



One-Step Modal Logics, Intuitionistic and Classical, Part 2

Harold T. Hodes¹

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Abstract

Hodes (2021) “looked under the hood” of the familiar versions of the classical propositional modal logic K and its intuitionistic counterpart (see Plotkin & Sterling 1986). This paper continues that project, addressing some familiar classical strengthenings of K (D, T, K4, KB, K5, Dio (the Diodorian strengthening of K) and GL), and their intuitionistic counterparts (see Plotkin & Sterling 1986 for some of these counterparts). Section 9 associates two intuitionistic one-step proof-theoretic systems to each of the just mentioned intuitionistic logics, this by adding for each a new rule to those which generated IK in Hodes (2021). For the systems associated with the intuitionistic counterparts of D and T, these rules are “pure one-step”: their schematic formulations does not use \Box or \Diamond . For the systems associated with the intuitionistic counterparts of K4, etc., these rules meet these conditions: neither \Box nor \Diamond is iterated; none use both \Box and \Diamond . The join of the two systems associated with each of these familiar logics is the full one-step system for that intuitionistic logic. And further “blended” intuitionistic systems arise from joining these systems in various ways. Adding the 0-version of Excluded Middle to their intuitionistic counterparts yields the one-step systems corresponding to the familiar classical logics. Each proof-theoretic system defines a consequence relation in the obvious way. Section 10 examines inclusions between these consequence relations. Section 11 associates each of the above consequence relations with an appropriate class of models, and proves them sound with respect to their appropriate class. This allows proofs of some failures of inclusion between consequence relations. (Sections 10 and 11 provide an exhaustive study of a variety of intuitionistic modal logics.) Section 12 proves that the each consequence relation is complete or (for those corresponding to GL) weakly complete, that relative to its appropriate class of models. The Appendix presents three further results about some of the intuitionistic consequence relations discussed in the body of the paper.

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✉ Harold T. Hodes
hth3@cornell.edu

¹ Sage School of Philosophy, Cornell University, Ithaca, NY 14853, USA

9 Additional Rules and Further Proof-theoretic Systems

9.1 Rules

What follows is a continuation of Hodes (2021).

In what follows, ‘ X ’ will be schematic for names of proof-theoretic systems. First, we have four “pure step-rules”. The first is a “thickening” (or if you prefer, a strengthening) rule.¹

1 \top Thickening If $C, v:1\top \Rightarrow_X \mathcal{D}_0:\mathbf{0}\theta$ then $C \Rightarrow_X \mathcal{D}:\mathbf{0}\theta$ for \mathcal{D} as pictured below.

$$\frac{[v:1\top] \quad \mathcal{D}_0}{\mathbf{0}\theta} \mathbf{1}\top T$$

Let $dpd(\mathcal{D}) = dpd(\mathcal{D}_0)$.

Strengthened 1 \perp E If $C, v:1\top \Rightarrow_X \mathcal{D}_0:\mathbf{1}\perp$, and $\chi \in MFml$, then $C \Rightarrow_X \mathcal{D}:\chi$ for \mathcal{D} as pictured below.

$$\frac{[v:1\top] \quad \mathcal{D}_0}{\mathbf{1}\perp} S\mathbf{1}\perp E$$

$$\chi^v$$

Let $dpd(\mathcal{D}) = \{[0]\hat{\ }s \mid s \in dpd(\mathcal{D}_0) \text{ and } \mathcal{D}_0(s) \neq v:1\top\}$.

0 Elimination If $C_1 \Rightarrow_X \mathcal{D}_1:\mathbf{0}\varphi$, $C_0, v:1\varphi \Rightarrow_X \mathcal{D}_0:\mathbf{0}\theta$, \mathcal{D}_0 has a barrier with exception for $v:1\varphi$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow_X \mathcal{D}:\mathbf{0}\theta$ for \mathcal{D} as pictured.

$$\frac{[v:1\varphi] \quad \mathcal{D}_1 \quad \mathcal{D}_0}{\mathbf{0}\varphi \quad \mathbf{0}\theta} \mathbf{0}E$$

$$\mathbf{0}\theta^v$$

Let $dpd(\mathcal{D}) = \{[0]\hat{\ }s \mid s \in dpd(\mathcal{D}_1)\} \cup \{[1]\hat{\ }s \mid s \in dpd(\mathcal{D}_0) \text{ and } \mathcal{D}_0(s) \neq v:1\varphi\}$.

0 Introduction If $C, v:1\top \Rightarrow_X \mathcal{D}_0:\mathbf{1}\varphi$ and \mathcal{D}_0 has a barrier with exception for $v:1\top$, then $C \Rightarrow_X \mathcal{D}:\mathbf{0}\varphi$ for \mathcal{D} as pictured.

$$\frac{[v:1\top] \quad \mathcal{D}_0}{\mathbf{1}\varphi} \mathbf{0}I$$

$$\mathbf{0}\varphi^v$$

Let $dpd(\mathcal{D}) = \{[0]\hat{\ }s \mid s \in dpd(\mathcal{D}_0), \mathcal{D}_0 \neq v:1\top\}$.

The next six rules concern a single occurrence of a modal operator. The asterisks on the names of the next first four rules below indicate that they are quasi-introduction and quasi-elimination rules; see §13 for a bit more on this.²

¹ $\mathbf{0}EM$ and $\mathbf{1}EM$ are also thickening rules.

²The “quasi-ness” of these rules consists in this: $\mathbf{1}\square E^*$ and $\mathbf{1}\diamond E^*$, respectively, do not Prawitz-invert $\mathbf{1}\square I^*$ and $\mathbf{1}\diamond I^*$.

1◇ Introduction* If $C_1 \Rightarrow_X \mathcal{D}_1:\mathbf{0}\varphi$, $C_0 \Rightarrow_X \mathcal{D}_0:\mathbf{1}\psi$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow_X \mathcal{D}:\mathbf{1}\diamond\varphi$ for this \mathcal{D} .

$$\frac{\begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_0 \\ \mathbf{0}\varphi & \mathbf{1}\psi \end{array}}{\mathbf{1}\diamond\varphi} \mathbf{1}\diamond I^*$$

Let $dpd(\mathcal{D}) = \bigcup_{i \in 2} \{[i]\wedge s \mid s \in dpd(\mathcal{D}_{1-i})\}$.

1□ Elimination* If $C \Rightarrow_X \mathcal{D}:\mathbf{1}\square\varphi$, then $C \Rightarrow_X \mathcal{D}:\mathbf{0}\varphi$ for this \mathcal{D} .

$$\frac{\begin{array}{c} \mathcal{D}_0 \\ \mathbf{1}\square\varphi \end{array}}{\mathbf{0}\varphi} \mathbf{1}\square E^*$$

Let $dpd(\mathcal{D}) = \{[0]\wedge s \mid s \in dpd(\mathcal{D}_0)\}$.

1◇ Elimination* If $C_0, v:\mathbf{1}\varphi \Rightarrow_X \mathcal{D}_0:\mathbf{0}\theta$, \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\varphi$, $C_1 \Rightarrow_X \mathcal{D}_1:\mathbf{1}\diamond\varphi$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow_X \mathcal{D}:\mathbf{0}\theta$, for \mathcal{D} as pictured.

$$\frac{\begin{array}{c} [v:\mathbf{1}\varphi] \\ \mathcal{D}_1 \quad \mathcal{D}_0 \\ \mathbf{1}\diamond\varphi \quad \mathbf{0}\theta \end{array}}{\mathbf{0}\theta} \mathbf{1}\diamond E^*$$

Let $dpd(\mathcal{D}) = \{[0]\wedge s \mid s \in dpd(\mathcal{D}_1)\} \cup \{[1]\wedge s \mid s \in dpd(\mathcal{D}_0), \mathcal{D}_0(s) \neq v:\mathbf{1}\varphi\}$.

1□ Introduction* If $C_0 \Rightarrow_X \mathcal{D}_0:\mathbf{1}\psi$, $C_1, v:\mathbf{1}\top \Rightarrow_X \mathcal{D}_1:\mathbf{1}\varphi$, \mathcal{D}_1 has a barrier with exception for $v:\mathbf{1}\top$, and $\{C_0, C_1\}$ is coherent, then $C_0 \cup C_1 \Rightarrow_X \mathcal{D}:\mathbf{1}\square\varphi$ for \mathcal{D} as pictured.

$$\frac{\begin{array}{c} [v:\mathbf{1}\top] \\ \mathcal{D}_1 \quad \mathcal{D}_0 \\ \mathbf{1}\varphi \quad \mathbf{1}\psi \end{array}}{\mathbf{1}\square\varphi} \mathbf{1}\square I^*$$

Let $dpd(\mathcal{D}) = \{[0]\wedge s \mid s \in dpd(\mathcal{D}_1), \mathcal{D}_1(s) \neq v:\mathbf{1}\top\} \cup \{[1]\wedge s \mid s \in dpd(\mathcal{D}_0)\}$.

The next two rules each concern two occurrences of a single modal operator.

0/1◇ Switching If $C_0 \Rightarrow_X \mathcal{D}_0:\mathbf{1}\psi$, $C_1 \Rightarrow_X \mathcal{D}_1:\mathbf{0}\diamond\varphi$ then $C_0 \cup C_1 \Rightarrow_X \mathcal{D}:\mathbf{1}\diamond\varphi$ for \mathcal{D} as pictured.

$$\frac{\begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_0 \\ \mathbf{0}\diamond\varphi & \mathbf{1}\psi \end{array}}{\mathbf{1}\diamond\varphi} \mathbf{0}/\mathbf{1}\diamond$$

Let $dpd(\mathcal{D}) = \bigcup_{i \in 2} \{[i]\wedge s \mid s \in dpd(\mathcal{D}_{1-i})\}$.

1/0□ Switching If $C \Rightarrow_X \mathcal{D}_0:\mathbf{1}\square\varphi$ then $C \Rightarrow_X \mathcal{D}:\mathbf{0}\square\varphi$ for \mathcal{D} as pictured.

$$\frac{\begin{array}{c} \mathcal{D}_0 \\ \mathbf{1}\square\varphi \end{array}}{\mathbf{0}\square\varphi} \mathbf{1}/\mathbf{0}\square$$

Let $dpd(\mathcal{D}) = \{[0]\wedge s \mid s \in dpd(\mathcal{D}_0)\}$.

Note: an instance of $\mathbf{1/0}\Box$ need not also be an instance of $\mathbf{0I}$, this because the latter rule requires the existence of a barrier.

The next two rules are semi-thickening rules in that the relevant logical constants occur only in discharged assumptions.

$\mathbf{1}\neg\Diamond$ Thickening If $C_0 \Rightarrow_X \mathcal{D}_0:\mathbf{1}\varphi$, $C_1, v_0:\mathbf{1}\varphi, v_1:\mathbf{1}\neg\Diamond\varphi \Rightarrow_X \mathcal{D}_1:\mathbf{0}\theta$, $\{C_0, C_1\}$ is coherent, and \mathcal{D}_1 contains a barrier with exception for $\{v_0:\mathbf{1}\varphi, v_1:\mathbf{1}\neg\Diamond\varphi\}$, then $C_0 \cup C_1 \Rightarrow_X \mathcal{D}:\mathbf{0}\theta$ for \mathcal{D} as pictured.

$$\frac{\begin{array}{cc} [v_0:\mathbf{1}\varphi][v_1:\mathbf{1}\neg\Diamond\varphi] \\ \mathcal{D}_1 & \mathcal{D}_0 \\ \mathbf{0}\theta & \mathbf{1}\varphi_{\mathbf{1}\neg\Diamond T} \end{array}}{\mathbf{0}\theta_{v_0, v_1}}$$

$$dpd(\mathcal{D}) = \{[1]\hat{s} \mid s \in dpd(\mathcal{D}_0)\} \cup \{[0]\hat{s} \mid s \in dpd(\mathcal{D}_1), \mathcal{D}_1(s) \notin \{v_0:\mathbf{1}\varphi, v_1:\mathbf{1}\neg\Diamond\varphi\}\}.$$

$\mathbf{1}\Box$ Thickening If $C_0 \Rightarrow_X \mathcal{D}_0:\mathbf{1}\psi$, $C_1, v:\mathbf{1}\Box\varphi \Rightarrow_X \mathcal{D}_1:\mathbf{1}\varphi$, $\{C_0, C_1\}$ is coherent, and \mathcal{D}_1 contains a barrier with exception for $v:\mathbf{1}\Box\varphi$, then $C_0 \cup C_1 \Rightarrow_X \mathcal{D}:\mathbf{1}\varphi$ for \mathcal{D} as pictured.

$$\frac{\begin{array}{cc} [v:\mathbf{1}\Box\varphi] \\ \mathcal{D}_1 & \mathcal{D}_0 \\ \mathbf{1}\varphi & \mathbf{1}\psi_{\mathbf{1}\Box T} \end{array}}{\mathbf{1}\varphi^v}$$

$$dpd(\mathcal{D}) = \{[1]\hat{s} \mid s \in dpd(\mathcal{D}_0)\} \cup \{[0]\hat{s} \mid s \in dpd(\mathcal{D}_1), \mathcal{D}_1(s) \neq v:\mathbf{1}\Box\varphi\}.$$

Up to now, all of the rules we have considered were schematically represented using at most one occurrence of one modal operator. The following particularly hairy additions to our menagerie of rules involve more than one such occurrence.

Diodorian $_{\Diamond}$ If $C_0, v_0:\mathbf{1}\varphi_0, v_1:\mathbf{1}\Diamond\varphi_1 \Rightarrow_X \mathcal{D}_0:\mathbf{0}\theta$, $C_1, v_2:\mathbf{1}\Diamond\varphi_0, v_3:\mathbf{1}\varphi_1 \Rightarrow_X \mathcal{D}_1:\mathbf{0}\theta$, for $i \in 2$ $C_{2+i} \Rightarrow_X \mathcal{D}_{2+i}:\mathbf{0}\Diamond\varphi_i$, $\{C_{i \in 4}\}$ is coherent, and there are barriers in \mathcal{D}_0 and in \mathcal{D}_1 with exceptions for $\{v_0:\mathbf{1}\varphi_0, v_1:\mathbf{1}\Diamond\varphi_1\}$ and for $\{v_2:\mathbf{1}\Diamond\varphi_0, v_3:\mathbf{1}\varphi_1\}$ respectively, then $\bigcup_{i \in 4} C_i \Rightarrow_X \mathcal{D}:\mathbf{0}\theta$ for \mathcal{D} as pictured.

$$\frac{\begin{array}{cccc} [v_0:\mathbf{1}\varphi_0] & [v_1:\mathbf{1}\Diamond\varphi_1] & [v_2:\mathbf{1}\Diamond\varphi_0] & [v_3:\mathbf{1}\varphi_1] \\ \mathcal{D}_2 & \mathcal{D}_3 & \mathcal{D}_0 & \mathcal{D}_1 \\ \mathbf{0}\Diamond\varphi_0 & \mathbf{0}\Diamond\varphi_1 & \mathbf{0}\theta & \mathbf{0}\theta_{Dio_{\Diamond}} \end{array}}{\mathbf{0}\theta_{v_0, v_1, v_2, v_3}}$$

$$dpd(\mathcal{D}) = \bigcup_{i \in 2} \{\{i\}\hat{s} \mid s \in dpd(\mathcal{D}_{2+i})\} \cup A_0 \cup A_1, \text{ for}$$

$$A_0 = \{\{2\}\hat{s} \mid s \in dpd(\mathcal{D}_0), \mathcal{D}_0(s) \notin \{v_0:\mathbf{1}\varphi_0, v_1:\mathbf{1}\Diamond\varphi_1\}\},$$

$$A_1 = \{\{3\}\hat{s} \mid s \in dpd(\mathcal{D}_1), \mathcal{D}_1(s) \notin \{v_2:\mathbf{1}\Diamond\varphi_0, v_3:\mathbf{1}\varphi_1\}\}.$$

Diodorian $_{\Box}$ If for $i \in 2$ $C_i, v_i:\mathbf{0}\Box\neg\varphi_i \Rightarrow_X \mathcal{D}_i:\mathbf{0}\theta$, $C_2, v_2:\mathbf{1}\varphi_0 \Rightarrow_X \mathcal{D}_2:\mathbf{1}\Box\neg\varphi_1$, $C_3, v_3:\mathbf{1}\varphi_1 \Rightarrow_X \mathcal{D}_2:\mathbf{1}\Box\neg\varphi_0$, $\{C_{i \in 4}\}$ is coherent, and there are barriers in \mathcal{D}_2

³I confess some unhappiness with this rule, because its indicated occurrence of \Diamond is embedded, and (even worse) is in the scope of \neg ; using $\neg\Diamond$ instead of $\Box\neg$ was “arbitrary”, a strained effort to put this rule “on the \Diamond side”. But I haven’t found more pleasing rule.

and in \mathcal{D}_3 with exceptions for $v_0:1\Box\neg\varphi_0$ and for $v_1:1\Box\neg\varphi_1$ respectively, then $\bigcup_{i \in 4} C_i \Rightarrow_X \mathcal{D}:\mathbf{0}\theta$ for \mathcal{D} as pictured.

$$\frac{\begin{array}{cccc} [v_2:1\varphi_0] & [v_3:1\varphi_1] & [v_0:0\Box\neg\varphi_0] & [v_1:0\Box\neg\varphi_1] \\ \mathcal{D}_2 & \mathcal{D}_3 & \mathcal{D}_0 & \mathcal{D}_1 \\ 1\Box\neg\varphi_1 & 1\Box\neg\varphi_0 & \mathbf{0}\theta & \mathbf{0}\theta \end{array}}{\mathbf{0}\theta \quad v_0, v_1, v_2, v_3} \text{Dio}_{\Box}$$

$dpd(\mathcal{D}) = \bigcup_{i \in 2} \{ (2+i) \wedge s \mid s \in dpd(\mathcal{D}_i) \mid \mathcal{D}_i(s) \neq v_i:0\Box\neg\varphi_i \} \cup A_0 \cup A_1$, for

$$\begin{aligned} A_0 &= \{ \langle 0 \rangle \wedge s \mid s \in dpd(\mathcal{D}_2), \mathcal{D}_2(s) \neq v_2:1\varphi_0 \}, \\ A_1 &= \{ \langle 1 \rangle \wedge s \mid s \in dpd(\mathcal{D}_3), \mathcal{D}_3(s) \neq v_3:1\varphi_1 \}. \end{aligned}$$

These Diodorian rules⁴ combine an aspect of \vee Elimination with an aspect of Strengthening rules (since both have a modal operator occurring in discharged assumptions).

9.2 Definitions

In what follows, I will modify the nomenclature used in [4] for modal logics that are stronger than K.⁵ Define the following proof-theoretic systems \Rightarrow_{IY} by adding rules to those generating \Rightarrow_{IK} as follows.

X	rules	X	rules	X	rules
ID _w	S1 $\perp E$	ID	1 $\top T$		
IT \diamond	0E	IT \Box	0I	IT	0E and 0I
IB \diamond	1 $\diamond I^*$	IB \Box	1 $\Box E^*$	IB	1 $\diamond I^*$ and 1 $\Box E^*$
I4 \diamond	1 $\diamond E^*$	I4 \Box	1 $\Box I^*$	I4	1 $\diamond E^*$ and 1 $\Box I^*$
I5 \diamond	0/1 \diamond	I5 \Box	1/0 \Box	I5	0/1 \diamond and 1/0 \Box
IGL \diamond	1 $\neg\diamond T$	IGL \Box	1 $\Box T$	IGL	1 $\neg\diamond T$ and 1 $\Box T$
IDio \diamond	<i>Dio</i> \diamond	IDio \Box	<i>Dio</i> \Box	IDio	<i>Dio</i> \diamond and <i>Dio</i> \Box

Form \Rightarrow_{IY}^- by removing $\diamond E^+$ from the rules generating \Rightarrow_{IY} .

⁴Named after Diodorus Cronus, died c. 284 B.C.E.; according to Alexander of Aphrodisias, Diodorus taught that there was only one possible future, i.e. the future was non-branching. Being an IDio \diamond -frame and being an IDio \Box -frame will both defined by conditions with non-branching flavors.

⁵Exceptions: (1) Since all of these systems are normal, i.e. (in this case) they include \Rightarrow_{IK}^- , I have omitted ‘K’ where Popkorn uses it. (2) Popkorn did not use ‘GL’, which abbreviates ‘Gödel and Löb’; see the article “Provability Logics” in the online Stanford Encyclopedia of Philosophy. In [1], the classical “no step” version is called G, for ‘Gödel’.

9.3 Definitions

The above proof-theoretic systems can be combined in obvious ways. The following blends are of obvious interest, using traditional names where possible.⁶

X	rules
ITB	$\mathbf{0}E, \mathbf{0}I, \mathbf{1}\Diamond I^*$ and $\mathbf{1}\Box E^*$
IS4	$\mathbf{0}E, \mathbf{0}I, \mathbf{1}\Diamond E^*$ and $\mathbf{1}\Box I^*$
IT5	$\mathbf{0}E, \mathbf{0}I, \mathbf{0}/\mathbf{1}\Diamond$ and $\mathbf{1}/\mathbf{0}\Box$
IS5	$\mathbf{0}E, \mathbf{0}I, \mathbf{1}\Diamond I^*, \mathbf{1}\Box E^*, \mathbf{1}\Diamond E^*$ and $\mathbf{1}\Box I^*$
IS4.3	$\mathbf{0}E, \mathbf{0}I, \mathbf{1}\Diamond E^*, \mathbf{1}\Box I^*, Dio_\Diamond$ and Dio_\Box

For other blends, I leave it to the reader to infer the rules from the names.

Form the classical correlates \Rightarrow_{CY} and $\Rightarrow_{\bar{C}Y}$ by adding $\mathbf{0}EM$ to the rules generating \Rightarrow_{IY} and $\Rightarrow_{\bar{I}Y}$, respectively.

In the obvious way, define \vdash_X and $\vdash_{\bar{X}}$ from \Rightarrow_X and $\Rightarrow_{\bar{X}}$ respectively.

10 Proof-theoretic Observations

10.1 Observations

- (1) $\vdash_{ID} \mathbf{1T}$;⁷ so if $\vdash_{ID} \subseteq \vdash_X, \vdash_{\bar{X}} = \vdash_X$.
For any $\varphi \in Fml$, the following are true.
- (2) $\mathbf{0}\Box\varphi \vdash_{ID}^- \mathbf{0}\Diamond\varphi$.
- (3) $\mathbf{0}\Box\perp \vdash_{ID_w}^- \mathbf{0}\perp$.
- (4) $\mathbf{0}\varphi \vdash_{IT_\Diamond}^- \mathbf{0}\Diamond\varphi$.
- (5) $\mathbf{0}\Box\varphi \vdash_{IT_\Box}^- \mathbf{0}\varphi$.
- (6) $\mathbf{0}\varphi \vdash_{IB_\Diamond}^- \mathbf{0}\Box\Diamond\varphi$;
- (7) $\mathbf{0}\Diamond\Box\varphi \vdash_{IB_\Box}^- \mathbf{0}\varphi$.
- (8) $\mathbf{0}\Diamond\Diamond\varphi \vdash_{I4_\Diamond}^- \mathbf{0}\Diamond\varphi$.
- (9) $\mathbf{0}\Box\varphi \vdash_{I4_\Box}^- \mathbf{0}\Box\Box\varphi$.
- (10) $\mathbf{0}\Diamond\varphi \vdash_{I5_\Diamond}^- \mathbf{0}\Box\Diamond\varphi$.
- (11) $\mathbf{0}\Diamond\Box\varphi \vdash_{I5_\Box}^- \mathbf{0}\Box\varphi$.
- (12) $\mathbf{0}\Diamond\varphi \vdash_{IGL_\Diamond}^- \mathbf{0}\Diamond(\varphi \& \neg\Diamond\varphi)$.
- (13) $\mathbf{0}\Box(\Box\varphi \supset \varphi) \vdash_{IGL_\Box}^- \mathbf{0}\Box\varphi$.
- (14) $\mathbf{0}\Diamond\varphi_0, \mathbf{0}\Diamond\varphi_1 \vdash_{IDio_\Diamond}^- \mathbf{0}(\Diamond(\varphi_0 \& \Diamond\varphi_1) \vee \Diamond(\Diamond\varphi_0 \& \varphi_1))$.
- (15) $\mathbf{0}\Box(\varphi_0 \supset \Box\neg\varphi_1), \mathbf{0}\Box(\varphi_1 \supset \Box\neg\varphi_0) \vdash_{IDio_\Box}^- \mathbf{0}(\Box\neg\varphi_0 \vee \Box\neg\varphi_1)$.

⁶I have honored tradition in this use of ‘S’ for \Rightarrow_{IS4} and \Rightarrow_{IS5} , recognizing that calling them \Rightarrow_{IT4} and \Rightarrow_{ITB4} would be more “logical”. Ditto for \Rightarrow_{CS4} and \Rightarrow_{CS5} . And for their associated consequence relations.

⁷This is the proof-theoretic correlate to the fact that for any ID-model \mathcal{M} , $W^{\mathcal{M}} \subseteq dom(R^{\mathcal{M}})$. $\mathbf{1T}$ could serve as an axiom for ID.

$$(16) \quad \vdash_{\overline{ID}} = \vdash_{ID} \text{ and } \vdash_{\overline{IT_\diamond}} = \vdash_{IT_\diamond} .$$

Proof (1), (2) and (3) are witnessed by the following, respectively.

$$\frac{\frac{[v:\mathbf{1}\top]_{\diamond I} \quad \mathbf{0}\diamond\top}{\mathbf{0}\diamond\top} \mathbf{1}\top\top \quad \frac{\mu:\mathbf{0}\Box\varphi \quad [v:\mathbf{1}\top]_{\Box E} \quad \mathbf{1}\varphi}{\mathbf{0}\diamond\varphi} \diamond I \quad \frac{\mu:\mathbf{0}\Box\perp \quad [v:\mathbf{1}\top]_{\Box E} \quad \mathbf{1}\perp}{\mathbf{0}\perp} \mathbf{S1}\perp E}{\mathbf{1}\top} \diamond E^+$$

(4) and (5) are witnessed by the following, respectively.

$$\frac{\mu:\mathbf{0}\varphi \quad \frac{[v:\mathbf{1}\varphi]_{\diamond I} \quad \mathbf{0}\diamond\varphi}{\mathbf{0}\diamond\varphi} \mathbf{0}E}{\mathbf{0}\diamond\varphi} \quad \frac{\mu:\mathbf{0}\Box\varphi \quad [v:\mathbf{1}\top]_{\Box E} \quad \mathbf{1}\varphi}{\mathbf{0}\varphi} \mathbf{0}I$$

In the deduction on the right, the barrier for the use of $\mathbf{0}I$ is $\{\}$. I leave proofs (6) and (7) as exercises. Hints: for (6) use $\mathbf{1}\diamond I^*$ followed by $\Box I$; for (7) use $\mathbf{1}\Box E^*$ followed by $\diamond E$. (8) and (9) are witnessed as follows.

$$\frac{\mu:\mathbf{0}\diamond\varphi \quad \frac{[v_1:\mathbf{1}\diamond\varphi] \quad \mathbf{0}\diamond\varphi}{\mathbf{0}\diamond\varphi} \mathbf{1}\diamond E^* \quad \frac{[v_0:\mathbf{1}\varphi]_{\diamond I} \quad \mu:\mathbf{0}\Box\varphi \quad [v_0:\mathbf{1}\top]_{\Box I} \quad \mathbf{1}\varphi}{\mathbf{0}\Box\varphi} \mathbf{1}\Box I^*}{\mathbf{0}\diamond\varphi} \mathbf{v}_1 \quad \frac{\mathbf{1}\Box\varphi \quad [v_1:\mathbf{1}\top]_{\mathbf{1}\Box I^*}}{\mathbf{0}\Box\varphi} \mathbf{v}_0 \quad \mathbf{v}_1 \quad \Box I$$

(10) and (11) are witnessed by the following.

$$\frac{\mu:\mathbf{0}\diamond\varphi \quad [v:\mathbf{1}\top]_{\mathbf{0}/\mathbf{1}\diamond} \quad \mathbf{1}\diamond\varphi}{\mathbf{0}\Box\varphi} \mathbf{v} \quad \Box I \quad \frac{[v:\mathbf{1}\Box\varphi]_{\mathbf{1}/\mathbf{0}\Box} \quad \mu:\mathbf{0}\Box\varphi \quad \mathbf{0}\Box\varphi}{\mathbf{0}\Box\varphi} \mathbf{v} \quad \diamond E$$

In the deduction on the left [right], the barrier for the use of $\Box I$ [$\diamond E$] is empty with vacuous exception for $v:\mathbf{1}\top$ (for any $v \in Var - \{\mu\}$) [for $v:\mathbf{1}\Box\varphi$].

(12) is witnessed by the following.

$$\frac{\frac{[v_0:\mathbf{1}\varphi] \quad [v_1:\mathbf{1}\neg\varphi]_{\mathbf{1}\&I} \quad \mathbf{1}(\varphi \& \neg\varphi)}{\mathbf{0}\diamond(\varphi \& \neg\varphi)} \diamond I \quad \frac{\mathbf{0}\diamond(\varphi \& \neg\varphi) \quad [v_2:\mathbf{1}\varphi]_{\mathbf{1}\neg\varphi T}}{\mathbf{0}\diamond(\varphi \& \neg\varphi)} \mathbf{v}_0, \mathbf{v}_1}{\mathbf{0}\diamond(\varphi \& \neg\varphi)} \mathbf{v}_2$$

I leave proofs of (13)-(16) as exercises. □

For $X \in \{ID, ID_w, IT_\diamond, IT_\Box, IB_\diamond, IB_\Box, I4_\diamond, I4_\Box, I5_\diamond, I5_\Box, IGL_\diamond, IGL_\Box, IDio_\diamond, IDio_\Box\}$, form \Rightarrow'_X by transforming the scheme given by each of 10.1(2)-(16) into a rule, and adding that rule to those defining \Rightarrow_{IK} . Define \vdash'_X from \Rightarrow'_X as usual.

10.2 Observations

$\vdash'_X = \vdash_X$. Similarly with \Rightarrow_{IK} in place of \Rightarrow_{IK} .

Proof By 10.1(2)-(16), $\vdash'_X \subseteq \vdash_X$. To prove that $\vdash_X \subseteq \vdash'_X$ we must show that the characteristic rule used to define \Rightarrow'_X is admissible (i.e. a derived rule) under \Rightarrow'_X .

For $X = ID$, we show that $\mathbf{1}\top\top$ is admissible in \Rightarrow'_{ID} . Assume that $C, v:\mathbf{1}\top \Rightarrow'_{ID} D_0:\mathbf{0}\theta$. Assume that for some μ and ψ , $\mu:\mathbf{1}\psi \in C$; fix \mathcal{D}_1 such that $\mu:\mathbf{1}\psi \Rightarrow_{IK} \mathcal{D}_1:\mathbf{1}\top$ (one use of *Trn*₀), and let $\mathcal{D}'_0 = [v := \mathcal{D}_1]D_0$; $C \Rightarrow'_{ID} \mathcal{D}'_0:\mathbf{0}\theta$, as required. Assume that there is no such μ and ψ ; so \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\top$, and $C \Rightarrow'_{ID} \mathcal{D}:\mathbf{0}\theta$ for \mathcal{D} as pictured below.

$$\frac{\frac{\overline{\mathbf{0}\top}}{\mathbf{0}\square\top}_{Nec} \quad [v:\mathbf{1}\top]}{\mathbf{0}\square\top}_{10.1(2)} \quad \mathcal{D}_0}{\mathbf{0}\diamond\top} \quad \mathbf{0}\theta_{\diamond E}$$

$$\mathbf{0}\theta^v$$

For $X = ID_w$, we show that $S\mathbf{1}\perp E$ is admissible in \Rightarrow'_{ID_w} . Assume that $C, v:\mathbf{1}\top \Rightarrow'_{ID_w} D_0:\mathbf{1}\perp$, \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\top$, and $\chi \in MFml$. Then $C \Rightarrow'_{ID_w} \mathcal{D}:\chi$ for \mathcal{D} as pictured below.

$$\frac{[v:\mathbf{1}\top] \quad \mathcal{D}_0}{\mathbf{1}\perp}_{\square I}$$

$$\frac{\mathbf{0}\square\perp^v}{\mathbf{0}\perp\mathbf{0}\perp E}_{10.1(3)}$$

$$\chi$$

For $X = IT_\diamond$, we prove that $\mathbf{0}E$ is admissible in $\Rightarrow'_{IT_\diamond}$. Assume that $C_1 \Rightarrow_{IT_\diamond} \mathcal{D}_1:\mathbf{0}\varphi$, $C_0, v:\mathbf{1}\varphi \Rightarrow_{IT_\diamond} \mathcal{D}_0:\mathbf{0}\theta$, \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\varphi$, and $\{C_0, C_1\}$ is coherent. $C \Rightarrow'_{IT_\diamond} \mathcal{D}:\mathbf{0}\theta$ for \mathcal{D} as pictured below.

$$\frac{\mathcal{D}_1 \quad [v:\mathbf{1}\varphi]}{\mathbf{0}\varphi}_{10.1(4)} \quad \mathcal{D}_0}{\mathbf{0}\diamond\varphi} \quad \mathbf{0}\theta_{\diamond E}$$

$$\mathbf{0}\theta^v$$

For $X = IT_\square$, we prove that $\mathbf{0}I$ is admissible in $\Rightarrow'_{IT_\square}$. If $C, v:\mathbf{1}\top \Rightarrow_{IT_\square} D_0:\mathbf{1}\varphi$ and \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\top$, then $C \Rightarrow'_{IT_\square} \mathcal{D}:\mathbf{0}\varphi$ for \mathcal{D} as pictured below.

$$\frac{[v:\mathbf{1}\top] \quad \mathcal{D}_0}{\mathbf{1}\varphi}_{\square I}$$

$$\frac{\mathbf{0}\square\varphi^v}{\mathbf{0}\varphi}_{10.1(5)}$$

For the remaining cases, the proofs are exercises. □

10.3 Observations

- (1) $\vdash_{ID_w} \subseteq \vdash_{ID}$. (2) $\vdash_{CD_w} = \vdash_{CD}$. (3) $\vdash_{ID_w} \subseteq \vdash_{IT_\square}$.

Proof For (1), we must prove that $S1\perp E$ is admissible in \Rightarrow_{ID} . Assume that $C, v:1\top \Rightarrow_{ID} \mathcal{D}_0:1\perp$ and $\theta \in Fml$. Fix \mathcal{D}_1 from the proof of 10.1(1) such that $\Rightarrow_{ID} \mathcal{D}_1:1\top$; let $\mathcal{D}_2 = [v := \mathcal{D}_1]\mathcal{D}_0$. So $C \Rightarrow_{ID} \mathcal{D}_2:1\perp$. One use of $1\perp E_m$ yields a \mathcal{D} so that $C \Rightarrow_{ID} \mathcal{D}:m\theta$, as required.

By (1), $\vdash_{CD_w} \subseteq \vdash_{CD}$. For (2) it suffices to show that $\vdash_{CD} \subseteq \vdash_{CD_w}$. For that, we show that $1\top\top$ is admissible in \Rightarrow_{CD_w} , i.e. that $\vdash_{CD_w} 1\top$. This is witnessed by the following.

$$\frac{\frac{\frac{[v_0:1\top]_{\diamond I}}{[v_2:0\neg\diamond\top]} \quad 0\diamond\top_{0\supset E}}{0\perp_{0\perp E}} \quad \frac{1\perp_{S1\perp E}}{0\diamond\top_{v_0} \quad 0EM}}{[v_1:0\diamond\top]} \quad 0\diamond\top_{v_1, v_2} \quad \diamond E^+}{1\top}$$

Note: although $S1\perp E$ and $0EM$ do not involve \diamond or \square , the use of \diamond in a witness for the above seems unavoidable.

To prove (3) we must show that $S1\perp E$ is admissible in $\vdash_{IT_{\square}}$. Assume that $C, v:1\top \Rightarrow_{IT_{\square}} \mathcal{D}_0:1\perp$, \mathcal{D}_0 has a barrier with exception for $v:1\top$, and $\chi \in MFml$. So $C \Rightarrow_{IT_{\square}} \mathcal{D}:\chi$ for \mathcal{D} as pictured.

$$\frac{\frac{[v:1\top]}{\mathcal{D}_0} \quad \frac{1\perp_{0I}}{0\perp_{v} \quad 0\perp E}}{\chi}$$

□

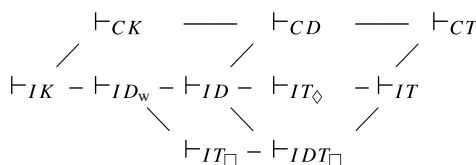
10.4 Observation

$$\vdash_{ID} \subseteq \vdash_{IT_{\diamond}}$$

Proof It suffices to show that $\vdash_{IT_{\diamond}} 0\diamond\top$, an easy exercise. □

10.5 Remark

The inclusion relations between the consequence relations generated by the pure one-step rules can be pictured as follows (with inclusion going from left to right).



10.6 Observations

(1) $\vdash_{ID\subseteq} \vdash_{IT_{\square}B_{\diamond}}$. (2) $\vdash_{ID\subseteq} \vdash_{IT_{\square}5_{\diamond}}$.

Proofs are good exercises.

10.7 Observations

(1) $\vdash_{IT_{\diamond}\subseteq} \vdash_{IDB_{\diamond}4_{\diamond}}$. (2) $\vdash_{IT_{\square}\subseteq} \vdash_{IDB_{\square}4_{\square}}$. (3) $\vdash_{IT\subseteq} \vdash_{IDB4}$. (4) $\vdash_{IT_{\diamond}\subseteq} \vdash_{IT_{\square}B_{\diamond}}$.
 (5) $\vdash_{IT_{\square}\subseteq} \vdash_{IT_{\diamond}B_{\diamond}}$.

Proof For (1), we show that $\mathbf{0}E$ is admissible under $\vdash_{IDB_{\diamond}4_{\diamond}}$. Assume that $C_1 \Rightarrow_{IDB_{\diamond}4_{\diamond}} \mathcal{D}_1:\mathbf{0}\varphi$, $C_0, v:\mathbf{1}\varphi \Rightarrow_{IDB_{\diamond}4_{\diamond}} \mathcal{D}_0:\mathbf{0}\theta$, \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\varphi$, and $\{C_0, C_1\}$ is coherent. $C_0 \cup C_1 \Rightarrow_{IDB_{\diamond}4_{\diamond}} \mathcal{D}:\mathbf{0}\theta$ for \mathcal{D} as pictured.

$$\frac{\mathcal{D}_1 \quad [v:\mathbf{1}\varphi] \quad \mathbf{0}\varphi \quad \mathbf{1}\top \quad \mathcal{D}_0}{\mathbf{1}\diamond\varphi \quad \mathbf{0}\theta} \frac{\text{10.1.(1)}}{\mathbf{1}\diamond I^*} \quad \frac{\mathbf{0}\theta}{\mathbf{0}\theta^v} \quad \mathbf{1}\diamond E^*$$

For (2), we show that $\mathbf{0}I$ is admissible under $\vdash_{IDB_{\square}4_{\square}}$. Assume that $C, v:\mathbf{1}\top \Rightarrow_{IDB_{\square}4_{\square}} \mathcal{D}_0:\mathbf{1}\varphi$ and \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\top$. Then $C \Rightarrow_{IDB_{\square}4_{\square}} \mathcal{D}:\mathbf{0}\varphi$ is witnessed as follows.

$$\frac{[v_0:\mathbf{1}\top] \quad \mathcal{D}_0 \quad \mathbf{1}\varphi \quad \mathbf{1}\top}{\mathbf{1}\square\varphi^{v_0} \quad \mathbf{1}\square E^*} \frac{\text{10.1.(2)}}{\mathbf{1}\square I^*} \quad \mathbf{0}\varphi$$

(3) follows from (1) and (2).

For (5), assume that $C, v:\mathbf{1}\top \Rightarrow_{IT_{\diamond}B_{\diamond}} \mathcal{D}_0:\mathbf{1}\varphi$ and \mathcal{D}_0 has a barrier with exception for $v:\mathbf{1}\top$. Then $\text{ran}(C) \vdash_{IT_{\diamond}B_{\diamond}} \mathbf{0}\varphi$ is witnessed as follows.

$$\frac{[v:\mathbf{1}\top] \quad \mathcal{D}_0 \quad \mathbf{1}\varphi \quad \square I \quad [v_1:\mathbf{1}\square\varphi] \quad \square E^*}{\mathbf{0}\square\varphi^{v_0} \quad \mathbf{0}\varphi_{\mathbf{0}E}} \quad \mathbf{0}\varphi^{v_1}$$

I leave (4), (6) and (7) as exercises. □

10.8 Observations

(1) $\vdash_{IB_{\diamond}\subseteq} \vdash_{IT_{\diamond}5_{\diamond}}$. (2) $\vdash_{IB_{\square}\subseteq} \vdash_{IT_{\square}5_{\square}}$. (3) $\vdash_{IB\subseteq} \vdash_{IT5}$.

Proof For (1), we show that $\mathbf{1}\diamond I^*$ is admissible in $\Rightarrow_{IT_{\diamond}5_{\diamond}}$. Assume that $C_1 \Rightarrow_{IT_{\diamond}5_{\diamond}} \mathcal{D}_1:\mathbf{0}\varphi$, $C_0 \Rightarrow_{IT_{\diamond}5_{\diamond}} \mathcal{D}_0:\mathbf{1}\psi$, and $\{C_0, C_1\}$ is coherent. For \mathcal{D} as pictured below, $C_0 \cup$

$C_1 \Rightarrow_{IT_{\diamond}5_{\diamond}} \mathcal{D} : \mathbf{1}\diamond\varphi$.

$$\frac{\frac{\mathcal{D}_1 \quad \frac{[v:\mathbf{1}\varphi]_{\diamond I}}{\mathbf{0}\diamond\varphi} \quad \mathcal{D}_0}{\mathbf{0}\diamond\varphi^v} \quad \mathbf{1}\psi_{\mathbf{0}/\mathbf{1}\diamond}}{\mathbf{1}\diamond\varphi}$$

Proof of (2) is a good exercise. (3) follows from (1) and (2). □

10.9 Observations

(1) $\vdash_{I4_{\diamond}} \subseteq \vdash_{IB_{\square}5_{\diamond}}$. (2) $\vdash_{I4_{\square}} \subseteq \vdash_{IB_{\diamond}5_{\square}}$. Surprisingly, (3) $\vdash_{I4_{\diamond}} \subseteq \vdash_{IB5_{\square}}$ and (4) $\vdash_{I4_{\square}} \subseteq \vdash_{IB5_{\diamond}}$. So (5) $\vdash_{I4} \subseteq \vdash_{IB5_{\square}}$ and (6) $\vdash_{I4} \subseteq \vdash_{IB5_{\diamond}}$.

Proof For (1), we show that $\mathbf{1}\diamond E^*$ is admissible in $\Rightarrow_{IB_{\square}5_{\diamond}}$. Assume that $C_1 \Rightarrow_{IB_{\square}5_{\diamond}} \mathcal{D}_1 : \mathbf{1}\diamond\varphi$, $C_0, v:\mathbf{1}\varphi \Rightarrow_{IB_{\square}5_{\diamond}} \mathcal{D}_0 : \mathbf{0}\theta$, \mathcal{D}_0 contains a barrier with exception for $v:\mathbf{1}\varphi$, and $\{C_0, C_1\}$ is coherent. For \mathcal{D} as pictured below, $C_0 \cup C_1 \Rightarrow_{IB_{\square}5_{\diamond}} \mathcal{D} : \mathbf{0}\theta$.

$$\frac{\frac{\frac{[v_0:\mathbf{0}\diamond\varphi]}{\mathbf{0}\diamond\varphi}{}_{10.1.(10)} \quad \mathcal{D}_1}{\mathbf{1}\diamond\varphi}{}_{Trn_1} \quad [v:\mathbf{1}\varphi]}{\frac{\frac{\mathbf{1}\diamond\varphi^v}{}{\mathbf{0}\diamond\varphi}{}_{1\Box E^*} \quad \mathcal{D}_0}{\mathbf{0}\theta^v}{}_{\diamond E}}$$

For (2), we show that $\mathbf{1}\square I^*$ is admissible in $\Rightarrow_{IB_{\diamond}5_{\square}}$. Assume that $C_0 \Rightarrow_{IB_{\diamond}5_{\square}} \mathcal{D}_0 : \mathbf{1}\psi$, $C_1, v:\mathbf{1}\top \Rightarrow_{IB_{\diamond}5_{\square}} \mathcal{D}_1 : \mathbf{1}\varphi$, \mathcal{D}_1 contains a barrier with exception for $v:\mathbf{1}\top$, and $\{C_0, C_1\}$ is coherent. For \mathcal{D} as pictured below, $C_0 \cup C_1 \Rightarrow_{IB_{\diamond}5_{\square}} \mathcal{D} : \mathbf{1}\square\varphi$.

$$\frac{\frac{\frac{[v_0:\mathbf{0}\diamond\square\varphi]}{\mathbf{0}\square\varphi}{}_{10.1.(11)} \quad \frac{\frac{[v:\mathbf{1}\top]}{\mathbf{1}\varphi}{}_{\square I} \quad \mathcal{D}_0}{\mathbf{0}\square\varphi^v}{}_{Trn_1} \quad \mathbf{1}\psi_{\mathbf{1}\diamond I^*}}{\mathbf{1}\square\varphi^v}{}_{Trn_1}}$$

For (3), we show that $\mathbf{1}\diamond E^*$ is admissible in $\Rightarrow_{IB5_{\square}}$. Assume that $C_1 \Rightarrow_{IB5_{\square}} \mathcal{D}_1 : \mathbf{1}\diamond\varphi$, $C_0, v:\mathbf{1}\varphi \Rightarrow_{IB5_{\square}} \mathcal{D}_0 : \mathbf{0}\theta$, \mathcal{D}_0 contains a barrier with exception for $v:\mathbf{1}\varphi$, and $\{C_0, C_1\}$ is coherent. Let \mathcal{D}_2 be as pictured.

$$\frac{\frac{\frac{[v_0:\mathbf{0}\varphi]}{\mathbf{0}\diamond\varphi}{}_{10.1.(6)} \quad [v_1:\mathbf{1}\varphi]}{\mathbf{1}\diamond\varphi}{}_{Trn_1} \quad \frac{\frac{[v_2:\mathbf{0}\diamond\varphi]}{\mathbf{0}\square\varphi}{}_{1/0\square} \quad \mathcal{D}_1}{\mathbf{1}\diamond\varphi}{}_{Trn_1}}{\mathbf{1}\square\varphi^v_2}$$

For \mathcal{D} as pictured below, $C_0 \cup C_1 \Rightarrow_{IB5\Box} \mathcal{D}:\mathbf{0}\theta$.

$$\frac{\frac{\mathcal{D}_2 \quad [v:\mathbf{1}\varphi]}{\mathbf{1}\Box\Diamond\varphi_{\mathbf{1}\Box E^*}} \quad \mathcal{D}_0}{\mathbf{0}\Diamond\varphi} \quad \mathbf{0}\theta}{\mathbf{0}\theta^v} \Diamond E$$

For (4), we show that $\mathbf{1}\Box I^*$ is admissible in $\Rightarrow_{IB5\Diamond}$. Assume that $C_1 \Rightarrow_{IB5\Diamond} \mathcal{D}_0:\mathbf{1}\psi$, $C_1, v:\mathbf{1}\top \Rightarrow_{IB5\Diamond} \mathcal{D}_1:\mathbf{1}\varphi$, \mathcal{D}_1 contains a barrier with exception for $v:\mathbf{1}\top$, and $\{C_0, C_1\}$ is coherent. So $C_0 \cup C_1 \Rightarrow_{IB5\Diamond} \mathcal{D}_2:\mathbf{1}\Box\Diamond\Box\varphi$ for \mathcal{D}_2 as pictured below.

$$\frac{\frac{[v:\mathbf{1}\top]}{\mathcal{D}_1} \quad \frac{\mathbf{1}\varphi_{\Box I}}{\mathbf{0}\Box\varphi^v} \quad \mathbf{1}\psi_{\mathbf{1}\Diamond I^*}}{\frac{[v_0:\mathbf{0}\Diamond\Box\varphi]}{\mathbf{0}\Box\Diamond\Box\varphi} \quad \mathbf{1}\Diamond\Box\varphi} \text{Trn}_2}{\mathbf{1}\Box\Diamond\Box\varphi^{v_0}} \text{10.1.(10)}$$

$C_0 \cup C_1 \Rightarrow_{IB5\Diamond} \mathcal{D}:\mathbf{1}\Box\varphi$ for \mathcal{D} as pictured below.

$$\frac{\frac{[v_1:\mathbf{0}\Diamond\Box\varphi]}{\mathbf{0}\varphi} \quad \frac{[v_2:\mathbf{0}\Box\Diamond\Box\varphi]}{\mathbf{1}\Diamond\Box\varphi} \text{Trn}_1}{\frac{\mathbf{1}\varphi^{v_1}_{\Box I}}{\mathbf{0}\Box\varphi} \quad \mathbf{1}\Box\Diamond\Box\varphi \text{Trn}_1} \text{Trn}_1}{\mathbf{1}\Box\varphi^{v_2}}$$

(5) follows from (2) and (3). (6) follows from (1) and (4). □

10.10 Observations

(1) $\vdash_{I5\Diamond} \subseteq \vdash_{IB\Diamond4\Diamond}$. (2) $\vdash_{I5\Box} \subseteq \vdash_{IB\Box4\Box}$. (3) $\vdash_{I5} \subseteq \vdash_{IB4}$. (4) $\vdash_{I5\Diamond} \subseteq \vdash_{IB\Diamond5\Box}$. (5) $\vdash_{I5\Box} \subseteq \vdash_{IB\Box5\Diamond}$.

Proof For (1), we show that $\mathbf{0}/\mathbf{1}\Diamond$ is admissible in $\Rightarrow_{IB\Diamond4\Diamond}$. Assume that $C_0 \Rightarrow_{IB\Diamond4\Diamond} \mathcal{D}_0:\mathbf{1}\psi$ and $C_1 \Rightarrow_{IB\Diamond4\Diamond} \mathcal{D}_1:\mathbf{0}\Diamond\varphi$. Then $C_0 \cup C_1 \Rightarrow_{IB\Diamond4\Diamond} \mathcal{D}:\mathbf{1}\Diamond\varphi$ for the following \mathcal{D} .

$$\frac{\frac{[v:\mathbf{0}\Diamond\Diamond\varphi]}{\mathbf{0}\Diamond\varphi} \quad \frac{\mathcal{D}_1 \quad \mathcal{D}_0}{\mathbf{0}\Diamond\varphi \quad \mathbf{1}\psi_{\mathbf{1}\Diamond I^*}}}{\mathbf{1}\Diamond\varphi^v} \text{Trn}_1 \text{10.1.(8)}$$

For (2), we show that $\mathbf{1}/\mathbf{0}\Box$ is admissible in $\Rightarrow_{IB\Box4\Box}$. Assume that $C \Rightarrow_{IB\Box4\Box} \mathcal{D}:\mathbf{1}\Box\varphi$. Then $C \Rightarrow_{IB\Box4\Box} \mathcal{D}:\mathbf{0}\Box\varphi$ for the following \mathcal{D} .

$$\frac{\frac{[v:\mathbf{0}\Box\varphi]}{\mathbf{0}\Box\Box\varphi} \quad \mathcal{D}_0}{\mathbf{1}\Box\varphi} \text{Trn}_1}{\mathbf{1}\Box\Box\varphi^v_{\mathbf{1}\Box E^*}} \text{10.1.(9)}$$

(3) follows from (1) and (2).

For (4), we show that $\mathbf{0}/\mathbf{1}\diamond$ is admissible in $\Rightarrow_{IB\diamond 5\Box}$. Assume that $C_0 \Rightarrow_{IB\diamond 5\Box} \mathcal{D}_0:\mathbf{1}\psi$ and $C_1 \Rightarrow_{IB\diamond 5\Box} \mathcal{D}_1:\mathbf{0}\diamond\varphi$. $C_0 \cup C_1 \Rightarrow_{IB\diamond 5\Box} \mathcal{D}:\mathbf{1}\diamond\varphi$ for the following \mathcal{D} .

$$\frac{\frac{\frac{\frac{[v_0:\mathbf{0}\varphi]}{10.1.(6)}}{\mathbf{0}\Box\diamond\varphi} \quad [v_1:\mathbf{1}\varphi]}{\mathbf{1}\Box\diamond\varphi^{v_0}} \quad \mathcal{D}_1}{\mathbf{0}\diamond\varphi} \quad \frac{\mathbf{0}\Box\diamond\varphi^{v_0}}{\mathbf{1}/\mathbf{0}\Box} \quad \mathcal{D}_0}{\mathbf{0}\Box\diamond\varphi^{v_1}} \quad \frac{\mathbf{0}\Box\diamond\varphi^{v_1}}{\mathbf{1}\psi_{\Box E}}}{\mathbf{1}\diamond\varphi}$$

I leave (5) as an exercise. □

10.11 Observations

(1) $\vdash_{IT\diamond B\diamond 4\diamond} = \vdash_{IDB\diamond 4\diamond}$. (2) $\vdash_{IB4\diamond} = \vdash_{IB5\diamond}$. (3) $\vdash_{IT\Box B\Box 4\Box} = \vdash_{IDB\Box 4\Box}$. (4) $\vdash_{IB4\Box} = \vdash_{IB5\Box}$. (5) $\vdash_{ITB5} = \vdash_{IS5} = \vdash_{IDB4}$.

Proof For (1): the inclusion from left to right uses 10.8(1); the inclusion from right to left uses 10.4 For (2): the inclusion from left to right uses 10.9(1); the inclusion from right to left uses 10.10(1). For (3): the inclusion from left to right uses 10.8(2); the inclusion from right to left uses 10.7(1). For (4): the inclusion from left to right uses 10.10(2). For (5), the leftmost identity uses (1) and (3); the rightmost uses (2) and (4). □

10.12 Observations

(1) $\vdash_{I4\Box} \subseteq \vdash_{IGL\Box}$. (2) $\vdash_{I4\diamond} \subseteq \vdash_{IGL\diamond}$.

Proof For (1), we show that $\mathbf{1}\Box I^*$ is admissible under $\Rightarrow_{IGL\Box}$. Assume that $C_0 \Rightarrow_{IGL\Box} \mathcal{D}_0:\mathbf{1}\psi$, $C_1, v:\mathbf{1}\top \Rightarrow_{IGL\Box} \mathcal{D}_1:\mathbf{1}\varphi$, \mathcal{D}_1 has a barrier with exception for $v:\mathbf{1}\top$, and $\{C_0, C_1\}$ is coherent. Fix \mathcal{D}_2 so that $v_0:\mathbf{1}\Box(\varphi\&\Box\varphi) \Rightarrow_{IK} \mathcal{D}_2:\mathbf{1}\top$. Let $\mathcal{D}'_1 = [v := \mathcal{D}_2]\mathcal{D}_1$; so $C_1, v_0:\mathbf{1}\Box(\varphi\&\Box\varphi) \Rightarrow_{IGL\Box} \mathcal{D}'_1:\mathbf{1}\varphi$. So $C_0 \cup C_1 \Rightarrow_{IGL\Box} \mathcal{D}:\mathbf{1}\Box\varphi$ for \mathcal{D} as pictured.

$$\frac{\frac{\frac{[v_0:\mathbf{1}\Box(\varphi\&\Box\varphi)]}{\mathcal{D}'_1} \quad \frac{[v_1:\mathbf{0}\Box(\varphi\&\Box\varphi)]}{\mathbf{0}\Box\varphi}}{\mathbf{1}\Box\varphi^{v_1}} \quad \mathcal{D}_0}{\mathbf{1}\Box\varphi^{v_1}} \quad \frac{\mathbf{1}\Box\varphi^{v_1}}{\mathbf{1}\&I} \quad \mathcal{D}_0}{\mathbf{1}(\varphi\&\Box\varphi)} \quad \frac{\mathbf{1}(\varphi\&\Box\varphi)}{\mathbf{1}\psi_{\Box E}}}{\mathbf{1}\Box\varphi}$$

For (2), we show that $\mathbf{1}\diamond E^*$ is admissible under $\Rightarrow_{IGL\diamond}$. Assume that $C_1, v:\mathbf{1}\varphi \Rightarrow_{IGL\diamond} \mathcal{D}_1:\mathbf{0}\theta$, \mathcal{D}_1 has a barrier with exception for $v:\mathbf{1}\varphi$, $C_0 \Rightarrow_{IGL\diamond} \mathcal{D}_0:\mathbf{1}\diamond\varphi$, and $\{C_0, C_1\}$ is coherent. Let ψ be $(\varphi \vee \diamond\varphi)$, $v_0, v_1, v_2 \in Var$ be fresh and

distinct. Let \mathcal{D}_2 be as pictured.

$$\frac{\frac{v_0:1\Diamond\varphi}{v_1:1\neg\Diamond\psi} \quad \frac{1\Diamond\psi}{1\supset E}}{1\perp} \quad \frac{1\perp}{1\varphi} \quad 1\supset E$$

Let $\mathcal{D}'_1 = [v := \mathcal{D}_2]\mathcal{D}_1$; so $C_1, v_0:1\Diamond\varphi, v_1:1\neg\Diamond\psi \Rightarrow_{IGL\Diamond} \mathcal{D}'_1:0\theta$. $C_0 \cup C_1 \Rightarrow_{IGL\Diamond} \mathcal{D}:0\theta$ for \mathcal{D} as pictured.

$$\frac{\frac{[v:1\varphi] \quad [v_1:1\neg\Diamond\psi] \quad [v_0:1\Diamond\varphi]}{\mathcal{D}_1 \quad \mathcal{D}'_1 \quad \mathcal{D}_0} \quad \frac{[v_2:1\psi] \quad 0\theta}{0\theta^{v,v_0}} \quad \frac{0\theta \quad 1\supset E}{1\psi} \quad \frac{1\Diamond\varphi \quad 1\vee I}{1\neg\Diamond T}}{0\theta^{v_2,v_1}}$$

□

10.13 Observation

Taking ‘Z’ so that ‘IZ \square ’ and ‘IZ \Diamond ’ are schematic for the names used above, $\vdash_{CZ\Diamond} = \vdash_{CZ\square}$.

This follows from the following: each characteristic rule used to define $\Rightarrow_{CZ\Diamond}$ is admissible under $\Rightarrow_{CZ\square}$; each characteristic rule used to define $\Rightarrow_{CZ\square}$ is admissible under $\Rightarrow_{CZ\Diamond}$. I leave the details to the reader.

10.14 Observations

10.3 - 10.14 remain true with ‘-’ superscripting ‘ \vdash_X ’.

Proof Check that $\Diamond E^*$ was not used in their proofs.

□

11 Appropriate Frames and More Soundness Theorems

We assign the logics introduced in §9 to classes of IK-frames as follows. In what follows, let $F = \langle W, R, \sqsubseteq \rangle$ be an IK-frame.

11.1 Definitions

F is an ID-frame [ID $_w$ -frame] iff for every $u \in W$ there is a v so that $uRv [uR^+v]$.⁸

In [3], Plotkin and Sterling define the classes of frames corresponding to a variety of intuitionistic modal logics. Most of the following definitions follow them.

⁸In the terminology of [4], p. 63, F is an ID frame iff it is serial.

F is an IT_{\diamond} -frame [IT_{\square} -frame] iff for every $u \in W$ there is a $u' \sqsupseteq u$ such that $uRu' [u'Ru]$.⁹ F is an IT -frame iff it is both an IT_{\square} - and an IT_{\diamond} -frame.

F is an IB_{\diamond} -frame [IB_{\square} -frame] iff for every $u, v \in W$ if uRv then for some $u' \sqsupseteq u$ vRu' [then vR^+u]. F is an IB -frame iff it is both an IB_{\diamond} - and an IB_{\square} -frame.

F is an $I4_{\diamond}$ -frame [$I4_{\square}$ -frame] iff for every u, v, w , if $uRvRw$ then for some $w' \sqsupseteq w$ uRw' [then uR^+w]. F is an $I4$ -frame iff it is both an $I4_{\diamond}$ - and an $I4_{\square}$ -frame.

F is an $I5_{\diamond}$ -frame [$I5_{\square}$ -frame] iff for every u, v, w , if uRv and uRw then for some $w' \sqsupseteq w$ vRw' [vR^+w]. F is an $I5$ -frame iff it is both an $I5_{\diamond}$ - and an $I5_{\square}$ -frame.

F is a super $I4_{\diamond}$ -frame iff for every u, v, w , if $uRvR^+w$ then for some $w' \sqsupseteq w$ uRw'

For any $u \in W$ let u be well-capped (in F) iff there is no infinite R^+ -chain in W starting from u . For well-capped members of W , define the norm $| \cdot |$ thus: $|u| = \sup\{|v| + 1 \mid uR^+v\}$. Note: $|u| = 0$ iff u is a dead-end⁺ (i.e. a dead-end for R^+). F is well-capped iff every $u \in W$ is.

F is an IGL_{\diamond} -frame [IGL_{\square} -frame] iff it is a well-capped super $I4_{\diamond}$ -frame [well-capped $I4_{\square}$ -frame]. F is an IGL frame iff it is both an IGL_{\diamond} - and an IGL_{\square} -frame.

F is an $IDio_{\diamond}$ -frame iff for any u and $v_i \in 2$, if uRv_0 and uRv_1 then there are $v'_0 \sqsupseteq v_0$ and $v'_1 \sqsupseteq v_1$ and either $uRv'_0Rv'_1$ or $uRv'_1Rv'_0$.¹⁰ F is an $IDio_{\square}$ -frame iff for any $u, u_i \in 2, v_i \in 2$, if $u \sqsubseteq u_i$ and u_iRv_i for both $i \in 2$ then either $v_0R^+v_1$ or $v_1R^+v_0$. F is an $IDio$ -frame iff it is both an $IDio_{\diamond}$ -frame and an $IDio_{\square}$ -frame,

F is an ITB -frame iff it is both an IT - and an IB -frame.

F is an $IS4$ -frame iff it is both an IT - and an $I4$ -frame.

F is an $IS5$ -frame iff it is an IT - and IB - and $I4$ -frame.

F is a CD - [CT -, CB -, $C4$ -, $C5$, CB -, $CS4$, $CS5$, CGB , $CDio$] frame iff it is a CK - and an ID - [IT -, IB -, $I4$ -, $I5$, IB -, $IS4$, $IS5$, IGL , $IDio$] frame.

Taking 'X' to be schematic for any of the above names, an X-model is an IK -model whose frame is an X-frame.

An inference is X-valid iff it is \mathcal{M} -valid for every X-model \mathcal{M} .

11.2 Soundness Theorems

Taking 'Y' so that 'IY' is schematic for any of the names of intuitionistic systems introduced in Section 9.2, \vdash_{IY} is sound with respect to \mathcal{M} -validity for IY-models \mathcal{M} . Furthermore $\vdash_{\overline{IY}}$ is sound with respect to \mathcal{M} -validity⁻ for IY-models.

Proof Consider any IY-model \mathcal{M} with frame $F = \langle W, R, \sqsubseteq \rangle$. We must prove this: for any C, \mathcal{D} and χ , if $C \Rightarrow_{IY} \mathcal{D} : \chi$ [$C \Rightarrow_{\overline{IY}} \mathcal{D} : \chi$] then $\langle A(\mathcal{D}), \chi \rangle$ is \mathcal{M} -valid [\mathcal{M} -valid⁻]. I leave the details for the square-bracket case to the reader.

We use induction on the stages of \Rightarrow_{IY} (i.e. on the depth of \mathcal{D}). The base case is trivial. Given $n \in \omega$, assume the obvious Induction Hypothesis. Consider C, \mathcal{D}, χ ; assume that $ht(\mathcal{D}) \leq n + 1$. The only cases that need discussion are those in which the root of \mathcal{D} , that is [], is entered by the distinctive rule (or one of the distinctive rules) that generate \Rightarrow_{IY} .

⁹So F is an IT_{\square} -frame iff for every $u \in W$ uR^+u .

¹⁰See p. 405 of [3].

For these arguments, consider a $u \in W$. We will show that (!) $\langle A(\mathcal{D}), \chi \rangle$ is \mathcal{M} -valid at u . Recall (V1) and (V2) from 2.5: (V1) if u is a dead-end and $\mathcal{M}, u \Vdash \Gamma$, then $\mathcal{M}, u \Vdash \chi$; (V2) for every v , if $\mathcal{M}, u, v \models \Gamma$ then $\mathcal{M}, u, v \models \chi$. Also, recall these abbreviations from the proof of 6.1: (A) u is a dead-end and $\mathcal{M}, u \Vdash A(\mathcal{D})$; (B) given v , $\mathcal{M}, u, v \Vdash A(\mathcal{D})$.

For $Y = D_w$, assume that $[]$ was entered into \mathcal{D} by a use of $S1\perp E$ as pictured in Section 9.1. By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{1}\perp \rangle$ is \mathcal{M} -valid. Assume (A). So $A(\mathcal{D}) \subseteq \mathbf{0}Fml$ and $\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D})$. Since \mathcal{M} is an ID_w -model, we may fix v so that uR^+v , and then fix u' so that $u \sqsubseteq u'Rv$. By the Persistence Lemma, $\mathcal{M}, u' \models \mathbf{0}^{-1}A(\mathcal{D})$. Since $\mathcal{M}, v \models \top$, $\mathcal{M}, u', v \Vdash A(\mathcal{D}_0)$. By (*), $\mathcal{M}, u', v \Vdash \mathbf{1}\perp$, a contradiction. (V1) vacuously follows. Assume (B). So $\mathcal{M}, u, v \Vdash A(\mathcal{D}_0)$; by (*) $\mathcal{M}, u, v \Vdash \mathbf{1}\perp$, a contradiction. (V2) vacuously follows, yielding (!).

For $Y = D$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{1}TT$ as pictured in Section 9.1; so χ is $\mathbf{0}\theta$. Since u is not a dead-end, (V1) follows vacuously. Assume (B). By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{0}\theta \rangle$ is \mathcal{M} -valid. Trivially $\mathcal{M}, u, v \Vdash \mathbf{1}\top$; so $\mathcal{M}, u, v \Vdash A(\mathcal{D}_0)$. By (*) $\mathcal{M}, u, v \Vdash \mathbf{0}\theta$. (V2) follows, yielding (!).

For $Y = T_\diamond$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{0}E$ as pictured in Section 9.1; so χ is $\mathbf{0}\theta$. Since T is a T_\diamond -frame, u is not a dead-end, and so (A) is false. Assume (B). Fix $\{s_{i \in (m)}\}$ to be a barrier in \mathcal{D}_0 with exception for $v: \mathbf{1}\varphi$. By the IH, (*) $\langle A(\mathcal{D}_1), \mathbf{1}\varphi \rangle$ and (**) $\langle A(\mathcal{D}_0), \mathbf{0}\theta \rangle$ are \mathcal{M} -valid. For distinct $v_1, \dots, v_m \in Var$, none occurring in \mathcal{D}_0 , let \mathcal{D}_0^S = the result of surgery on \mathcal{D}_0 at s_1, \dots, s_m using v_1, \dots, v_m . Fix $\theta_{i \in (m)}$ and $\mathcal{D}'_{i \in (m)}$ as we have done several times in Section 6.1. As in previous arguments, $\langle A(\mathcal{D}'_i), \mathbf{0}\theta_i \rangle$ is \mathcal{M} -valid at u . By choice of $\{s_{i \in (m)}\}$ and \mathcal{D}_0^S ,

$$A(\mathcal{D}_0^S) \subseteq (A(\mathcal{D}_0) \cap \mathbf{0}Fml) \cup \{\mathbf{0}\theta_{i \in (m)}\} \cup \{\mathbf{1}\varphi\}.$$

Since \mathcal{M} is an IT_\diamond -model, we may fix a u' so that $u \sqsubseteq u'$ and uRu' . By the right-completeness of F we may fix a v' so that $v \sqsubseteq v'$ and $u'Rv'$. Since $\mathcal{M}, u, v \Vdash A(\mathcal{D}_0)$, $\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D}_0)$ and $\mathcal{M}, v \models \mathbf{1}^{-1}A(\mathcal{D}_0)$; by the Persistence Lemma, $\mathcal{M}, u' \models \mathbf{0}^{-1}A(\mathcal{D}_0)$ and $\mathcal{M}, v' \models \mathbf{1}^{-1}A(\mathcal{D}_0)$; so $\mathcal{M}, u', v' \Vdash A(\mathcal{D}_1)$. By (**) $\mathcal{M}, u', v' \Vdash \mathbf{0}\varphi$. So $\mathcal{M}, u' \models \varphi$. As in those previous arguments, for $i \in (m)$ we have that $\mathcal{M}, u, v \Vdash A(\mathcal{D}'_i)$. So $\mathcal{M}, u, v \Vdash \mathbf{0}\theta_i$. So $\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D}_0^S)$. So $\mathcal{M}, u, u' \Vdash A(\mathcal{D}_0^S)$. Since $ht(\mathcal{D}_0^S) \leq ht(\mathcal{D}_0) \leq n$, by the IH $\langle A(\mathcal{D}_0^S), \mathbf{0}\theta \rangle$ is \mathcal{M} -valid. So $\mathcal{M}, u, u' \Vdash \mathbf{0}\theta$; so $\mathcal{M}, u \models \theta$; so $\mathcal{M}, u, v \Vdash \mathbf{0}\theta$. (V2) follows, yielding (!).

For $Y = T_\square$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{0}I$ as pictured in Section 9.1; so χ is $\mathbf{0}\varphi$. By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{1}\varphi \rangle$ is \mathcal{M} -valid. Since \mathcal{M} is an IT_\square -model, we may fix a u' so that $u \sqsubseteq u'$ and $u'Ru$. Assume (A). So $A(\mathcal{D}) \subseteq \mathbf{0}Fml$. Since $\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D})$, by the Persistence Lemma $\mathcal{M}, u' \Vdash A(\mathcal{D})$. So $\mathcal{M}, u', u \Vdash A(\mathcal{D}_0)$. By (*), $\mathcal{M}, u', u \Vdash \mathbf{1}\varphi$; so $\mathcal{M}, u \models \varphi$; so $\mathcal{M}, u \Vdash \mathbf{0}\varphi$. (V1) follows. Assume (B). Fix $\{s_i\}_{i \in (m)}$ to be a barrier in \mathcal{D}_0 with exception for $v: \mathbf{1}\top$. Fix distinct $v_1, \dots, v_m \in Var$ as above, and let \mathcal{D}_0^S = the result of surgery on \mathcal{D}_0 at s_1, \dots, s_m using v_1, \dots, v_m . Fix $\theta_{i \in (m)}$ and $\mathcal{D}'_{i \in (m)}$ as usual. For any $i \in (m)$, $\mathcal{M}, u, v \Vdash A(\mathcal{D}'_i)$, and (by the IH) $\langle A(\mathcal{D}'_i), \mathbf{0}\theta_i \rangle$ is \mathcal{M} -valid at u . So $\mathcal{M}, u, v \Vdash \mathbf{0}\theta_i$. So $\mathcal{M}, u \models$

$\mathbf{0}^{-1}A(\mathcal{D}_0^{\mathcal{S}})$. As usual,

$$A(\mathcal{D}_0^{\mathcal{S}}) \subseteq (A(\mathcal{D}_0) \cap \mathbf{0}Fml) \cup \{\mathbf{0}\theta_{i \in (m)}\} \cup \{\mathbf{1}\top\}.$$

By the Persistence Lemma $\mathcal{M}, u' \models \mathbf{0}^{-1}A(\mathcal{D}_0^{\mathcal{S}})$. So $\mathcal{M}, u', u \Vdash A(\mathcal{D}_1^{\mathcal{S}})$. Since $ht(\mathcal{D}_1^{\mathcal{S}}) \leq n$, by the IH $\langle A(\mathcal{D}_0^{\mathcal{S}}), \mathbf{1}\varphi \rangle$ is \mathcal{M} -valid; so $\mathcal{M}, u', u \Vdash \mathbf{1}\varphi$. So $\mathcal{M}, u \models \varphi$. (V2) follows, yielding (!).

For $Y = B_{\diamond}$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{1}\diamond I^*$ as pictured in Section 9.1; so χ is $\mathbf{1}\diamond\varphi$. By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{1}\psi \rangle$ and (**) $\langle A(\mathcal{D}_0), \mathbf{0}\varphi \rangle$ are \mathcal{M} -valid. Assume (A). So $\mathcal{M}, u \Vdash A(\mathcal{D}_0)$; by (*) $\mathcal{M}, u \Vdash \mathbf{1}\psi$, a contradiction. (V1) vacuously follows. Assume (B). So $\mathcal{M}, u, v \Vdash A(\mathcal{D}_1)$. By (**) $\mathcal{M}, u, v \Vdash \mathbf{0}\varphi$; so $\mathcal{M}, u \models \varphi$. Since F is an IB_{\diamond} -frame, we may fix a $u' \sqsupseteq u$ so that vRu' . By the Persistence Lemma $\mathcal{M}, u' \models \varphi$. So $\mathcal{M}, v \models \diamond\varphi$. So $\mathcal{M}, u, v \Vdash \chi$. (V2) follows, yielding (!).

For $Y = B_{\square}$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{1}\square E^*$ as pictured in Section 9.1; so χ is $\mathbf{0}\varphi$ and $A(\mathcal{D}) = A(\mathcal{D}_0)$. By the IH, (*) $\langle A(\mathcal{D}), \mathbf{1}\square\varphi \rangle$ is \mathcal{M} -valid. Assume (A). By (*) $\mathcal{M}, u \Vdash \mathbf{1}\square\varphi$, a contradiction. (V1) vacuously follows. Assume (B). By (*) $\mathcal{M}, u, v \Vdash \mathbf{1}\square\varphi$; so $\mathcal{M}, v \models \square\varphi$. Since F is an IB_{\square} -frame we may fix a v' so that $v \sqsubseteq v'$ and $v'Ru$. By the Persistence Lemma, $\mathcal{M}, v' \models \square\varphi$. Since $v'R^+u$, $\mathcal{M}, u \models \varphi$. So $\mathcal{M}, u, v \Vdash \mathbf{0}\varphi$. (V2) follows, yielding (!).

For $Y = 4_{\diamond}$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{1}\diamond E^*$ as pictured in Section 9.1; so χ is $\mathbf{0}\theta$. By the IH, (*) $\langle A(\mathcal{D}_1), \mathbf{1}\diamond\varphi \rangle$ is \mathcal{M} -valid. Assume (A). Since $\mathcal{M}, u \Vdash A(\mathcal{D}_1)$, by (*) $\mathcal{M}, u \Vdash \mathbf{1}\diamond\varphi$, a contradiction. (V1) vacuously follows. Assume (B). Fix $\{s_{i \in (m)}\}$ to be a barrier in \mathcal{D}_0 with exception for $v:\mathbf{1}\varphi$. For distinct $v_1, \dots, v_m \in Var$, none occurring in \mathcal{D} , let $\mathcal{D}_0^{\mathcal{S}}$ = the result of surgery on \mathcal{D}_0 at s_1, \dots, s_m using v_1, \dots, v_m . Fix $\theta_{i \in (m)}$ and \mathcal{D}'_i as in previous arguments. Since $\mathcal{M}, u, v \Vdash A(\mathcal{D}_1)$, by (*) $\mathcal{M}, u, v \Vdash \mathbf{1}\diamond\varphi$; so $\mathcal{M}, v \models \diamond\varphi$. Fix w so that vRw and $\mathcal{M}, w \models \varphi$. Since F is an $I4_{\diamond}$ -frame, we may fix a w' so that uRw' and $w \sqsubseteq w'$. By the Persistence Lemma $\mathcal{M}, w' \models \varphi$. Again, for any $i \in (m)$, $\mathcal{M}, u, v \Vdash A(\mathcal{D}'_i)$, and (by the IH) $\langle A(\mathcal{D}'_i), \mathbf{0}\theta_i \rangle$ is \mathcal{M} -valid at u . So $\mathcal{M}, u, v \Vdash \mathbf{0}\theta_i$. As in the argument under the case for $Y=T_{\diamond}$, (!!) follows. So $\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D}_0^{\mathcal{S}})$. So $\mathcal{M}, u, w' \Vdash A(\mathcal{D}_0^{\mathcal{S}})$.

Since $ht(\mathcal{D}_0^{\mathcal{S}}) \leq n$, by the IH $\langle A(\mathcal{D}_0^{\mathcal{S}}), \mathbf{0}\theta \rangle$ is \mathcal{M} -valid. So $\mathcal{M}, u, w' \Vdash \mathbf{0}\theta$; so $\mathcal{M}, u \models \theta$; so $\mathcal{M}, u, v \Vdash \mathbf{0}\theta$. (V2) follows, yielding (!).

For $Y = 4_{\square}$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{1}\square I^*$ as pictured in Section 9.1; so χ is $\mathbf{1}\square\varphi$. Assume (A). Since $\mathcal{M}, u \Vdash A(\mathcal{D}_0)$, by (*) $\mathcal{M}, u \Vdash \mathbf{1}\psi$, a contradiction. (V1) vacuously follows. Assume (B). Fix $\{s_{i \in (m)}\}$ to be a barrier in \mathcal{D}_1 with exception for $v:\mathbf{1}\top$. Fix $v_1, \dots, v_m \in Var$ as above and let $\mathcal{D}_1^{\mathcal{S}}$ = the result of surgery on \mathcal{D}_1 at s_1, \dots, s_m using v_1, \dots, v_m . For each $i \in (m)$ fix θ_i and \mathcal{D}'_i as above (except cut out of \mathcal{D}_1 rather than \mathcal{D}_0). By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{1}\psi \rangle$ is \mathcal{M} -valid. Since $\mathcal{M}, u, v \Vdash \mathbf{1}\top$, $\mathcal{M}, u, v \Vdash A(\mathcal{D}_1)$. Claim: $\mathcal{M}, v \models \square\varphi$. Given w , assume that vR^+w . Fix v' so that $v \sqsubseteq v'Rw$. Since uRv and F satisfies left-completeness, we may fix u' so that $u \sqsubseteq u'Rv'$; since F is an $I4_{\square}$ -frame we may fix u'' so that $u' \sqsubseteq u''Rw$. Consider any $i \in (m)$. As before, $\mathcal{M}, u, v \Vdash A(\mathcal{D}'_i)$, and (by the IH)

$\langle A(\mathcal{D}'_1), \mathbf{0}\theta_i \rangle$ is \mathcal{M} -valid at u . So $\mathcal{M}, u, v \Vdash \mathbf{0}\theta_i$. Since

$$A(\mathcal{D}_1^{\mathfrak{S}}) \subseteq (A(\mathcal{D}_1) \cap \mathbf{0Fml}) \cup \{\mathbf{0}\theta_{i \in (m)}\} \cup \{\mathbf{1}\top\},$$

$\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D}_1^{\mathfrak{S}})$. By the Persistence Lemma $\mathcal{M}, u'' \models \mathbf{0}^{-1}A(\mathcal{D}_1^{\mathfrak{S}})$. By the construction of $\mathcal{D}_1^{\mathfrak{S}}$, $\mathcal{M}, u'', w \Vdash A(\mathcal{D}_1^{\mathfrak{S}})$. Since $ht(\mathcal{D}_1^{\mathfrak{S}}) \leq n$, by the IH $\langle A(\mathcal{D}_1^{\mathfrak{S}}), \mathbf{1}\varphi \rangle$ is \mathcal{M} -valid. So $\mathcal{M}, u'', w \Vdash \mathbf{1}\varphi$. So $\mathcal{M}, w \models \varphi$. Thus $\mathcal{M}, u, v \Vdash \mathbf{1}\Box\varphi$. (V2) follows, yielding (!).

For $Y = 5\Diamond$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{0}/\mathbf{1}\Diamond$ as pictured in Section 9.1; so χ is $\mathbf{1}\Diamond\varphi$. By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{1}\psi \rangle$ and (**) $\langle A(\mathcal{D}_1), \mathbf{0}\Diamond\varphi \rangle$ are \mathcal{M} -valid. Assume (A). As usual, (*) yields a contradiction; (V1) vacuously follows. Assume (B). By (**) $\mathcal{M}, u, v \Vdash \mathbf{0}\Diamond\varphi$; so $\mathcal{M}, u \models \Diamond\varphi$. Fix a w so that uRw and $\mathcal{M}, w \models \varphi$. Since F is an $I5\Diamond$ -frame, we may fix a w' so that $w \sqsubseteq w'$ and vRw' . By the Persistence Lemma, $\mathcal{M}, w' \models \varphi$; so $\mathcal{M}, v \models \Diamond\varphi$; so $\mathcal{M}, u, v \Vdash \mathbf{1}\Diamond\varphi$. (V2) follows, yielding (!).

For $Y = 5\Box$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{1}/\mathbf{0}\Box$ as pictured in Section 9.1; so χ is $\mathbf{0}\Box\varphi$. By the IH, (*) $\langle A(\mathcal{D}), \mathbf{1}\Box\varphi \rangle$ is \mathcal{M} -valid. Assuming (A), $\mathbf{1}\Box\varphi \in \mathbf{0Fml}$ for a contradiction; (V1) vacuously follows. Assume (B). By (*) $\mathcal{M}, u, v \Vdash \mathbf{1}\Box\varphi$; so $\mathcal{M}, v \models \Box\varphi$. Given any w , assume that uR^+w ; fix $u' \sqsupseteq u$ so that $u'Rw$. By right-completeness we may fix a $v' \sqsupseteq v$ so that $u'Rv'$. Since F is an $I5\Box$ -frame, $v'R^+w$; so vR^+w ; so $\mathcal{M}, w \models \varphi$. So $\mathcal{M}, u \models \Box\varphi$; so $\mathcal{M}, u, v \Vdash \mathbf{0}\Box\varphi$. (V2) follows, yielding (!).

For $Y = GL\Diamond$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{1}\neg\Diamond T$ as pictured in Section 9.1. So χ is $\mathbf{0}\theta$. By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{1}\varphi \rangle$ is \mathcal{M} -valid. Assume (A). Since $\mathcal{M}, u \Vdash A(\mathcal{D}_0)$, by (*) $\mathcal{M}, u \Vdash \mathbf{1}\varphi$ for a contradiction. (V1) vacuously follows. Assume (B). Fix $\{s_{i \in (m)}\}$ to be a barrier in \mathcal{D}_1 with exception for $\{v_0: \mathbf{1}\neg\Diamond\varphi, v_1: \varphi\}$. Construct $\mathcal{D}_1^{\mathfrak{S}}$ from \mathcal{D}_1 using $\{s_{i \in (m)}\}$ and fresh variables in the usual way; fix $\theta_{i \in (m)}$ as in previous such arguments. By now familiar arguments,

$$A(\mathcal{D}_1^{\mathfrak{S}}) \subseteq (A(\mathcal{D}_1) \cap \mathbf{0Fml}) \cup \{\mathbf{0}\theta_{i \in (m)}\} \cup \{\mathbf{1}\neg\Diamond\varphi\}$$

and $\mathcal{M}, u \models \mathbf{0}^{-1}A(\mathcal{D}_1^{\mathfrak{S}})$. Claim: for every x , if uRx and $\mathcal{M}, x \models \varphi$, then for some y , uRy , $\mathcal{M}, y \models \varphi$ and $\mathcal{M}, y \models \neg\Diamond\varphi$. Proof is by induction on $|x|$. If $|x| = 0$, $\mathcal{M}, x \models \neg\Diamond\varphi$; so x is as needed. Assume the obvious IH. Given x , assume the if-clause. If $\mathcal{M}, x \models \neg\Diamond\varphi$; again x is as needed. Assume that $\mathcal{M}, x \not\models \neg\Diamond\varphi$. So we may fix a z so that xR^+z and $\mathcal{M}, z \models \varphi$. Since $|z| < |x|$, the inner IH applies to z , yielding the existence of a y as needed. The Claim follows. Since uRv and $\mathcal{M}, v \models \varphi$, we may fix a y so that uRy , $\mathcal{M}, y \models \varphi$ and $\mathcal{M}, y \models \neg\Diamond\varphi$. So $\mathcal{M}, u, y \Vdash A(\mathcal{D}_1^{\mathfrak{S}})$. Since $ht(\mathcal{D}_1^{\mathfrak{S}}) \leq n$, by the IH, $\langle A(\mathcal{D}_1^{\mathfrak{S}}), \mathbf{0}\theta \rangle$ is \mathcal{M} -valid. So $\mathcal{M}, u, y \Vdash \mathbf{0}\theta$. So $\mathcal{M}, u \models \theta$; so $\mathcal{M}, u, v \Vdash \mathbf{0}\theta$. (V2) follows, yielding (!).

For $Y = IGL\Box$, assume that $[]$ was entered into \mathcal{D} by a use of $\mathbf{1}\Box T$ as pictured in Section 9.1. So χ is $\mathbf{1}\varphi$. By the IH, (*) $\langle A(\mathcal{D}_0), \mathbf{1}\psi \rangle$ is \mathcal{M} -valid. Assume (A). Since $\mathcal{M}, u \Vdash A(\mathcal{D}_0)$, by (*) $\mathcal{M}, u \Vdash \mathbf{1}\psi$, a contradiction. (V1) vacuously follows. Assume (B). Fix $\{s_{i \in (m)}\}$ to be a barrier in \mathcal{D}_1 with exception for $v: \mathbf{1}\Box\varphi$. Construct $\mathcal{D}_1^{\mathfrak{S}}$ from \mathcal{D}_1 using $\{s_{i \in (m)}\}$ and fresh variables in the usual way; fix $\theta_{i \in (m)}$ as in

previous such arguments. As usual,

$$A\left(\mathcal{D}_1^S\right) \subseteq \left(A\left(\mathcal{D}_1\right) \cap \mathbf{0}Fml\right) \cup \left\{\mathbf{0}\theta_{i \in (m)}\right\} \cup \left\{\mathbf{1}\square\varphi\right\},$$

$\mathcal{M}, u \models \mathbf{0}^{-1}A\left(\mathcal{D}_1^S\right)$ and $ht\left(\mathcal{D}_1^S\right) \leq n$. By the IH, $\langle A\left(\mathcal{D}_1^S\right), \mathbf{1}\varphi \rangle$ is \mathcal{M} -valid. We now prove that for every $x \in W$, if uR^+x then $\mathcal{M}, x \models \varphi$, using induction on $|x|$. Given a dead-end⁺ x , since $\mathcal{M}, x \models \square\varphi$, $\mathcal{M}, u, x \models A\left(\mathcal{D}_1^S\right)$; by $(**)$ $\mathcal{M}, u, v \Vdash \mathbf{1}\varphi$. Given $x \in W$ with $|x| > 0$, consider any y so that xR^+y . Since F is an $I4_{\square}$ -frame, R^+ is transitive; so uR^+y . Since $|y| < |x|$, by the inner IH $\mathcal{M}, y \models \varphi$. So $\mathcal{M}, x \models \square\varphi$. So $\mathcal{M}, u, x \Vdash A\left(\mathcal{D}_1^S\right)$; by $(**)$ $\mathcal{M}, u, x \Vdash \mathbf{1}\varphi$. The claim follows. Since uRv , $\mathcal{M}, v \models \varphi$. (V2) follows, yielding (!).

For $Y = IDio_{\diamond}$, assume that $[]$ was entered into \mathcal{D} by a use of Dio_{\diamond} as pictured in Section 9.1. By the IH, for both $i \in 2$, $(*_i) \langle A(\mathcal{D}_{i+2}), \mathbf{0}\diamond\varphi_i \rangle$ is \mathcal{M} -valid. Assume (A). Since $\mathcal{M}, u \Vdash A(\mathcal{D}_{i+2})$, by $(*_i)$ $\mathcal{M}, u \Vdash \mathbf{0}\diamond\varphi_i$; so u is not a dead-end, a contradiction, which yields (V1). Assume (B). Fix $\{s_{i,j \in (m)}\}$ to be a barrier in \mathcal{D}_i with exception for $\{v_{2i}:\mathbf{1}\varphi_i, v_{2i+1}:\mathbf{1}\diamond\varphi_{1-i}\}$ for both $i \in 2$. Construct \mathcal{D}_i^S from \mathcal{D}_i using $\{s_{i,j \in (m)}\}$ and fresh variables in the usual way; For both $i \in 2$, by $(*_i)$ $\mathcal{M}, u \models \diamond\varphi_i$; so we may fix v_i such that $\mathcal{M}, v_i \models \varphi_i$ and uRv_i . Fix v'_0, v'_1 and $k \in 2$ so that $v_i \sqsubseteq v'_i$ for both $i \in 2$, and $uRv'_kRv'_{1-k}$. So $\mathcal{M}, v'_i \models \varphi_i$ for both $i \in 2$; so $\mathcal{M}, v'_k \models \diamond\varphi_{1-k}$. So $\mathcal{M}, u, v'_k \Vdash \mathbf{1}\varphi_j$ and $\mathcal{M}, u, v'_k \Vdash \mathbf{1}\diamond\varphi_{1-k}$. By familiar reasoning, $\mathcal{M}, u, v'_k \Vdash A(\mathcal{D}_i^S)$. I leave the rest to the reader.

For $Y = IDio_{\square}$, assume that $[]$ was entered into \mathcal{D} by a use of Dio_{\square} as pictured in Section 9.1; so χ is $\mathbf{0}\theta$. By the IH, for both $i \in 2$, $(*_i) \langle A(\mathcal{D}_i), \mathbf{0}\theta \rangle$ is \mathcal{M} -valid. Assume (B). For each $i \in 2$, fix m_i and the barrier $\{s_{i,j \in (m_i)}\}$ in \mathcal{D}_{i+2} with exception for $v_{i+2}:\mathbf{1}\varphi_i$. Then fix $\{\theta_{i,j \in (m_i)}\}$ and amputate to construct \mathcal{D}_{i+2}^S from \mathcal{D}_{i+2} in the usual way. So

$$A\left(\mathcal{D}_{i+2}^S\right) \subseteq \left(A\left(\mathcal{D}_{i+2}\right) \cap \mathbf{0}Fml\right) \cup \left\{\mathbf{0}\theta_{i,j \in (m_i)}\right\} \cup \left\{\mathbf{1}\varphi_i\right\}.$$

Claim: for some $i \in 2$, $\mathcal{M}, u \models \square\neg\varphi_i$. Assume otherwise. Fix v_i such that uR^+v_i and $\mathcal{M}, v_i \not\models \neg\varphi_i$, and then fix $u_i \supseteq u$ so that u_iRv_i , this for both $i \in 2$. Fix $v'_i \supseteq v_i$ so that $\mathcal{M}, v'_i \models \varphi_i$, and then fix $u'_i \supseteq u_i$ so that $u'_iRv'_i$, again for both $i \in 2$. So $uR^+u'_0$ and $uR^+u'_1$. Since \mathcal{M} is an $IDio_{\square}$ -model, we may fix a $k \in 2$ so that $v'_kR^+v'_{1-k}$. Consider either $i \in 2$. By a familiar argument (using (B), and in particular v), $\mathcal{M}, u \models \{\theta_{i,j \in (m_i)}\}$. By the Persistence Lemma, $\mathcal{M}, u'_i \models \mathbf{0}^{-1}A(\mathcal{D}_{i+2}^S)$, and so $\mathcal{M}, u'_i, v'_i \Vdash A\left(\mathcal{D}_{i+2}^S\right)$. By the IH applied to \mathcal{D}_{i+2}^S , $\langle A(\mathcal{D}_{i+2}^S), \mathbf{1}\square\neg\varphi_{1-i} \rangle$ is \mathcal{M} -valid. So $\mathcal{M}, u'_i, v'_i \Vdash \mathbf{1}\square\neg\varphi_{1-i}$; so $\mathcal{M}, v'_i \models \square\neg\varphi_{1-i}$. Since $\mathcal{M}, v'_k \models \square\neg\varphi_{1-k}$, $\mathcal{M}, v'_{1-k} \models \neg\varphi_{1-k}$, a contradiction. The Claim follows. Fixing such a i , $\mathcal{M}, u, v \Vdash A(\mathcal{D}_i)$. By $(*_i)$, $\langle A(\mathcal{D}_i), \mathbf{0}\theta \rangle$ is \mathcal{M} -valid. So $\mathcal{M}, u, v \Vdash \mathbf{0}\theta$, proving (V2). Now assume (A). The argument under the (B)-case applies, with this simplification: since $\mathcal{M}, u \Vdash A(\mathcal{D}_{i+2}) - \{\mathbf{1}\square\neg\varphi_i\}$, $m_i = 0$ and $\{s_{i,j \in (m_i)}\} = \{\}$; so $\mathcal{D}_{i+2}^S = \mathcal{D}_{i+2}$. So (V1), and thus (!).

For the “blended” systems (T, B, etc.), use the arguments for their “ingredient” systems. □

For each of the above choices of X , the “furthermore” follows by straightforward revisions to the above proof.

We will now consider some non-inclusions. In the following specifications of $F = \langle W, R, \sqsubseteq \rangle$, \sqsubseteq will be reflexive on whatever we take as our W and transitive, and R might be specified in part just by giving positive R -facts.

11.3 Corollaries

$$(1) \vdash_{ID_w}^- \not\vdash_{CB45}. (2) \vdash_{ID}^- \not\vdash_{ID_w B45}. (3) \vdash_{ID}^- \not\vdash_{IT_{\square} B_{\square} 45_{\square}}.$$

Proof For (1) let $W = 1$ and $R = \{\}$; check that $F = \langle W, R, \sqsubseteq \rangle$ is a CB45-frame. For any valuation function \mathcal{V} on $W \times S$, $\langle \mathbf{0} \sqsubseteq \perp, \mathbf{0} \perp \rangle$ is not $\langle F, \mathcal{V} \rangle$ -valid at 0; by 11.2 $\mathbf{0} \sqsubseteq \perp \not\vdash_{CB45} \mathbf{0} \perp$. Then use 10.1(3).

For (2) let $W = 2, 0 \sqsubseteq 1$, and $1R1$. Check that F is an $ID_w B45$ -frame. For any valuation function \mathcal{V} on $W \times S$, $\mathbf{0} \diamond \top$ is not $\langle F, \mathcal{V} \rangle$ -valid at 0; by 11.2, $\not\vdash_{ID_w B45} \mathbf{0} \diamond \top$. Then use 10.1(2).

For (3) let $W = 2, 0 \sqsubseteq 1, 1R0$ and $1R1$. Check that F is an $IT_{\square} B_{\square} 45_{\square}$ -frame. For any valuation function \mathcal{V} , $\mathbf{0} \diamond \top$ is not $\langle F, \mathcal{V} \rangle$ -valid at 0. By 11.2, $\not\vdash_{IT_{\square} B_{\square} 45_{\square}} \mathbf{0} \diamond \top$. Then use 10.1(2). □

For the remaining corollaries, assume that $S \neq \{\}$; fix $\pi \in S$.

11.4 Corollaries

$$(1) \vdash_{IT_{\diamond}}^- \not\vdash_{IDT_{\square} B_{\square} 45}. (2) \vdash_{IT_{\square}}^- \not\vdash_{IT_{\diamond} B_{\diamond} 45}.$$

Proof Proof of (1) takes some effort. Let $E = \{2n \mid n \in \omega\}$, $O = \{2n + 1 \mid n \in \omega\}$, and $\sqsubseteq = (\leq |E) \cup (\leq |O)$. Fix an $R_0 : E - \{0\} \rightarrow E$ so that for $n \in E - \{0\}$ $R_0(n) > n$ and $\{n' \in E \mid R_0(n') = R_0(n)\}$ is infinite.¹¹ Let $R_1 : E - \{0\} \rightarrow O$ such that for $n \in E - \{0\}$ $R_1(n) = R_0(n) + 1$. For each $m, n, n' \in \omega$ let $T_{m,n,n'} =$

$$\{\langle 2m, 2n + 1 \rangle, \langle 2n + 1, 2m \rangle, \langle 2m, 2n' + 1 \rangle, \langle 2n' + 1, 2m \rangle, \langle 2n + 1, 2n' + 1 \rangle\},$$

and let $R_2 = \bigcup \{T_{m,n,n'} \mid R_0(2n) = R_0(2n') = 2m\}$. Set $R = R_0 \cup R_1 \cup R_2 \cup \{\langle 0, 1 \rangle, \langle 1, 1 \rangle\}$ and $F = \langle \omega, R, \sqsubseteq \rangle$. Claim: F is an $IDT_{\square} B_{\square} 45_{\square}$ -frame. Clearly it is an ID-frame. R_0 insures that for each $n \in E$ there is an $m \in E$ such that mRn ; R_2 insures that for each $n \in O$ there is an $m \in O$ such that mRn . So F is an IT_{\square} -frame. R_0 insures that for each $n, m \in E$, if nRm then there is a $p \in E$ such that pRm ; R_2 insures that for each $n, m \in O$, if nRm then there is a $p \in O$ such that pRm . Also, for any $n \in E$, $\{m \in O \mid nRm \text{ and } mRn\}$ is infinite; so if $nR2k + 1$ then for some $j > k$ $2j + 1Rn$. Also, for any $n \in O$ and $m \in E$, if nRm then mRn . So F is an IB_{\square} -frame. I will leave the tedious verification of the I45-frame conditions, and of left- and right-completeness, to the reader. (A picture, say with even numbers $2n$ in

¹¹For example, let $R_0(2n) = 2(n - m)$ for $m =$ the greatest triangular number $\leq n$. (A triangular number is one of the form $2k^2 + k$ or $2k^2 + 3k + 1$.)

a column on the left, and odd numbers $2n + 1$ on the right, e.g. for $n \leq 8$, counting upward, and arrows to indicate R restricted to these numbers, will be helpful. Also note that if $m \in O$, $n, n' \in E$, mRn and mRn' then mR_2n and mR_2n' , and so $n = n'$.) Let $\mathcal{V}(u, \pi) = 1$ iff $u = 0$; so $\langle \{0\pi\}, 0\Diamond\pi \rangle$ is not $\langle F, \mathcal{V} \rangle$ -valid at 0. By 11.2, $0\pi \not\vdash_{IT_{\square}B_{\square}45} 0\Diamond\pi$. Now use 10.1(4).

In contrast to (1), (2)'s proof is almost trivial. Let $W = 2$, $0 \sqsubseteq 1$, $0R1$ and $1R1$; check that F is an $IT_{\Diamond}B_{\Diamond}45$ -frame. Let $\mathcal{V}(x, \pi) = 1$ iff $x = 1$. $\langle \{0\Box\pi\}, 0\pi \rangle$ is not $\langle F, \mathcal{V} \rangle$ -valid at 0. By 11.2, $0\Box\pi \not\vdash_{IT_{\Diamond}B_{\Diamond}45} 0\pi$. Then use 10.1(5). \square

No finite model will be a counter-model witnessing that $0\pi \not\vdash_{IT_{\square}B_{\square}45} 0\Diamond\pi$. Remarkably, no finite $IT_{\square}5_{\Diamond}$ -model even witnesses that $0\pi \not\vdash_{IT_{\square}5_{\Diamond}} 0\Diamond\pi$. To see this, assume that F is a finite $IT_{\square}5_{\Diamond}$ -frame and $u \in W^F$ so that $\langle F, \mathcal{V} \rangle, u \models \pi$ and $\langle F, \mathcal{V} \rangle, u \not\models \Diamond\pi$. Since W^F is finite and F is an IT_{\square} -frame, there is an n and $u = u_0, \dots, u_n$ so that for each $i < n$ $u_i \sqsubseteq u_{i+1}$, $u_{i+1}Ru_i$, and there is no $v \sqsupseteq u_n$ so that $v \neq u_n$. Since u_nRu_n , there is an $m \leq n$ and a $u' \sqsupseteq u_m$ so that u_mRu' . Let m be the least such; fix the corresponding u' . If $m = 0$, by the Persistence Lemma $\langle F, \mathcal{V} \rangle, u' \models \pi$, and so $\langle F, \mathcal{V} \rangle, u \models \Diamond\pi$, a contradiction. So $m > 0$. So u_mRu_{m-1} and u_mRu' . Since F is a finite $I5_{\Diamond}$ -frame, for some $v \sqsupseteq u'$, $u_{m-1}Rv$. Since $u_m \sqsubseteq v$ this contradicts choice of m . So there is no such F .

11.5 Corollaries

(1) $\vdash_{IB_{\Diamond}}^- \not\vdash_{ITB_{\square}45_{\square}}$. (2) $\vdash_{IB_{\square}}^- \not\vdash_{ITB_{\Diamond}45_{\Diamond}}$. (3) $\vdash_{IB_{\Diamond}}^- \not\vdash_{IT_{\square}45}$. (4) $\vdash_{IB_{\square}}^- \not\vdash_{IT_{\Diamond}45}$.

Proof For (1) let $W = 4$, $2 \sqsubseteq 0$, $3 \sqsubseteq 1$, $R = \{0, 1\}^2 \cup \{2, 3\}^2 \cup \{(i, j) \mid i \in \{0, 1\}, j \in \{2, 3\}\}$. Check that F is an $ITB_{\square}45_{\square}$ -frame. Let $\mathcal{V}(x, \pi) = 1$ iff $x = 0$. $\langle \{0\pi\}, 0\Diamond\pi \rangle$ is not $\langle F, \mathcal{V} \rangle$ -valid at 0; by 11.2 $0\pi \not\vdash_{ITB_{\square}45_{\square}} 0\Diamond\pi$. Then use 10.1(6).

For (2), let $W = 4$, $0 \sqsubseteq 2$, $1 \sqsubseteq 3$, $R = \{2, 3\}^2 \cup id \setminus \{0, 1\} \cup \{(0, i) \mid i \in \{1, 2, 3\}\} \cup \{(1, 2), (1, 3)\}$. Check that F is an $ITB_{\Diamond}45_{\Diamond}$ -frame. Let $\mathcal{V}(x, \pi) = 1$ iff $x \in \{1, 2, 3\}$. $\langle \{0\Diamond\pi\}, 0\pi \rangle$ is not $\langle F, \mathcal{V} \rangle$ -valid at 0; by 11.2 $0\Diamond\pi \not\vdash_{ITB_{\Diamond}45_{\Diamond}} 0\pi$. Then use 10.1(7).¹²

(3) and (4) are easy exercises. Hint: take $W = 2$. \square

11.6 Corollaries

(1) $\vdash_{I4_{\Diamond}}^- \not\vdash_{IT_{\Diamond}B_{\Diamond}4_{\square}5}$. (2) $\vdash_{I4_{\square}}^- \not\vdash_{ITB_{\Diamond}4_{\square}5_{\Diamond}}$. (3) $\vdash_{I4_{\Diamond}}^- \not\vdash_{ITB_{\square}4_{\square}5_{\square}}$. (4) $\vdash_{I4_{\square}}^- \not\vdash_{IT_{\square}B_{\square}4_{\Diamond}5}$. (5) $\vdash_{I4_{\square}}^- \not\vdash_{ITB_{\square}4_{\Diamond}5_{\square}}$. (6) $\vdash_{I4_{\square}}^- \not\vdash_{ITB_{\Diamond}4_{\Diamond}5_{\Diamond}}$.

Proof For (1)-(3) let $W = 4$, $0 \sqsubseteq 3$, and for any $u \in 4$, $\mathcal{V}(u, \pi) = 1$ iff $u = 2$. At the end, use 10.1.8.

For (1), let $R = \{1, 2, 3\}^2 \cup \{(0, 1), (0, 3)\}$; check that F is an $IT_{\Diamond}B_{\Diamond}4_{\square}5$ -frame. $\langle \{0\Diamond\Diamond\pi\}, 0\Diamond\pi \rangle$ is not $\langle F, \mathcal{V} \rangle$ -valid at 0; by 11.2, $1\Diamond\pi \not\vdash_{IT_{\Diamond}B_{\Diamond}4_{\square}5} 0\Diamond\pi$.

¹²These examples are due to Philip Sink.

For (2), let $R = \{1, 2, 3\}^2 \cup \{\langle 0, 0 \rangle, \langle 0, 1 \rangle\}$; check that F is an $ITB_{\diamond 4 \square 5 \diamond}$ -frame. $\langle \{0 \diamond \diamond \pi\}, 0 \diamond \pi \rangle$ is not not $\langle F, \mathcal{V} \rangle$ -valid at 0; by 11.2, $1 \diamond \pi \not\vdash_{IT_{\diamond} B_{\diamond 4 \square 5}} 0 \diamond \pi$.

For (3), let $R = \{0, 1, 3\}^2 \cup \{1, 2, 3\}^2$; check that F is an $ITB_{\square 4 \square 5 \square}$ -frame. $\langle \{0 \diamond \diamond \pi\}, 0 \diamond \pi \rangle$ is not not $\langle F, \mathcal{V} \rangle$ -valid at 0; by 11.2, $1 \diamond \pi \not\vdash_{IT B_{\square 4 \square 5 \square}} 0 \diamond \pi$.

(4)-(6) are good exercises. □

11.7 Corollaries

(1) $\vdash_{I5_{\diamond}}^- \not\vdash_{IT B_{\square 45 \square}}$. (2) $\vdash_{I5_{\square}}^- \not\vdash_{IT B_{\diamond 45 \diamond}}$.

Proof For (1), let F and \mathcal{V} be as they were for 11.5(1). Check that $\langle \{0 \diamond \pi\}, 0 \square \diamond \pi \rangle$ is not $\langle F, \mathcal{V} \rangle$ -valid at 0; so by 11.2, $0 \diamond \pi \not\vdash_{IT B_{\square 45 \square}} 0 \square \diamond \pi$. Then use 10.1(10). For (2), let F and \mathcal{V} be as they were for 11.5(2); a similar argument applies. □

I leave non-inclusions involving the IGL and IDio systems to another occasion, or another logician.

11.8 Theorems

Taking ‘Y’ so that ‘CY’ is schematic for any of the names for classical systems introduced in Section 9.2, \vdash_{CY} is sound with respect to CY-models.

Proofs are straightforward.

11.9 Corollaries

For ‘Y’ replaceable as above, the restriction of \vdash_{CY} to $0Fml$ is just the result of prefixing 0 to all formulas in the familiar no-step classical consequence relation \vdash_Y .

Proof Consider any $\Sigma \subseteq Fml$ and $\varphi \in Fml$. Assume that $\Sigma \vdash_Y \varphi$; fix a deduction \mathcal{D} witness this in the no-step (Prawitz-format) Natural Deduction proof-theoretic system formalizing \vdash_Y . Prefixing 0 to every formula-label in \mathcal{D} yields a deduction witnessing that $0\Sigma \vdash_{CY} 0\varphi$. Assume that $0\Sigma \vdash_{CY} 0\varphi$. By 11.8, $\langle 0\Sigma, 0\varphi \rangle$ is CY-valid. It is easy to see that then φ is a Y-consequence of Σ according to the standard Kripkean model-theoretic semantics for the no-step classical logic Y. So $\Sigma \vdash_Y \varphi$.¹³ □

11.10 Definition

Consider an IK-frame F and a formula φ ; let $S =$ the set formula-constants occurring in φ . Let $F \models \varphi$ iff for every valuation \mathcal{V} on $W^F \times S$, φ is $\langle F, \mathcal{V} \rangle$ -valid. φ defines the class of IK-frames F such that $F \models \varphi$.

¹³If one defines \vdash_Y model-theoretically in terms of Kripke-models, this is immediate; if one defines it proof-theoretically, use the completeness of that proof-theoretic system with respect to Kripke-models.

11.11 Appropriateness Observations

We will now show that the classes of IK-frames assigned in Section 11.1 to the logics under consideration are “appropriate” (in the sense of [1], pp 80-81).

- (1) $\diamond\top$ defines the class of ID-frames. (2) $\neg\Box\perp$ defines the class of ID_w -frames.
- Fix a formula-constant π . (3) $(\pi \supset \diamond\pi)$ defines the class of IT_\diamond -frames. (4) $(\Box\pi \supset \pi)$ defines the class of IT_\Box -frames. (5) $(\pi \supset \Box\diamond\pi)$ defines the class of IB_\diamond -frames. (6) $(\diamond\Box\pi \supset \pi)$ defines the class of IB_\Box -frames. (7) $(\diamond\diamond\pi \supset \diamond\pi)$ defines the class of $I4_\diamond$ -frames (8) $(\Box\pi \supset \Box\Box\pi)$ defines the class of $I4_\Box$ -frames. (9) $(\diamond\pi \supset \Box\diamond\pi)$ defines the class of $I5_\diamond$ -frames. (10) $(\diamond\Box\pi \supset \Box\pi)$ defines the class of $I5_\Box$ -frames. (11) $(\Box(\Box\pi \supset \pi) \supset \Box\pi)$ defines the class of IGL_\Box -frames. (12) $(\diamond\pi \supset \diamond(\pi \ \& \ \neg\diamond\pi))$ defines the class of IGL_\diamond -frames.

Assume that $\pi_0, \pi_1 \in S$ are distinct. (13) $((\diamond\pi_0 \ \& \ \diamond\pi_1) \supset (\diamond(\pi_0 \ \& \ \pi_1) \vee (\diamond(\diamond\pi_0 \ \& \ \pi_1)))$ defines the class of $IDio_\diamond$ -frames. (14) $((\Box(\pi_0 \supset \Box\neg\pi_1) \ \& \ \Box(\pi_1 \supset \Box\neg\pi_0)) \supset (\Box\neg\pi_0 \vee \Box\neg\pi_1))$ defines the class of $IDio_\Box$ -frames. (15) $(\pi \vee \neg\pi)$ defines the class of CK-frames.

Proof Consider any IK-frame $F = \langle W, R, \sqsubseteq \rangle$.

(1) If F is an ID-frame, $F \models \diamond\top$. Assume that $F \not\models \diamond\top$. Let \mathcal{V} be the valuation on $W \times S$ assigning every $\langle x, \pi \rangle$ to 0. For any $u \in W$, $\langle F, \mathcal{V} \rangle, u \not\models \diamond\top$; so u is not a dead-end. So F is an ID-frame.

(2) If F is an ID_w -frame, $F \models \neg\Box\perp$. Assume that $F \not\models \neg\Box\perp$. For \mathcal{V} as above and any $u \in W$, $\langle F, \mathcal{V} \rangle, u \not\models \neg\Box\perp$; so u is not a dead-end⁺. So F is an ID_w -frame.

For what follows, set $S = \{\pi\}$.

(3) By 10.1(4) and 11.2, $\mathbf{0}(\pi \supset \diamond\pi)$ is IT_\diamond -valid. So if F is an IT_\diamond -frame then $F \models (\pi \supset \diamond\pi)$. Assume that $F \not\models (\pi \supset \diamond\pi)$. Given $u \in W$, let \mathcal{V} be the valuation on $W \times S$ so that for any $x \in W$ $\mathcal{V}(x, \pi) = 1$ iff $u \sqsubseteq x$. Clearly \mathcal{V} is persistent (with respect to \sqsubseteq). Thus $\langle F, \mathcal{V} \rangle, u \models \pi$. So $\langle F, \mathcal{V} \rangle, u \not\models \diamond\pi$; so for some $u' \ uRu'$ and $u \sqsubseteq u'$. So F is an IT_\diamond -frame.

(4) By 10.1(5) and 11.2, $\mathbf{0}(\Box\pi \supset \pi)$ is IT_\Box -valid. So if F is an IT_\Box -frame then $F \models (\Box\pi \supset \pi)$. Assume that $F \not\models (\Box\pi \supset \pi)$. Given $u \in W$, let \mathcal{V} be the valuation on $W \times S$ so that for any $x \in W$ $\mathcal{V}(x, \pi) = 1$ iff uR^+x . Claim: \mathcal{V} is persistent. Assume that $\mathcal{V}(v, \pi) = 1$ and $v \sqsubseteq v'$. Since uR^+v we may fix u' so that $u \sqsubseteq u'Rv$; by the left-completeness of F we may fix u'' so that $u' \sqsubseteq u''Rv'$; thus uR^+v' ; so $\mathcal{V}(v', \pi) = 1$. The claim follows. Since $\langle F, \mathcal{V} \rangle, u \models \Box\pi$, $\langle F, \mathcal{V} \rangle, u \not\models \pi$. So uR^+u . So F is an IT_\Box -frame.

(5) By 10.1(6) and 11.2, $\mathbf{0}(\pi \supset \Box\diamond\pi)$ is IB_\diamond -valid. So if F is an IB_\diamond -frame then $F \models (\pi \supset \Box\diamond\pi)$. Assume that $F \not\models (\pi \supset \Box\diamond\pi)$. Given $u \in W$, let \mathcal{V} be the valuation on $W \times S$ as in the proof of (3). So $\langle F, \mathcal{V} \rangle, u \models \pi$. So $\langle F, \mathcal{V} \rangle, u \not\models \Box\diamond\pi$. Given v , assume that uRv . Since uR^+v , $\langle F, \mathcal{V} \rangle, v \models \diamond\pi$. Fix u' so that vRu' and $\langle F, \mathcal{V} \rangle, u' \models \pi$; so $u \sqsubseteq u'$. So F is an IB_\diamond -frame.

(6) By 10.1(7) and 11.2, $\mathbf{0}(\diamond\Box\pi \supset \pi)$ is IB_\Box -valid. So if F is an IB_\Box -frame then $F \models (\diamond\Box\pi \supset \pi)$. Assume that $F \not\models (\diamond\Box\pi \supset \pi)$. Given $u, v \in W$, assume that uRv . Let \mathcal{V} be the valuation on $W \times S$ so that for any $x \in W$ $\mathcal{V}(x, \pi) = 1$ iff vR^+x . As in the argument for (4), \mathcal{V} is persistent. So $\langle F, \mathcal{V} \rangle, v \models \Box\pi$; so $\langle F, \mathcal{V} \rangle, u \not\models \diamond\Box\pi$. So $F \not\models \pi$. So vR^+u . So F is an IB_\Box -frame.

(7) Using 10.1(8) and 11.2 as above, if F is an $I4_{\diamond}$ -frame then $F \models (\diamond\diamond\pi \supset \diamond\pi)$. Assume that $F \models (\diamond\diamond\pi \supset \diamond\pi)$. Given $u, v, w \in W$, assume that $uRvRw$. Let \mathcal{V} be the valuation on $W \times S$ so that for any $x \in W$ $\mathcal{V}(x, \pi) = 1$ iff $w \sqsubseteq x$. \mathcal{V} is persistent. Also $\langle F, \mathcal{V} \rangle, u \models \diamond\diamond\pi$; so $\langle F, \mathcal{V} \rangle, u \models \diamond\pi$. Fix a w' so that uRw' and $\langle F, \mathcal{V} \rangle, w' \models \pi$. So $w \sqsubseteq w'$. So F is an $I4_{\diamond}$ -frame.

(8) Using 10.1(9) and 11.2 as above, if F is an $I4_{\square}$ -frame then $F \models (\square\pi \supset \square\square\pi)$ is easy. Assume that $F \models (\square\pi \supset \square\square\pi)$. Given $u, v, w \in W$, assume that $uRvRw$. Let \mathcal{V} be the valuation on $W \times S$ as in the proof of (4); as in the argument for (4), \mathcal{V} is persistent. Since $\langle F, \mathcal{V} \rangle, u \models \square\pi$, $\langle F, \mathcal{V} \rangle, u \models \square\square\pi$. So $\langle F, \mathcal{V} \rangle, w \models \pi$; so uR^+w . So F is an $I4_{\square}$ -frame.

(9) Using 10.1(10) and 11.2 as above, if F is an $I5_{\diamond}$ -frame then $F \models (\diamond\pi \supset \square\diamond\pi)$. Assume that $F \models (\diamond\pi \supset \square\diamond\pi)$. Given $u, v, w \in W$, assume that uRv and uRw . Let \mathcal{V} be the valuation on $W \times S$ as in the proof of (7). Since $\langle F, \mathcal{V} \rangle, u \models \diamond\pi$, $\langle F, \mathcal{V} \rangle, u \models \square\diamond\pi$. So $\langle F, \mathcal{V} \rangle, v \models \diamond\pi$; fix w' so that vRw' and $\langle F, \mathcal{V} \rangle, w' \models \pi$; so $w \sqsubseteq w'$. So F is an $I5_{\diamond}$ -frame.

(10) Using 10.1(11) and 11.2 as above, if F is an $I5_{\square}$ -frame then $F \models (\diamond\square\pi \supset \square\pi)$. Assume that $F \models (\diamond\square\pi \supset \square\pi)$. Given $u, v, w \in W$, assume that uRv and uRw . Let \mathcal{V} be the valuation on $W \times S$ so that for any $x \in W$ $\mathcal{V}(x, \pi) = 1$ iff vR^+x . Since $\langle F, \mathcal{V} \rangle, v \models \square\pi$, $\langle F, \mathcal{V} \rangle, u \models \diamond\square\pi$. So $\langle F, \mathcal{V} \rangle, u \models \square\pi$. So $\langle F, \mathcal{V} \rangle, w \models \pi$. So vR^+w . So F is an $I5_{\square}$ -frame.

(11) Using 10.1(13) and 11.2 as above, if F is an IGL_{\square} -frame then $F \models (\square(\square\pi \supset \pi) \supset \square\pi)$. Assume that $F \models (\square(\square\pi \supset \pi) \supset \square\pi)$. The proof that F is an IGL_{\square} -frame recapitulates the argument for classical GL given in [1], pp. 82-83, using R^+ in place of R . Given u, v, w , assume that uR^+vR^+w ; to prove that uR^+w , let $\mathcal{V}(x, \pi) = 1$ iff uR^+x and for every y if xR^+y then uR^+y . It suffices to show that $\langle F, \mathcal{V} \rangle, u \models \square(\square\pi \supset \pi)$, since then $\langle F, \mathcal{V} \rangle, u \models \square\pi$. Details are left to the reader. Given $u \in W$, to show that u is well-capped let $\mathcal{V}(x, \pi) = 1$ iff for every $x^+ \supseteq x$ x^+ is well-capped. Check that \mathcal{V} is persistent. Check that for any $v \in W$, $\langle F, \mathcal{V} \rangle, v \models (\square\pi \supset \pi)$. Given $u \in W$, it follows that $\langle F, \mathcal{V} \rangle, u \models \square(\square\pi \supset \pi)$; so $\langle F, \mathcal{V} \rangle, u \models \square\pi$. So for every v , if uR^+v then v is well-capped. So u is well-capped.

(12) Using 10.1(12) and 11.2 as above, if F is an IGL_{\diamond} -frame then $F \models (\diamond\pi \supset \diamond(\pi \& \neg\diamond\pi))$. Assume that $F \models (\diamond\pi \supset \diamond(\pi \& \neg\diamond\pi))$. Claim 1: F is a super $I4_{\diamond}$ -frame. Given u, v, w , assume that $uRvR^+w$; fix $v' \supseteq v$ so that $v'Rw$. Let \mathcal{V} be the valuation on $W \times S$ such that $\mathcal{V}(x, \pi) = 1$ iff either (i) for some $y \supseteq w$ xR^+y or (ii) $x \supseteq w$. Check that \mathcal{V} is persistent. Since vR^+w , $\langle F, \mathcal{V} \rangle, v \models \pi$; so $\langle F, \mathcal{V} \rangle, u \models \diamond\pi$; so $\langle F, \mathcal{V} \rangle, u \models \diamond(\pi \& \neg\diamond\pi)$. Fix an x so that uRx and $\langle F, \mathcal{V} \rangle, x \models (\pi \& \neg\diamond\pi)$. Assume (i), for a contradiction. Fix $y \supseteq w$ so that xR^+y . So $\langle F, \mathcal{V} \rangle, y \models \pi$; so $\langle F, \mathcal{V} \rangle, x \models \diamond\pi$. Since $\langle F, \mathcal{V} \rangle, x \models \neg\diamond\pi$ we have a contradiction. (ii) follows. Claim 1 follows. Claim 2: there is no infinite R -chain. Assume that $\langle u_i \rangle_{i \in \omega}$ is an R -chain. Let \mathcal{V} be the valuation on $W \times S$ so that for any $x \in W$ $\mathcal{V}(x, \pi) = 1$ iff for some $i \in \omega$ $u_i \sqsubseteq x$. So \mathcal{V} is persistent. Since $\langle F, \mathcal{V} \rangle, u_1 \models \pi$, $\langle F, \mathcal{V} \rangle, u_0 \models \diamond\pi$; so $\langle F, \mathcal{V} \rangle, u_0 \models \diamond(\pi \& \neg\diamond\pi)$. Fix an x so that uRx and $\langle F, \mathcal{V} \rangle, x \models (\pi \& \neg\diamond\pi)$. Fix $i \in \omega$ so that $u_i \sqsubseteq x$. Since F is right-complete, we may fix a y so that xRy and $y \supseteq u_{i+1}$. So $\langle F, \mathcal{V} \rangle, y \models \pi$; so $\langle F, \mathcal{V} \rangle, x \models \diamond\pi$, contrary to $\langle F, \mathcal{V} \rangle, x \models \neg\diamond\pi$. Claim 2 follows. Claim 3: if there is an infinite R^+ -chain then there is an infinite R -chain. Assume that $\langle u_i \rangle_{i \in \omega}$ is an R^+ -chain. For each $i \in \omega$ fix $u'_i \supseteq u_i$ so that u'_iRu_i . Since

$u'_{2i+1}Ru_{2i+2}$, by the left-completeness of F there is a $v \sqsupseteq u'_{2i+1}$ so that vRu'_{2i+2} ; so $u'_{2i+1}R^+u'_{2i+2}$. Using claim 1, for each $i \in \omega$ we may fix a $u_{2i+2}^+ \sqsupseteq u'_{2i+2}$ so that $u'_{2i}Ru_{2i+2}^+$. Let $u_0^* = u'_0$ and $u_2^* = u_2^+$. With u_{2i+2}^* defined so that $u'_{2i+2} \sqsubseteq u_{2i+2}^*$, since $u'_{2i+2}Ru_{2i+4}^+$ the right-completeness of F lets us fix a $u_{2i+4}^* \sqsupseteq u_{2i+4}^+$ so that $u_{2i+2}^*Ru_{2i+4}^*$. So $\langle u_{2i}^* \rangle_{i \in \omega}$ is an R -chain, proving claim 3. By claims 2 and 3, F is well-capped.

(13) Using 10.1(14) and 11.2 as above, if F is an IDio_\diamond -frame then

$$(*) F \models ((\diamond\pi_0 \& \diamond\pi_1) \supset (\diamond(\pi_0 \& \diamond\pi_1) \vee (\diamond(\diamond\pi_0 \& \pi_1))).$$

Assume (*). Given $u, v_i \in 2$ assume that for both $i \in 2$ uRv_i . Fix \mathcal{V} so that $\mathcal{V}(x, \pi) = 1$ iff for some $i \in 2$ π is π_i and $v_i \sqsubseteq x$. \mathcal{V} is persistent and $\langle F, \mathcal{V} \rangle, u \models (\diamond\pi_0 \& \diamond\pi_1)$. Thus $\langle F, \mathcal{V} \rangle, u \models (\diamond(\pi_0 \& \diamond\pi_1) \vee \diamond(\diamond\pi_0 \& \pi_1))$. Assume that $\langle F, \mathcal{V} \rangle, u \not\models \diamond(\pi_0 \& \diamond\pi_1)$. Fix v'_0 so that uRv'_0 and $\langle F, \mathcal{V} \rangle, v'_0 \models (\pi_0 \& \diamond\pi_1)$. So $v_0 \sqsubseteq v'_0$ and $\langle F, \mathcal{V} \rangle, v'_0 \models \diamond\pi_1$. Fix v'_1 so that $v'_0Rv'_1$ and $\langle F, \mathcal{V} \rangle, v'_1 \models \pi_1$. So $v_1 \sqsubseteq v'_1$. So F is an IDio_\diamond -frame. If $\langle F, \mathcal{V} \rangle, u \models \diamond(\diamond\pi_0 \& \pi_1)$ then a similar argument applies.

(14) Using 10.1(15) and 11.2 as above, F is an IDio_\square -frame then

$$(**) F \models ((\square(\pi_0 \supset \square\neg\pi_1) \& \square(\pi_1 \supset \square\neg\pi_0)) \supset (\square\neg\pi_0 \vee \square\neg\pi_1)).$$

Assume (**). Given $u_i \in 2$ and $v_i \in 2$ assume that $u_0 \sqsubseteq u_1$ and u_iRv_i . Fix \mathcal{V} so that $\mathcal{V}(x, \pi) = 1$ iff for some $i \in 2$ π is π_i , $v_i \sqsubseteq x$, and there is no $x^+ \sqsupseteq x$ so that $v_{1-i}R^+x^+$. Check that \mathcal{V} is persistent. Consider either $i \in 2$. Claim: $\langle F, \mathcal{V} \rangle, u_0 \models \square(\pi_i \supset \square\neg\pi_{1-i})$. It suffices to show that for any x , if u_0R^+x and $\langle F, \mathcal{V} \rangle, x \models \pi_i$ then $\langle F, \mathcal{V} \rangle, x \models \square\neg\pi_{1-i}$. Given x , assume the if-clause. So $v_i \sqsubseteq x$. Given y assume that xR^+y . So v_iR^+y . Assume for a contradiction that $\langle F, \mathcal{V} \rangle, y \not\models \neg\pi_{1-i}$. Fix a $y' \sqsupseteq y$ so that $\langle F, \mathcal{V} \rangle, y' \models \pi_{1-i}$. So there is no $y^+ \sqsupseteq y'$ so that $v_iR^+y^+$. But since F is left-complete, we may fix a $v'_i \sqsupseteq v_i$ so that v'_iRy' ; so v_iR^+y' , a contradiction. So $\langle F, \mathcal{V} \rangle, x \models \square\neg\pi_{1-i}$. The claim follows. By (**), $\langle F, \mathcal{V} \rangle, u_0 \models (\square\neg\pi_0 \vee \square\neg\pi_1)$. Fix $j \in 2$ so that $\langle F, \mathcal{V} \rangle, u \models \square\neg\pi_j$. So $\langle F, \mathcal{V} \rangle, v_j \models \neg\pi_j$; so $\langle F, \mathcal{V} \rangle, v_j \not\models \pi_j$. Since $v_j \sqsubseteq v_j$, there is an $x^+ \sqsupseteq v_j$ so that $v_{1-j}R^+x^+$. So F is an IDio_\square -frame.

(15) Proof is an exercise. □

12 More Canonical Frames and More Completeness Theorems

12.1 Canonical Model Theorems

For ‘X’ replaceable by the names of the logics introduced above other than ‘ IGL_\square ’, ‘ IGL_\diamond ’ and ‘ CGL ’, the canonical frame for \vdash_X is an X-frame, and so the canonical model for \vdash_X is an X-model.

Proof $X = \text{ID}$: Let $\Phi \in W_{\vdash_{\text{ID}}}$. Since $\vdash_{\text{ID}} \mathbf{0} \diamond \top, \diamond \top \in \Phi$. By the Diamond Lemma 7.13, there is a $\Psi \in W_{\vdash_{\text{ID}}}$ such that $\Phi R_{\vdash_{\text{ID}}} \Psi$.

$X = \text{ID}_w$: To avoid clutter let $\vdash = \vdash_{\text{ID}_w}$. Consider any $\Phi \in W_{\vdash}$. Claim: $\mathbf{0} \Phi \cup \{\mathbf{0} \diamond \top\} \not\models \mathbf{0} \perp$. Assume that $\mathbf{0} \Phi \cup \{\mathbf{0} \diamond \top\} \vdash \mathbf{0} \perp$; fix C, v and \mathcal{D}_0 . so that

$C, v:\mathbf{0}\diamond\top \Rightarrow_{ID_w} \mathcal{D}_0:\mathbf{0}\perp$ and $ran(C) \subseteq \mathbf{0}\Phi$. Let \mathcal{D}_1 look thus for $v_1 \in Var - dom(C)$.

$$\frac{v_1:\mathbf{1}\top}{\mathbf{0}\diamond\top} \diamond I$$

Let $\mathcal{D}_2 = [v := \mathcal{D}_1]\mathcal{D}_0$. So \mathcal{D}_2 looks thus.

$$\frac{\frac{v_1:\mathbf{1}\top}{\mathbf{0}\diamond\top} \diamond I}{\mathcal{D}_0} \mathbf{0}\perp$$

Construct \mathcal{D} as pictured.

$$\frac{\frac{[v_1:\mathbf{1}\top]}{\mathcal{D}_2} \frac{\mathbf{0}\perp \mathbf{0}\perp E}{\mathbf{1}\perp} S1\perp E}{\mathbf{0}\perp} \nu_1$$

Since $C \Rightarrow_{ID_w} \mathcal{D}:\mathbf{0}\perp, \mathbf{0}\Phi \vdash \mathbf{0}\perp$, a contradiction that proves the claim. By Lindenbaum’s Lemma for \vdash , we may fix a $\Sigma \in W_{\vdash}$ so that $\Phi \subseteq \Sigma$ and $\diamond\top \in \Sigma$. By the argument above for $X=ID$, there is a $\Psi \in W_{\vdash}$ so that $\Phi R_{\vdash} \Psi$.

$X = IT_{\diamond}$: *The Special Diamond Lemma for IT_{\diamond}* . If $\Phi \in W_{IT_{\diamond}}$ then for some $\Psi, \Phi R_{IT_{\diamond}} \Psi$ and $\Phi \subseteq \Psi$.

For clutter-control, set $\vdash = \vdash_{IT_{\diamond}}, W = W_{\vdash}, R = R_{\vdash}$. Let $\Gamma = \mathbf{0}\Phi \cup \mathbf{1}\Phi$. Redo the definition of q and $\langle \Psi_j \rangle_{j \in q}$ from the proof of 7.13. Claim 1: for every $j \in \omega$, (i) $j \in q$, (ii) $\Gamma \cup \mathbf{1}\Psi_j$ is $\mathbf{0}$ -closed under \vdash , (iii) for every $\sigma \in \Sigma, \diamond(\sigma \& \wedge \Psi_j) \in \Phi$, and (iv) if $j > 0$ and j is even, the bad case for j does not obtain. Proof is by induction. For the base step, only (ii) deserves our attention. Given $\delta \in Fml$, assume that $\Gamma \vdash \mathbf{0}\delta$. Since Φ is closed under conjunction we may fix a $\varphi \in \Phi$ so that $\mathbf{0}\Phi \cup \{\mathbf{1}\varphi\} \vdash \mathbf{0}\delta$. Since $\mathbf{0}\varphi \in \mathbf{0}\Phi$, using $\mathbf{0}E$ yields $\mathbf{0}\Phi \cup \{\mathbf{0}\varphi\} \vdash \mathbf{0}\delta$, i.e. $\mathbf{0}\Phi \vdash \mathbf{0}\delta$. Since $\Phi \in W$, it is $\mathbf{0}$ -closed under \vdash ; so $\mathbf{0}\delta \in \mathbf{0}\Phi \subseteq \Gamma$. So 0 satisfies (ii). The rest of the proof of Claim 1, and then the rest of the argument, imitates that used to prove 7.13.

$X = IT_{\square}$ *The Special Unbox Lemma for IT_{\square}* . If $\Psi \in W_{IT_{\square}}$ then for some $\Phi, \Phi R_{IT_{\square}} \Psi$ and $\Psi \subseteq \Phi$.

For clutter-control set $\vdash = \vdash_{IT_{\square}}, W = W_{\vdash}, R = R_{\vdash}$. Let $\Gamma = \mathbf{0}\Psi \cup \mathbf{1}\Psi$. Redo the definition of q and $\langle \Phi_j \rangle_{j \in q}$ from the proof of 7.11. Claim 1: for every $j \in \omega$, (i) $j \in q$; (ii) $\Gamma \cup \mathbf{0}\Phi_j$ is $\mathbf{1}$ -closed under \vdash , (iii) for every $\rho \in Fml$, if $\Gamma \cup \mathbf{0}\Phi_j \vdash \mathbf{0}\square\rho$ then $\rho \in \Psi$; (iv) if $j > 0$ and j is even, the bad case for j does not obtain. Proof is by induction. For the base step, only (ii) and (iii) deserves discussion. Given $\delta \in Fml$, assume that $\Gamma \vdash \mathbf{1}\delta$. Since Ψ is closed under conjunction, we may fix a $\psi \in \Psi$ so that $\mathbf{0}\Psi \cup \{\mathbf{1}\psi\} \vdash \mathbf{1}\delta$. So $\mathbf{0}\Psi \cup \{\mathbf{1}\top\} \vdash \mathbf{1}(\psi \supset \delta)$. Using $\mathbf{0}I, \mathbf{0}\Psi \vdash \mathbf{0}(\psi \supset \delta)$. So $\mathbf{0}\Psi \vdash \mathbf{0}\delta$. Since $\Psi \in W$ it is $\mathbf{0}$ -closed under \vdash : so $\mathbf{0}\delta \in \mathbf{0}\Psi \subseteq \Gamma$. Thus 0 satisfies (ii). Given $\rho \in Fml$, assume that $\Gamma \vdash \mathbf{0}\square\rho$. By Section 10.2(4), $\Gamma \vdash \mathbf{0}\rho$. Since 0 satisfies (ii), $\mathbf{0}\rho \in \Gamma$; so $\rho \in \Psi$. The rest of the proof of Claim 1, and then the rest of the argument, imitates that used to prove 7.11.

$X = IB_{\diamond}$: *The Special Diamond Lemma for IB_{\diamond}* . If $\Sigma R_{IB_{\diamond}} \Phi$ then for some Ψ , $\Phi R_{IB_{\diamond}} \Psi$ and $\Sigma \subseteq \Psi$.

Set $\vdash = \vdash_{IB_{\diamond}}$, $W = W_{\vdash}$, $R = R_{\vdash}$. Assume the if-clause. Let $\Gamma = \mathbf{0}\Phi \cup \mathbf{1}\Sigma$. Redo the definition of q and $\langle \Psi_j \rangle_{j \in q}$ from the proof of 7.13. Claim 1: for every $j \in \omega$, (i) $j \in q$, (ii) $\Gamma \cup \mathbf{1}\Psi_j$ is $\mathbf{0}$ -closed under \vdash , (iii) for every $\sigma \in \Sigma$, $\diamond(\sigma \& \bigwedge \Psi_j) \in \Phi$, and (iv) if $j > 0$ and j is even, the bad case for j does not obtain. Proof is by induction. For the base step, only (ii) deserves our attention. Assume that $\Gamma \vdash \mathbf{0}\delta$. Since $\Sigma \in W$, Σ is closed under conjunction. So we can fix $\sigma \in \Sigma$ so that $\mathbf{0}\Phi \cup \{\mathbf{1}\sigma\} \vdash \mathbf{0}\delta$. Using $\diamond E$ once, $\mathbf{0}\Phi \cup \{\mathbf{0}\diamond\sigma\} \vdash \mathbf{0}\delta$. Since $\mathbf{0}\Sigma \cup \mathbf{1}\Phi \vdash \mathbf{0}\sigma$ and $\Phi \neq \{\}$, using $\mathbf{1}\diamond I^*$ once yields that $\mathbf{0}\Sigma \cup \mathbf{1}\Phi \vdash \mathbf{1}\diamond\sigma$. Since $\Sigma R\Phi$, $\mathbf{0}\Sigma \cup \mathbf{1}\Phi$ avoids $\mathbf{0}\perp$ under \vdash , by the $\mathbf{0}\perp$ -Avoidance Lemma for \vdash ; so $\mathbf{0}\Sigma \cup \mathbf{1}\Phi$ is closed under \vdash ; so $\diamond\sigma \in \Phi$. Cutting $\mathbf{0}\diamond\sigma$ yields $\mathbf{0}\Phi \vdash \mathbf{0}\delta$. Since $\Phi \in W$, $\mathbf{0}\delta \in \mathbf{0}\Phi \subseteq \Gamma$. Thus 0 satisfies (ii). The rest of the proof of Claim 1, and then the rest of the argument, imitates that used to prove 7.13.

$X = IB_{\square}$: *The Special Unbox Lemma for IB_{\square}* . If $\Psi R_{IB_{\square}} \Sigma$ then for some Φ , $\Phi R_{IB_{\square}} \Psi$ and $\Sigma \subseteq \Phi$.

Set $\vdash = \vdash_{IB_{\square}}$, $W = W_{\vdash}$, $R = R_{\vdash}$. Assume the if-clause. Let $\Gamma = \mathbf{0}\Sigma \cup \mathbf{1}\Psi$. Redo the definition of q and $\langle \Phi_j \rangle_{j \in q}$ from the proof of 7.11. Claim 1: for every $j \in \omega$, (i) $j \in q$; (ii) $\Gamma \cup \mathbf{0}\Phi_j$ is $\mathbf{1}$ -closed under \vdash , (iii) for every $\rho \in Fml$, if $\Gamma \cup \mathbf{0}\Phi_j \vdash \mathbf{0}\square\rho$ then $\rho \in \Psi$, (iv) if $j > 0$ and j is even, the bad case for j does not obtain. Proof is by induction. For the base step, only (ii) and (iii) deserves discussion. Assume that $\Gamma \vdash \mathbf{1}\delta$. Since $\Psi \in W$, Ψ is closed under conjunction; so we may fix $\psi \in \Psi$ so that $\mathbf{0}\Sigma \cup \{\mathbf{1}\psi\} \vdash \mathbf{1}\delta$. So $\mathbf{0}\Sigma \cup \{\mathbf{1}\top\} \vdash \mathbf{1}(\psi \supset \delta)$; so $\mathbf{0}\Sigma \vdash \mathbf{0}\square(\psi \supset \delta)$. Since $\Sigma \in W$, it is $\mathbf{0}$ -closed under \vdash ; so $\square(\psi \supset \delta) \in \Sigma$. So $\mathbf{0}\Psi \cup \mathbf{1}\Sigma \vdash \mathbf{1}\square(\psi \supset \delta)$. Using $\mathbf{1}\square E^*$ once yields that $\mathbf{0}\Psi \cup \mathbf{1}\Sigma \vdash \mathbf{0}(\psi \supset \delta)$. Since $\mathbf{0}\psi \in \mathbf{0}\Psi$, $\mathbf{0}\Psi \cup \mathbf{1}\Sigma \vdash \mathbf{0}\delta$. Since $\Psi R\Sigma$, the $\mathbf{0}\perp$ -Avoidance Lemma for \vdash implies that $\mathbf{0}\Psi \cup \mathbf{1}\Sigma$ avoids $\mathbf{0}\perp$ under \vdash , and so is closed under \vdash ; so $\delta \in \Psi$. So $\mathbf{1}\delta \in \Gamma$. Thus 0 satisfies (ii). Assume that $\Gamma \vdash \mathbf{0}\square\rho$. As above we may fix $\psi \in \Psi$ so that $\mathbf{0}\Sigma \cup \{\mathbf{1}\psi\} \vdash \mathbf{0}\square\rho$. using $\diamond E$ once, $\mathbf{0}\Sigma \cup \{\mathbf{0}\diamond\psi\} \vdash \mathbf{0}\square\rho$; so $\mathbf{0}\Sigma \vdash \mathbf{0}(\diamond\psi \supset \square\rho)$. Using 4.2.(4), $\mathbf{0}\Sigma \vdash \mathbf{0}\square(\psi \supset \rho)$. Since Σ is $\mathbf{0}$ -closed under \vdash , $\square(\psi \supset \rho) \in \Sigma$. So $\mathbf{0}\Psi \cup \mathbf{1}\Sigma \vdash \mathbf{1}\square(\psi \supset \rho)$. Using $\mathbf{1}\square E^*$, $\mathbf{0}\Psi \cup \mathbf{1}\Sigma \vdash \mathbf{0}(\psi \supset \rho)$. Since $\mathbf{0}\psi \in \mathbf{0}\Psi$, $\mathbf{0}\Psi \cup \mathbf{1}\Sigma \vdash \mathbf{0}\rho$. So $\rho \in \Psi$. Thus 0 satisfies (iii). The rest of the proof of Claim 1, and then the rest of the argument, imitates that used to prove 7.11.

$X = I4_{\diamond}$: *The Special Diamond Lemma for $I4_{\diamond}$* . If $\Phi_0 R_{I4_{\diamond}} \Phi_1 R_{I4_{\diamond}} \Sigma$ then for some Ψ , $\Phi_0 R_{I4_{\diamond}} \Psi$ and $\Sigma \subseteq \Psi$.

Set $\vdash = \vdash_{I4_{\diamond}}$, $W = W_{\vdash}$, $R = R_{\vdash}$. Assume the if-clause. Let $\Gamma = \mathbf{0}\Phi_0 \cup \mathbf{1}\Sigma$. Redo the definition of q and $\langle \Psi_j \rangle_{j \in q}$ from the proof of 7.13. Claim 1: for every $j \in \omega$, (i) $j \in q$, (ii) $\Gamma \cup \mathbf{1}\Psi_j$ is $\mathbf{0}$ -closed under \vdash , (iii) for every $\sigma \in \Sigma$, $\diamond(\sigma \& \bigwedge \Psi_j) \in \Phi$, and (iv) if $j > 0$ and j is even, the bad case for j does not obtain. Proof is by induction. For the base step, only (ii) merits attention. Assume that $\Gamma \vdash \mathbf{0}\delta$. Since $\Sigma \in W$, Σ is closed under conjunction. So we can fix $\sigma \in \Sigma$ so that $\mathbf{0}\Phi_0 \cup \{\mathbf{1}\sigma\} \vdash \mathbf{0}\delta$. Using $\diamond E$, $\mathbf{0}\Phi_0 \cup \{\mathbf{0}\diamond\sigma\} \vdash \mathbf{0}\delta$. So $\mathbf{1}\diamond\sigma \in \mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1$. By one use of $\mathbf{1}\diamond E^*$, $\mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1 \vdash \mathbf{0}\diamond\sigma$. Cutting $\mathbf{0}\diamond\sigma$ we get $\mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1 \vdash \mathbf{0}\delta$. Since $\Phi_0 R\Phi_1$, the $\mathbf{0}\perp$ -Avoidance Theorem for \vdash implies that $\mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1$ is $I4_{\diamond}$ -closed under \vdash ; so $\mathbf{0}\delta \in \mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1$; so $\delta \in \Phi_0$; so $\mathbf{0}\delta \in \Gamma$. Thus 0 satisfies (ii). The rest of the proof of Claim 1, and then the rest of the argument, imitates that used to prove 7.13.

$X = I4_{\square}$: *The Special Unbox Lemma for $I4_{\square}$* . If $\Sigma R_{I4_{\square}} \Psi_0 R_{I4_{\square}} \Psi_1$ then for some Φ , $\Phi R_{I4_{\square}} \Psi_1$ and $\Sigma \subseteq \Phi$.

Set $\vdash = \vdash_{I4_{\square}} W = W_{\vdash}$, $R = R_{\vdash}$. Assume the if-clause. Let $\Gamma = \mathbf{0}\Sigma \cup \mathbf{1}\Psi_1$. Redo the definition of q and $\langle \Phi_j \rangle_{j \in q}$ from the proof of 7.11. Claim 1: for every $j \in \omega$, (i) $j \in q$, (ii) $\Gamma \cup \mathbf{0}\Phi_j$ is $\mathbf{1}$ -closed under \vdash , (iii) for every $\rho \in Fml$, if $\Gamma \cup \mathbf{0}\Phi_j \vdash \mathbf{0}\square\rho$ then $\rho \in \Psi_1$, (iv) if $j > 0$ and j is even, the bad case for j does not obtain. Proof is by induction. For the base step, only (ii) and (iii) deserve discussion. Assume that $\Gamma \vdash \mathbf{1}\delta$. Since $\Psi_1 \in W$, Ψ_1 is closed under conjunction. So we may fix $\psi \in \Psi_1$ so that $\mathbf{0}\Sigma \cup \{\mathbf{1}\psi\} \vdash \mathbf{1}\delta$. So $\mathbf{0}\Sigma \cup \{\mathbf{1}\top\} \vdash \mathbf{1}(\psi \supset \delta)$; so $\mathbf{0}\Sigma \vdash \mathbf{0}\square(\psi \supset \delta)$. Since $\Sigma \in W$, it is $\mathbf{0}$ -closed under \vdash ; so $\square(\psi \supset \delta) \in \Sigma$. By one use of $\square E$ and the fact that $\Psi_0 \neq \{\}$, $\mathbf{0}\Sigma \cup \mathbf{1}\Psi_0 \vdash \mathbf{1}(\psi \supset \delta)$. By one use of $\mathbf{1}\square I^*$ and the fact that $\Psi_0 \neq \{\}$, $\mathbf{0}\Sigma \cup \mathbf{1}\Psi_0 \vdash \mathbf{1}\square(\psi \supset \delta)$. Since $\Sigma R\Psi_0$, 7.5 implies that $\mathbf{0}\Sigma \cup \mathbf{1}\Psi_0$ is closed under \vdash ; so $\square(\psi \supset \delta) \in \Psi_0$. So $\mathbf{0}\Psi_0 \cup \mathbf{1}\Psi_1 \vdash \mathbf{0}\square(\psi \supset \delta)$. Since $\Psi_1 \neq \{\}$, one use of $\square E$ yields that $\mathbf{0}\Psi_0 \cup \mathbf{1}\Psi_1 \vdash \mathbf{1}(\psi \supset \delta)$. Since $\mathbf{1}\psi \in \mathbf{1}\Psi_1$, $\mathbf{0}\Psi_0 \cup \mathbf{1}\Psi_1 \vdash \mathbf{1}\delta$. Since $\Psi_0 R\Psi_1$, $\mathbf{0}\Psi_0 \cup \mathbf{1}\Psi_1$ is $I4_{\square}$ -closed under \vdash , (7.5 again), $\delta \in \Psi_1$. So 0 satisfies (ii). Assume that $\Gamma \vdash \mathbf{0}\square\rho$. Fix $\psi \in \Psi_1$ so that $\mathbf{0}\Sigma \cup \{\mathbf{1}\psi\} \vdash \mathbf{0}\square\rho$. So $\mathbf{0}\Sigma \cup \{\mathbf{0}\diamond\psi\} \vdash \mathbf{0}\square\rho$; so $\mathbf{0}\Sigma \vdash \mathbf{0}(\diamond\psi \supset \square\rho)$; using 4.2.(4), $\mathbf{0}\Sigma \vdash \mathbf{0}\square(\psi \supset \rho)$; so $\square(\psi \supset \rho) \in \Sigma$. As in the argument just given for (ii), $\rho \in \Psi_1$. So 0 satisfies (iii). The rest of the proof of Claim 1, and then the rest of the argument, imitates that used to prove 7.11.

$X = I5_{\diamond}$: *The Special Diamond Lemma for $I5_{\diamond}$* . If $\Phi_0 R_{I5_{\diamond}} \Phi_1$ and $\Phi_0 R_{I5_{\diamond}} \Sigma$ then for some Ψ , $\Phi_1 R_{I5_{\diamond}} \Psi$ and $\Sigma \subseteq \Psi$.

Set $\vdash = \vdash_{I5_{\diamond}} W = W_{\vdash}$, $R = R_{\vdash}$. Let $\Gamma = \mathbf{0}\Phi_1 \cup \mathbf{1}\Sigma$. Redo the definition of q and $\langle \Psi_j \rangle_{j \in q}$ from the proof of 7.13. Claim 1: for every $j \in \omega$, (i) $j \in q$, (ii) $\Gamma \cup \mathbf{1}\Psi_j$ is $\mathbf{0}$ -closed under \vdash , (iii) for every $\sigma \in \Sigma$, $\diamond(\sigma \& \wedge \Psi_j) \in \Phi$, and (iv) if $j > 0$ and j is even, the bad case for j does not obtain. Proof is by induction. For the base case, only (ii) merits attention. Assume that $\Gamma \vdash \mathbf{0}\delta$. Since Σ is closed under conjunction, we may fix a $\sigma \in \Sigma$ so that $\mathbf{0}\Phi_1 \cup \mathbf{1}\sigma \vdash \mathbf{0}\delta$. Using $\diamond E$, $\mathbf{0}\Phi_1 \cup \{\mathbf{0}\diamond\sigma\} \vdash \mathbf{0}\delta$. Since $\Phi_0 R\Sigma$, $\diamond\sigma \in \Phi_0$. So using $\mathbf{0}/\mathbf{1}\diamond$, $\mathbf{0}\Phi_0 \vdash \mathbf{1}\diamond\sigma$. Since $\Phi_0 R\Phi_1$, 7.5 implies that $\mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1$ is closed under \vdash ; so $\mathbf{1}\diamond\sigma \in \mathbf{0}\Phi_0 \cup \mathbf{1}\Phi_1$; so $\diamond\sigma \in \Phi_1$. Cutting $\mathbf{0}\diamond\sigma$, $\mathbf{0}\Phi_1 \vdash \mathbf{0}\delta$. Since $\Phi_1 \in W$, Φ_1 is $\mathbf{0}$ -closed under \vdash . So $\mathbf{0}\delta \in \mathbf{0}\Phi_1 \subseteq \Gamma$. Thus 0 satisfies (ii). The rest of the induction, and then the rest of the argument, follows the proof of 7.13.

$X = I5_{\square}$: *The Special Unbox Lemma for $I5_{\square}$* . If $\Theta R_{I5_{\square}} \Sigma$ and $\Theta R_{I5_{\square}} \Psi$ then for some Φ , $\Phi R_{I5_{\square}} \Psi$ and $\Sigma \subseteq \Phi$.

Set $\vdash = \vdash_{I5_{\square}} W = W_{\vdash}$, $R = R_{\vdash}$. Assume the if-clause. Let $\Gamma = \mathbf{0}\Sigma \cup \mathbf{1}\Psi$. Redo the definition of q and $\langle \Phi_j \rangle_{j \in q}$ from the proof of 7.11. Claim 1: for every $j \in \omega$, (i) $j \in q$, (ii) $\Gamma \cup \mathbf{0}\Phi_j$ is $\mathbf{1}$ -closed under \vdash , (iii) for every $\rho \in Fml$, if $\Gamma \cup \mathbf{0}\Phi_j \vdash \mathbf{0}\square\rho$ then $\rho \in \Psi$, (iv) if $j > 0$ and j is even, the bad case for j does not obtain. Proof is by induction. For the base step, only (ii) and (iii) deserve discussion. Assume that $\Gamma \vdash \mathbf{1}\delta$. Since $\Sigma \in W$, it is closed under conjunction. So we may fix $\psi \in \Psi$ so that $\mathbf{0}\Sigma \cup \{\mathbf{1}\psi\} \vdash \mathbf{1}\delta$. So $\mathbf{0}\Sigma \cup \{\mathbf{1}\top\} \vdash \mathbf{1}(\psi \supset \delta)$; so $\mathbf{0}\Sigma \vdash \mathbf{0}\square(\psi \supset \delta)$. Since Σ is $\mathbf{0}$ -closed under \vdash , $\square(\psi \supset \delta) \in \Sigma$. So $\mathbf{0}\Theta \cup \mathbf{1}\Sigma \vdash \mathbf{1}\square(\psi \supset \delta)$. Using $\mathbf{1}/\mathbf{0}\square$, $\mathbf{0}\Theta \cup \mathbf{1}\Sigma \vdash \mathbf{0}\square(\psi \supset \delta)$. Since $\Theta R\Sigma$, 7.5 implies that $\mathbf{0}\Theta \cup \mathbf{1}\Sigma$ is closed under \vdash ; so $\square(\psi \supset \delta) \in \Theta$. Since $\Theta R\Psi$, 7.5 implies that $\mathbf{0}\Theta \cup \mathbf{1}\Psi$ is closed under \vdash . Also, since $\mathbf{0}\Theta \cup \mathbf{1}\Psi \vdash \mathbf{0}\square(\psi \supset \delta)$ and $\Psi \neq \{\}$, one use of $\square E$ shows that $\mathbf{0}\Theta \cup \mathbf{1}\Psi \vdash \mathbf{1}(\psi \supset \delta)$. So $\mathbf{0}\Theta \cup \mathbf{1}\Psi \vdash \mathbf{1}\delta$. Since $\mathbf{0}\Theta \cup \mathbf{1}\Psi$ is closed under \vdash , $\delta \in \Psi$; so $\mathbf{1}\delta \in \mathbf{1}\Psi \subseteq \Gamma$. Thus

0 satisfies (ii). Assume that $\Gamma \vdash \mathbf{0}\Box\rho$. Since $\Psi \neq \{\}$, $\Gamma \vdash \mathbf{1}\rho$, using $\Box E$. Because 0 satisfies (ii), $\mathbf{1}\rho s \in \Gamma$; so $\rho \in \Psi$. Thus 0 satisfies (iii). The rest of the proof of Claim 1, and then the rest of the argument, imitates that used to prove 7.11.

$X = \text{Dio}_\diamond$: *The Special Lemma for IDio* $_\diamond$. If $\Sigma R_{IDio_\diamond} \Sigma_i$ for both $i \in 2$ then for each $i \in 2$ there is a Σ'_i so that for both $i \in 2$ $\Sigma_i \subseteq \Sigma'_i$, and either (A) $\Sigma R_{IDio_\diamond} \Sigma'_0 R_{IDio_\diamond} \Sigma'_1$ or (B) $\Sigma R_{IDio_\diamond} \Sigma'_1 R_{IDio_\diamond} \Sigma'_0$.

Set $\vdash = \vdash_{IDio_\diamond}$. Assume the if-clause. For both $i \in 2$ let $\Gamma_i = \mathbf{0}\Sigma \cup \mathbf{1}\Sigma_i$ and $\Gamma_{i,1-i} = \mathbf{0}\Sigma_i \cup \mathbf{1}\Sigma_{1-i}$. By 7.4, $\Gamma_i \not\vdash \mathbf{0}\perp$. Claim 1: for some $i \in 2$ $\Gamma_{i,1-i} \not\vdash \mathbf{0}\perp$. Assume otherwise. So we can fix $\sigma_{00}, \sigma_{10} \in \Sigma_0$ and $\sigma_{11}, \sigma_{01} \in \Sigma_1$ so that $\{\mathbf{0}\sigma_{00}, \mathbf{1}\sigma_{11}\} \vdash \mathbf{0}\perp$ and $\{\mathbf{0}\sigma_{01}, \mathbf{1}\sigma_{10}\} \vdash \mathbf{0}\perp$. Using $\diamond E$, $\{\mathbf{0}\sigma_{00}, \mathbf{0}\diamond\sigma_{11}\} \vdash \mathbf{0}\perp$ and $\{\mathbf{0}\sigma_{01}, \mathbf{0}\diamond\sigma_{10}\} \vdash \mathbf{0}\perp$. Since $(\sigma_{00} \& \sigma_{10}) \in \Sigma_0$, $(\sigma_{11} \& \sigma_{01}) \in \Sigma_1$, $\Sigma R_{IDio_\diamond} \Sigma_0$ and $\Sigma R_{IDio_\diamond} \Sigma_1$, and also $\diamond(\sigma_{00} \& \sigma_{10}), \diamond(\sigma_{11} \& \sigma_{01}) \in \Sigma$. By one use of *Dio* $_\diamond$,

$$\mathbf{0}\diamond(\sigma_{00} \& \sigma_{10}), \mathbf{0}\diamond(\sigma_{11} \& \sigma_{01}) \vdash \mathbf{0}(\diamond(\sigma_{00} \& \sigma_{10} \& \diamond(\sigma_{11} \& \sigma_{01})) \vee \diamond(\sigma_{11} \& \sigma_{01} \& \diamond(\sigma_{00} \& \sigma_{10}))).$$

Since Σ is closed $_{\vdash}$,

$$(\diamond(\sigma_{00} \& \sigma_{10} \& \diamond(\sigma_{11} \& \sigma_{01})) \vee \diamond(\sigma_{11} \& \sigma_{01} \& \diamond(\sigma_{00} \& \sigma_{10}))) \in \Sigma.$$

So $\diamond(\sigma_{00} \& \sigma_{10} \& \diamond(\sigma_{11} \& \sigma_{01})) \in \Sigma$ or $\diamond(\sigma_{11} \& \sigma_{01} \& \diamond(\sigma_{00} \& \sigma_{10})) \in \Sigma$. Assume the left disjunct. Since

$$\mathbf{0}\diamond(\sigma_{00} \& \sigma_{10} \& \diamond(\sigma_{11} \& \sigma_{01})) \vdash \mathbf{0}\diamond(\sigma_{00} \& \diamond\sigma_{11}),$$

$\mathbf{0}\diamond(\sigma_{00} \& \diamond\sigma_{11}) \in \Sigma$. But $\mathbf{0}(\sigma_{00} \& \diamond\sigma_{11}) \vdash \mathbf{0}\perp$; using *Trn* $_1$, $\mathbf{1}(\sigma_{00} \& \diamond\sigma_{11}) \vdash \mathbf{1}\perp$; so $\mathbf{1}(\sigma_{00} \& \diamond\sigma_{11}) \vdash \mathbf{0}\perp$ using $\mathbf{1}\perp E_1$. So $\mathbf{0}\diamond(\sigma_{00} \& \diamond\sigma_{11}) \vdash \mathbf{0}\perp$ using $\diamond E$, contrary to $\Sigma \in W$. A similar argument yields a contradiction assuming the right disjunct. Claim 1 follows. We will now assume that $\Gamma_{0,1} \not\vdash \mathbf{0}\perp$, to prove (A). A similar argument will apply assuming that $\Gamma_{1,0} \not\vdash \mathbf{0}\perp$, to prove (B).

We will construct a $q \in \omega + 1$ and a double sequence $\langle \Phi_j, \Psi_j \rangle_{j \in q}$, and prove that $q = \omega$. For each $j \in q$ we will have $\Phi_j, \Psi_j \subseteq Fml$, both finite. For what follows, let $A(j, \sigma_0, \sigma_1)$ be $\diamond(\sigma_0 \& \bigwedge \Phi_j \& \diamond(\sigma_1 \& \bigwedge \Psi_j))$. We will insure that (*) for every $\sigma_0 \in \Sigma_0$ and $\sigma_1 \in \Sigma_1$, $\mathbf{0}\Sigma \cup \{\mathbf{0}A(j, \sigma_0, \sigma_1)\} \not\vdash \mathbf{0}\perp$. Claim 2: (*) will insure that (*) $\Gamma_0 \cup \mathbf{1}\Phi_j \not\vdash \mathbf{0}\perp$ and (*) $\Gamma_{0,1} \cup \mathbf{0}\Phi_j \cup \{\mathbf{1}\Psi_j\} \not\vdash \mathbf{0}\perp$. Assume (*). Assume that $\Gamma_0 \cup \mathbf{1}\Phi_j \vdash \mathbf{0}\perp$. We can fix a $\sigma_0 \in \Sigma_0$ so that $\mathbf{0}\Sigma \cup \{\mathbf{1}(\sigma_0 \& \bigwedge \Phi_j)\} \vdash \mathbf{0}\perp$. So $\mathbf{0}\Sigma \cup \{\mathbf{1}(\sigma_0 \& \bigwedge \Phi_j \& \diamond \bigwedge \Psi_j)\} \vdash \mathbf{0}\perp$. Using $\diamond E$,

$$\mathbf{0}\Sigma \cup \left\{ \mathbf{0}\diamond \left(\sigma_0 \& \bigwedge \Phi_j \& \diamond \bigwedge \Psi_j \right) \right\} \vdash \mathbf{0}\perp$$

contrary to (*). Assume that $\Gamma_{0,1} \cup \mathbf{0}\Phi_j \cup \{\mathbf{1}\Psi_j\} \vdash \mathbf{0}\perp$. We can fix $\sigma_i \in \Sigma_i$ for both $i \in 2$ so that $\{\mathbf{0}\sigma_0, \mathbf{1}\sigma_1, \mathbf{0}\bigwedge \Phi_j, \mathbf{1}\bigwedge \Psi_j\} \vdash \mathbf{0}\perp$. Using $\mathbf{0}\&E$, $\mathbf{1}\&E$ and $\diamond E$, $\mathbf{0}((\sigma_0 \& \bigwedge \Phi_j) \& \diamond(\sigma_1 \& \bigwedge \Psi_j)) \vdash \mathbf{0}\perp$. Using *Trn* $_1$ and then $\mathbf{1}\perp E_1$, $\mathbf{1}((\sigma_0 \& \bigwedge \Phi_j) \& \diamond(\sigma_1 \& \bigwedge \Psi_j)) \vdash \mathbf{0}\perp$; using $\diamond E$, $\mathbf{0}A(j, \sigma_0, \sigma_1) \vdash \mathbf{0}\perp$, contrary to (*). Claim 2 follows.

We now imitate the proof of 7.7. Let $0 \in q$ and $\Phi_0 = \Psi_0 = \{\}$. Given $j \in \omega$, assume that $j \in q$ and for some $n \in \omega$, $j \in [4n, 4n + 3]$. Fix that n . Let $4n + 1 \in q$ and $\Psi_{4n+1} = \Psi_{4n}$. If $\Gamma_0 \cup \mathbf{1}\Phi_{4n} \not\vdash \mathbf{1}\zeta_n$ and $\Gamma_{0,1} \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \not\vdash \mathbf{0}\zeta_n$, let $4n + 2 \in q$, $\Phi_{4n+2} = \Phi_{4n+1} = \Phi_{4n}$ and $\Psi_{4n+2} = \Psi_{4n+1}$. Assume that either $\Gamma_0 \cup \mathbf{1}\Phi_{4n} \vdash \mathbf{1}\zeta_n$ or

$\Gamma_{0,1} \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \vdash \mathbf{0}\zeta_n$. Let $\Phi_{4n+1} = \Phi_{4n} \cup \{\zeta_n\}$. If ζ_n is not a disjunction let $4n + 2 \in q$, $\Phi_{4n+2} = \Phi_{4n+1}$ and $\Psi_{4n+2} = \Psi_{4n+1}$. Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. If for some $i \in 2$, for every $\sigma_0 \in \Sigma_0$ and $\sigma_1 \in \Sigma_1$ we have $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n+1, (\sigma_0 \& \varphi_i), \sigma_1)\} \not\vdash \mathbf{0}\perp$, fix such an i and let $4n + 2 \in q$, $\Phi_{4n+2} = \Phi_{4n+1} \cup \{\varphi_i\}$, and $\Psi_{4n+2} = \Psi_{4n+1}$. Otherwise (the bad case for $4n + 2$) let $q = 4n + 2$, and we are done. Now assume that $4n + 2 \in q$ (and so the bad case for $4n + 2$ did not obtain). Let $4n + 3 \in q$ and $\Phi_{4n+3} = \Phi_{4n+2}$. If $\Gamma_{0,1} \cup \mathbf{0}\Phi_{4n+2} \cup \mathbf{1}\Psi_{4n+2} \vdash \mathbf{1}\zeta_n$, let $\Phi_{4n+3} = \Phi_{4n+4} = \Phi_{4n+2}$. Assume that $\Gamma_{0,1} \cup \mathbf{0}\Phi_{4n+2} \cup \mathbf{1}\Psi_{4n+2} \vdash \mathbf{1}\zeta_n$. Let $\Psi_{4n+3} = \Psi_{4n+2} \cup \{\zeta_n\}$. If ζ_n is not a disjunction let $4n + 4 \in q$, $\Psi_{4n+4} = \Psi_{4n+3}$ and $\Phi_{4n+4} = \Phi_{4n+3}$. Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. If for some $i \in 2$ $\Gamma_{0,1} \cup \mathbf{0}\Phi_{4n+2} \cup \mathbf{1}\Psi_{4n+2} \cup \{\mathbf{1}\varphi_i\} \not\vdash \mathbf{0}\perp$, fix such an i ; let $4n + 4 \in q$, $\Psi_{4n+4} = \Psi_{4n+1} \cup \{\varphi_i\}$ and $\Phi_{4n+4} = \Phi_{4n+3}$. Otherwise (the bad case for $4n + 4$) let $q = 4n + 4$, and we are finished.

Claim 3: for every $j \in \omega$, (i) $j \in q$, (ii) (*) is true, and (iii) if $j > 0$ and j is even then the bad case for j does not obtain. Proof by induction on j . The base-step. $\mathbf{0}$ satisfies (i) by stipulation and (iii) vacuously. Consider any $\sigma_i \in \Sigma_i$ for both $i \in 2$. Assume that $\mathbf{0}\Sigma \cup \{\mathbf{0}\diamond(\sigma_0 \& \diamond\sigma_1)\} \vdash \mathbf{0}\perp$. Using $\diamond I$, $\mathbf{0}\Sigma \cup \{\mathbf{1}(\sigma_0 \& \diamond\sigma_1)\} \vdash \mathbf{0}\perp$; so $\mathbf{0}\Sigma \cup \{\mathbf{1}\top\} \vdash \mathbf{1}\neg(\sigma_0 \& \diamond\sigma_1)$; using $\square I$, $\mathbf{0}\Sigma \vdash \mathbf{0}\square\neg(\sigma_0 \& \diamond\sigma_1)$; so $\square\neg(\sigma_0 \& \diamond\sigma_1) \in \Sigma$. So $\neg(\sigma_0 \& \diamond\sigma_1) \in \Sigma_0$. Since $\sigma_0 \in \Sigma_0$, $\neg\diamond\sigma_1 \in \Sigma_0$. So $\mathbf{0}\neg\diamond\sigma_1 \in \Gamma_{0,1}$; so $\Gamma_{0,1} \vdash \mathbf{0}\square\neg\sigma_1$; so $\Gamma_{0,1} \vdash \mathbf{1}\neg\sigma_1$. Since $\sigma_1 \in \Sigma_1$, $\mathbf{1}\sigma_1 \in \Gamma_{0,1}$. So $\Gamma_{0,1} \vdash \mathbf{0}\perp$, contrary to assumption. So for $j = 0$, (ii) follows.

The induction step. Given j , assume the obvious IH. Fix $n \in \omega$ so that $4n \leq j \leq 4n + 3$.

Case 1: $j = 4n$. By stipulation $4n + 1 \in q$, i.e. $4n + 1$ satisfies (i) and vacuously satisfies (iii). If $\Gamma_0 \cup \mathbf{1}\Phi_{4n} \not\vdash \mathbf{1}\zeta_n$ and $\Gamma_{0,1} \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \not\vdash \mathbf{0}\zeta_n$, the IH implies that $j+1$ satisfies (ii). Assume that either (a) $\Gamma_0 \cup \mathbf{1}\Phi_{4n} \vdash \mathbf{1}\zeta_n$ or (b) $\Gamma_{0,1} \cup \mathbf{0}\Phi_{4n} \cup \mathbf{1}\Psi_{4n} \vdash \mathbf{0}\zeta_n$. Given $\sigma_i \in \Sigma_i$ for both $i \in 2$, assume that $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 1, \sigma_0, \sigma_1)\} \vdash \mathbf{0}\perp$. Assuming (a), fix $\tau \in \Sigma_0$ so that $\mathbf{0}\Sigma \cup \{\mathbf{1}\tau, \mathbf{1}\wedge \Phi_{4n}\} \vdash \mathbf{1}\zeta_n$. Let σ'_0 be $(\tau \& \sigma_0)$. So $\mathbf{0}\Sigma \cup \{\mathbf{1}(\sigma'_0 \& \wedge \Phi_{4n})\} \vdash \mathbf{1}(\sigma_0 \& \Phi_{4n+1})$; so

$$\mathbf{0}\Sigma \cup \left\{ \mathbf{1}(\sigma'_0 \& \wedge \Phi_{4n} \& \diamond(\sigma_1 \& \wedge \Psi_{4n})) \right\} \vdash \mathbf{1}(\sigma_0 \& \Phi_{4n+1} \& \diamond(\sigma_1 \& \wedge \Psi_{4n+1})).$$

Using $\diamond I$ followed by $\diamond E$, $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n, \sigma'_0, \sigma_1)\} \vdash \mathbf{0}A(4n + 1, \sigma_0, \sigma_1)$. So $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n, \sigma_0, \sigma_1)\} \vdash \mathbf{0}\perp$. Assuming (b), fix $\tau_i \in \Sigma_i$ for both $i \in 2$ so that $\{\mathbf{0}\tau_0, \mathbf{1}\tau_1, \mathbf{0}\wedge \Phi_{4n}, \mathbf{1}\wedge \Psi_{4n}\} \vdash \mathbf{0}\zeta_n$. Let σ'_i be $(\tau_i \& \sigma_i)$ for both $i \in 2$; so $\sigma'_i \in \Sigma_i$. So the following follow:

$$\begin{aligned} &\mathbf{0}\sigma'_0, \mathbf{1}\sigma'_1, \mathbf{0}\wedge \Phi_{4n}, \mathbf{1}\wedge \Psi_{4n} \vdash \mathbf{0}(\sigma_0 \& \wedge \Phi_{4n+1} \& \diamond(\sigma_1 \& \wedge \Psi_{4n+1})). \\ &\mathbf{0}(\sigma'_0 \& \wedge \Phi_{4n}), \mathbf{1}(\sigma'_1 \& \wedge \Psi_{4n}) \vdash \mathbf{0}(\sigma_0 \& \wedge \Phi_{4n+1} \& \diamond(\sigma_1 \& \wedge \Psi_{4n+1})). \\ &\mathbf{0}(\sigma'_0 \& \wedge \Phi_{4n}), \mathbf{0}\diamond(\sigma'_1 \& \wedge \Psi_{4n}) \vdash \mathbf{0}(\sigma_0 \& \wedge \Phi_{4n+1} \& \diamond(\sigma_1 \& \wedge \Psi_{4n+1})). \\ &\mathbf{1}(\sigma'_0 \& \wedge \Phi_{4n}), \mathbf{0}\diamond(\sigma'_1 \& \wedge \Psi_{4n}) \vdash \mathbf{1}(\sigma_0 \& \wedge \Phi_{4n+1} \& \diamond(\sigma_1 \& \wedge \Psi_{4n+1})), \end{aligned}$$

the last using Trn_1 . So using $\diamond I$ followed by $\diamond E$, $\mathbf{0}A(4n, \sigma'_0, \sigma'_1) \vdash \mathbf{0}A(4n + 1, \sigma_0, \sigma_1)$. So $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n, \sigma_0, \sigma_1)\} \vdash \mathbf{0}\perp$. In both cases we have contradicted (ii) of the IH. So $4n + 1$ satisfies (ii).

Case 2: $j = 4n + 1$. If ζ_n is not a disjunction, clearly $4n + 2$ satisfies (i)-(iii). Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. Claim: either for every $\sigma_0 \in \Sigma_0$ and $\sigma_1 \in \Sigma_1$ $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 1, (\sigma_0 \& \varphi_0), \sigma_1)\} \not\vdash \mathbf{0}\perp$, or for every $\sigma_0 \in \Sigma_0$ and $\sigma_1 \in \Sigma_1$

$\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 1, (\sigma_0 \& \varphi_1), \sigma_1)\} \not\vdash \mathbf{0}\perp$. Assume otherwise. Fix $\sigma_{00}, \sigma_{01} \in \Sigma_0$ and $\sigma_{10}, \sigma_{11} \in \Sigma_1$ so that $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 1, (\sigma_{00} \& \varphi_0), \sigma_{10})\} \vdash \mathbf{0}\perp$ and $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 1, (\sigma_{01} \& \varphi_1), \sigma_{11})\} \vdash \mathbf{0}\perp$. Let σ_0 be $(\sigma_{00} \& \sigma_{01})$ and σ_1 be $(\sigma_{10} \& \sigma_{11})$; so $\sigma_0 \in \Sigma_0$ and $\sigma_1 \in \Sigma_1$. Let B be $\diamond(\sigma_1 \& \wedge \Psi_j)$. By some deductive work,

$$\mathbf{0}A(4n + 1, \sigma_0, \sigma_1) \vdash \mathbf{0}\diamond \left((\varphi_0 \& \sigma_0 \& \wedge \Phi_j \& B) \vee (\varphi_1 \& \sigma_0 \& \wedge \Phi_j \& B) \right).$$

Using 4.2.(5),

$$\mathbf{0}\diamond \left((\varphi_0 \& \sigma_0 \& \wedge \Phi_j \& B) \vee (\varphi_1 \& \sigma_0 \& \wedge \Phi_j \& B) \right) \vdash \mathbf{0}(A(j, (\sigma_0 \& \varphi_0), \sigma_1) \vee A((\sigma_0 \& \varphi_1), \sigma_1)).$$

For each $i \in 2$ a little work shows that

$$\mathbf{0}A(4n + 1, (\sigma_0 \& \varphi_i), \sigma_1) \vdash \mathbf{0}A(4n + 1, (\sigma_{0i} \& \varphi_i), \sigma_{1i}),$$

and so $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 1, (\sigma_0 \& \varphi_i), \sigma_1)\} \vdash \mathbf{0}\perp$. So $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 1, \sigma_0, \sigma_1)\} \vdash \mathbf{0}\perp$. But we have shown that $4n + 1$ satisfies (ii), for a contradiction. The claim follows. Thus $4n + 2$ satisfies (ii), and with that, (i) and (iii) as well.

Case 3: $j = 4n + 2$. By stipulation $4n + 3$ satisfies (i), and vacuously satisfies (iii). If $\Gamma_{0,1} \cup \mathbf{0}\Phi_{4n+2} \cup \mathbf{1}\Psi_{4+2n} \not\vdash \mathbf{1}\zeta_n$, the IH implies that $j + 1$ satisfies (ii). Assume that $\Gamma_{0,1} \cup \mathbf{0}\Phi_{4n+2} \cup \mathbf{1}\Psi_{4n+2} \vdash \mathbf{1}\zeta_n$. Given $\sigma_i \in \Sigma_i$ for both $i \in 2$, assume that $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 3, \sigma_0, \sigma_1)\} \vdash \mathbf{0}\perp$. Fix $\tau_i \in \Sigma_i$ for both $i \in 2$ so that $\{\mathbf{0}\tau_0, \mathbf{1}\tau_1, \mathbf{0}\wedge \Phi_{4n+2}, \mathbf{1}\wedge \Psi_{4n+2}\} \vdash \mathbf{1}\zeta_n$. Let σ'_i be $(\tau_i \& \sigma_i)$ for both $i \in 2$; so $\sigma'_i \in \Sigma_i$. So the following follow:

$$\begin{aligned} &\mathbf{0}\sigma'_0, \mathbf{1}\sigma'_1, \mathbf{0}\wedge \Phi_{4n+2}, \mathbf{1}\wedge \Psi_{4n+2} \vdash \mathbf{1}(\sigma_1 \& \wedge \Psi_{4n+3}). \\ &\mathbf{0}\sigma'_0, \mathbf{1}\sigma'_1, \mathbf{0}\wedge \Phi_{4n+2}, \mathbf{1}\wedge \Psi_{4n+2} \vdash \mathbf{0}(\sigma_0 \& \wedge \Phi_{4n+3} \& \diamond(\sigma_1 \& \wedge \Psi_{4n+3})). \\ &\mathbf{0}(\sigma'_0 \& \wedge \Phi_{4n+2}), \mathbf{1}(\sigma'_1 \& \wedge \Psi_{4n+2}) \vdash \mathbf{0}(\sigma_0 \& \wedge \Phi_{4n+3} \& \diamond(\sigma_1 \& \wedge \Psi_{4n+3})). \\ &\mathbf{0}(\sigma'_0 \& \wedge \Phi_{4n+2}), \mathbf{0}\diamond(\sigma'_1 \& \wedge \Psi_{4n+2}) \vdash \mathbf{0}(\sigma_0 \& \wedge \Phi_{4n+3} \& \diamond(\sigma_1 \& \wedge \Psi_{4n+3})). \\ &\mathbf{1}(\sigma'_0 \& \wedge \Phi_{4n+2}), \mathbf{0}\diamond(\sigma'_1 \& \wedge \Psi_{4n+2}) \vdash \mathbf{1}(\sigma_0 \& \wedge \Phi_{4n+3} \& \diamond(\sigma_1 \& \wedge \Psi_{4n+3})), \end{aligned}$$

Using $\diamond I$ followed by $\diamond E$, $\mathbf{0}A(4n + 2, \sigma'_0, \sigma'_1) \vdash \mathbf{0}A(4n + 3, \sigma_0, \sigma_1)$. So $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 2, \sigma_0, \sigma_1)\} \vdash \mathbf{0}\perp$, contrary to $4n + 2$ satisfying (ii). So $4n + 3$ satisfies (ii).

Case 4: $j = 4n + 3$. If ζ_n is not a disjunction, clearly $4n + 4$ satisfies (i)-(iii). Assume that ζ_n is $(\varphi_0 \vee \varphi_1)$. Claim: either for every $\sigma_0 \in \Sigma_0$ and $\sigma_1 \in \Sigma_1$ $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 3, \sigma_0, (\sigma_1 \& \varphi_0))\} \not\vdash \mathbf{0}\perp$ or for every $\sigma_0 \in \Sigma_0$ and $\sigma_1 \in \Sigma_1$ $\mathbf{0}\Sigma \cup \{\mathbf{0}A(4n + 3, \sigma_0, (\sigma_1 \& \varphi_1))\} \not\vdash \mathbf{0}\perp$. Assume otherwise; I leave details to the reader. The crucial points: for appropriately defined σ_0 and σ_1 , taking B_i to be $(\varphi_i \& \sigma_1 \& \wedge \Psi_{4n+3})$ for $i \in 2$,

$$\begin{aligned} &\mathbf{0}A(4n + 3, \sigma_0, \sigma_1) \vdash \mathbf{0}\diamond (\sigma_0 \& \wedge \Phi_{4n+3} \& \diamond(B_0 \vee B_1)). \\ &\mathbf{0}\diamond (\sigma_0 \& \wedge \Phi_{4n+3} \& \diamond(B_0 \vee B_1)) \vdash \mathbf{0}(A(4n + 2, \sigma_0, (\varphi_0 \& \sigma_1)) \vee A(4n + 2, \sigma_0, (\varphi_1 \& \sigma_1))). \end{aligned}$$

The second of these uses 4.2.(5) twice. Thus $4n + 4$ satisfies (ii), and with that (i) and (iii) as well. Claim 3 follows. Thus $q = \omega$.

Let $\Sigma'_0 = \bigcup_{j \in \omega} \Phi_j$, $\Sigma'_1 = \bigcup_{j \in \omega} \Psi_j$, $\Gamma'_0 = \mathbf{0}\Sigma \cup \mathbf{1}\Sigma'_0$, and $\Gamma'_{0,1} = \mathbf{0}\Sigma'_0 \cup \mathbf{1}\Sigma'_1$. Check that for each $n \in \omega$, if $\zeta_n \in \Sigma_0$ then $\zeta_n \in \Phi_{4n+1}$, and if $\zeta_n \in \Sigma_1$ then $\zeta_n \in \Psi_{4n+3}$. We have insured that Γ'_0 and $\Gamma'_{0,1}$ avoid $\mathbf{0}\perp$. So Σ'_0 and Σ'_1 are as required.

After all the work for the previous lemma, the next is surprisingly easy.

$X = \text{Dio}\Box$: *The Special Lemma for IDio* \Box . If for both $i \in 2$ $\Sigma \subseteq \Sigma_i$ and $\Sigma_i R_{IDio}\Box \Theta_i$ then for both $i \in 2$ there is a $\Theta_i^+ \supseteq \Theta_i$ so that either $\Theta_0^+ R_{IDio}\Box \Theta_1^+$ or $\Theta_1^+ R_{IDio}\Box \Theta_0^+$.

Set $\vdash = \vdash_{IDio}\Box$. Assume the if-clause. For $i \in 2$ let $\Gamma_i = \mathbf{0}\Theta_0 \cup \mathbf{1}\Theta_1$. Claim 1: either $\Gamma_0 \not\vdash \mathbf{0}\perp$ or $\Gamma_1 \not\vdash \mathbf{0}\perp$. Assume otherwise. There are $\sigma_{0i}, \sigma_{1i} \in \Theta_i$ for both $i \in 2$ so that $\mathbf{0}\sigma_{00}, \mathbf{1}\sigma_{11} \vdash \mathbf{0}\perp$ and $\mathbf{0}\sigma_{01}, \mathbf{1}\sigma_{10} \vdash \mathbf{0}\perp$. So $\mathbf{0}(\sigma_{00}\&\sigma_{10}), \mathbf{1}(\sigma_{11}\&\sigma_{01}) \vdash \mathbf{0}\perp$ and $\mathbf{0}(\sigma_{11}\&\sigma_{01}), \mathbf{1}(\sigma_{00}\&\sigma_{10}) \vdash \mathbf{0}\perp$. So $\mathbf{0}(\sigma_{00}\&\sigma_{10}), \mathbf{1}\top \vdash \mathbf{1}\neg(\sigma_{11}\&\sigma_{01})$ and $\mathbf{0}(\sigma_{11}\&\sigma_{01}), \mathbf{1}\top \vdash \mathbf{1}\neg(\sigma_{00}\&\sigma_{10})$. Using $\Box I$, $\mathbf{0}(\sigma_{00}\&\sigma_{10}) \vdash \mathbf{0}\Box\neg(\sigma_{11}\&\sigma_{01})$ and $\mathbf{0}(\sigma_{11}\&\sigma_{01}) \vdash \mathbf{0}\Box\neg(\sigma_{00}\&\sigma_{10})$. Using *Trn*₁, $\mathbf{1}(\sigma_{00}\&\sigma_{10}) \vdash \mathbf{1}\Box\neg(\sigma_{11}\&\sigma_{01})$ and $\mathbf{1}(\sigma_{11}\&\sigma_{01}) \vdash \mathbf{1}\Box\neg(\sigma_{00}\&\sigma_{10})$; fix \mathcal{D}_2 and \mathcal{D}_3 to witness these, respectively. Let ψ be $(\Box\neg(\sigma_{11}\&\sigma_{01}) \vee \Box\neg(\sigma_{00}\&\sigma_{10}))$. Consider the following deduction.

$$\frac{\frac{\frac{[v_2:\mathbf{1}(\sigma_{00}\&\sigma_{10})] \quad [v_3:\mathbf{1}(\sigma_{11}\&\sigma_{01})]}{\mathbf{1}\Box\neg(\sigma_{11}\&\sigma_{01})} \mathcal{D}_2 \quad \frac{\mathbf{1}\Box\neg(\sigma_{00}\&\sigma_{10})}{\mathbf{1}\Box\neg(\sigma_{00}\&\sigma_{10})} \mathcal{D}_3 \quad \frac{[v_0:\mathbf{0}\Box\neg(\sigma_{00}\&\sigma_{10})]_{\mathbf{0}\vee I}}{\psi} \quad \frac{[v_1:\mathbf{0}\Box\neg(\sigma_{11}\&\sigma_{01})]_{\mathbf{0}\vee I}}{\psi_{Dio}\Box}}{\psi^{v_0, v_1, v_2, v_3}}$$

Since Σ is closed $_{\vdash}$, $\psi \in \Sigma$. Since Σ is \vee -complete, either $\Box\neg(\sigma_{11}\&\sigma_{01}) \in \Sigma$ or $\Box\neg(\sigma_{00}\&\sigma_{10}) \in \Sigma$. Assume that $\Box\neg(\sigma_{11}\&\sigma_{01}) \in \Sigma$. So $\Box\neg(\sigma_{11}\&\sigma_{01}) \in \Sigma_1$; so $\neg(\sigma_{11}\&\sigma_{01}) \in \Theta_1$; since $\sigma_{01}, \sigma_{11} \in \Theta_1 \in W_{\vdash}$ we have a contradiction. Similarly assuming that $\Box\neg(\sigma_{00}\&\sigma_{10}) \in \Sigma$. Claim 1 follows. Fix $j \in 2$ so that $\Gamma_j \not\vdash \mathbf{0}\perp$. By 7.7 (the Avoidance Theorem for \vdash) there are $\Theta_{i \in 2}^+$ as desired.

For the blended intuitionistic logics (most prominently IT, IB, I4, I5, IB, IS4, IS5, IS4.3), the proofs just combine that proofs for their ingredient logics. Similarly for the classical logics CD, CT, CKB, CK4, CK5, CB, CS4, CS5, CS4.3. \square

12.2 Completeness Theorems

For ‘X’ schematic for the above names, \vdash_X is complete (i.e. inference-complete) with respect to X-models.

Proof Replace ‘IK’ by ‘X’ in the proof of 7.18 and use 12.1. \square

12.3 Observations

Let $X \in \{\text{IGL}\Box, \text{IGL}\Diamond, \text{CGL}\}$, and $\vdash_X^* = \{\langle \Gamma, \chi \rangle \mid \langle \Gamma, \chi \rangle \text{ is } X\text{-valid}\}$. (1) For any signature S , \vdash_X^* is not finitary. (2) For a class C of frames, let \mathcal{M} be a C -model iff \mathcal{M} is an IK-model and $F^{\mathcal{M}} \in C$. Let $\vdash^C = \{\langle \Gamma, \chi \rangle \mid \text{for every } C\text{-model } \mathcal{M}, \langle \Gamma, \chi \rangle \text{ is } \mathcal{M}\text{-valid}\}$. If $S \neq \{\}$ then \vdash_X is not complete (i.e. inference-complete) with respect to C -models, i.e. $\vdash_X \neq \vdash^C$.

Proof For (1), it suffices to consider $S = \{\}$. Let θ be $\Diamond\Box\perp$, and $\Sigma = \{\Box^n(\theta \supset \Diamond\theta) \mid n \in \omega\}$. The well-cappedness of X-frames insures that no pointed X-model that makes Σ true; so $\mathbf{0}\Sigma \vdash_X^* \mathbf{0}\perp$. But for any finite $\Sigma' \subseteq \Sigma$, it is easy to construct a pointed X-model that makes Σ' true; so $\mathbf{0}\Sigma' \not\vdash_X^* \mathbf{0}\perp$, proving the claim.

For (2), fix $\pi \in S$. Given a class C of frames, assume that $(*) \vdash_X = \vdash^C$. Claim: every $F \in C$ is an X -frame. Consider an $F \in C$. For $X=IGL_{\square}$: since $\vdash_{IGL_{\square}} \mathbf{0}(\square(\square\pi \supset \pi) \supset \square\pi)$, by $(*) \vdash^C \mathbf{0}(\square(\square\pi \supset \pi) \supset \square\pi)$; so $F \models \mathbf{0}(\square(\square\pi \supset \pi) \supset \square\pi)$; by 11.14, F is an IGL_{\square} -frame. A similar argument applies for $X=IGL_{\diamond}$ using $\mathbf{0}(\diamond\pi \supset \diamond(\pi \& \neg \diamond\pi))$. For $X=CGL$, conjoin $(\pi \vee \neg \pi)$ to either of the above formulas to show that F is a CGL -frame. Thus $\vdash_X^* \subseteq \vdash^C$. So $\mathbf{0}\Sigma \vdash^C \mathbf{0}\perp$. By $(*) \mathbf{0}\Sigma \vdash_X \mathbf{0}\perp$. Since \vdash_X is finitary, we may fix a finite $\Sigma' \subseteq \Sigma$ so that $\mathbf{0}\Sigma' \vdash_X \mathbf{0}\perp$. By the Soundness Theorem for X , $\mathbf{0}\Sigma' \vdash_X^* \mathbf{0}\perp$. But again, this is false. \square

12.4 Theorem

\vdash_{CGL} is weakly (i.e. formula-) complete (with respect to CGL -models).

The proof of the weak completeness of no-step classical GL given in [1] can be modified to prove this.

Proof Set $\vdash = \vdash_{CGL}$. Given $\chi \in Mfml$, assume that $\not\vdash \chi$. If $\chi \in \mathbf{0}Fml$, by Lindenbaum’s Lemma for \vdash we may fix a $\Phi \in W_{\vdash}$ with $\mathbf{0}^{-1}\chi \notin \Phi$. We can “carve out” a finite $W \subseteq W_{\vdash}$ so that $\Phi \in W$, the restriction of F_{\vdash} to W , call it F , is an $CK4$ -frame, and the restriction of \mathcal{M}_{\vdash} to F , call it \mathcal{M} , is such that $\mathcal{M}, \Phi \not\models \mathbf{0}^{-1}\chi$. If $\chi \in \mathbf{1}Fml$, by the Unbox Lemma $\Phi, \Psi \in W_{\vdash}$ with $\mathbf{1}^{-1}\chi \notin \Psi$ and $\Phi R_{\vdash} \Psi$. We can “carve out” a finite $W \subseteq W_{\vdash}$ so that $\Phi, \Psi \in W$, the restriction of F_{\vdash} to W , call it F , is an $CK4$ -frame. In both cases, classicality makes the left- and right-completeness of W is trivial. Since W is finite, R is well-capped. The restriction of \mathcal{M}_{\vdash} to F , call it \mathcal{M} , is such that $\mathcal{M}, \Psi \not\models \mathbf{1}^{-1}\chi$. Details are left to the reader. \square

12.5 Conjecture

For $X \in \{IGL_{\square}, IGL_{\diamond}\}$, \vdash_X is weakly (i.e. formula-) complete (with respect to X -models).

Note that the technique used for 12.4 cannot be straightforwardly applied to \vdash_X . The sticking-point: getting a finite W for which the restriction of F_{\vdash_X} to W is left- and right-complete.

13 Looking Ahead

Assign modal depth to a rule in an obvious way – the maximum depth of modal operators in the schematic presentation of that rule. So the first four rules introduced in Section 9.1 have modal depth 0, while the “no step” rules considered in the first four cases from Section 10.3 have modal depth 1. The remaining rules introduced in Section 9.1 have modal depth 1, while the “no step” rules considered in the remaining cases from Section 10.3 have modal depth 2. So using the step-marker $\mathbf{1}$ in addition to $\mathbf{0}$ allows us to formulate “one step” rules that decrease by 1 the modal depth of the “no step” rules considered above. Similarly for “no step” rules that we have not considered.

In this paper, we have not fully “lifted the hood” on the rules of depth greater than 0. Doing that will involve adding another step-marker **2**, which would allow us to lower modal depth by 2. Carrying this idea to its obvious extreme will lead us to consider languages which have a step-marker n for each $n \in \omega$. We could then extend \Rightarrow_X for replacements for ‘X’ considered above, to such languages. Work to be done!

Appendix: More About Intuitionistic Modal Logics

For A.1 and A.2, let $\vdash = \vdash_{IY}$ for $\vdash_{IGL_\diamond} \not\subseteq \vdash_{IY}$ and $\vdash_{IGL_\square} \not\subseteq \vdash_{IY}$.

A.1 Observation

\vdash has the disjunction property, i.e. for any $\varphi_{i \in 2} \in Fml$, if $\vdash \mathbf{0}(\varphi_0 \vee \varphi_1)$ then $\vdash \mathbf{0}\varphi_0$ or $\vdash \mathbf{0}\varphi_1$.

Proof Assume the if-clause. Assume that for both $i \in 2 \not\vdash \mathbf{0}\varphi_i$. By Completeness for IY we may fix an IY-model \mathcal{M}_i and u_i so that $\mathcal{M}_i, u_i \not\vdash \varphi_i$. Let $F^{\mathcal{M}_i} = \langle W_i, R_i, \sqsubseteq_i \rangle$. Without loss of generality we may assume that $W_0 \cap W_1 = \{\}$ and $0 \notin W_0 \cup W_1$. Let $W = \{0\} \cup W_0 \cup W_1$, $R = R_0 \cup R_1 \cup \{(0, 0)\}$, and

$$\sqsubseteq = \{\langle 0, u \rangle \mid u_0 \sqsubseteq_0 u \text{ or } u_1 \sqsubseteq_0 u\} \cup \sqsubseteq_0 \cup \sqsubseteq_1 .$$

Check that $F = \langle W, R, \sqsubseteq \rangle$ is an IY-frame.¹⁴ For $v \in W$ and $\pi \in S$, let

$$\mathcal{V}(v, \pi) = \begin{cases} \mathcal{V}_i(v, \pi) & \text{if } v \in W_i, i \in 2, \\ 0 & \text{if } v = 0. \end{cases}$$

Let $\mathcal{M} = \langle F, \mathcal{V} \rangle$. For both $i \in 2$ and any $u \in W_i$ and v, uRv iff $uR_i v$, and also $u \sqsubseteq v$ iff $u \sqsubseteq_i v$. So $\mathcal{M}, u_i \not\vdash \varphi_i$. By the if-clause and the soundness of \vdash with respect to IY-models, $\mathcal{M}, u \not\vdash (\varphi_0 \vee \varphi_1)$. Fix i so that $\mathcal{M}, u \not\vdash \varphi_i$. By the Persistence Lemma $\mathcal{M}, u_i \not\vdash \varphi_i$, a contradiction. The then-clause follows. \square

If 12.5 is true, the previous argument applies for $Y = GL_\diamond$ and $Y = GL_\square$.

A.2 Observation

Consider any $\varphi \in Fml$. (1) If $\not\vdash \mathbf{0}\diamond\top$, then $\mathbf{0}\neg\square\varphi \vdash \mathbf{0}\diamond\neg\varphi$ iff $\vdash \mathbf{0}\neg\neg\square\varphi$. (2) Assume that the replacement for ‘Y’ is constructed using ‘ T_\diamond ’ and any of the above names. Then $\mathbf{0}\neg\square\varphi \vdash \mathbf{0}\diamond\neg\varphi$ iff $\mathbf{0}\neg\square\varphi \vdash \mathbf{0}\neg\varphi$. (3) Assume that the replacement for ‘Y’ is constructed using ‘D’, ‘ T_\square ’, ‘T’, ‘ 4_\diamond ’, ‘ 4_\square ’, ‘4’ or ‘Dio \square ’. Then $\mathbf{0}\neg\square\varphi \vdash \mathbf{0}\diamond\neg\varphi$ iff either $\vdash \mathbf{0}\diamond\neg\varphi$ or $\vdash \mathbf{0}\neg\neg\square\varphi$.

Proof For (1), assume that $\not\vdash \mathbf{0}\diamond\top$; so IY-frames can have dead-ends. Right to left is trivial. Assume the left-side. Assume that $\not\vdash \mathbf{0}\neg\neg\square\varphi$. By Completeness for IY we may fix an IY-model \mathcal{M} and u so that $\mathcal{M}, u \not\vdash \neg\neg\square\varphi$. Let $F^{\mathcal{M}} = \langle W, R, \sqsubseteq \rangle$

¹⁴If $\not\vdash \mathbf{0}\top$ we didn’t need to have $OR0$. If Y contains Dio \square , to show that F is an IDio \square -frame we use this fact: if $u_i R_i v$ for $i \in 2$, then OR^+v .

and $\mathcal{V} = \mathcal{V}^{\mathcal{M}}$. So we may fix a $u' \sqsupseteq u$ so that $\mathcal{M}, u' \models \neg \Box \varphi$. Without loss of generality assume that $0 \notin W$. Let $W' = \{0\} \cup W$, $\sqsubseteq' = \sqsubseteq \cup \{(0, v) \mid u' \sqsubseteq v\}$, and $F' = \langle W', R, \sqsubseteq' \rangle$. For $v \in W'$ and $\pi \in S$, let

$$\mathcal{V}'(v, \pi) = \begin{cases} \mathcal{V}(v, \pi) & \text{if } v \in W, \\ 0 & \text{if } v = 0. \end{cases}$$

Let $\mathcal{M}' = \langle F', \mathcal{V}' \rangle$. By an easy induction, (*) for every $v \in W$ and every $\psi \in Fml$, $\mathcal{M}', v \models \psi$ iff $\mathcal{M}, v \models \psi$. If $\mathcal{M}', 0 \models \Box \varphi$ then $\mathcal{M}', u' \models \Box \varphi$, and then by (*) $\mathcal{M}, u' \models \Box \varphi$, a contradiction. So $\mathcal{M}', 0 \not\models \Box \varphi$. Assume that $0 \sqsubseteq' v$ and $\mathcal{M}', v \models \Box \varphi$; so $v \neq 0$, and so $v \sqsupseteq u'$; so by (*) $\mathcal{M}, v \models \Box \varphi$, a contradiction. So $\mathcal{M}', 0 \models \neg \Box \varphi$. Since 0 is a dead-end in F' , $\mathcal{M}', 0 \not\models \Diamond \neg \varphi$, contrary to the left-side. (1) follows.

For (2) going right to left, use 10.1(4). Assume the left-side. Assume that $0 \neg \Box \varphi \not\models \Box \neg \varphi$. By Completeness for IY we may fix an IY-model \mathcal{M} and u so that $\mathcal{M}, u \models \neg \Box \varphi$ but $\mathcal{M}, u \not\models \Box \neg \varphi$. For W, R, \sqsubseteq and \mathcal{V} as above, let $R' = R \cup \{(0, u)\}$ and let W', \sqsubseteq' and \mathcal{V}' be as above; let $\mathcal{M}' = \langle F', \mathcal{V}' \rangle$. (*) carries over from the previous paragraph. As above, $\mathcal{M}', 0 \models \neg \Box \varphi$. For any v , if $0R'v$ then $v = u$; so $\mathcal{M}, u \not\models \Diamond \neg \varphi$, contrary to the left-side. (2) follows.

For (3), assume that 'Y' is replaced appropriately. Right to left is trivial. Assume the left side. By (1) we lose no generality by assuming that $\vdash \mathbf{0} \Diamond \top$. Assume that $\not\models \mathbf{0} \neg \neg \Box \varphi$ and $\not\models \mathbf{0} \Diamond \neg \varphi$. By Completeness for IY we may fix IY-models $\mathcal{M}_{i \in 2}$ and $u_{i \in 2}$ so that $\mathcal{M}_0, u_0 \not\models \Diamond \neg \varphi$ and $\mathcal{M}_1, u_1 \not\models \neg \neg \Box \varphi$. Let $F^{\mathcal{M}_i} = \langle W_i, R_i, \sqsubseteq_i \rangle$ and $\mathcal{V}_i = \mathcal{V}^{\mathcal{M}_i}$ for both $i \in 2$. Without loss of generality let $W_0 \cap W_1 = \{ \}$ and $0, 1 \notin W_0 \cup W_1$. As above we may fix a $u'_1 \sqsupseteq_1 u_1$ so that $\mathcal{M}_1, u'_1 \models \neg \Box \varphi$; fix v_1 so that $u'_1 R_1^+ v_1$ and $\mathcal{M}_1, v_1 \not\models \varphi$. Since $\vdash \mathbf{0} \Diamond \top$, we may fix a v so that $u_0 R_0 v$. Since $\mathcal{M}_0, v \not\models \neg \varphi$, we may fix a $v_0 \sqsupseteq_0 v$ so that $\mathcal{M}_0, v_0 \models \varphi$. Let $W = \{0, 1\} \cup W_0 \cup W_1$,

$$\begin{aligned} \sqsubseteq &= \{ \langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 1 \rangle \} \cup \sqsubseteq_0 \cup \sqsubseteq_1, \\ R^* &= \{ \langle 0, v_0 \rangle, \langle 1, v_0 \rangle, \langle 1, v_1 \rangle \} \cup R_0 \cup R_1, \\ R_2 &= \{ \langle 0, w \rangle \mid v_0 R_0 w \} \cup \{ \langle j, w \rangle \mid v_j R_j w, j \in 2 \}, \end{aligned}$$

$R = R^* \cup R_2 \cup \{ \langle 1, 0 \rangle \}$,¹⁵ $F = \langle F, R, \sqsubseteq \rangle$, $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1$, and $\mathcal{M} = \langle F, \mathcal{V} \rangle$. Check that F is an IY-frame.¹⁶ By easy inductions, (*) for both $i \in 2$ for every $v \in W_i$ and every $\psi \in Fml$, $\mathcal{M}, v \models \psi$ iff $\mathcal{M}_i, v \models \psi$.¹⁷ If $\mathcal{M}, 1 \models \Box \varphi$ then $\mathcal{M}, v_1 \models \varphi$, and by (*) $\mathcal{M}_1, v_1 \models \varphi$, a contradiction. So $\mathcal{M}, 1 \not\models \Box \varphi$. If $\mathcal{M}, 0 \models \Box \varphi$ then by the Persistence Lemma $\mathcal{M}, 1 \models \Box \varphi$, a contradiction. So $\mathcal{M}, 0 \not\models \Box \varphi$; so $\mathcal{M}, 0 \models \neg \Box \varphi$. For any

¹⁵We could have kept R a little smaller for certain choices of Y, as follows:

$$R = \begin{cases} R^* & \text{if } Y=D \text{ or } Y=Dio\Box, \\ R^* \cup R_2 & \text{if } Y \text{ contains } 4\% \text{ but not } T\Box, \\ R^* \cup \{ \langle 1, 0 \rangle \} & \text{if } Y \text{ contains } T\Box \text{ but not } 4\%, \\ R^* \cup \{ \langle 1, 0 \rangle \} \cup R_2 & \text{if } Y \text{ contains } T\Box \text{ and } 4\%. \end{cases}$$

(Above replace ‘%’ by ‘ \Diamond ’, ‘ \Box ’ or make it blank.)

¹⁶We need $1Rv_0$ for right-completeness.

¹⁷Were we to define F so as to make it a B_{\Diamond} -, B_{\Box} -, 5_{\Diamond} -, 5_{\Box} -, or Dio_{\Diamond} -frame, it isn't clear how we could insure (*).

v , if $0Rv$ then $v = v_0$.¹⁸ So since $\mathcal{M}, v_0 \models \varphi$, $\mathcal{M}, 0 \not\models \diamond\neg\varphi$. This contradicts the left-side. (3) follows. \square

A.3 Observation

A formula is non-modal iff it contains no occurrences of \square or \diamond . Let $\vdash_I =$ non-modal intuitionistic consequence.

For any choice of Y as in §11, \vdash_{IY} is conservative over \vdash_I . In other words, for any set Γ of non-modal formulas and a non-modal formula φ , $0\Gamma \vdash_{IY} 0\varphi$ iff $\Gamma \vdash_I \varphi$.

Proof Consider Γ and φ as described. Right to left is trivial. Assume that $\Gamma \not\vdash_I \varphi$. By the completeness of (non-modal) intuitionist logic with respect to intuitionistic Kripke models, we may fix an intuitionistic Kripke-model $\mathcal{M} = \langle W, \sqsubseteq, \mathcal{V} \rangle$ with signature \mathcal{S} and a $u \in W$ such that $\mathcal{M}, u \models \Gamma$ but $\mathcal{M}, u \not\models \varphi$.

Consider $Y = B, 5, GL$ or Dio .¹⁹ Set $R_0 = \{ \}$ and $\mathcal{M}_0 = \langle W, R_0, \sqsubseteq, \mathcal{V} \rangle$. \mathcal{M}_0 is an IY-model, and $\mathcal{M}_0, u \models \Gamma$ but $\mathcal{M}_0, u \not\models \varphi$; so $\mathcal{M}_0, u \Vdash 0\Gamma$ but $\mathcal{M}_0, u \not\Vdash 0\varphi$. So $0\Gamma \not\vdash_{IY} 0\varphi$.

Consider $Y = S5$.²⁰ Set $R_1 = id|W$ and $\mathcal{M}_1 = \langle W, R_1, \sqsubseteq, \mathcal{V} \rangle$. \mathcal{M}_1 is an IS5-model, and as above $\mathcal{M}_1, u \Vdash 0\Gamma$ but $\mathcal{M}_1, u \not\Vdash 0\varphi$. So $0\Gamma \not\vdash_{IY} 0\varphi$. \square

Thus the modal apparatus in IY has not surreptitiously strengthened the “background” (i.e. non-modal) logic from intuitionistic logic to classical logic or an intermediate logic.

A.4 Observation

We can push this idea further. Consider $Y = S4, GL$ or Dio ,²¹ a set Δ of formulas such that for each $\delta \in \Delta$ $0\delta \not\vdash_{IY} 0\perp$, and a set Θ of formulas such that $0\Theta \not\vdash_{IY} 0\perp$.

For any Γ and φ as in A.3, if $0(\Gamma \cup \diamond\Delta \cup \square\Theta) \vdash_{IY} 0\varphi$ then $\Gamma \vdash_I \varphi$.

Proof Given Γ and φ as described, assume that $\Gamma \not\vdash_I \varphi$. Let \mathcal{M}, u and \mathcal{M}_i for $i \in 2$ be as in A.3; set $u^{\mathcal{M}} = u$ and $\mathcal{M}' = \mathcal{M}_i$. Without loss of generality, assume that $u^{\mathcal{M}}$ is the unique initial element of $\langle W^{\mathcal{M}}, \sqsubseteq^{\mathcal{M}} \rangle$. By our model-existence theorems, for each $\delta \in \Delta$ fix an IY-model \mathcal{M}_δ and a $u_\delta \in W^{\mathcal{M}_\delta}$ such that $\mathcal{M}_\delta, u_\delta \models \delta$. Also fix an IY-model \mathcal{M}_Γ and a $u_\Gamma \in W^{\mathcal{M}_\Gamma}$ such that $\mathcal{M}_\Gamma, u_\Gamma \models \Gamma$. Without loss of generality, we can make sure that $W^{\mathcal{M}}, W^{\mathcal{M}_\Gamma}$ and the $W^{\mathcal{M}_\delta}$ for $\delta \in \Delta$ are all disjoint from one another. Let

$$\begin{aligned} W^* &= W^{\mathcal{M}} \cup \bigcup_{\delta \in \Delta} W^{\mathcal{M}_\delta} \cup (W^{\mathcal{M}} \times W^{\mathcal{M}_\Gamma}); \\ W_\delta &= \{v \mid u^{\mathcal{M}_\delta} \sqsubseteq^{\mathcal{M}_\delta} v\}; \\ W_\Gamma &= \{v \mid u^{\mathcal{M}_\Gamma} \sqsubseteq^{\mathcal{M}_\Gamma} v\}. \end{aligned}$$

¹⁸If the replacement for ‘Y’ contained ‘ T_\diamond ’ we would need to have $0R0$ or $0R1$, which would block this point.

¹⁹Recall that $\vdash_{IY} \subseteq \vdash_{IGL}$ if $Y = K$ or 4 ; so those choices of Y are also covered by this case.

²⁰This case covers those Y such that $\vdash_{IT\%} \subseteq \vdash_{IS5}$ for $\% \in \{ \square, \diamond \}$.

²¹ $Y = GL$ covers the $Y = 4$ case, since $\vdash_{I4} \subseteq \vdash_{IGL}$; but since $\vdash_{IS4} \not\subseteq \vdash_{IGL}$ we need to list $S4$ separately.

Assume that $Y = T$ or Dio . Let

$$R' = \bigcup_{\delta \in \Delta} (W^{\mathcal{M}} \times W_{\delta}) \cup \bigcup_{v \in W^{\mathcal{M}}} (\{v\} \times (\{v\} \times W_{\Gamma})).$$

Note: for each $v \in W^{\mathcal{M}}$ and $w \in W_{\Gamma}$, $vR' \langle v, w \rangle$. We will construct an IY-model $\mathcal{M}^* = \langle W^*, R^*, \sqsubseteq^*, \mathcal{V}^* \rangle$ by “sewing” each $\mathcal{M}_{\delta \in \Delta}$ and copies of \mathcal{M}_{Γ} onto \mathcal{M}' , with R' as the “seam”. For each $v \in W^{\mathcal{M}}$ form $\mathcal{M}_{\Gamma, v}$ from \mathcal{M}_{Γ} by replacing each $w \in W^{\mathcal{M}_{\Gamma}}$ by $\langle v, w \rangle$. Let

$$R^* = R^{\mathcal{M}'} \cup R' \cup \bigcup_{\delta \in \Delta} R^{\mathcal{M}_{\delta}} \cup \bigcup_{v \in W^{\mathcal{M}}} R^{\mathcal{M}_{\Gamma, v}}.$$

For $v, w \in W^*$ let $v \sqsubseteq^* w$ iff either (i) $v \sqsubseteq^{\mathcal{M}'} w$, or (ii) for some $\delta \in \Delta$ $v \sqsubseteq^{\mathcal{M}_{\delta}} w$, or (iii) for some $x \in W^{\mathcal{M}}$ and $y, y' \in W^{\mathcal{M}_{\Gamma}}$, $v = \langle x, y \rangle$, $w = \langle x, y' \rangle$ and $y \sqsubseteq^{\mathcal{M}_{\Gamma}} y'$. Claim 1: the frame $\langle W^*, R^*, \sqsubseteq^* \rangle$ is left- and right-complete. Given v, w , assume that vR^*w . The only interesting case: $v \in W^{\mathcal{M}}$ and $w \in \bigcup_{\delta \in \Delta} W_{\delta} \cup (W^{\mathcal{M}} \times W^{\mathcal{M}_{\Gamma}})$. If $v' \sqsupseteq v$ then $v'R^*w \sqsupseteq w$; this suffices for right-completeness. If $w' \sqsupseteq w$ then $v \sqsubseteq v'R^*w'$; this suffices for left-completeness.

For each $v \in W^*$ and $\gamma \in \mathcal{S}$, let

$$\mathcal{V}^*(v, \gamma) = \begin{cases} \mathcal{V}^{\mathcal{M}'}(v, \gamma) & \text{if } v \in W^{\mathcal{M}}, \\ \mathcal{V}^{\mathcal{M}_{\delta}}(v, \gamma) & \text{if } v \in W^{\mathcal{M}_{\delta}} \text{ for } \delta \in \Delta, \\ \mathcal{V}^{\mathcal{M}_{\Gamma}}(w, \gamma) & \text{if } v = \langle x, w \rangle \text{ for } x \in W^{\mathcal{M}}, w \in W^{\mathcal{M}_{\Gamma}}. \end{cases}$$

Check that \mathcal{M}^* is an IY-model.

Assume that $Y = S4$ or GL . It will be convenient to have a transitive frame, which requires “sewing a wider seam”. Let

$$R'' = \bigcup_{\delta \in \Delta} (W^{\mathcal{M}} \times W^{\mathcal{M}_{\delta}}) \cup \bigcup_{v \in W^{\mathcal{M}}} (\{v\} \times (\{v\} \times W^{\mathcal{M}_{\Gamma}})),$$

$$R^* = R^{\mathcal{M}'} \cup R'' \cup \bigcup_{\delta \in \Delta} R^{\mathcal{M}_{\delta}} \cup \bigcup_{v \in W^{\mathcal{M}}} R^{\mathcal{M}_{\Gamma, v}}.$$

Note: for each $v \in W^{\mathcal{M}}$ and $w \in W^{\mathcal{M}_{\Gamma, v}}$, $vR' \langle v, w \rangle$. Define \sqsubseteq^* as in the previous case. Claim 2: $\langle W^*, R^*, \sqsubseteq^* \rangle$ is left- and right-complete, and also transitive. Left- and right-completeness follow much as claim 1 did. Given x, y, z , assume that xR^*yR^*z . If $xR^{\mathcal{M}'}y$ then $Y = S4$ and $x = y$ (since $i = 1$), yielding xR^*z . In the other interesting case, $xR''y$ and $y \left(\bigcup_{\delta \in \Delta} R^{\mathcal{M}_{\delta}} \cup \bigcup_{v \in W^{\mathcal{M}}} R^{\mathcal{M}_{\Gamma, v}} \right) z$; then $xR'z$; so xR^*z . Define \mathcal{V}^* as in the previous case. Check that $\mathcal{M}^* = \langle W^*, R^*, \sqsubseteq^*, \mathcal{V}^* \rangle$ is an IY-model.

Claim under both cases: for any formula σ and $u \in W^*$: (i) if $u \in W^{\mathcal{M}}$, $\mathcal{M}^*, u \models \sigma$ iff $\mathcal{M}', u \models \sigma$; (ii) if for $\delta \in \Delta$ $u \in W^{\mathcal{M}_{\delta}}$, $\mathcal{M}^*, u \models \sigma$ iff $\mathcal{M}_{\delta}, u \models \sigma$; (iii) if for $x \in W^{\mathcal{M}}$ and $y \in W^{\mathcal{M}_{\Gamma}}$ $u = \langle x, y \rangle$, these are equivalent: $\mathcal{M}^*, u \models \sigma$; $\mathcal{M}_{\Gamma, x}, u \models \sigma$; $\mathcal{M}_{\Gamma}, y \models \sigma$. Proof: induction on the construction of σ .

Thus $\mathcal{M}^*, u^{\mathcal{M}} \models (\Gamma \cup \diamond \Delta \cup \square \Theta)$, but $\mathcal{M}^*, u^{\mathcal{M}} \not\models \varphi$. So $\mathbf{0}(\Gamma \cup \diamond \Delta \cup \square \Theta) \not\prec_{IY} \mathbf{0}\varphi$. □

A.5 Observation

The observation in A.4 does not extend to $Y = B_{\square}$.

Example. Consider any $\pi \in \mathcal{S}$. Let $\Gamma = \Theta = \{\}$, $\varphi = (\pi \vee \neg\pi)$, $\Delta = \{\Box\varphi\}$. Recall that $\mathbf{0} \diamond \Box\varphi \vdash_{IB\Box} \mathbf{0}\varphi$. So for any $IB\Box$ -model \mathcal{M} and $u \in W^{\mathcal{M}}$, if $\mathcal{M}, u \models \diamond\Box\varphi$ then $\mathcal{M}, u \models \varphi$. But $\not\vdash_I \varphi$.

Question: Does the observation in A.4 extend to $Y = B_{\diamond}$?

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