv (\varphi \supset \psi) \text{ iff } \mathscr{A} \models \varphi \text{ and } \mathscr{A} \dashv \psi;$$
$$\mathscr{A} \models (\exists \nu)\theta \text{ iff for some } a \in \mathscr{U}\ \mathscr{A} \models \mathrm{Sub}(\mathbf{a}, \nu, \theta);$$
$$\mathscr{A} \dashv (\exists \nu)\theta \text{ iff for each } a \in \mathscr{U}\ \mathscr{A} \dashv \mathrm{Sub}(\mathbf{a}, \nu, \theta).$$ [13]

For a sentence $\varphi$ based on $S$ we adopt these definitions:

$\varphi$ is valid iff for all models $\mathscr{A}$ for $S$, $\mathscr{A} \models \varphi$;

$\varphi$ is bivalent iff for all models $\mathscr{A}$ for $S$ either

$$\mathscr{A} \models \varphi \quad \text{or} \quad \mathscr{A} \dashv \varphi.$$ [14]

Let $\mathscr{A} | \varphi$ iff $\mathscr{A} \not\models \varphi$ and $\mathscr{A} \not\dashv \varphi$; i.e., iff $\varphi$ is neither true nor false in $\mathscr{A}$. Our definition allows for this, e.g., if $\varphi$ is $\gamma(\tau)$ or $\tau = \sigma$ and $\tau^{\mathscr{A}}\uparrow$ and $\sigma^{\mathscr{A}}\uparrow$.[15] Introduce '$\neg$', '&', '$\vee$', '$\equiv$', and '$\forall$' by the familiar abbreviations. Let $E(\tau)$ for a term $\tau$ abbreviate $(\exists \nu)\nu = \tau$, where $\nu$ may be any variable distinct from $\tau$; read it as "$\tau$ exists." We then have:

$$\mathscr{A} \models E(\tau) \text{ iff } \tau^{\mathscr{A}}\downarrow; \quad \mathscr{A} \dashv E(\tau) \text{ iff } \tau^{\mathscr{A}}\uparrow.$$

Imagine a sense-bearing language (or fragment thereof) $\mathscr{L}$ containing no vocabulary enabling the speakers to speak about mathematical objects (or any other potentially problematic abstract objects like possibilities or the like). Suppose its alethic underpinnings can be adequately represented by models of the sort introduced above: its closed singular terms are all built for designation, its predicates for applying and not applying to (tuples of) objects, and its only resources for the construction of sentences are truth-functional connectives and first-order universal and existential quantification. We now consider three ways in which $\mathscr{L}$ could be enriched with mathematical vocabulary, and two ways of modeling such enrichments: one in keeping with the Mathematical-Object theory, the other in keeping with the Alternative theory.

## 3

Case 1: $\mathscr{L}$ is enriched to $\mathscr{L}^p$ to allow talk "about" ordered pairs. We'll model this enrichment with the uninterpreted language $L^p$ formed from $L$ by adding the expressions '**p**' and '**OP**', governed by the following new clauses into the definitions of termhood and formulahood, respectively:

for any terms $\tau$ and $\sigma$, $\mathbf{p}(\tau, \sigma)$ is a term;
for any term $\tau$, $\mathbf{OP}(\tau)$ is a formula.

Given a model $\mathscr{A}$ for $S$, let $\wp$ be a pairer for $\mathscr{A}$ iff $\wp$ is a function from $|\mathscr{A}|^2$ one-one into $|\mathscr{A}|$. Assuming the Axiom of Choice, $\mathscr{A}$ has a pairer iff card($|\mathscr{A}|$) is either one or infinite. For any term $\tau$ we define $\tau^{\mathscr{A},\wp}$ by relativizing the corresponding definition in §2 to $\wp$ and adding:

$$\mathbf{p}(\tau,\sigma)^{\mathscr{A},\wp} = \wp(\tau^{\mathscr{A},\wp}, \sigma^{\mathscr{A},\wp});$$
$$\mathbf{p}(\tau,\sigma)^{\mathscr{A},\wp}\uparrow \quad \text{iff either } \tau^{\mathscr{A},\wp}\uparrow \quad \text{or} \quad \sigma^{\mathscr{A},\wp}\uparrow.$$

For sentences $\varphi$ of $L_{\mathscr{A}}$ we define $\mathscr{A},\wp \overset{p}{\models} \varphi$ and $\mathscr{A},\wp \overset{p}{=\!\!|} \varphi$ by relativizing the corresponding definition in §2 to $\wp$ and adding the clause:

$$\mathscr{A},\wp \overset{p}{\models} \mathbf{OP}(\tau) \quad \text{iff } \tau^{\mathscr{A},\wp}\downarrow \text{ and } \quad \tau^{\mathscr{A},\wp} \in \text{rng}(\wp);$$
$$\mathscr{A},\wp \overset{p}{=\!\!|} \mathbf{OP}(\tau) \quad \text{iff } \tau^{\mathscr{A},\wp}\downarrow \text{ and } \quad \tau^{\mathscr{A},\wp} \notin \text{rng}(\wp).$$

Case 2: $\mathscr{L}$ is enriched to $\mathscr{L}^\#$ to allow talk "about" the natural numbers, with number-terms, and predicates for numberhood and order on the numbers. We'll model this enrichment with the uninterpreted language $L^\#$, formed from $L$ by adding '#' (a variable-binding term-forming operator on formulae), 'N' and '$\leq$' (syntactically like one- and two-place predicates, respectively). A representor $\kappa$ for a model $\mathscr{A}$ is a one-one function from the finite cardinality quantifiers on $\mathscr{A}$ into $|\mathscr{A}|$; $\overset{\#}{\models}$ and $\overset{\#}{=\!\!|}$ relate pairs $\langle \mathscr{A}, \kappa \rangle$ to sentences of $L^\#_{\mathscr{A}}$. Details are given in [11], so are omitted here.

Case 3: $\mathscr{L}$ is enriched to $\mathscr{L}^\wedge$ with talk "about" sets. Let's suppose that $\mathscr{L}$'s speakers adopt the "limitation of size" conception, first articulated by Cantor and implicit in standard *ZF*-like set-theories: there are absolutely infinitely many objects, indeed, absolutely infinitely many sets; no sets are absolutely infinite; that is, no set has as many members as there are objects, though of course there are infinite (i.e., relatively infinite) sets.

We enrich $L$ with the new symbols '**Set**', '$\in$', and '$\hat{}$' to form $L^{\hat{}}$. Termhood and formulahood are now defined by a simultaneous induction, with these new clauses:

> For any formula $\varphi$ and variable $\nu$, $\hat{\nu}\varphi$ is a term;
> For any term $\tau$, **Set**$(\tau)$ is a formula;
> For any terms $\tau$ and $\sigma$, $\tau \in \sigma$ is a formula.

For any set $x$, let card$(x)$ be the cardinality of $x$. Given a model $\mathscr{A}$ for $S$, let $\kappa = \text{card}(|\mathscr{A}|)$. Let an extensor for $\mathscr{A}$ be a one-one function from Power$^{\kappa}(|\mathscr{A}|) = \{A \subseteq |\mathscr{A}| : \text{card}(\mathscr{A}) < \kappa\}$ into $|\mathscr{A}|$. Assuming the Axiom of Choice, $\mathscr{A}$ has an extensor iff $|\mathscr{A}|$ is infinite.

Given an extensor $e$ for $\mathscr{A}$, we simultaneously define a partial function from closed terms into $|\mathscr{A}|$ and two relations $\hat{\models}$ and $\hat{=\!|}$, writing $\tau^{\mathscr{A},e}$, $\mathscr{A}, e \mathrel{\hat{\models}} \varphi$ and $\mathscr{A}, e \mathrel{\hat{=\!|}} \varphi$ where $\tau$ is a term and $\varphi$ is a sentence of $L^{\hat{}}_{\mathscr{A}}$, by relativizing to $e$ the clauses used in the previous section, and adding these clauses:

| | |
|---|---|
| $\hat{\nu}\varphi^{\mathscr{A},e} = e(A)$ | if $A = \{a : \mathscr{A}, e \mathrel{\hat{\models}} \text{Sub}(\mathbf{a}, \nu, \varphi)\} \in \text{dom}(e)$; |
| $\hat{\nu}\varphi^{\mathscr{A},e}\uparrow$ | if there is no such $A$; |
| $\mathscr{A}, e \mathrel{\hat{\models}} \mathbf{Set}(\tau)$ | iff $\tau^{\mathscr{A},e}\downarrow$ and $\tau^{\mathscr{A},e} \in \text{rng}(e)$; |
| $\mathscr{A}, e \mathrel{\hat{=\!|}} \mathbf{Set}(\tau)$ | iff $\tau^{\mathscr{A},e}\downarrow$ and $\tau^{\mathscr{A},e} \notin \text{rng}(e)$; |
| $\mathscr{A}, e \mathrel{\hat{\models}} \tau \in \sigma$ | iff $\tau^{\mathscr{A},e}\downarrow$, $\sigma^{\mathscr{A},e}\downarrow$, and for some $A$ $e(A) = \sigma^{\mathscr{A},e}$ and $\tau^{\mathscr{A},e} \in A$; |
| $\mathscr{A}, e \mathrel{\hat{=\!|}} \tau \in \sigma$ | iff $\tau^{\mathscr{A},e}\downarrow$, $\sigma^{\mathscr{A},e}\downarrow$, and there is no $A$ with $e(A) = \sigma^{\mathscr{A},e}$ and $\tau^{\mathscr{A},e} \in A$. |

From now on we restrict our attention to infinite models.

## 4

According to the Mathematical-Object theory, in each of these three cases, the speakers of $\mathscr{L}$ have acquired access to objects of a peculiarly mathematical kind. They have acquired a term-forming operator to construct singular terms designating such objects, and a one-place predicate to apply to all and only such objects. In the second and third cases they have also acquired a two-place predicate that stands for an important relation involving such objects.

To understand the importance of these relations, we must recognize the connection between certain higher-order entities and certain objects. Each natural number $n$ corresponds to a cardinality-quantifier, represented by expressions of the form 'there are exactly $n$ many $x$s'; each set $s$ corresponds to the level-one Fregean concept represented by predicates of the form 'belongs to $s$'. Furthermore these correspondences are, in some mysterious way, intrinsic to the numbers and sets. In other words, the Mathematical-Object theory claims that there is a "standard representor" assigning each cardinality-quantifier to its corresponding number, and a "standard extensor" assigning to each level-one Fregean concept fortunate enough to have an extension that extension.[16] The first case almost fits this mold: the Mathematical-Object theory holds that any two objects given in a definite order correspond intrinsically to an ordered pair, as determined by the "standard pairer."

Thus, according to the Mathematical-Object theory, models for $S$ are no longer adequate to represent the alethic underpinnings for $\mathscr{L}^{p}$, $\mathscr{L}^{\#}$, or $\mathscr{L}^{\wedge}$. Consider $\mathscr{L}^{\wedge}$. By itself a model for $S$ doesn't represent anything about the work done by locutions for set-abstraction, sethood, and membership. These additions require a new notion of modelhood; For example, in case 3 we need pairs of the form $\langle \mathscr{A}, \mathscr{e} \rangle$ where $\mathscr{A}$ is a model of the old sort appropriate to $L$, and $\mathscr{e}$ is an extensor for $\mathscr{A}$; $\mathscr{e}$ is needed to represent the standard extensor. Relative to a choice of $\mathscr{A}$ and $\mathscr{e}$, the assignment of a term $\tau$ to $\tau^{\mathscr{A},\mathscr{e}}$ represents designation; the truth of an interpreted sentence parsed by $\varphi$ is represented by $\mathscr{A}, \mathscr{e} \hat{\models} \varphi$, and its falsehood by $\mathscr{A}, \mathscr{e} \hat{\Dashv} \varphi$. Since the models of the new sort, pairs $\langle \mathscr{A}, \mathscr{e} \rangle$, tell us what to do with '$\hat{\ }$', '**Set**', and '$\in$', these expressions should be classified as non-logical vocabulary.

Because this theory takes terms like 'the empty set' to be genuine and successful designators, it suggests that sentences like 'The empty set is blue' or 'The empty set = Julius Ceasar' have definite truth-values.[17] This is mirrored by the modeling under case 3: if $\varphi$ is '$\mathbf{B}(\hat{x}(x \neq x))$' or '$\hat{x}(x \neq x) = \tau$', where '$\mathbf{B}$' $\in S$ is a one-place predicate-constant and $\tau^{\mathscr{A}}\downarrow$, either $\mathscr{A}, \mathscr{e} \hat{\models} \varphi$ or $\mathscr{A}, \mathscr{e} \hat{\Dashv} \varphi$.

According to the Alternative theory in none of these cases have the speakers stumbled across peculiarly mathematical objects. Nor have they constructed or otherwise manufactured objects that somehow hadn't existed before. Instead they have developed ways of extending (or in case 1, of simplifying) the expressive resources of $\mathscr{L}$. So in all cases a model for $S$ by itself, without supplementation by a pairer, representor, or extensor, can still model the alethic underpinnings of $\mathscr{L}^p$, $\mathscr{L}^\#$, and $\mathscr{L}^\wedge$.

Here is how that modeling works for $\mathscr{L}^\wedge$. For any infinite model $\mathscr{A}$, any sentence $\varphi$ and term $\tau$ of $L^\wedge_{\mathscr{A}}$, let:

$$\mathscr{A} \models \varphi \text{ iff for every extensor } e \text{ for } \mathscr{A}, \mathscr{A}, e \models \varphi;$$

$$\mathscr{A} \dashv \varphi \text{ iff for every extensor } e \text{ for } \mathscr{A}, \mathscr{A}, e \dashv \varphi;$$

$$\tau^{\mathscr{A}} = a \text{ iff for every extensor } e \text{ for } \mathscr{A}, \tau^{\mathscr{A}, e} = a;$$

$$\tau^{\mathscr{A}}\uparrow \quad \text{iff there is no such } a.$$

As just defined, $\models$ and $\dashv$ extend the corresponding relations defined in §2: if $\varphi$ is a sentence of $L$, then $\mathscr{A} \models \varphi$ holds in the sense of §2 iff $\mathscr{A} \models \varphi$ holds as defined above; similarly for $\dashv$, and for the assignment of $\tau$ to $\tau^{\mathscr{A}}$. Definitions of validity and bivalance carry over from §2, with one significant change: the quantifier over models is restricted to infinite models. Let $\text{Biv}(L^\wedge)$ be the set of bivalent sentences of $L^\wedge$. An the following logical notion now merits attention:

$\varphi$ is logically truth-valueless iff

for all infinite models $\mathscr{A}$ for $S$, $\mathscr{A} | \varphi$.

Analogous definitions apply to $\varphi$ and $\tau$ in $L^p_{\mathscr{A}}$ and $L^\#_{\mathscr{A}}$.

According to the Alternative theory, relative to a choice of a model $\mathscr{A}$, $\models$ and $\dashv$ model being true and being false in $\mathscr{L}^\wedge$, and the assignment of terms $\tau$ to $\tau^{\mathscr{A}}$ models designation in $\mathscr{L}^\wedge$. There is no "standard" extensor that needs to be represented in a model-theoretic semantics adequate to model the underpinnings of set-theoretic discourse.

Thus 'The empty set', for example, is not a designator, and 'The empty set = Julius Ceasar' has no truth-value. Correspondingly, '$\hat{x}(x \neq x)$'$^{\mathscr{A}}$ is undefined for every model $\mathscr{A}$, and '$\hat{x}(x \neq x) = \mathbf{a}$' is logically truth-valueless. And a sentence like 'The empty set is blue' need not have a truth-value; '$\mathbf{B}(\hat{x}(x \neq x))$' is not bivalent; indeed it will be true iff '$(\forall x)\mathbf{B}(x)$' is true, and it will be false iff '$(\exists x)\mathbf{B}(x)$' is false. Furthermore the semantic job of words expressing sethood and membership is not to apply or fail to apply to (tuples of) objects: so a sentence like 'Julius Ceasar is a set' has no truth-value; correspondingly, '**Set(a)**' is logically truth-valueless.

Such sentences are peculiar in that they employ mathematical vocabulary, but clearly lack mathematical content. The Alternative theory characterizes the semantic basis of their peculiarity. Nonetheless 'The empty set exists' and 'There are sets' are true, even logically true;

correspondingly, '$E(\hat{x}(x \neq x))$' and '$(\exists x)\mathrm{Set}(x)$' are valid. It should be emphasized that these sentences do not rely on any peculiarly mathematical construal of 'exists' or its synonyms; our semantics handles occurrences of '$\exists$' uniformly through $L^{\wedge}$.

Thus $\mathscr{L}^{\wedge}$ carries commitment to the existence of sets. But it is a thin commitment, for the truth of 'There are sets' is not based on the applicability of 'is a set' to some objects. Similarly for $\mathscr{L}^{p}$ and ordered pairs, and $\mathscr{L}^{\#}$ and natural numbers.

On this view, the assignment of term $\tau$ to $\tau^{\mathscr{A}, e}$, and the relations $\hat{\models}$ and $\hat{=\!\!|}$ relative to a pair $\langle \mathscr{A}, e \rangle$ are supervaluations, doing no representational work; there are no facts about $\mathscr{L}^{\wedge}$ for them to model. They are, in Kaplan's phrase,[18] artifacts of the model-theory, mere stepping-stones to the definition of $\models$, $=\!\!|$, and $\tau^{\mathscr{A}}$. The semantic roles of abstraction terms, and of expressions for sethood and membership, are modeled by the role of $^{\wedge}$-terms, '**Set**', and '$\in$' in the latter definitions. A model $\mathscr{A}$ itself tells us nothing about that role. Thus for the Alternative theory '$^{\wedge}$', '**Set**', and '$\in$' are logical constants. Similar remarks apply to $L^{p}$ and $L^{\#}$.

The virtues of the Alternative theory are particularly clear when we consider ontological reduction. Suppose that the enriched language from case 1 is now further enriched by set-theoretic talk. (The discussion to follow easily carries over to case 2.) As is well known, the mathematical work done by ordered pairs can be done as well by certain sets, e.g., by Wiener-Kuratowski pairs. For the Mathematical-Object theorist, either (1) some such "reduction" of ordered pairs to sets is right, or (2) none are right: ordered pairs are *sui generis*, and in particular, are not sets.

Option (1) is indefensible.[19] According to option (2), in replacing the *sui generis* notion of ordered-pairhood by a set-theoretic one, we choose to ignore a portion of mathematical reality. It's universally agreed that this loss is of no mathematical interest. If the point of mathematics is to describe mathematical reality, why should mathematicians glibly ignore a part of this reality? For the Mathematical-Object theorist, this aspect of mathematical practice should seem unreasonable.

The Alternative theory avoids this uncomfortable dilemma: It sees "no fact of the matter" to ontological reduction.[20] Enrich $L^{p}$ to $L^{p,\wedge}$ in the obvious way. Given a model $\mathscr{A}$, a pairer $p$, and an extensor $e$ for $\mathscr{A}$, define $\tau^{\mathscr{A}, p, e}$, $\overset{p^{\wedge}}{\models}$, and $\overset{p^{\wedge}}{=\!\!|}$ in the obvious way. To model truth, falsity, and designation in $L^{p,\wedge}$ define $\tau^{\mathscr{A}}$, $\mathscr{A} \models \varphi$, and $\mathscr{A} =\!\!| \varphi$ according to the pattern set for $L^{p}$ and $L^{\wedge}$, but now universally quantifying over both pairers and extensors. Equations of the form '$\mathbf{p}(\tau, \sigma) =$

'$\{\{\tau\}, \{\tau, \sigma\}\}$' (in primitive notation the right-hand side would be written out using '$\hat{\ }$') are logically truth-valueless, as is '$(\exists x)(\mathbf{Set}(x) \,\&\, \mathbf{OP}(x))$'. Indeed, it is precisely the sentences that intuitively do have mathematical content that come out bivalent. Unlike its rival, the Alternative theory offers a semantic basis for the obvious lack of mathematical content to the choice between different set-theoretic definitions of ordered pairs.

5

According to the Mathematical-Object theory, the point of enriching $\mathscr{L}$ in cases 2 and 3 is straightforward: to allow the speakers of $\mathscr{L}$ to talk about mathematical objects, real things that, for one reason or another, might merit talking about. The Alternative theory sees a different point: a way of encoding statements in higher-order languages into a first-order syntax, making them both notationally and conceptually more tractable.

We'll now look at such a language, a dyadic second-order language that corresponds exactly to Biv($L^{\wedge}$).[21] Form $L'$ from $L$ by introducing a single variable '$X$' of type (0, 0), letting $X(\tau, \sigma)$ be a formula for any terms $\tau$ and $\sigma$, and letting '$\exists$' bind '$X$' as usual. Relative to a model $\mathscr{A}$, define $\models$ and $=\!\mid$ by letting '$X$' range over Power($|\mathscr{A}| \times |\mathscr{A}|$). If $e$ is an extensor for $\mathscr{A}$, let

$$e' = \{\langle a, b\rangle : \text{for some } A \in \mathrm{dom}(e),\ a \in A \text{ and } e(A) = b\}.$$

It's easy to write down a formula Ext($X$) in which '$X$' is the only free variable such that for any model $\mathscr{A}$ and $E \subseteq |\mathscr{A}| \times |\mathscr{A}|$,

$$\mathscr{A} \models \mathrm{Ext(E)} \text{ iff } E = e' \quad \text{for some extensor } e \text{ for } \mathscr{A}.$$

We may then syntactically specify a translation $s$ from sentences of $L^{\wedge}$ to formulae of $L'$ in which '$X$' is the only free variable so that for any infinite model $\mathscr{A}$:

$$\text{for } \varphi \in \text{Sent } (L^{\wedge}) \quad \mathscr{A} \models \varphi \quad \text{iff } \mathscr{A} \models (\forall X)(\mathrm{Ext}(X) \supset s(\varphi)).$$

So for $\varphi \in$ Biv($L^{\wedge}$), $\mathscr{A} =\!\mid \varphi$ iff $\mathscr{A} =\!\mid (\forall X)(\mathrm{Ext}(X) \supset s(\varphi))$.[22] The logic of $L^{\wedge}$ is a fragment of second-order logic.

For $\varphi \in$ Biv($L^{\wedge}$), $(\forall X)(\mathrm{Ext}(X) \supset s(\varphi))$ represents $\varphi$'s semantic form, since its syntax makes plain, more perspicuously than does $\varphi$ itself, the role of basic semantic facts in fixing $\varphi$'s truth-value. Why this asymmetry between $\varphi$ and this sentence?

Let a language (sense-bearing or model-theoretic) be semantically uniform iff for each of its logico-syntactic lexical categories all items of that category have semantic jobs of the same sort. $L'$ is semantically uniform. $L^{\wedge}$ is not. Some singular terms are in $L^{\wedge}$ to designate; but

$\hat{}$-terms are built to encode Fregean concepts. Some predicates are in $L^{\hat{}}$ to apply, or fail to apply, to (tuples of) objects; but '**Set**' and '$\in$' do quite different work. Correspondingly, $\mathscr{L}^{\hat{}}$ is not semantically uniform; it runs more risk of philosophic misconstrual than would an enrichment of $\mathscr{L}$ whose logical-syntax were modeled by $L'$. This danger is the price of $\mathscr{L}^{\hat{}}$'s practical advantages. Thick ontological commitment is determined by semantic form. When limning the ultimate furniture of reality, we do best to speak, or at least think, in terms of a semantically uniform language. $L'$ is better than $L^{\hat{}}$; and in $L'$ there is no talk of sets.

On the other hand, $\varphi$ wears its logical form on its surface: $(\forall X)(\mathrm{Ext}(X) \supset s(\varphi))$ does not represent $\varphi$'s logical form. A sentence's logical form is a matter of its potential roles in inferences. Some of the structure of $(\forall X)(\mathrm{Ext}(X) \supset s(\varphi))$, e.g. the initial universal quantifications and the conditional structure of what follows it, is irrelevant to the inferential practice of the speakers of $\mathscr{L}^{\hat{}}$. The practice does not involve speakers making reference to particular extensors, and so doesn't involve their instancing such an initial quantification, or applying modus ponens to statements of the form $\mathrm{Ext}(\mathbf{E}) \supset \mathrm{Sub}(\mathbf{E}, X, s(\varphi))$. (For statements parsed as $(\forall x)(\mathbf{Set}(x) \supset \psi)$, inferential practice would include instancing of the initial quantifier and applying modus ponens to the result.) Parsing is a matter of making logico-syntactic (i.e., logical) form perspicuous. Thin ontological commitments are to be determined merely from logico-syntactic form, which need not coincide with semantic form.

Points analogous to these apply to $L^{\#}$. Unlike $L^{\hat{}}$ and $L^{\#}$, the bivalent fragment of $L^{p}$ does not extend the expressive power of $L$ (though it might yield greater expressive convenience).[23] But if $L$ is first enriched to a monadic second-order language $L^{1}$ by introducing infinitely many type-1 variables, and then $L^{1}$ is enriched by $L^{1,p}$, the latter is in effect a dyadic (and thus a full) second-order language; so $L^{1,p}$ does extend the expressive power of $L^{1}$.

## 6

Let a sentence of $L^{p}$, $L^{\#}$, or $L^{\hat{}}$ be pure if $S$ is empty. By a symmetry argument, all pure sentences are bivalent. Many familiar mathematical principles may be expressed as pure validities in these languages.

For example, consider the Basic Fact about ordered pairs:

$$(\forall x)(\forall y)(\forall u)(\forall v)(\mathbf{p}(x,y) = \mathbf{p}(u,v) \supset (x = u \,\&\, y = v)).$$

This Basic Fact seems to express "all there is to ordered pairing." The Mathematical-Object theorist can't agree: for example, '⟨France, England⟩ = Julius Ceasar' is not decided by the Basic Fact. But the

Alternative theory offers a precise statement of this thesis, one that is a theorem.[24]

No principles expressible in $L^{\#}$ or $L^{\wedge}$ are complete for the natural numbers or for sets as the Basic Fact is for ordered pairs. This is because the expressive powers of these languages are greater than that of any first-order language. As the other side of this coin, some of their pure bivalent sentences are neither valid nor have valid negations: the cardinality of a model can determine truth-value. Note that the axioms of extensionality, of pairs, the axioms of separation, and the existence of the null-set are all expressible by valid sentences in $L^{\wedge}$.[25]

Does the Alternative theory deserve the label 'Logicism'? This issue is made delicate by our need to restrict attention to infinite models. Although 'logical' is a vague label, few are willing to regard 'There are infinitely many objects' as a logical truth. And unless we do so, the Alternative theory is only logicist "modulo actual infinity."

An initially promising way to avoid assuming that there are infinitely many objects is to go modal, assuming only a "possible infinity."[26] Unfortunately, if we allow quantification over mathematical objects within the scope of the modal operator, it will have to be a special sort of quantifier. For more detail, see [11]. This rather weakens the appeal of going modal.

## 7

Confusion of the semantic notion of real truth with the set-theoretic notion of truth-in-a-model has contributed to some confusion about the role of set-theory in a definition or theory of real truth. Although this matter is tangential to this essay, the apparatus introduced here will, in §8, reconnect with our main thread. Impatient readers may skip both sections.

Sets do play a small but essential role in a definition of truth for a quantificational language: such a definition will use the notion of satisfaction, and so presuppose the existence of variable-assignments, which are functions, which (for our purposes) are sets.

In this section we'll model the relation between a sense-bearing first-order language and a metalanguage for it by the relation between uninterpreted languages $L$ and $M^*$. Working in appropriate models for $M^*$, we'll consider a definition of truth for sentences of $L$ that models a definition of real truth for statements of a first-order object-language.

Suppose that $1P$ and $2P$ are sets of one- and two-place predicate-constants, respectively, and $IC$ is a finite set of individual-constants; let $S = 1P \cup 2P \cup IC$, $L = L(S)$. Let $S_0$ be the set of quote-names for all elements of $S$, for the logical constants '$\bot$', '$\supset$', '$\exists$', '$=$', for '(' and ')', and for all variables. These quote-names are individual-constants; let $\delta'$ be $\delta$'s quote-name for $\delta \in S$. Let $S_1$ be:

$$S_0 \cup \{`*', `\mathbf{1P}', `\mathbf{2P}', `\mathbf{IC}', `\mathbf{Var}', `\mathbf{Fml}', `\mathbf{FV}', \mathbf{Sent}'\},$$

where '$*$' is a two-place function-constant introduced to represent concatenation, '**FV**' is a two-place predicate-constant, and the rest are one-place predicate-constants. Suppose $\mathscr{A}$ is a model for $S'$ with $S_1 \subseteq S'$; under the following conditions we may think of $\mathscr{A}$ as containing $L$: all $\mathscr{A}$-values of members of $S_0$ are distinct; $`\mathbf{1P}'^{\mathscr{A}} = \{\gamma'^{\mathscr{A}} : \gamma \in 1P\}$; analogously for $`\mathbf{2P}'^{\mathscr{A}}$ and $`\mathbf{IC}'^{\mathscr{A}}$; $`\mathbf{Var}'^{\mathscr{A}} = `\nu'^{\mathscr{A}} : \nu$ is a variable}; $`\mathbf{Fml}'^{\mathscr{A}}$ is the set of "formulae" generated appropriately from $S_0$ using $`*'^{\mathscr{A}}$; $`\mathbf{Sent}'^{\mathscr{A}}$ is the corresponding set of "sentences"; $`*'^{\mathscr{A}}$ is cycle-free on the closure of $S_0$ under it; $`\mathbf{FV}'^{\mathscr{A}} = \{\langle a, b\rangle : a \in `\mathbf{Var}'^{\mathscr{A}}, b \in `\mathbf{Fml}'^{\mathscr{A}}$, and "$a$ occurs free in $b$"}. Under these conditions we'll say that $\mathscr{A}$ is syntactically adequate for $L$.

Let $S_2 = S_1 \cup \{`\mathbf{Unv}', `\mathbf{1Ap}', `\mathbf{2Ap}', `\mathbf{Des}'\}$, where '**1Ap**' and '**Des**' are two-place predicate-constants and '**2Ap**' is a three-place predicate-constant. First, let's confine our attention to total models. Let a model $\mathscr{A}$ for $S_2$ be adequate iff: it is syntactically adequate, and the following "basic semantic axioms" are true in $\mathscr{A}$:

$$(\exists x)\mathbf{Unv}(x);$$
$$(\forall x)(\forall y)(\mathbf{1Ap}(x, y) \supset [\mathbf{1P}(x) \;\&\; \mathbf{Unv}(y)]);$$
$$(\forall x)(\forall y)(\forall z)(\mathbf{2Ap}(x, y, z) \supset [\mathbf{2P}(x) \;\&\; \mathbf{Unv}(y) \;\&\; \mathbf{Unv}(z)]);$$
$$(\forall x)(\forall y)(\mathbf{Des}(x, y) \supset [\mathbf{IC}(x) \;\&\; \mathbf{Unv}(y)]);$$
$$(\forall x)(\forall y)(\forall y')([\mathbf{Des}(x, y) \;\&\; \mathbf{Des}(x, y')] \supset y = y');$$
$$\text{"Totality"}: (\forall x)(\mathbf{IC}(x) \supset (\exists y)\mathbf{Des}(x, y)).$$

Such a model determines a total model $\mathscr{B}$ for $L$ with $|\mathscr{B}| = `\mathbf{Unv}'^{\mathscr{A}}$ and such that for any $\gamma \in S$:

$$\text{if } \gamma \in 1P \text{ then } \gamma^{\mathscr{B}} = \{a : \langle \gamma', a\rangle \in `\mathbf{1Ap}'^{\mathscr{A}}\};$$
$$\text{if } \gamma \in 2P \text{ then } \gamma^{\mathscr{B}} = \{\langle a, b\rangle : \langle \gamma', a, b\rangle \in `\mathbf{2Ap}'^{\mathscr{A}}\};$$
$$\text{if } \gamma \in IC \text{ then } \gamma^{\mathscr{B}} \text{ is the unique } a \text{ such that } \langle \gamma', a\rangle \in `\mathbf{Des}'^{\mathscr{A}}.$$

Letting $M = L(S_2)$, $M$ expresses (relative to $\mathscr{A}$) the basic semantic facts about $L$ (relative to $\mathscr{B}$).[27] Relative to $\mathscr{A}$ we want to define truth for sentences of $L$ (relative to $\mathscr{B}$). The inductive definition of satisfaction will have to be represented by axioms involving the new expression '**Sat**'; but some set-theoretic machinery is also needed.

Form $M^*$ by adding '**Sat**' to the logical lexicon of $M^{\hat{}}$, with '**Sat**' behaving syntactically as a two-place predicate-constant. Since we want our theory to apply in any adequate model, even a countable one, and since our sets will conform to the limitation-of-size conception, we must permit all sets to be finite; in particular we must allow variable-assignments that are finite. Relative to $\mathscr{A}$, let a variable-assignment be simply a function into '**Unv**'$^{\mathscr{A}}$ such that all members of its domain are variables. Thus the empty set is a variable-assignment. Let Asgmt($x$) be the formula of $M^*$ saying "$x$ is a variable-assignment." Let Vrnt($x, z, w, x'$) be the formula saying "$x'$ is the variant of $x$ assigning $z$ to $w$."

Given an infinite model $\mathscr{A}$, let $e$ be an extensor for $\mathscr{A}$ and $s \subseteq |\mathscr{A}| \times |\mathscr{A}|$. We now define $\tau^{\mathscr{A}, e, s}$ for terms $\tau$ of $M^*$, and $\mathscr{A}, e, s \mathrel{\hat{\models}} \varphi$ and $\mathscr{A}, e, s \mathrel{\hat{=\!|}} \varphi$ as usual, taking $s$ as the extension of '**Sat**'. Let $s = \mathrm{Sat}^{\mathscr{A}, e}$ iff the appropriate axioms governing '**Sat**' come out true in the sense of $\mathrel{\hat{\models}}$ relative to $\langle \mathscr{A}, e \rangle$. First we'll need a bookkeeping axiom:

$$(\forall x)(\forall y)(\mathbf{Sat}(x, y) \supset$$
$$[\mathrm{Asgmt}(x) \;\&\; \mathbf{Fml}(y) \;\&\; (\forall u)(\mathbf{FV}(u, y) \supset u \in \mathrm{dom}(x))]).$$

Then we need formulations of the familiar clauses from the recursive definition of satisfaction, e.g.:

$$(\forall x)(\forall p)(\forall y)([\mathrm{Asgmt}(x) \;\&\; \mathbf{1P}(p) \;\&\; \mathbf{Var}(y) \;\&\; y \in \mathrm{dom}(x)]$$
$$\supset [\mathbf{Sat}(x, p * y) \equiv \mathbf{1Ap}(p, x(y))]);$$
$$(\forall x)(\forall p)(\forall i)(\forall z)([\mathrm{Asgmt}(x) \;\&\; \mathbf{1P}(p) \;\&\; \mathbf{IC}(i) \;\&\; \mathbf{Des}(i, z)]$$
$$\supset [\mathbf{Sat}(x, p * i) \equiv \mathbf{1A}(p, z)]);$$
$$(\forall x)(\forall y)(\forall z)([\mathrm{Asgmt}(x) \;\&\; \mathbf{Var}(y) \;\&\; \mathbf{Fml}(z)] \supset$$
$$[\mathbf{Sat}(x, \text{'('} * \text{'}\exists\text{'} * y * \text{')'} * z) \equiv$$
$$(\exists v)(\exists x')(\mathbf{Unv}(v) \;\&\; \mathrm{Vrnt}(x', x, y, v) \;\&\; \mathbf{Sat}(x', z))]).$$

From these samples, the reader should be able to figure out what the remaining axioms should be. Relative to $\mathscr{A}, e$ these axioms implicitly define $\mathrm{Sat}^{\mathscr{A}, e}$.

As usual, we let:

$\mathscr{A} \models \varphi$ iff for every extensor $e$ for $\mathscr{A}$: $\mathscr{A}, e, \mathrm{Sat}^{\mathscr{A}, e} \mathrel{\hat{\models}} \varphi$;
$\mathscr{A} =\!| \varphi$ iff for every extensor $e$ for $\mathscr{A}$: $\mathscr{A}, e, \mathrm{Sat}^{\mathscr{A}, e} \mathrel{\hat{=\!|}} \varphi$.

We may now define truth for sentences of $L$. For any adequate $\mathscr{A}$, fix $\mathscr{B}$ as above; for any $\psi \in \mathrm{Sent}(L)$, let $a \in$ '**Sent**'$^{\mathscr{A}}$ "be" $\psi$; then:

$$\mathscr{B} \models \psi \text{ iff } \mathscr{A} \models \mathbf{Sat}(\hat{x}(x \neq x), \mathbf{a}).$$

We abbreviate 'Sat($\hat{x}(x \neq x), z$)' as 'True($z$)'. (By our bookkeeping axiom, 'True($z$)' entails '**Sent**($z$)'.)

To accommodate non-total models and represent a three-valued semantics, we must drop the "totality" axiom used in the definition of adequacy. We also change $M^*$, supplementing '**Sat**' with '**Frus**' to represent frustration (= anti-satisfaction), and introducing appropriate axioms to implicitly define $\mathrm{Frus}^{\mathscr{A},*}$. We may then define 'False($z$)' in terms of '**Frus**'. Details are left to the reader.

Notice that $\mathscr{A}$ only represents basic semantic facts of designation and application. '**Sat**' and '**Frus**', like '**Set**' and '$\in$', are used "supervaluationally"; their semantic roles are not that of an ordinary two-place predicate constant. But since their semantic roles are tied to those of the members of $S_2$, which are not logical constants, we can't consider them full-blooded logical constants; at best they are hybrids of the logical and the non-logical. On the other hand, '**1Ap**', '**2Ap**', and '**Des**' are ordinary predicate-constants. If we wanted to make heavier use of mathematical machinery, we could avoid using distinct predicate-constants to handle application for predicate-constants in $S$ of different "adicity": we could use a single two-place predicate-constant '**Ap**' to handle all predicate-constants in $S$, taking a two-place predicate-constant as applying to ordered pairs, etc. '**Ap**' would be a logical expression. Building in ordered pairs "at the ground floor" encourages the illusion that basic semantic facts involve mathematical objects, and that basic semantic relations have the logical status of mathematical relations; thus it is best avoided. Of course we could also have built in mathematical objects "at the basement," construed syntactic objects (e.g., formulae), as tuples or as sets.

## 8

According to the Alternative theory, the semantics for mathematical notions differs significantly from that for less arcane notions. Our model-theoretic semantics for $L^{\wedge}$, for example, is intended to model the alethic underpinnings of talk "about" sets, to show how set-theoretic statements get truth-values even though 'set' does not stand for a genuine kind, '$\in$' doesn't stand for a genuine relation, and set-terms do not designate.

It is tempting to formulate this Alternative theory as the claim "Sets are not objects." The spirit may be right; but taken literally, this claim is trivially false; '$x$ is an object' may be parsed as '$x = x$' (or even as '$\neg \bot$').[28] Clearly '$(\forall x)(\mathbf{Set}(x) \supset x = x)$' is valid.

This is why the Alternative theory is a theory about the semantics of mathematical discourse: it cannot be "pulled down," using Carnap's phrases, from the formal to the material mode. But it may appear that the Alternative theory is undercut by similar reasons: once it has been conceded that, for example, the empty set is an object, why not say that 'the empty set' designates the empty set? Since it is set, why not say that 'set' applies to it?

In fact, this can be said; but in so saying the notions of designation and application are being stretched. 'Julius Caesar' robustly designates, and 'Roman' robustly applies to, a famous general; but to say that 'the empty set' designates and 'set' applies to a famous set is to use 'designates' and 'applies to' in a non-robust disquotational way.[29] To clarify the difference, we consider how to define truth for sentences of $\hat{L}$.

First we expand $S_0$ and add to the notion of syntactic adequacy of a model $\mathscr{A}$ to handle the syntax of $\hat{L}$. In particular, we'll need a one-place predicate-constant '$\mathbf{\hat{T}rm}$' to apply to exactly the $\hat{}$-terms of $\hat{L}$, and the right-domain of '$\mathbf{FV}$'$^{\mathscr{A}}$ will include $\hat{}$-terms. But we would not have to expand the $\mathscr{A}$-values of '$\mathbf{1Ap}$', '$\mathbf{2Ap}$', and '$\mathbf{Des}$' so as to make $\mathscr{A}$ represent applicative facts about '$\mathbf{Set}$' and '$\in$', for there are no such facts! Form $M^+$ from $M^*$ by adding the two-place predicate-constant '$\mathbf{1Ap}^+$' and the three-place predicate-constant '$\mathbf{2Ap}^+$' and '$\mathbf{Des}^+$'. Given $\mathscr{A}$, $e$, let $1\mathrm{Ap}^{+\mathscr{A},e}$ be implicitly defined by this axiom:

$$(\forall x)(\forall y)(\mathbf{1AP}^+(x,y) \supset [\mathbf{1P}(x) \vee x = \text{'}\mathbf{Set}\text{'}];$$
$$\text{for each } \gamma \in 1P\text{: } (\forall y)(\mathbf{1Ap}^+(\gamma', y) \equiv \mathbf{1Ap}(\gamma', y));$$
$$(\forall y)(\mathbf{1Ap}^+(\text{'}\mathbf{Set}\text{'}, y) \equiv \mathbf{Set}(y)).$$

Analogous axioms, using '$\in$' rather than '$\mathbf{Set}$', implicitly define $2\mathrm{Ap}^{+\mathscr{A},e}$. '$\mathbf{1Ap}^+$' and '$\mathbf{2Ap}^+$' "stretch" '$\mathbf{1AP}$' and '$\mathbf{2Ap}$'. Unlike the latter two, they do not play the semantic role of ordinary predicate-constants; they are partly non-logical like '$\mathbf{1Ap}$' and '$\mathbf{2Ap}$', and partly logical like '$\mathbf{Set}$' and '$\in$'. Notice that, for example, '$(\forall y)(\mathbf{1Ap}^+(\text{'}\mathbf{Set}\text{'}, y) \equiv \mathbf{Set}(y))$' comes out valid with respect to adequate models.

Let $e$ be a '$\mathbf{Unv}$'-extensor for $\mathscr{A}$ iff $e$ is an extensor for $\mathscr{A}$ and for every $A \subseteq \text{'}\mathbf{Unv}\text{'}^{\mathscr{A}}$, if $A \in \mathrm{dom}(e)$ then $e(A) \in \text{'}\mathbf{Unv}\text{'}^{\mathscr{A}}$. For $e$ a '$\mathbf{Unv}$'-extensor for $\mathscr{A}$, we implicitly define $\mathrm{Sat}^{\mathscr{A},e}$, $\mathrm{Frus}^{\mathscr{A},e}$, and $\mathrm{Des}^{+\mathscr{A},e}$

with axioms, including these bookkeeping axioms:

$$(\forall x)(\forall y)(\forall z)(\mathbf{Des}^+(x, y, z) \supset [\text{Asgmt}(x) \,\&\, (\text{`}\hat{\ }\text{-}\mathbf{Trm}(y) \vee \mathbf{IC}(y)) \\ \&(\forall v)(\mathbf{FV}(v, y) \supset v \in \text{dom}(x))]);$$
$$(\forall x)(\forall y)(\forall z)(\forall z')([\mathbf{Des}^+(x, y, z) \,\&\, \mathbf{Des}^+(x, y, z')] \supset z = z');$$
$$(\forall x)(\forall y)(\forall z)([\text{Asgmt}(x) \,\&\, \mathbf{IC}(y)] \supset \\ [\mathbf{Des}^+(x, y, z) \equiv \mathbf{Des}(y, z)]).$$

We'll also need the following axiom:

$$(\forall x)(\forall y)(\forall z)(\forall u)([\text{Asgmt}(x) \,\&\, \mathbf{Fml}(y) \,\&\, \mathbf{Var}(z) \\ \&(\forall v)(\mathbf{FV}(v, y) \supset v \in \text{dom}(x))] \supset \\ [\mathbf{Des}^+(x, \text{`}\hat{\ }\text{'} * z * y, u) \equiv (\forall w)(w \in u \equiv [\mathbf{Unv}(w) \\ \&(\exists x')(\text{Vrnt}(x', x, z, w) \,\&\, \mathbf{Sat}(x', y))])]).$$

Otherwise the definition of truth for sentences of $L^{\hat{}}$ within $M^+$ runs as it did for $L$; details are left to the reader. Unlike '**Des**', '$\mathbf{Des}^+$' does not play the semantic role of an ordinary three-place predicate; like '$\mathbf{1Ap}^+$', '$\mathbf{2Ap}^+$', '**Sat**', and '**Frus**', it is a hybrid of the logical and the non-logical.

We define $\mathscr{A} \models \varphi$ and $\mathscr{A} =\!\mid \varphi$ for $\varphi$ a sentence of $M^+_{\mathscr{A}}$ as before. There is a slight twist to the definition of truth for sentences of $L^{\hat{}}$. Let 'True($z$)' abbreviate:

$$(\forall y)((\forall u)[u \in y \supset \mathbf{Unv}(u)] \supset \mathbf{Unv}(y)) \supset \mathbf{Sat}(\hat{x}(x \neq x), z).$$

We define 'False($z$)' analogously. The antecedent of these conditionals has the effect of "restricting our attention" to '**Unv**'-extensors as we unpack $\mathscr{A} \models$ True(**a**) and $\mathscr{A} =\!\mid$ False(**a**).

The moral: we can define truth in a set-theoretic language in terms of partially non-robust notions of designation and application, without compromising the Alternative theory. Of course, this definition uses the meta-language's set-theoretic apparatus more heavily than did the definition of truth for sentences of a non-set-theoretic language: we don't only need variable-assignments; we also need to use '**Set**' and '$\in$' in the disquotational semantic axioms governing '**Set**', and '$\in$', and '$\hat{\ }$'.

To think in terms of the disquotational semantics for mathematical discourse is to adopt a certain picture of that discourse: the Mathematical-Object picture. We picture truth for such discourse as structurally analogous to truth for robustly referential discourse. For mathematical purposes, this picture is fine. Indeed, that picture is overwhelmingly natural, given the logical syntax of our mathematical discourse. As the applications of model-theory to algebra show, it has mathematical value; models can represent non-robust semantic "facts" as well as robust ones. The Mathematical-Object theory transforms this picture into a theory of the alethic underpinnings of mathematical discourse. A natural error; but an error nonetheless.

## 9

According to our Alternative theory, mathematical theories are first-order encoding of higher-order logics. The ontological commitments to mathematical objects that the Mathematical-Object theory takes to be thick, the Alternative theory takes to be merely thin. Unlike thin commitments, thick commitments are preserved by encoding (indeed, by any sort of paraphrase that preserves semantic form). So the alternative theory saddles mathematical discourse with whatever thick commitments are carried by the higher-order quantification in the encoded logic. Why should we be more comfortable with, for example, second-order quantification than with quantification that purports to be over mathematical objects?

First of all, second-order quantification is found even in central areas of linguistic practice. For example, Dummett offers 'There is something that Plato was and Socrates was not'. The second-order nature of its initial quantifier-phrase is made especially clear by its instancings. Suppose a speaker uses this sentence assertively, and then backs up her assertion with 'For example, a dramatist', shorthand for 'For example, Plato was a dramatist and Socrates was not'; here the syntactic role of 'a dramatist' is certainly predicative. The first assertion carries at least a thin commitment to the existence of something that Plato was and Socrates wasn't. I can see no reason to deny that this commitment is also thick, that for this statement logical and semantic form coincide.[30] Of course, the syntax of natural languages forces us to complete the phrase 'is an example of the things to which that assertion is committed' with a noun-phrase such as 'being a dramatist' or a variant of Frege's preferred form 'the concept *dramatist*'. But these are not designators; indeed, they strike me as ersatz singular terms, as English make-do for constructions whose logical form involves variable binding.[31]

Second-order quantification can be rather subtle. Some plural noun-phrase constructions involve second-order quantification, e.g. the Geach-Kaplan example 'Some critics admire only one another'. But even for these statements, logical and semantic form appear to coincide. Suffice then to say: since central parts of our linguistic practice already saddle us with whatever thick commitments second-order quantification carries, exchanging an additional thick commitment to problematic mathematical objects for those of second-order quantification is a good deal.

George Boolos has argued that second-order quantification need carry no ontological commitments, I take it not even thin commitments, beyond those carried by first-order quantification.[32] If this is right, the advantage in exchanging thick commitment to mathematical objects for second-order quantification is clear. But suppose we accept Frege's doctrine that second-order quantification carried commitment to unsaturated entities, Fregean concepts and relations. Is thick commitment to such entities better than thick commitment to mathematical objects?

I think so. There are psychological and sociological facts about our linguistic practice, some historical in form, some structural in form, that constitute a supervenience base for facts about the application and non-application of predicates to objects. There seems to be no such base for purported facts about designation of mathematical objects.[33] Furthermore, as Furth claims, facts about the application and non-application of predicates are really all there is to facts about reference for predicates.[34] Even when thick, commitments to Fregean concepts and relations are light. But commitments to mathematical objects are heavy, too heavy, I contend, to be borne by a reasonable theory of reference.[35]

Are mathematical objects fictions? At least in part the answer depends on the extent of analogy between mathematical and meta-fictional statements. Quite unlike mathematical discourse, the telling of tales, performance of plays, and so forth largely consist of pretended assertion, whereas mathematical discourse involves genuine assertion. But meta-fictional discourse is different: attributions of fictional content (e.g., 'Hamlet was Danish', or even 'Hamlet existed'), construed as if prefixed with 'According to Shakespeare's *Hamlet*', can have truth-values; some singular terms in them (e.g., 'Hamlet' in the above example) contribute to determining that truth-value, but do so without designating anything.

The semantic form of such attributions of fictional content starts off with the "According to . . . ' prefix, though in actual speech it's usually omitted. One might press on with the analogy as follows: the semantic form of statements that appear to be about mathematical objects should also start off with a prefix whose force is "Construe the following within the mathematical-object picture," and whose semantic role is simply to indicate that what follows is to be evaluated in the supervaluational way modeled in §4. But this would be both unnecessary and misleading.

The distinctively fictional vocabulary of fiction has an attenuated life in our assertive practices outside of attributions of fictional content;

'Hamlet existed' can be asserted under a construal that lacks an operator like 'According to *Hamlet*'; and there would be a point to such assertions, for example, to make the (false) claim that *Hamlet* was based on Danish history. But our actual mathematical vocabulary has absolutely no life outside of the mathematical-object picture; so occurrence of such vocabulary in a statement is a sufficient cue that it is to be construed within that picture, that is, as the Alternative theory would have it. An operator explicitly indicating this is unnecessary.

But more importantly: 'According to *Hamlet*' shows that the statement with which it starts is about *Hamlet*. What could be to mathematical statements as Shakespeare's play is to statements construed as starting with 'According to *Hamlet*'? As far as I can see, nothing. Mathematical discourse itself may be the only remotely plausible answer, and such reflexivity strikes me as still quite implausible. In short, the analogy between mathematical statements and attributions of fictional content remains rather limited.

I can imagine that some philosophers will react to the Alternative theory by impatiently asking for the bottom line: "Are there really numbers, sets, and so forth?" In his testimony before the McCarthy committee, Dalton Trumbo responded to a well-known question by saying, "Many questions can be answered 'yes' or 'no' only by a moron or a slave." Presumably he thought that the question he had been asked, whether he was or had ever been a member of the Communist Party, was such a question. I doubt that he took membership in such a party to involve borderline cases. More likely, his point was this: even though there is a correct "Yes or No" answer, such an answer can easily give a wrong impression and by itself is unilluminating.

Perhaps Trumbo was wrong on this matter. But if the question had been "Are there really sets?" or "Does the number 3 really exist?" his response would have been right on target.

These questions are formulated within a language representable by $L^{\wedge}$ and $L^{\#}$. Understood straightforwardly, their answers are straightforwardly, indeed trivially, "Yes." But these commitments are thin. Mathematics does not require the Mathematical-Objective picture. In a semantically uniform language for mathematics, of a sort representable by, for example, $L'$, there would be no talk of mathematical objects. One might clumsily express this by saying that sets and numbers are unreal, not part of the furniture of the universe. Perhaps the point of the 'really' in these questions is to try to bend them into addressing this issue. But this strains our language; use of the words 'set' or 'number' pushes us into the Mathematical-Object picture. Once this picture is seen right, even the ontologically scrupulous philosopher should be comfortable with it.[36]

## Notes

1 Chihara's distinction between ontological and mythological platonism from [4] may be a gesture toward this distinction.
2 I am excluding from the scope of semantics matters not relevant to truth-conditions of statements, even matters that are relevant to understanding.
3 In the original version of "On What There Is," Quine uses 'ontological presupposition'; later, e.g., in *Word and Object*, he prefers 'ontic commitment'.
4 This point in made in [5] and [3].
5 One further point. In general we state commitments by specifying kinds, e.g., numbers, electrons, cabbages, kings. But if there is only one thing of that kind, we may say that the commitment is to that thing. (A looser use of 'ontological commitment' has some currency, according to which a piece of discourse is committed to each thing in the range of its variables.)
6 Of course for some singular terms designating is their semantic job, though they fail to do their job: They are empty. And a singular term containing free variables, e.g., 'the capital of $x$', is not a designator at all (though its semantic role is parasitic on the roles of its instances). Frege would say that it stands for a function; but such standing-for is *not* what I'm calling designation, because functions of this sort are not objects. I use 'designates' to mean reference at level-zero, the sort that only holds between closed singular terms and objects.
7 In [9] I called the Alternative theory 'Coding-Fictionalism'. The root 'fiction' seemed to encourage misunderstanding; hence the change.
8 I ignore the more general topos-theoretic notion of modelhood.
9 To bear sense is (schematically summing Wittgenstein and Putnam) to have a use in a niche. Natural languages are sense-bearing; sometimes formal languages bear sense, especially among contemporary mathematicians. Sense-bearing languages are often called 'interpreted'. This usage encourages conflation of understanding and interpreting; to interpret is to try to re-express in more understandable words.
10 Of course *a specification* of a model could give sense to a previously non-sense-bearing language. Also, given any model $\mathscr{A}$ there could be a population such that, for example, $\alpha^{\mathscr{A}}$ was the designatum in their language of $\alpha$ for every individual-constant $\alpha$ in the vocabulary of $\mathscr{A}$'s a model, and so forth. Then the language of that model would be used, and so sense-bearing; the model itself would be an especially "lifelike" representation of its alethic underpinnings.
11 A statement is true or false only as a statement in a sense-bearing language. Truth is a genuine semantic property; truth-in-a-model is a set-theoretic ersatz-semantic relation. A definition of truth-in-a-model models a definition of truth, though this modeling can be confusing. Here's why. For a sense-bearing quantificational language, truth must be defined in terms of satisfaction (or the like). Truth-in-a-model can be defined in terms of satisfaction in a model, making the relation between these definitions clear. But other definitions are notationally simpler, and so are often preferred (e.g., in this essay); philosophically, they are best conceived as shorthand for a definition in terms of satisfaction in a model.
12 Non-total models could be made more partial: we could have allowed them to assign predicate-constants partial extensions. Our models are, in the terminology of [10], extension-wise total.
13 Here $\mathrm{Sub}(\mathbf{a}, \nu, \theta)$ is the result of substituting $\mathbf{a}$ for all occurrences of $\nu$ free in $\theta$.
14 For more on the logic determined by this model-theoretic semantics, see [10].

15 '$=$' is not handled as a two-place predicate-constant with extension $\{\langle a, a\rangle : a \in |\mathscr{A}|\}$. Motivation: given that 'Venus' designates something and 'Vulcan' doesn't, an equation like 'Vulcan = Venus' seems false, not merely truth-valueless.

16 Russell's paradox showed that not every Fregean concept of level one can be fortunate enough to have an extension. When restricted to sets, it becomes Cantor's Theorem: in effect, that no $\mathscr{A}$ has an extensor with domain Power($|\mathscr{A}|$).

17 Proponents of the Mathematical-Object theory usually think that such sentences are false. But on what basis? Do they describe brute facts, knowable only by pure intuition? Their falsity is not required by mathematics. That alone should render suspect any theory requiring them to have a truth-value. Of course some have claimed that a predicate like 'blue' is partial, only applying, or genuinely failing to apply, within the appropriate "category." Are these categorial constraints brute facts? How do we find out about them? Unlike the Mathematical-Object theory, the Alternative theory gives some basis for so-called intuitions about category-errors.

18 See [13]. Kaplan's actual phrase is 'artifact of the model'; but a model-theoretic semantics is itself a model, as explained in note 11.

19 See the discussion of sets and the natural numbers in [1].

20 Although plainly incompatible with the Mathematical-Object theory, this view has found proponents.

21 [11] presents the fifth-order language $L^{0,4}(exactly)$ and a translation $t$ from its sentences into Biv($L^{\#}$), such that for every $\varphi \in$ Sent($L^{0,4}(exactly)$) and any infinite model $\mathscr{A}$:

$$\mathscr{A} \models \varphi \text{ iff } \mathscr{A} \models t(\varphi); \quad \mathscr{A} \not\models \varphi \text{ iff } \mathscr{A} \dashv t(\varphi).$$

In [12] I introduce the $\omega$-order language $L^{\omega}$ under what I there called the weak higher-order semantics, with a translation $t$ from its sentences into Biv($L^{\wedge}$) that meets an analog for Sent($L^{\omega}$) of the above condition. The sentences in the images of these translations can be regarded as formulations of their inverse-values within a first-order syntax. For each of these translations it is an open question whether they are onto Biv($L^{\#}$) and Biv($L^{\wedge}$) up to equivalence; I conjecture that in both cases this is not the case.

22 Form $L_u'$ from $L_u$ of [10] as above. With $L_u'$ in place of $L'$, there is a translation $t$ for which we could replace Biv($L^{\wedge}$) by Sent($L^{\wedge}$) in this result.

23 This follows by a three-valued interpolation lemma.

24 Let a sentence $\varphi$ of $L^{p}$ be truth-valueless relative to a set of sentences $\Delta$ iff for any infinite model $\mathscr{A}$, if $\mathscr{A} \models \Delta$ then $\mathscr{A} | \varphi$. This is the theorem: for any $\neg$-complete set $\Delta$ of sentences of $L$ and $\varphi$ as above, either $\varphi$ is truth-valueless relative to $\Delta$ or $\Delta \cup$ {the basic fact} decides $\varphi$ in first-order logic. In particular, if $\varphi$ is pure then either $\varphi$ is logically truth-valueless or the basic fact decides $\varphi$.

25 Where $C$ is a class of models, let $\varphi$ be valid with respect to $C$ iff $\varphi$ is true in all members of $C$. The union axiom and the axioms of replacement are valid with respect to models of regular cardinality; the axiom of infinity is valid with respect to models of uncountable cardinality; the power-set axiom is valid with respect to models of strong-limit cardinality. So, except for the axiom of regularity, all axioms of $ZF$ are valid with respect to models of inaccessible cardinality. This suggests that the status of these axioms is not as straightforward as that of those axioms that are valid simpliciter. Acceptance of these less self-evident axioms is, I suggest, an expression of a view about the size of the universe. (The power-set, infinity, and replacement axioms were never as self-evident as extensionality, pairs, and even separation.) The axiom of Regularity (and perhaps even Union?) derives its appeal from the iterative conception of sethood. From the Alternative view, that conception is unnecessarily restrictive. Regularity is a restrictive axiom. In many contexts we could live without it, faking its effect by restricting quantification over sets to well-founded sets. See [12] for more discussion.

26 For each natural number $n$, let $\varphi_n$ say "There are at least $n$ objects." To assume actual infinity is to assume $\{\varphi_1, \varphi_2, \ldots\}$. To assume potential infinity (against an *S5* background) is to assume $\{\Diamond\varphi_1, \Diamond\varphi_2, \ldots\}$.

27 If $S$ is finite and we replace $S_1$ by $S_0 \cup S$, then '1Ap($x$, $y$)' could be replaced by a conjunction:

$$(x = \text{'P'} \,\&\, \mathrm{P}(y)) \vee (x = \text{'Q'} \,\&\, \mathrm{Q}(y)) \vee \ldots$$

and similarly for '2Ap($x$, $y$, $z$)'.

28 Objecthood is implicit in '$x$''s syntactic status as a type-0 variable. This is why in the *Tractatus* Wittgenstein calls objecthood a formal property.

29 This distinction brings out what some might call the "transcendental realism" implicit in the Alternative theory.

30 This example is from [6, p. 219]. One might argue that this sentence is logically equivalent to 'Socrates ≠ Plato', and so its commitments are those of the latter as well. This presupposes a fixed and broad range for the second-order quantifier (including at least being Socrates or being Plato); but the contexts in which this sentence would most comfortably be used severely restrict the range of that quantifier. Indeed, the truth of a statement made with Dummett's sentence would be sensitive to that feature of context; 'Socrates ≠ Plato' shows no such sensitivity. That difference alone indicates enough non-equivalence to defeat this move.

31 Fregean reference for predicates is a relation of type (?, 1); so in a perspicuous notation we might express the referential work of the predicate 'is a horse' by: (Stands-for $x$) ('is a horse', $x$ is a horse). (There is a question mark in the preceding because the type of level-one predicates is a matter of controversy; in [8] Geach may be read as arguing that they are not objects; in the *Tractatus* Wittgenstein appears to agree.) Similarly, we might say: (Witnesses $x$) (the Dummett sentence, $x$ is a dramatist). I agree with Frege's assessment of the so-called paradox of the concept *horse*: that it is not a serious problem for the Fregean doctrine of reference for incomplete expressions.

32 See [2]. Boolos would take the Geach-Kaplan example to carry commitment only to the existence of some critics. As an assessment of thin commitment, this requires that thin commitments be preserved by only very tight paraphrase. 'There is something that some critics are, only critics are, and all who are admire only others who are' is a paraphrase of the Geach-Kaplan example; but I'd guess that Boolos would think that it carried different thin commitments. Suppose that the Mutual Admiration Society (MAS) is a club of critics who admire only one another. One might back up a statement made with the last sentence by going on to say 'For example, a member of the MAS'. This suggests that being a member of the MAS is in the range of the initial second-order quantifier. If the above paraphrase does preserve thin commitments, then even the Geach-Kaplan sentence could carry commitment to being a member of the MAS (and would if all critics who admire only one another are members of the MAS, and the MAS has exactly two members, each admiring the other).

33 See [9].

34 See [7], a remarkable paper that has been remarkably ignored.

35 Given the Furthean (Pickwickian?) view of what reference to Fregean concepts and relations amounts to, the disagreement expressed in [2] between Boolos and myself may be merely verbal.

36 The sort of approach here applied to mathematical discourse can, modulo vagueness about identity conditions, also be applied to talk "about" fact-like entities (states-of-affairs) including possible worlds, and "about" some meaning-like entities. A rather different approach applies to talk "about" Peircean types. Although the view presented here is undoubtedly at odds with his own, Hilary Putnam, through his writing and teaching, had an enormous influence on the thinking that led to this essay.