Springer

Where Do the Natural Numbers Come from? In Memory of Geoffrey Joseph

Author(s): Harold T. Hodes

Source: *Synthese*, Sep., 1990, Vol. 84, No. 3, The Philosophy of Mathematics, Part II (Sep., 1990), pp. 347-407

Published by: Springer

Stable URL: https://www.jstor.org/stable/20116824

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at https://about.jstor.org/terms

Springer is collaborating with JSTOR to digitize, preserve and extend access to *Synthese*

JSTOR

This content downloaded from
132.174.252.179 on Thu, 23 Man Thu, 01 Jan 1976 12:34:56 UTC
All use subject to https://about.jstor.org/terms

HAROLD T. HODES

WHERE DO THE NATURAL NUMBERS COME FROM?[1]

In memory of Geoffrey Joseph

This paper offers a model-theoretic development of the ideas presented in (Hodes 1984). In Section 1, Section 5 and Section 10 I will discuss the philosophical motivation for this project. For more, see (Hodes 1984) and (Hodes 1990). More information about some of the higher-order logics that will be introduced below may be found in (Hodes 1988a and 1988b).[2]

1.

Throughout his philosophical career, Frege maintained that numbers were objects.[3] In part,[4] this thesis reflects facts about the syntactic form of sentences containing arithmetical expressions, the sorts of sentences uttered by infants learning to count things, children learning sums and simple algebra, and mathematicians teaching or advancing their science. With respect to their syntactic roles in the formation of sentences, expressions like '2', '2 + 3', 'the number of moons of Jupiter' and 'the least prime greater than 10' closely resemble paradigmatic singular terms, expressions like 'is prime' and 'is greater than' are much like paradigmatic predicate-phrases, and expressions like 'some number' and 'all numbers' resemble first-order quantifier-phrases from the other corners of our language. Why fight our inclination for generalization? Let's classify expressions of the first sort as singular terms, those of the second sort as predicate-phrases, and those of the third sort as first-order quantifier-phrases.[5]

This classification of lexical items doesn't merely help us understand the formation of individual sentences; it supports a characterization of what Quine called "the interanimation of sentences", especially within chunks of discourse containing inferences. This characterization permits our reasoning "about" numbers to be adequately regimented within any complete formalization of first-order logic. Mathematical predicate-phrases play a proof-theoretic role like that played by paradigmatic level-one predicates; mathematical quantifier-phrases behave proof-theoretically like paradigmatic first-order quantifier-expressions. In-

Synthese **84**: 347–407, 1990.
© 1990 *Kluwer Academic Publishers. Printed in the Netherlands.*

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

deed, this fit made use of the phrase 'our reasoning "about" numbers' virtually irresistible, though scare-quotes are needed to prevent this logico-syntactic point from collapsing to the semantic point about to be considered. The former point may be put into this slogan: mathematical discourse, as we know it, conforms to the Mathematical-Object Picture.

Frege's doctrine draws its motivation from that picture but goes beyond the broadly syntactic facts just considered to make a claim about the semantics for arithmetic discourse. In the paradigm cases, the logico-syntactic status of singular-termhood goes with a particular sort of semantic role – that of designating what Frege called "objects" (and others sometimes call "individuals"). Similarly, paradigmatic predicate-phrases play the semantic role of applying or failing to apply to objects. (Frege says that they stand for certain level-one functions. In (Furth 1968) the author argues, persuasively I think, that this is not to accord to them a role beyond the preceding.) And similarly, first-order quantifier phrases usually do the familiar job of "ranging over" objects. The full force of Frege's thesis is this: to specify logico-syntactic status is to specify semantic role. So arithmetic singular terms designate numbers; arithmetic predicates apply or fail to apply to numbers (or to tuples of numbers); arithmetic quantifier-phrases range over numbers. 'Number theory is "about" numbers' may be a truism for everybody. But for Frege, it is a truism to be construed literally; scare-quotes on 'about' may be dropped. Thus the Mathematical-Object Picture becomes the Mathematical-Object Theory: a semantic doctrine, a chapter in the theory of truth, one that takes mathematical discourse to carry thick ontological commitments to specifically mathematical objects.

Frege, of course, didn't walk this primrose path alone. But he did develop it in a direction not favored by all who accepted the Mathematical-Object theory. Frege did not draw distinctions of type between objects;[6] in his formalism predication (or more generally, function-application) was the only source of typing. Frege really invented the theory of types; but he only looked at levels zero, one, and two. Russell and Whitehead got the credit for the theory of types because they considered all finite levels. Of course their type-theory primarily typed proposition functions (rather than Fregean entities), and secondarily typed sets. Their primary type-structure, like the Fregean type-structure, was a matter of saturation versus varieties of unsaturation. Their secondary typing was distinctively anti-Fregean in spirit: it introduced distinctions of type between objects.[7] Frege did not treat even very

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

basic sortal concepts that figure importantly in logic, of which being a natural number was one, as logical categories in the sense in which type-theoretic categories are logical. Sentences like '1 = Julius Caesar' or '1 is located on the surface of the Earth' may be peculiar; but for Frege this peculiarity is obvious falsity, not ill-formedness. Russell and Whitehead parted company with Frege on this point, as did Carnap, some devotees of ordinary language in the 1950s, and those linguists who emphasized selectional restrictions in generative syntax.

The slide from the Mathematical-Object picture to the Mathematical-Object theory is so natural that most would not even see it as a move. Firstly, truths "about" natural numbers entail that they exist. Secondly, the following principle may seem self-evident:

> After all, what we say metalinguistically by "$\underline{a}$' has a reference' is just the object-language, '$(\exists x)x = \underline{a}$' . . . (Wright 1983, p. 83)

Misguided Meingongians have denied that in all cases this principle holds from left to right. I shall reject it in the other direction; the model-theoretic semantics to be presented in Section 5 show how the left side may fail while the right side holds.

The above Supposedly Self-Evident Principle is an ingredient in Dummett's interpretation of Frege's context principle. Dummett writes:

> But what the context principle, interpreted as a thesis about reference, and applied to proper names, tells most immediately against is the conception that an expression can behave *exactly* like a singular term and yet be denied a reference. . . . to say that an expression behaves exactly like a singular term is to say that it has just the connections with the use of predicates, general terms and quantifiers that will render the denial of a reference to it incoherent. [1981, p. 384][8]

I won't address the question of whether this was Frege's view. My claim is this: one who uses this view to defend the Mathematical-Object Theory is taking an overly narrow view of both the synthetic facts and the semantic possibilities concerning arithmetic discourse.

On the syntactic side: in addition to the syntactic facts that encouraged us to classify certain numerical expressions as singular terms, further syntactic facts show them to be rather special singular terms. Unlike our paradigmatic singular terms, a word like 'four' leads a double life, appearing adjectively in quantifier-phrases, e.g. 'There are exactly four moons of Jupiter', as well as in contexts like 'The number of moons of Jupiter equals four'. The behavior of 'the number four' suggests that the adjectival use of 'four' is more basic than its use as a

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

singular term. 'There are exactly the number four moons of Jupiter' is ungrammatical; but it may be argued that in contexts like 'The number of moons of Jupiter equals four', 'four' abbreviates 'the number four'.[9] One might conclude that 'four' by itself is not a genuine singular term. All in all, both conclusions go too far.[9a] Indeed number-words make unusual adjectives: they don't enter into comparative or grading constructions; they form further adjectives (the ordinal-words), and only the latter transform into adverbs; they're mildly archaic as predicate adjectives, a use that requires plural subjects. (In 'She is four', 'years old' was deleted.)

These slight syntactic peculiarities of numerical singular terms are at best clues to their semantic peculiarity, as well as to the semantic peculiarity of arithmetic predicates. Even if we conclude that such a term leads the syntactic life of a singular term as fully as do our favorite paradigms of singular-termhood, we are not forced to apply to them the Supposedly Self-Evident Principle going from right to left.

For the reason sketched in my 1984 article, I want to resist the slide from the Mathematical-Object Picture to the Mathematical-Object Theory. Natural numbers are, loosely speaking, fictions created to encode cardinality-quantifiers, thereby clothing a certain higher-order logic in the attractive garments of lower-order logic. More precisely: arithmetic singular terms that appear to do the semantic job of *designating* numbers really do the different job of *encoding* cardinality-quantifiers; quantifier-phrases that appear to quantify over numbers really encode higher-order quantification over cardinality-quantifiers; predicate-phrases, whose logico-syntactic behavior make them of level one, really do the semantic work of expressions of higher levels.

Assimilation of quantification to singular predication, e.g. of the logical form of 'Socrates was bald' to that of 'Some Greek was bald', was a major barrier to an adequate understanding of logical syntax. By clarifying the difference between singular terms and quantifier-phrases, thus subdividing a category that was previously thought homogeneous, Frege took a giant step for logic. I'm proposing a somewhat analogous step: that within the category of singular terms we further distinguish between designators and encoders of, for example, quantifiers.

In this paper, I'll present a model-theoretic "picture" of the Mathematical-Object Theory, and one of my Alternative Theory. This exercise is intended to help us grasp what these theories amount to, and the reasons favoring the latter.

To give a model-theoretic semantics is to specify a class of uninterpre-

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

ted languages, a class of models for appropriate such languages, and a definition of truth and falsity (or, more generally, of satisfaction and frustration) "in" (i.e., relative to) models in that class for sentences (or, more generally, formulae) of the appropriate languages. A model-theoretic semantics permits set-theoretic definitions of logical relations, like entailment and equivalence, and logical properties, like validity, definitions that apply primarily to sentences of uninterpreted languages. Why think that such definitions explicate prior logical notions? When will the entailments, equivalences, etc., so defined on uninterpreted languages parse (i.e., reflect under appropriate parsing) the entailments, equivalences, etc., implicit in the practice that animates discourse in the sense-bearing languages being modeled? This will happen insofar as (1) the models in question can model (in the engineering sense) the basic referential relations between the world (including both what there is and how it is) and languages – "real live" sense-bearing (sometimes called interpreted) languages. It will also happen insofar as (2) the relations of truth and falsity in-a-model model (again in the engineering sense) "real live" truth and falsity for statements in such languages. Thus much of the philosophical value of a definition of truth-in-a model lies in the way it could guide us in constructing a definition, or perhaps a theory, of real live truth for interpreted languages, or at least for fragments thereof.

2.

The semantics that will interest us most, to be presented in Section 5, will be three-valued. This suggests that we work with a three-valued semantics from the beginning. But doing so would require attention to some side-issues peculiar to three-valued semantics, distracting from the main themes of this paper. Therefore discussion of a thoroughly three-valued approach is relegated to Appendix 2. By Section 4 we'll have to deal with non-designating terms; so we'll allow them from the start, handling them under the Falsehood Convention as in (Burge 1980).

Let our basic logical lexicon consist of the expressions '$\perp$', '$\supset$', '$\exists$', '$=$'. Fix disjoint countable sets of variables of types 0, 1 and (0, 0). (As usual, as a type-symbol '1' abbreviates '(0)', '2' abbreviates '((0)', etc.) Let S be a set of predicate-constants and function-constants, each associated with a member of ω giving its number of places.[10] (As usual, 0-place function-constants will be called "individual constants".) S de-

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

termines the language $L^{(0,0)}(S)$; usually we'll omit mention of S. The class of type-0 terms based on S is defined by the usual induction:

> all type-0 variables and all 0-place members of S are type-0 terms;
> for any type-0 terms $\tau_0, \ldots, \tau_n$ and any $n+1$-place function-constant $\xi \in S$, $\xi(\tau_0, \ldots, \tau_n)$ is a type-0 term.

We define $\mathrm{Fml}(L^{(0,0)})$, the set of formulae of $L^{(0,0)}$, by the usual induction; perhaps these clauses deserve mention:

> '$\perp$' is a formula;
> if τ and σ are terms of type 0, α is a variable of type 1 and γ is a variable of type $(0,0)$ then $\tau = \sigma$, $\alpha(\tau)$ and $\gamma(\tau, \sigma)$ are formulae;
> if φ is a formula and ν is a variable of type 0, 1, or $(0,0)$ then $(\exists \nu)\varphi$ is a formula.

Sometimes parentheses will be dropped to decrease clutter. Let $\mathrm{Sent}(L^{(0,0)})$ be the set of sentences, i.e. formulae containing no occurrences of free variables, of $L^{(0,0)}$. Introduce '$\neg$', '&', '$\vee$', '$\equiv$' and '$\forall$' by the usual abbreviations. For a formula φ, a type-0 variable ν and type-0 term τ, $\varphi(\tau/\nu)$ is the result of substituting τ for all occurrences of ν free in φ; when a formula is indicated as $\varphi(\nu, \ldots)$, instead of $\varphi(\tau/\nu)$ we'll write $\varphi(\tau, \ldots)$. Similarly for variables and corresponding constants of other types.

A model $\mathscr{A}$ for S consists of a set $|\mathscr{A}|$ and a partial function on S such that:

> for a 0-place predicate-constant ζ, $\zeta^{\mathscr{A}} \in \{0, 1\}$;
> for an $n+1$-place predicate-constant ζ, $\zeta^{\mathscr{A}}: |\mathscr{A}|^{n+1} \to \{0, 1\}$;
> for an individual-constant τ, either $\tau^{\mathscr{A}} \uparrow$ or $\tau^{\mathscr{A}} \in |\mathscr{A}|$;
> for an $n+1$-place function-constant ξ, $\xi^{\mathscr{A}}$ is a function from a subset of $|\mathscr{A}|^{n+1}$ into $|\mathscr{A}|$.

$\mathscr{A}$ is total iff

> for each individual-constant τ, $\tau^{\mathscr{A}} \downarrow$;
> for each $n+1$-place function constant ξ, $\mathrm{dom}(\xi^{\mathscr{A}}) = |\mathscr{A}|^{n+1}$.

We'll frequently identify a subset of $|\mathscr{A}|$ with its characteristic function. Enrich $L^{(0,0)}$ to $L^{(0,0)}_{\mathscr{A}}$ by introducing an individual-constant $\underline{a}$ for each $a \in |\mathscr{A}|$, a 1-place predicate-constant $\underline{A}$ for each $A \subseteq |\mathscr{A}|$, and a 2-place

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

predicate-constant $\underline{B}$ for each $B \subseteq |\mathscr{A}|^2$. We define a partial function $\mathrm{des}^{\mathscr{A}}$ on closed type-0 terms of $L_{\mathscr{A}}^{(0,0)}$ as usual:

$$\mathrm{des}^{\mathscr{A}}(\tau) \simeq \tau^{\mathscr{A}};$$
$$\mathrm{des}^{\mathscr{A}}(\underline{a}) = a \text{ for } a \in |\mathscr{A}|;$$
$$\mathrm{des}^{\mathscr{A}}(\xi(\tau_0, \ldots, \tau_n)) \simeq \xi^{\mathscr{A}}(\mathrm{des}^{\mathscr{A}}(\tau_0), \ldots, \mathrm{des}^{\mathscr{A}}(\tau_n)),$$

for any individual-constant τ from S, any $n+1$-place function-constant ξ from S, and any terms $\tau_0, \ldots, \tau_{n11}$ based on S. We'll frequently write $\mathrm{des}^{\mathscr{A}}(\tau)$ as $\tau^{\mathscr{A}}$. Of course if for some $i \leqslant n$ $\tau_i^{\mathscr{A}}\uparrow$, then $\xi(\tau_0, \ldots, \tau_n)^{\mathscr{A}}\uparrow$. We define $\mathscr{A} \models \varphi$ for $\varphi \in \mathrm{Sent}(L_{\mathscr{A}}^{(0,0)})$ as usual, with these clauses deserving mention:

$\mathscr{A} \not\models$ '$\perp$';
$\mathscr{A} \models \tau = \sigma$ iff $\tau^{\mathscr{A}} \simeq \sigma^{\mathscr{A}}$;
$\mathscr{A} \models \zeta(\tau_0, \ldots, \tau_n)$ iff for all $i \leqslant n$ $\tau_i^{\mathscr{A}}\downarrow$ and $\zeta^{\mathscr{A}}(\tau_0^{\mathscr{A}}, \ldots, \tau_n^{\mathscr{A}}) = 1$, for ζ an $n+1$-place predicate constant;
$\mathscr{A} \models (\exists\alpha)\varphi$ iff for some $B \subseteq |\mathscr{A}|$ $\mathscr{A} \models \varphi(\underline{B}/\alpha)$ for a variable α of type 1;
$\mathscr{A} \models (\exists\gamma)\varphi$ iff for some $B \subseteq |\mathscr{A}|^2$ $\mathscr{A} \models \varphi(\underline{B}/\gamma)$, for a variable γ of type $(0,0)$.

Notice: we are counting an equation as true when both terms flanking '=' are "non-designating"; we count an atomic predication as false when at least one term following the predicate-constant is "non-designating" (the Falsehood Convention). Let $E(\tau)$ abbreviate $(\exists\nu)\nu = \tau$, where ν is any type-0 variable not free in τ. If τ is closed, $\mathscr{A} \models E(\tau)$ iff $\tau^{\mathscr{A}}\downarrow$; so '$E$' parses 'exists' in its "predicative" use. We are interested in sentences of $L^{(0,0)}$; the detour through $L_{\mathscr{A}}^{(0,0)}$ was used to simplify our definitions, i.e., to avoid defining truth-in-$\mathscr{A}$ in terms of satisfaction-in-$\mathscr{A}$.

We could enrich $L^{(0,0)}$ with the definite-description operator 't', and the clause:

$$\mathrm{des}^{\mathscr{A}}((t\nu)\varphi) \simeq \text{the unique } a \in |\mathscr{A}| \text{ so that } \mathscr{A} \models \varphi(\underline{a}/\nu).$$

But use of 't' can be avoided by the usual Russellian transformation, always giving 't' "smallest scope".

Form L^1 from $L^{(0,0)}$ by eliminating variables of type $(0,0)$. Form L^0 from L^1 by eliminating variables of type 1. In what follows, 'x' may be replaced by '0', '1' or '$(0,0)$'. For any possible sense-bearing language

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

$\mathscr{L}$ modeled by the uninterpreted language L^x, a model $\mathscr{A}$ for S can model $\mathscr{L}$'s alethic underpinnings, in which case $\mathscr{A} \models$, i.e., bearing the converse of $\models$ to $\mathscr{A}$, models truth for statements in $\mathscr{L}$.

3.

We now consider some third- and fifth-order enrichments of L^x. For 'x' replaced by '1' or '(0, 0)' enrich L^x to $L^{x,4}$ by introducing countably infinite sets of variables of type 2 and of type 4, and adding these to the formation-rules of L^x, for $i \in \{0, 1\}$:

> if μ is a variable of type $2i + 2$, ρ a variable of type $2i$, and φ a formula, then $(\mu\rho)\varphi$ is a formula;
> if μ is a variable of type 2 or type 4 and φ is a formula then $(\exists\mu)\varphi$ is a formula.

In $(\mu\rho)\varphi$, the prefix $(\mu\rho)$ binds all occurrences of ρ free in φ, but the indicated occurrence of μ is free in the $(\mu\nu)\varphi$.

Let $\mathscr{A}$ be a model for S. Suppose $\mathscr{D}_2 \subseteq \text{Power}^2(|\mathscr{A}|)$ and $\mathscr{D}_4 \subseteq \text{Power}^2(\mathscr{D}_2)$. Form $L^{(0,0),4}_{\mathscr{A},\mathscr{D}_2,\mathscr{D}_4}$ as in Section 2, with the addition of a type-2 [type-4] constant $\underline{Q}$ for each $Q \in \mathscr{D}_2$ [$Q \in \mathscr{D}_4$]. For $\varphi \in \text{Sent}(L^{(0,0),4}_{\mathscr{A},\mathscr{D}_2,\mathscr{D}_4})$ we define $\mathscr{A}, \mathscr{D}_2, \mathscr{D}_4 \models \varphi$ by taking type-2 [type-4] variables to range over $\mathscr{D}_2$ [$\mathscr{D}_4$] and adding this clause:

$$\mathscr{A}, \mathscr{D}_2, \mathscr{D}_4 \models (Q\nu)\varphi \text{ if } \hat{\nu}\varphi^{\mathscr{A},\mathscr{D}_2,\mathscr{D}_4} \in Q.$$

Here we have:

$$\hat{\nu}\varphi^{\mathscr{A},\mathscr{D}_2,\mathscr{D}_4} = \begin{cases} \{a \in |\mathscr{A}|: \mathscr{A}, \mathscr{D}_2, \mathscr{D}_4 \models \varphi(\underline{a}/\nu)\} & \text{if } \nu \text{ is type 0;} \\ \{Q \in \mathscr{D}_2: \mathscr{A}, \mathscr{D}_2, \mathscr{D}_4 \models \varphi(\underline{Q}/\nu)\} & \text{if } \nu \text{ is type 2.} \end{cases}$$

For $n \in \omega$, let:

$$\begin{aligned} &{}^2Q(n) = \{A \subseteq |\mathscr{A}|: \text{card}(A) = n\}; \\ &{}^2\text{EXACTLY} = \{{}^2Q(n): n < \omega\}; \\ &{}^4Q(n) = \{A \subseteq {}^2\text{EXACTLY}: \text{card}(A) = n\}; \\ &{}^4\text{EXACTLY} = \{{}^4Q(n): n < \omega\}. \end{aligned}$$

(Here members of ω are used merely as indexes; we could do without them.) We'll be most concerned with the case in which $\mathscr{D}_{2i} = {}^{2i}\text{EXACTLY}$, for $i \in \{1, 2\}$. In that situation, we won't mention $\mathscr{D}_2$ and $\mathscr{D}_4$. To indicate we're in this case, we'll write $(\mu\rho)\varphi$ as (EXACTLY $\mu\rho)\varphi$

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

and ${}^2\underline{Q(n)}$ and ${}^4\underline{Q(n)}$ as $\underline{n}$, and call $L^{x,4}$ "$L^{x,4}$(EXACTLY)", as in (Hodes 1988a and 1988b).

Where μ and μ' are variables of type 2, we adopt the following abbreviation:

$$\mu \leqslant \mu' : (\exists\alpha)(\exists\alpha')((\text{EXACTLY } \mu\alpha)\alpha\nu \ \& \ (\text{EXACTLY } \mu'\nu)\alpha'\nu \ \& \ (\forall\nu)(\alpha\nu \supset \alpha'\nu));$$

here α and α' are any distinct variables of type 1, and ν is of type 0. Thus for any $n, m < \omega$, if $n, m \leqslant \text{card}(\mathscr{A})$:

$$\mathscr{A} \models {}^2\underline{Q(n)} \leqslant {}^2\underline{Q(m)} \text{ iff } n \leqslant m.$$

Let $\mu < \mu'$ be $\mu \leqslant \mu' \ \& \ \neg\mu' \leqslant \mu$; let $\mu = \mu'$ be $\mu \leqslant \mu' \ \& \ \mu' \leqslant \mu$.

Notice that the range of the type-2 variables of $L^{x,4}$(EXACTLY) is disjoint from the range of the type-4 variables. Nonetheless, we can express "cross-type identity", as follows. For variables μ and ρ of types 4 and 2 respectively, adopt this abbreviation:

$$\rho = \mu : (\text{EXACTLY } \mu\nu)\nu < \rho,$$

where ν is any variable of type 2 distinct from ρ.[11] So for any $\mathscr{A}$ and n, m as above: $\mathscr{A} \models {}^2\underline{Q(n)} = {}^4\underline{Q(m)}$ iff $n = m$. Let $\mu = \rho$ be $\rho = \mu$. With our "cross-type equality" we can define $\mu \leqslant \mu'$ where the types of μ and μ' are 2 or 4 and at least one is of type 4.

Because cross-type equality can be expressed, we gain no further expressive power by adding variables of type 6 to form $L^{x,6}$(EXACTLY). A sentence φ of such a language is equivalent to a sentence of $L^{x,6}$(EXACTLY) formed as follows: for each type-6 variable μ in φ introduce a distinct new variable μ' of type 4; replace each subformula of the form (EXACTLY $\mu\rho)\theta$ by

$$(\text{EXACTLY } \mu'\nu)(\exists\nu)(\nu = \rho \ \& \ \theta),$$

where ν is a new type-2 variable, and replace all other occurrences of μ in φ by μ'. This is a special case of the more general Collapsing Theorem of (Hodes 1988a).

Our definition of $\mu \leqslant \mu'$ involved variables of type 1 in an essential way; see Observation 3.5 of (Hodes 1988a). So if we were to form $L^{0,4}$ merely by enriching L^0 with variables of types 2 and 4, subject to the formation and semantic rules used with $L^{1,4}$ and $L^{(0,0),4}$, we could not then define '$\leqslant$'. Since we will want it, we form $L^{0,4}$ by also adding '$\leqslant$' as a primitive logical constant, subject to these formation and semantic

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

rules, for $i \in \{1, 2\}$ and $n, m \in \omega$:

for variables μ, μ' of type $2i$, $\mu \leqslant \mu'$ is a formula;
$\mathscr{A} \models {}^{2i}\underline{Q(n)} \leqslant {}^{2i}\underline{Q(m)}$ iff $n \leqslant m$.

Again, when $\mathscr{D}_{2i}$ is taken to be ^{2i}EXACTLY for $i \in \{1, 2\}$, we write $(\mu\rho)\varphi$ as (EXACTLY $\mu\rho)\varphi$, call $L^{0,4}$ "$L^{0,4}$(EXACTLY)", etc. Warning: this is *not* how '$L^{0,4}$(exactly)' is used in my (1988a and 1988b).

We'll be most interested in infinite models. For $\Delta \cup \{\varphi\} \subseteq \mathrm{Sent}(L^{x,4}(\text{EXACTLY}))$, let Δ ∞-entail φ iff:

for any infinite model $\mathscr{A}$, if $\mathscr{A} \models \Delta$ then $\mathscr{A} \models \varphi$.

We define ∞-validity, ∞-equivalence and other ∞-logical notations as usual, with the above restriction to infinite models.

We adopt these abbreviations:

$(\underline{\text{EXACTLY } 0}\nu)\varphi : \neg(\exists\nu)\varphi$;
$(\underline{\text{EXACTLY } n+1}\nu)\varphi : (\exists\rho)(\varphi(\rho/\nu)\ \&\ (\underline{\text{EXACTLY } n}\nu)(\varphi\ \&\ \nu \neq \rho))$,

where ρ is any variable distinct from ν, and both are either of type 0 or type 2. So $(\underline{\text{EXACTLY } n}\nu)\varphi$ and (EXACTLY $\underline{n}\nu)\varphi$ are equivalent. The syntactic distinction between such a pair of sentences is not meant to model any real difference between sense-bearing statements. The former notation, e.g., $(\underline{\text{EXACTLY } n+1}\nu)\varphi$, shows how a quantifier can be carved out of $(\exists\rho)(\varphi(\rho/\nu)\ \&\ (\underline{\text{EXACTLY } n}\nu)(\varphi\ \&\ \nu \neq \rho))$. Rewriting the prefix as $(\underline{\text{EXACTLY } n+1}\nu)$ makes it plain that this quantifier is itself in the range of quantifiers of higher type. Within, for example, (EXACTLY $\underline{1}\nu)\varphi$, '$\underline{1}$' is not a term of type 0; it doesn't represent a singular term, not even one occurring syncategoromatically. It indicates that the entire prefix (EXACTLY $\underline{1}\nu$), which abbreviates (EXACTLY ${}^{2i}\underline{Q(1)}$), stands for the quantifier also represented by $(\underline{\text{EXACTLY } 1}\nu)$ namely ${}^{2i}Q(1)$. The thesis "Numbers are quantifiers", urged in (Grossman 1973), is not false; it's incoherent on grounds of logical syntax alone.

By Observation 2.2 of my (1988a), assuming a little Choice (all we need is that all Dedekind-finite sets are finite, which we assume from now on), $L^{(0,0),4}$(EXACTLY) and $L^{(0,0)}$ are expressively equivalent; i.e., every sentence of the former can be translated into an equivalent sentence of the latter. So the entire discussion in this section (as well as that in Section 5) concerns fragments of $L^{(0,0)}$.

For 'x' replaceable by '0', '1' or '(0, 0)', form $L^{x,2}$(EXACTLY) from

This content downloaded from
:ffff:ffff:ffff:ffff:ffff:ffff on Thu, 01 Jan 1976 12:34:56 UTC
All use subject to https://about.jstor.org/terms

$L^{x,4}$(EXACTLY) by eliminating use of variables of type 4. By theorem 3.1 of my (1988a), $L^{0,2}$(EXACTLY) is expressively weaker than $L^{0,4}$(EXACTLY). Problem: is $L^{1,2}$(EXACTLY) expressively weaker than $L^{1,4}$(EXACTLY)? See Conjecture C of the same article.

For variables μ_0, μ_1, μ_2 of type-2, let Add(μ_0, μ_1, μ_2) abbreviate:

$$(\forall\alpha)(\forall\alpha')([(\text{EXACTLY } \mu_0\nu)\alpha\nu \ \& \ (\text{EXACTLY } \mu_1\nu)\alpha'\nu \ \& \ \neg(\exists\nu)(\alpha\nu \ \& \ \alpha'\nu)] \supset (\text{EXACTLY } \mu_2\nu)(\alpha\nu \vee \alpha'\nu));$$

here α and α' are any distinct variables of type 1. Thus for any infinite model $\mathscr{A}$ and $n, m, p < \omega$:

$$(*) \qquad \mathscr{A} \models \text{Add}(\underline{n}, \underline{m}, \underline{p}) \text{ iff } n + m = p.$$

So if μ_0, μ_1 and μ_2 are distinct, this formula defines addition of finite cardinalities. If μ_0, μ_1, μ_2 are of type 2 or 4, and at least one is of type 4, then Add(μ_0, μ_1, μ_2) can be defined using "cross-type equality" to the same effect.

This definition of addition may be plausibly called "analytic" in that it provides an analysis of our understanding of addition. In the following argument-form the premises may be shown to entail the conclusion by very simple and obvious moves ($\forall$-elimination, &-introduction, then *modus ponens*):

$$\frac{\begin{array}{l}\text{Add}(\mu_0, \mu_1, \mu_2)\\ (\text{EXACTLY } \mu_0\nu)\varphi\\ (\text{EXACTLY } \mu_1\nu)\varphi\\ \neg(\exists\nu)(\varphi \ \& \ \varphi)\end{array}}{(\text{EXACTLY } \mu_2\nu)(\varphi \vee \varphi)}$$

The proof-theoretic simplicity of such an argument is, perhaps, part of the content of the claim that this definition of addition is analytical.

The restriction to infinite models was needed in (*); if card($\mathscr{A}$) $< n + m$ then for any $p < \omega$ we have $\mathscr{A} \models$ Add($\underline{n}, \underline{m}, \underline{p}$). This feature of our definition would be avoided if to Add(. . .) we conjoined:

$$(\exists\alpha)(\exists\alpha')((\text{EXACTLY } \mu_0\nu)\alpha\nu \ \& \ (\text{EXACTLY } \mu_1\nu)\alpha'\nu \ \& \ \neg(\exists\nu)(\alpha\nu \ \& \ \alpha'\nu)).$$

Doing so would not seriously decrease the simplicity of the above inference-form.

For variables μ_0, μ_1,μ_2 of type 2, let Mult(μ_0, μ_1, μ_2) abbreviate:

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

$$(\forall\alpha)(\forall\gamma)([(\text{EXACTLY } \mu_1\nu)\alpha\nu \;\&\; (\forall\nu)(\alpha\nu \supset (\text{EXACTLY } \mu_0\rho)\gamma\nu\rho) \;\&\; (\forall\nu)(\forall\nu')([\alpha\nu \;\&\; \alpha\nu']) \supset \neg(\exists\rho)[\gamma\nu\pi \;\&\; \gamma\nu'\rho])] \supset (\text{EXACTLY } \mu_2\rho)(\exists\nu)[\alpha\nu \;\&\; \gamma\nu\rho]),$$

where α is of type 1, γ is of type $(0, 0)$, and ν, ν' and ρ are distinct variables of type 0. Again, if μ_0, μ_1 and μ_2 are distinct, $\text{Mult}(\mu_0, \mu_1, \mu_2)$ defines multiplication of finite cardinalities. In spite of its complexity, this definition of multiplication has a claim to being counted as analytical. In my (1988b) it is shown that multiplication is not definable in $L^{1,4}(\text{EXACTLY})$; so the use of a type-$(0, 0)$ variable above was unavoidable.

A variation of the preceding semantics deserves mention. Let:

$$\begin{aligned}
&{}^2Q_\infty = \{A \subseteq |\mathscr{A}|: A \text{ is infinite}\};\\
&{}^2\text{EXACTLY}_\infty = {}^2\text{EXACTLY} \cup \{{}^2Q_\infty\};\\
&{}^4Q(n)_\infty = \{A \subseteq {}^2\text{EXACTLY}_\infty: \text{card}(A) = n\} \text{ for } n < \omega;\\
&{}^4Q_\infty = \{A \subseteq {}^2\text{EXACTLY}_\infty: A \text{ is infinite}\};\\
&{}^4\text{EXACTLY}_\infty = \{{}^4Q(n)_\infty: n < \omega\} \cup \{{}^4Q_\infty\}.
\end{aligned}$$

When we take $\mathscr{D}_{2i}$ to be ${}^{2i}\text{EXACTLY}_\infty$ for $i \in \{1, 2\}$, we'll omit mention of $\mathscr{D}_{2i}$, write $(\mu\rho)\varphi$ as $(\text{EXACTLY}^\infty\ \mu\rho)\varphi$, and call $L^{x,4}$ "$L^{x,4}(\text{EXACTLY}^\infty)$". All abbreviations carry over from $L^{x,4}(\text{EXACTLY})$. These semantics are, in the following sense, equivalent.

Observation 1: there are transitions s^∞ and s,

$$\begin{aligned}
&s^\infty: \text{Sent}(L^{x,4}(\text{EXACTLY})) \to \text{Sent}(L^{x,4}(\text{EXACTLY}^\infty)),\\
&s: \text{Sent}(L^{x,4}(\text{EXACTLY}^\infty)) \to \text{Sent}(L^{x,4}(\text{EXACTLY})),
\end{aligned}$$

so that for any infinite model $\mathscr{A}$, $\varphi \in \text{Sent}(L^{x,4}(\text{EXACTLY}))$, $\psi \in \text{Sent}(L^{x,4}(\text{EXACTLY}^\infty))$:

$$\mathscr{A} \models \varphi \text{ iff } \mathscr{A} \models s^\infty(\varphi);\ \mathscr{A} \models \psi \text{ iff } \mathscr{A} \models s(\psi).$$

For convenient notation, we'll apply s^∞ and s to formulae. Let s^∞ be homomorphic, except that for μ a variable of type 2 or 4: $s^\infty((\exists\mu)\theta)$ is $(\exists\mu)(\neg(\text{exactly}^\infty\ \mu\nu)\neg\bot \;\&\; s^\infty(\theta))$, for ν any variable of type 2 or 0. To define s, we use '$\underline{\infty}$' as a constant of types 2 and 4. Define s' on $\text{Fml}(L^{x,4}(\text{EXACTLY}^\infty))$ to replace 'EXACTLY^∞' by 'EXACTLY' and to replace $(\exists\mu)\theta$, where μ is of type-2 or type-4, by $(\exists\mu)\theta \vee \theta(\underline{\infty}/\mu)$. Form $s(\varphi)$ from $s'(\varphi)$ by replacing each subformula of the form $(\text{EXACTLY } \underline{\infty}\ \nu)\theta$ by $\neg(\exists\mu)(\text{EXACTLY } \mu\nu)\theta$, thereby eliminating all uses of '$\underline{\infty}$'. It's easy to see that these translations are as claimed.

Ordinal uses of natural numbers can also be expressed in these

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

languages. We could have introduced the logical constant 'th' with the following formation and designation rules:

> if φ is a formula and ν and ρ are distinct variables of type-$2i$ and μ is a variable of type $2i + 2$, for $i \in \{0, 1\}$, then $(\mu\text{th}\ \nu\rho)\varphi$ is a term of type-$2i$;
> $\text{des}^{\mathscr{A}}((\mu\text{th}\ \nu\rho)\varphi) \simeq$ the nth element in $\{\langle a, b\rangle : \mathscr{A} \models \varphi(\underline{a}, \underline{b}/\nu, \rho)\}$, provided that set has a unique linearly ordered inition segment of length n.

In fact, such an enrichment can be definitional: use of 'th' can be eliminated in favor of the definite description operator 't', which then in turn can be eliminated.

4.

For 'x' replaceable as before, we'll now enrich L^x to a language whose syntax approximates mathematical practice more closely than did the syntax of the languages presented in Section 3. These languages model arithmetic discourse "within" the Mathematical-Object picture.

Given a model $\mathscr{A}$, members of $|\mathscr{A}|$ represent objects, and members of ^{2i}EXACTLY (and ${}^{2i}\text{EXACTLY}_{\infty}$) represent, in Fregean terms, concepts of level $2i$, for $i \in \{1, 2\}$. Taking $i = 1$, Frege wrote:

> These second level concepts form a series and there is a rule in accordance with which, if one of these concepts is given, we can specify the next.

Similarly for the corresponding concepts of level four, which Frege doesn't explicitly consider. He continues:

> But still we do not have the numbers of arithmetic; we do not have objects, but concepts. How can we get from these concepts to the numbers of arithmetic in a way that cannot be faulted? Or are there simply no numbers in arithmetic? Could the numerals help to form signs for these second-level concepts, and yet not be signs in their own right? (Frege 1979), p. 256–7.

Frege's point can be clarified by looking at our notation. In $(\underline{\text{EXACTLY}\ 1}\nu)$, for ν a variable of type 0, we have a numeral helping to form a sign for a second-level concept. More generally, our definition of $(\underline{\text{EXACTLY}\ n+1}\nu)$ exhibits the rule "in accordance with which, if one of these concepts is given, we can specify the next". In claiming that numbers are objects, Frege answered his last two questions negatively. Rather, he thought that for each such concept there is, or at least we

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

should make sure that there is,[12] a corresponding object, a natural number (i.e., a finite cardinal) that represents that concept. He believed in what in my (1984) I called "the standard representor", assigning each concept represented by the phrase 'there are exactly n' to the number n. His question "How can we get from these concepts to the numbers of arithmetic in a way that cannot be faulted?" may be read as a call for a more informative specification of the standard representor. On the other hand, the Alternative theory, reflected in the semantics to be presented in Section 5, rejects the existence, and even the need for, a standard representor.

For 'x' replaceable as above, we enrich L^x to $L^{x,\#}$ by adding to the logical lexicon the expressions '$\#$', '$\underline{N}$' and '$\leqslant$', governed by formation rules that treat '$\underline{N}$' as a 1-place predicate-constant and '$\leqslant$' as a 2-place predicate-constant of type $(0, 0)$; furthermore:

> if φ is a formula of $L^{x,\#}$ and ν is a variable of type 0 then $(\# \nu)\varphi$ is a term of type 0.

(Of course the terms and formulae of $L^{x,\#}$ are defined by a simultaneous induction.)

Let $\varkappa$ be a representor for a model $\mathscr{A}$ iff $\varkappa$ is a one-one function from ${}^2\mathrm{EXACTLY}$ into $|\mathscr{A}|$. $\mathscr{A}$ has a representor iff $|\mathscr{A}|$ is infinite; we'll restrict our attention to such models. Let a Frege-model have the form $(\mathscr{A}, \varkappa)$ where $\varkappa$ is a representor for $\mathscr{A}$.[13] We now consider a semantics that models the Mathematical-Object theory of arithmetic discourse, in its Fregean version.

Expand $L^{x,\#}$ to $L^{x,\#}_{\mathscr{A}}$ as usual. We define $\mathrm{des}^{\mathscr{A},\varkappa}$ and $A, \varkappa \overset{\#}{\models}$ by relativizing the usual clauses to $\varkappa$ and adding these:

$$\mathrm{des}^{\mathscr{A}\varkappa}((\# \nu)\varphi) \simeq \varkappa(Q), \text{ if } \hat{\nu}\varphi^{\mathscr{A},\varkappa} \in Q \text{ for } Q \in {}^2\mathrm{EXACTLY};$$
$$\mathscr{A}, \varkappa \overset{\#}{\models} \underline{N}(\tau) \text{ iff } \mathrm{des}^{\mathscr{A},\varkappa}(\tau) \in \mathrm{Rng}(\varkappa);$$
$$\mathscr{A}, \varkappa \overset{\#}{\models} \tau \leqslant \sigma \text{ iff for some } n, m < \omega,\ \mathrm{des}^{\mathscr{A},\varkappa}(\tau) = \varkappa({}^2Q(n)),$$
$$\mathrm{des}^{\mathscr{A},\varkappa}(\sigma) = \varkappa({}^2Q(m)), \text{ and } n \leqslant m.$$

Notice that even if $\mathscr{A}$ is total there are type-0 terms of $L^{x,\#}$ that don't designate relative to $\mathscr{A}$, e.g. '$(\# x) \neg \bot$'.

In $(\mathscr{A}, \varkappa)$, $\varkappa$ represents the Mathematical-Object theorist's purported "standard representor". Thus members of $\mathrm{Rng}(\varkappa)$ represent the natural numbers, '$\underline{N}$' parses 'is a natural number', and '$\leqslant$' parses 'is less than or equal to' (standing for the ordering of the natural numbers induced by the "rule" that generates Frege's sequence of cardinality-quantifi-

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

ers). The relation between phrases like 'there are exactly four' and 'the number four' is modeled in $(\mathscr{A}, \varkappa)$ by that between ${}^2Q(4)$ and $\varkappa({}^2Q(4))$. The distinctively Fregean aspect of a Frege-model is that numbers are, along with all other objects, in the range of variables of type 0. '$\underline{N}$' represents an ordinary predicate: relative to $(\mathscr{A}, \varkappa)$ it definitely applies, or fails to apply, to any $a \in |\mathscr{A}|$. Similarly, the semantic-role of '$\leqslant$' consists in its applying or not applying to pairs of objects.

With a variation on this model-theoretic semantics, we can avoid introducing non-designating terms, provided of course that $\mathscr{A}$ is total. Let $\varkappa$ be a representor$^\infty$ for $\mathscr{A}$ iff $\varkappa$ is a one–one function from ${}^2\mathrm{EXACTLY}_\infty$ into $|\mathscr{A}|$. Let a Frege$^\infty$-model be have the form $(\mathscr{A}, \varkappa)$ for $\varkappa$ a representor$^\infty$ for $\mathscr{A}$. We extend our previous semantics with this further clause:

$$\mathrm{des}^{\mathscr{A},\varkappa}((\#\nu)\varphi) = \varkappa({}^2Q_\infty) \text{ iff } \hat{\nu}\varphi^{\mathscr{A},\varkappa} \text{ is infinite.}$$

This reconstructs a pre-Cantorian version of the Mathematical Object Theory, according to which infinity is a single object, which we may even call a number, and which is the cardinality of all infinite collections.[14] Thus in the Frege$^\infty$-model $(\mathscr{A}, \varkappa)$, $\varkappa({}^2\mathrm{Q}_\infty)$ represents this infinity. The rest of the definition of truth in such models runs as usual.

We may introduce numerals by following Frege's original definitions:

'$\bar{0}$' abbreviates $(\#\nu)\bot$;
$\overline{n+1}$ abbreviates $(\#\nu)\nu \leqslant \bar{n}$;
'$\bar{\infty}$' abbreviates $(\#\nu)\neg\bot$,

where ν is any type-0 variable. We may define successor by letting $\underline{s}(\tau)$ abbreviate $(\#\rho)(\rho \leqslant \tau \vee \neg\underline{N}(\tau))$, where τ is any term and ρ any variable of type-0. Note that if $\varkappa$ is a representor [representor$^\infty$] for $\mathscr{A}$ and $a \notin \mathrm{Rng}(\varkappa)$, $\mathrm{des}^{\mathscr{A},\varkappa}(\underline{s}(\underline{a})) \uparrow$ $[\mathrm{des}^{\mathscr{A},\varkappa}(\underline{s}(\underline{a})) = \varkappa({}^2\mathrm{Q}_\infty)]$.

Where $\Delta \cup \{\varphi\} \subseteq \mathrm{Sent}(L^{x,\#})$, let Δ Frege-entail [Frege$^\infty$-entail] φ iff:

> for every Frege-model [Frege$^\infty$-model] $(\mathscr{A}, \varkappa)$, if $\mathscr{A}, \varkappa \overset{\#}{\models} \Delta$ then $\mathscr{A}, \varkappa \overset{\#}{\models} \varphi$.

Define Frege-validity [Frege$^\infty$-validity] and other logical notions as usual. For each $n \in \omega$,

$E(\bar{n})$ is Frege-valid and Frege$^\infty$-valid;
$\neg E(\bar{\infty})$ is Frege-valid;
$E(\bar{\infty})$ is Frege$^\infty$-valid.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

We may now make precise the sense in which $L^{x,\#}$ under the first [second] of the above semantics encodes $L^{x,4}(\text{EXACTLY})$ $[L^{x,4}(\text{EXACTLY}^{\infty})]$.

Observation 2: there is a translation t_0 $[t_0^{\infty}]$,

$$t_0\colon \text{Sent}(L^{x,4}(\text{EXACTLY})) \to \text{Sent}(L^{x,\#})$$
$$[t_0^{\infty}\colon \text{Sent}(L^{x,4}(\text{EXACTLY}^{\infty})) \to \text{Sent}(L^{x,\#})]$$

so that for any Frege-model $(\mathscr{A}, \varkappa)$ and $\varphi \in \text{Sent}(L^{x,4}(\text{EXACTLY}))$ [any Frege$^{\infty}$-model $(\mathscr{A}, \varkappa)$ and $\varphi \in \text{Sent}(L^{x,4}(\text{EXACTLY}^{\infty}))$]:

$$\mathscr{A} \models \varphi \text{ iff } \mathscr{A}. \varkappa \overset{\#}{\models} t_0(\varphi)\ [t_0^{\infty}(\varphi)].$$

To form $t_0(\varphi)$, to each variable μ of type-2 or type-4 in φ associate a distinct new type-0 variable μ' not occurring in φ; replace subformulae of φ of the form $(\text{EXACTLY } \mu\nu)\theta$ by $\mu' = (\#\nu)\theta(\mu'/\mu))$; replace subformulae of the form $(\exists\mu)\psi$ by $(\exists\mu')(\underline{N}(\mu') \ \&\ \psi(\mu'/\mu))$; replace any terms of the form $(\#\mu)\theta$ for μ of type 2 that we may have created by $(\#\mu')\theta$; thus we eliminate all variables of types 2 and 4. The result is both $t_0(\varphi)$ and $t_0^{\infty}(\varphi)$.

There are no translations in the opposite direction. For example, if τ is an individual-constant, there is no $\varphi \in \text{Sent}(L^{(0.0),4}(\text{EXACTLY}))$ so that for any Frege-model $(\mathscr{A}, \varkappa)$:

$$\mathscr{A} \models \varphi \text{ iff } \mathscr{A}, \varkappa \overset{\#}{\models} \tau = \bar{0}.$$

Similarly for $L^{(0,0),4}(\text{EXACTLY}^{\infty})$ and Frege$^{\infty}$-models.

Notice that for any term τ of L^0 of type 0 and any $n \in \omega$, both $\bar{n} = \tau$ and $\bar{n} \neq \tau$ are not Frege-valid [nor Frege$^{\infty}$-valid], though they are well-formed and, of course, bivalent. Is this in keeping with the spirit of Frege's writings? Although he wanted '1 ≠ Julius Caesar' to come out true within his reconstruction of arithmetic, his permutation argument in Section 10 of *The Basic Laws* in effect (applied to numbers rather than to sets) showed that it was not a logical truth, at least as he had originally conceived of logic, and would have to be added as a separate stipulation.[15] Suffice to say: the model-theoretic reconstruction of arithmetic just presented shows why such identities should be sources of discomfort for Frege: his refusal to make type distinctions between objects makes such sentences well-formed; but they are not encodings of sentences of $L^{(0,0),4}(\text{EXACTLY})$. If our sole grasp of numbers is as encodings of finite-cardinality quantifiers (as claimed in my 1984a article, p. 136), then we have no justification for regarding '1 = Julius Caesar' as true or as false.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

Observation 3. There is a translation s^{∞},

$$s^{\infty}\colon \mathrm{Sent}(L^{x,\#}) \to \mathrm{Sent}(L^{x,\#}),$$

so that for any Frege$^{\infty}$-model $(\mathscr{A}, \varkappa)$ and any $\varphi \in \mathrm{Sent}(L^{x,\#})$:

$$\mathscr{A}, \varkappa \restriction {}^{2}\mathrm{EXACTLY} \overset{\#}{\models} \varphi \text{ iff } \mathscr{A}, \varkappa \overset{\#}{\models} s^{\infty}(\varphi).$$

For any term τ of $L^{x,\#}$, let D_τ be the disjunction of all formula of the form $\sigma = \bar{\infty}$ for all $\#$-terms σ that are subterms of τ. So $\mathrm{des}^{\mathscr{A}\varkappa \restriction {}^{2}\mathrm{EXACTLY}}(\tau)\uparrow$ iff $\mathscr{A}, \varkappa \overset{\#}{\models} \mathscr{D}_\tau \vee \neg E(\tau)$. Given $\varphi \in \mathrm{Sent}(L^{x,\#})$, form $s^{\infty}(\varphi)$ by replacing all subformulae of φ of the form $\tau = \sigma$ by:

$$[(D_\tau \vee \neg E(\tau)) \;\&\; (\mathrm{D}_\sigma \vee \neg \mathrm{E}(\sigma))] \vee [\neg \mathrm{D}_\tau \;\&\; \neg D_\sigma \;\&\; \tau = \sigma],$$

and replacing all subformulae of the form $\zeta(\tau_0, \ldots, \tau_{n-1})$ by:

$$\zeta(\tau_0, \ldots, \tau_{n-1}) \;\&\; \neg D_{\tau_0} \;\&\; \cdots \;\&\; \neg D_{\tau_{n-1}},$$

where ζ is either a predicate-constant, '$\underline{N}$', '$\leqslant$', or a variable of type 1 or $(0, 0)$.

If S is non-empty then s^{∞} cannot be reversed: there is a $\varphi \in \mathrm{Sent}(L^{0,\#})$ and a Frege$^{\infty}$-model $(\mathscr{A}, \varkappa)$ so that for any $\varphi \in \mathrm{Sent}(L^{(0,0),\#})$ we do *not* have:

$$\mathscr{A}, \varkappa \overset{\#}{\models} \varphi \text{ iff } \mathscr{A}, \varkappa \restriction {}^{2}\mathrm{EXACTLY} \overset{\#}{\models} \psi.$$

For example, for a one-place predicate-constant $\zeta \in S$, take φ to be $\zeta(\bar{\infty})$. Fix $\mathscr{A}$, a_0 and a_1 so that $\mathscr{A} \models \zeta(\underline{a}_0)$ and $\mathscr{A} \not\models \zeta(\underline{a}_1)$; let $\varkappa$ be a representor for $\mathscr{A}$ with $a_0, a_1 \notin \mathrm{Rng}(\varkappa)$; let $\varkappa_i = \varkappa \cup \{\langle {}^{2}Q_{\infty}, a_i\rangle\}$ for $i \in 2$. If there is a ψ as above,

$$\mathscr{A}. \varkappa_0 \overset{\#}{\models} \zeta(\bar{\infty}) \text{ iff } \mathscr{A}, \varkappa_1 \overset{\#}{\models} \zeta(\bar{\infty}),$$

contrary to the construction.

5.

The remarks of Section 1 may now be made more precise. To adopt the Mathematical-Object Picure of "the numbers of arithmetic" is to speak, think and reason (1) in a sense-bearing language $\mathscr{L}$ whose logical syntax is modeled (in the engineering sense) by non-sense-bearing languages of the form $L^{x,\#}$, and (2) with a logic whose entailment relation is at least partially modeled by Frege-entailment [or Frege$^{\infty}$-entailment]. The last condition amounts to this: if a valid inference in $\mathscr{L}$ can be

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

regimented by Δ/φ in $L^{x,\#}$ then Δ Frege-entails [or Frege$^{\infty}$-entails, depending on how infinity is handled in $\mathscr{L}$] φ.

To accept the Fregean version of the Mathematical-Object Theory is, at least in part, to think that (3) Frege-models [or Frege$^{\infty}$-models] model the basic referential facts the distribution of truth and falsity for statements in $\mathscr{L}$, and so that (3.1) designation relative to a Frege-model [or a Frege$^{\infty}$-model] models "real live" designation for singular terms of $\mathscr{L}$, and (3.2) truth relative to a Frege-model [Frege$^{\infty}$-model] models "real live" truth for statements in $\mathscr{L}$. Perhaps that theory should also maintain that (4) entailment in $\mathscr{L}$ is completely modeled by Frege-entailment [or Frege$^{\infty}$-entailment]; that is, if a non-entailment in $\mathscr{L}$ can be parsed in $L^{x,\#}$ as Δ/φ then Δ doesn't Frege-entail [Frege$^{\infty}$-entail] φ.

The Mathematical-Object theory is committed to the existence of a standard representor. For the reasons given in (Hodes 1984a), this is an unreasonable doctrine. (That it makes the question of whether 1 is identical to Julius Caesar well-defined is a hint that it's unreasonable.) According to the Alternative theory, the numbers of finite arithmetic "are" merely devices for encoding higher-order statements, statements that could be parsed in a language of the form $L^{x,4}$(EXACTLY) [or $L^{x,4}$(EXACTLY$^{\infty}$)], into a lower-order syntactic form, namely statements that could be parsed in a language of the form $L^{x,\#}$. For that task any one-one assignment of cardinality-quantifiers to objects will do. This thought suggests a model-theoretic semantics that represents the Fregean version of the Mathematical-Object picture under its Alternative construal.

We restrict attention to infinite models. For an infinite model $\mathscr{A}$, a type-0 term τ of $L^{x,\#}$, and $\varphi \in \mathrm{Sent}(L^{x,\#})$, let:

$\mathscr{A} \models \varphi$ iff for every representor $\varkappa$ for $\mathscr{A}$ $\mathscr{A}, \varkappa \overset{\#}{\models} \varphi$;
$\mathscr{A} \dashv \varphi$ iff for every representor $\varkappa$ for $\mathscr{A}$ $\mathscr{A}, \varkappa \overset{\#}{\not\models} \varphi$;
$\mathrm{des}^{\mathscr{A}}(\tau) = a$ iff for every representor $\varkappa$ for $\mathscr{A}$ $\mathrm{des}^{\mathscr{A},\varkappa}(\tau) = a$;
$\mathrm{des}^{\mathscr{A}}(\tau) \uparrow$ iff there is no a so that $\mathrm{des}^{\mathscr{A},\varkappa}(\tau) = a$.

This model-theoretic semantics is three-valued: $\mathscr{A} \models$ represents truth, $\mathscr{A} \dashv$ represents falsity. Let:

$\mathscr{A} \mid \varphi$ iff $\mathscr{A} \not\models \varphi$ and $\mathscr{A} \not\dashv \varphi$.

It also extends the semantics given in Section 2: for $\varphi \in \mathrm{Sent}(L^{(0,0)})$ and a term τ based on S:

$\mathscr{A} \models \varphi$ [$\mathscr{A} \dashv \varphi$] under the semantics of Section 2 iff $\mathscr{A} \models \varphi$ [$A \not\models \varphi$] under this definition;

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

$\mathrm{des}^{\mathscr{A}}(\tau) = a$ under the semantics of Section 2 iff $\mathrm{des}^{\mathscr{A}}(\tau)$ under this definition.

For $\Delta \cup \{\varphi\} \subseteq \mathrm{Sent}(L^{x,\#})$, let Δ ∞-entail φ iff:

for every infinite model $\mathscr{A}$, if $\mathscr{A} \models \Delta$ then $\mathscr{A} \models \varphi$.

We adopt these definitions:

φ is ∞-valid iff $\{\ \}$ ∞-entails φ;
φ is ∞-invalid iff $\{\ \}$ ∞-entails $\neg\varphi$;
φ is ∞-truth-valueless iff for any infinite total model $\mathscr{A}$, $\mathscr{A} \mid \varphi$;
φ is ∞-bivalent iff for any infinite total model $\mathscr{A}$, either $\mathscr{A} \models \varphi$ or $\mathscr{A} \dashv \varphi$;
φ is positively ∞-equivalent to ψ iff for any infinite model $\mathscr{A}$, $\mathscr{A} \models \varphi$ iff $\mathscr{A} \models \psi$;
φ is ∞-equivalent to ψ iff φ is positively ∞-equivalent to ψ and $\neg\varphi$ is ∞-positively ∞-equivalent to $\neg\psi$.

Let ∞-Biv$(L^{x,\#})$ be the set of ∞-bivalent sentences of $L^{x,\#}$.

Analogously, where $\mathscr{A}$ is infinite, let:

$\mathscr{A} \overset{\infty}{\models} \varphi$ iff for every representor$^{\infty}$ $\varkappa$ for $\mathscr{A}$ $\mathscr{A}, \varkappa \overset{\#}{\models} \varphi$;
$\mathscr{A} \overset{\infty}{\dashv} \varphi$ iff for every representor$^{\infty}$ $\varkappa$ for $\mathscr{A}$ $\mathscr{A}, \varkappa \overset{\#}{\not\models} \varphi$.

Define $\mathrm{des}^{\infty\mathscr{A}}$ analogously. ∞-Entailment$^{\infty}$, ∞-validity$^{\infty}$, etc. are to be defined in the obvious way.

For the Alternative theorist, infinite models, rather than Frege-models [or Frege$^{\infty}$-models] model the basic semantic facts underlying the distribution of truth and falsity to statements in arithmetic discourse. Given an infinite model $\mathscr{A}$, $\mathscr{A} \models$ and $\mathscr{A} \dashv$, defined as above, represent real live truth and falsity, and $\mathrm{des}^{\mathscr{A}}$ represents designation. Notice: under this semantics, a #-term does not contribute to determining the truth-value-in-$\mathscr{A}$ of a sentence in which it occurs by designating something! For a #-term τ, $\mathrm{des}^{\mathscr{A}}(\tau)$ is undefined; τ makes its contribution in the more roundabout way implicit in the above definitions of truth and falsity in $\mathscr{A}$. Rather than designating objects, we may say that #-terms encode quantifiers; for example, for every $n \in \omega$, $\bar{n}$ encodes ${}^{2}Q(n)$ and ${}^{4}Q(n)$.[16]

Of course $E(\bar{n})$ is ∞-valid, even though $\bar{n}$ does not designate. Furthermore '$\neg E(\bar{\infty})$' is ∞-valid and '$E(\bar{\infty})$' is ∞-valid$^{\infty}$. Keep in mind that under this semantics 'The number one exists' and 'The Eiffel Tower exists' would both be parsed with 'E'; there is no basis for saying that

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

'exists' bears two different senses in these two truths. The obvious parsing of 'There are natural numbers', $(\exists\nu)\underline{N}(\nu)$ $[(\exists\nu)(\underline{N}(\nu)\ \&\ \bar{\infty} \neq \nu)]$, is trivially ∞-valid [∞-valid$^{\infty}$].

Similarly, '$\underline{N}$' ['$\leqslant$'] does not stand for a level one concept or property [a relation], or even have an extension in any model. Rather it encodes a concept [relation] of type 3 [type (2, 2)] (and thereby also one of type 5 [type (4, 4)]). And the quantificational context $(\exists\nu)(\underline{N}(\nu)\ \&\ \ldots)$ doesn't express quantification over a restricted class of objects. It's best thought of as encoding quantification over cardinality-quantifiers of type 2 (and thereby also of type 4).

Our definitions of $\models$ and $\dashv$ relative to $\mathscr{A}$ use $\overset{\#}{\models}$ relative to Frege-models of the form $(\mathscr{A}, \varkappa)$, and thereby also using $\mathrm{des}^{\mathscr{A},\varkappa}$. This might lead one to object to the contention that the above model-theoretic semantics gives the Alternative theorist a suitable picture of the semantics of real arithmetic discourse. Must there be something wrong with the claim that $\mathscr{A}\models$, rather than $\mathscr{A}, \varkappa \overset{\#}{\models}$, represents truth? Or the claim that $\mathrm{des}^{\mathscr{A}}$, rather than $\mathrm{des}^{\mathscr{A},\varkappa}$, represents designation? If so, we have not captured the semantic thesis central to the Alternative theory – that singular terms may contribute to determining the truth-value of sentences without having to be designators: after all, functions of the form $\mathrm{des}^{\mathscr{A},\varkappa}$ are defined on some #-terms.

This worry rests on a misunderstanding. It is up to whoever offers a model (in the engineering sense) for some purported phenomenon or range of facts to decide which features of that model are to represent features of what is being modeled, and which features are mere artifacts. The Alternative theorist rules truth in Frege-models and designation in Frege-models to be, varying David Kaplan's phrase, artifacts of his model-theory.[17] Artifacts of a model can do essential work in enabling that model to do its modeling thing, but be artifacts nonetheless. (Think of a model of a molecule constructed from sticks and styrofoam balls; the balls and the spatial relations between them represent atoms and the spatial relations between the atoms; the sticks play an important role in permitting this model to serve that function, though they need not represent any aspect of the molecules.) For each infinite $\mathscr{A}$, such work is done by $\mathscr{A}, \varkappa \overset{\#}{\models}$ and by $\mathrm{des}^{\mathscr{A},\varkappa}$, for each representor $\varkappa$ on $\mathscr{A}$. For the Mathematical-Object theorist, they represent truth and designation. But according to the Alternative theory, they are supervaluational artifacts, stepping-stones to the definition of what really matters: $\mathscr{A}\models$ and $\mathscr{A}\dashv$. This difference is reflected in differences between

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

the logic of the Fregean and of the Alternative theorist: the latter's logic allows non-∞-bivalent and even ∞-truth-valueless sentences; furthermore, the Alternative theory's notion of ∞-entailment [∞-entailment$^{\infty}$] properly includes that of Frege-entailment [Frege-entailment$^{\infty}$].

Our translation t_0 [t_0^{∞}] still encodes $L^{x,4}$(EXACTLY) into $L^{x,\#}$; but the Alternative theorist does not think of this encoding as involving any special sort of objects.

Observation 4: for any $\varphi \in \mathrm{Sent}(L^{x,4}(\text{EXACTLY}))$ [$\mathrm{Sent}(L^{x,4}(\text{EXACTLY}^{\infty}))$] and any infinite model $\mathscr{A}$:

$$\mathscr{A} \models \varphi \text{ iff } \mathscr{A} \models t_0(\varphi) \; [\mathscr{A} \overset{\infty}{\models} t_0^{\infty}(\varphi)];$$

furthermore t_0 [t_0^{∞}] maps into ∞-Biv($L^{x,\#}$).[18]

Thus the switch from $\overset{\#}{\models}$ to $\models$ doesn't lose what we most want: the encoding power of $L^{x,\#}$. But some of what we don't want is lost. For example, for any individual constant τ, $\overline{1} = \tau$ is ∞-truth-valueless. (Notice that if $\mathrm{des}^{\mathscr{A}}(\tau)\uparrow$ then $\mathscr{A} \dashv \overline{1} = \tau$; this phenomenon suggested putting the restriction to total models into the definitions of ∞-truth-valueless and ∞-bivalence.) We now can say exactly what is peculiar about a sentence like '1 = Julius Caesar': on grounds of logic alone it lacks truth-value. More generally, if σ is a closed #-term and τ is a closed type-0 term based on S, $\sigma = \tau$ is ∞-truth-valueless; but if σ and τ are either both closed #-terms or both closed type-0 terms based on S, $\sigma = \tau$ is not, and in the latter case it's ∞-bivalent. The ∞-truth-valuelessness of equations between closed #-terms and closed terms based on S explains at least some of the motivation for the doctrine that numbers constitute a separate logical type. But this semantics makes that doctrine false: for τ based on S, $\overline{1} = \tau$ is a sentence of $L^{0,\#}$; correspondingly '1 = Julius Caesar' should be considered a sentence of English, though one lacking a truth-value for logical reasons. Note: if τ is not a #-term then $\underline{N}(\tau)$ is ∞-truth-valueless; so under this semantics '$\underline{N}$' does not represent an ordinary predicate. Indeed, under this semantics the semantic role of 'N' ['$\leqslant$'] does not consist in applying or not applying to [tuples of] objects. Similar remarks apply to the semantics based on $\overset{\infty}{\models}$ and $\overset{\infty}{\dashv}$.

In fact, in a sense made precise by the following, all of what we don't want (viz., the baggage carried by the Mathematical-Object theory and mentioned in the paragraph after Observation 3) is lost.

Observation 5: there is a translation t_1 [t_1^{∞}] with

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

$$t_1\ [t_1^{\infty}]\colon \mathrm{Sent}(L^{(0,0),\#}) \to \mathrm{Sent}(L^{(0,0)}),$$

so that for any $\varphi \in \mathrm{Sent}(L^{(0,0),\#})$ and any infinite model $\mathscr{A}$:

$$(*)\qquad \begin{array}{l} \mathscr{A} \models \varphi \text{ iff } \mathscr{A} \models t_1(\varphi) \\ [\mathscr{A} \overset{\infty}{\models} \varphi \text{ iff } \mathscr{A} \models t_1^{\infty}(\varphi)]; \end{array}$$

in particular, for $\varphi \in \infty\text{-Biv}(L^{(0,0),\#})$ $[\infty\text{-Biv}^{\infty}(L^{(0,0),\#})]$, we also have:

$$(**)\qquad \begin{array}{l} \mathscr{A} \dashv \varphi \text{ iff } \mathscr{A} \not\models t_1(\varphi) \\ [\mathscr{A} \overset{\infty}{\dashv} \varphi \text{ iff } \mathscr{A} \not\models t_1^{\infty}(\varphi)]. \end{array}$$

Proof. For $\psi, \theta \in \mathrm{Fml}(L_{\mathscr{A}}^{(0,0)})$ in which at most the type-0 variable ν is free, let $(\leqslant\nu)(\psi, \theta)$ be a sentence of $L_{\mathscr{A}}^{(0,0)}$ that is true in $\mathscr{A}$ iff $\mathrm{card}(\hat{\nu}\psi^{\mathscr{A}}) \leqslant \mathrm{card}(\hat{\nu}\theta^{\mathscr{A}})$. Where $\varkappa$ is a representor for $\mathscr{A}$ and $n \in \omega$, let $a_n = \varkappa({}^2Q(n))$. Let R code $\varkappa$ iff $R = \{\langle a_n, a_m\rangle : n < m < \omega\}$. For $\gamma \in \mathrm{Var}((0,0))$ there is a formula $\mathrm{Std}(\gamma)$ of $L^{(0,0)}$ so that for any model $\mathscr{A}$ and $R \subseteq |\mathscr{A}|^2$:

$$\mathscr{A} \models \mathrm{Std}(\underline{R}) \text{ iff } R \text{ codes a representor for } \mathscr{A}.$$

Given $\varphi \in \mathrm{Sent}(L^{(0,0),\#})$, fix $\gamma \in \mathrm{Var}((0,0))$ not occurring in φ. We'll construct a formula $\varphi'(\gamma)$ with γ free so that for any Frege-model $(\mathscr{A}, \varkappa)$ and any R coding $\varkappa$:

$$\mathscr{A}, \varkappa \overset{\#}{\models} \varphi \text{ iff } \mathscr{A} \models \varphi'(\underline{R}).$$

We may then let $t_2(\varphi)$ be $(\forall\gamma)(\mathrm{Std}(\gamma) \supset \varphi'(\gamma))$.

Construct φ' as follows. First transform φ so that for every atomic subformula $\zeta(\tau_0, \ldots, \tau_{n-1})$, none of the τ_is are #-terms; e.g. replace $\zeta(\ldots, (\#\rho)\theta, \ldots)$ by

$$(\exists\nu)(\nu = (\#\rho)\theta \ \&\ \zeta(\ldots, \nu, \ldots)),$$

where ν is a type-0 variable not occurring in φ. Make sure that if τ is a #-term in φ then τ only occurs in the contexts $\tau = \rho$ where ρ is a type-0 variable, or $\tau = \sigma$ or $\sigma = \tau$ where σ is a #-term. Also make sure that each variable is bound at most once. Make these replacements in the resulting sentence:

$\underline{N}(\tau)$ by $(\exists\nu)\gamma\tau\nu$;
$\tau \leqslant \sigma$ by $(\exists\nu)\gamma\tau\nu \ \&\ (\exists\nu)\gamma\sigma\nu \ \&\ (\leqslant\nu)(\gamma\nu\tau, \gamma\nu\sigma)$;
$(\#\nu)\theta = \rho$ by $(\exists\nu)\gamma\rho\nu \ \&\ (\leqslant\nu)(\theta, \gamma\nu\rho) \ \&\ (\leqslant\nu)(\gamma\nu\gamma, \theta)$;
$(\#\nu)\theta = (\#\rho)\theta'$ by $(\leqslant\delta)(\theta(\delta/\nu),\ \theta'(\delta/\rho)) \ \&\ (\leqslant\delta)(\theta'(\delta/\rho), \theta(\delta/\nu))$.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

Here ν is a type-0 variable not occurring in τ, σ or ρ, and δ is a type-0 variable not occurring in θ or θ'. The construction of $t_1^{\infty}(\varphi)$ is similar.

Obviously we cannot add in Observation 5 the further requirement that both (*) and (**) hold for all infinite $\mathscr{A}$ and all $\varphi \in \mathrm{Sent}(L^{(0,0),\#})$. Our semantics for $L^{(0,0)}$, is two-valued; but φ may lack a truth-value in $\mathscr{A}$, in which case (*) and (**) don't both hold.

Suppose we were to adopt a three-valued semantics for $L^{(0,0)}$, giving '$\supset$' (and hence '&' and '$\vee$') the so-called "strong Kleene" truth-table and including '$\underline{u}$' as a primitive formula governed by the clauses

$$\text{for any model } \mathscr{A},\ \mathscr{A} \not\models \text{`}\underline{u}\text{'} \text{ and } \mathscr{A} \not\dashv \text{`}\underline{u}\text{'}.$$

(Keep in mind that in a three-valued semantics, truth and falsity ($\dashv$) are defined together in one simultaneous induction; see Appendix 2.) Then letting $t_1(\varphi)$ be:

$$(\forall\gamma)(\mathrm{Std}(\gamma) \supset \varphi'(\gamma)) \vee (\underline{u}\ \&\ (\exists\gamma)(\mathrm{Std}(\gamma)\ \&\ \varphi'(\gamma)))$$

we have (*) and also:

$$\mathscr{A} \dashv \varphi \text{ iff } \mathscr{A} \dashv t_1(\varphi).$$

Further facets of Observation 5 are discussed in Section 6.

Observations 4 and 5 are important for the Alternative theory. The Alternative theory maintains that the sole point of discourse within the Mathematical-Object picture is to conveniently encode higher-order statements, statements that could be parsed by sentences in a language that is really a fragment of $L^{(0,0)}$. By Observation 4, bivalent statements do the encoding, even though the syntax of the encoding language will generate non-bivalent statements. The failure of Observations 5 would have suggested that our mathematical practice, which uses bivalent statements that may be parsed in $L^{x,\#}$, involved more than such encoding.

We now must look at an apparent difficulty. According to the Alternative theory, to accept of the Mathematical-Object Picture amounts to using a language modeled by $L^{x,\#}$ under the three-valued semantics just given, instead of one of the form $L^{x,4}$(EXACTLY). But is our reasoning even approximated by the logic our semantics imposes on $L^{x,\#}$? For example, our mathematical practice apparently sanctions use of '$\supset$'-introduction and '$\vee$'-elimination. But these rules are not sound with respect to ∞-entailment [∞-entailment$^{\infty}$]! That is to say, the following principles do *not* hold for all $\Delta \cup \{\varphi, \psi, \theta\} \subseteq \mathrm{Sent}(L^{x,\#})$:

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

If $\Delta \cup \{\varphi\}$ ∞-entails θ then Δ ∞-entails $(\varphi \supset \theta)$;
If Δ ∞-entails $(\varphi \vee \psi)$, $\Delta \cup \{\varphi\}$ ∞-entails θ, and $\Delta \cup \{\psi\}$ ∞-entails θ then Δ ∞-entails θ.

[Similarly for ∞-entailment$^{\infty}$.] For example, let '$\underline{P}$' be a one-place predicate-constant. Then '$(\underline{P}(\bar{0})) \vee \neg \underline{P}(\bar{0}))$' is ∞-valid; furthermore: '$\underline{P}(\bar{0})$' ∞-entails '$(\forall x)\underline{P}(x)$', and '$\neg \underline{P}(\bar{0})$' ∞-entails '$(\forall x) \neg \underline{P}(x)$'. (In fact '$\underline{P}(\bar{0})$' is positively ∞-equivalent to '$(\forall x)\underline{P}(x)$', and similarly for the second pair.) So '$\vee$'-elimination would require that '$(\forall x)\underline{P}(x) \vee (\forall x) \neg \underline{P}(x)$' be ∞-valid; and it isn't. Similarly, '$\exists$'-elimination is not sound.

Can the Alternative theorist simply reject these rules? No: the Alternative theory is offered as an account of the semantic underpinnings of mathematical practice within the Mathematical-Object picture, not a program to change it. If our reasoning in a language modeled by $L^{x,\#}$ uses these rules, the Alternative theory should revise its account of the basis for the use of such rules, not tell people to abandon them.

But does mathematical practice within the Mathematical-Object Picture involve acceptance of these rules *in full generality*? As mentioned above, our actual mathematical reasoning makes no use of sentences parsed by non-∞-bivalent sentences of $L^{x,\#}$. On the Alternative theory, that is no surprise: such sentences do not encode sentences representable in $L^{x,4}$(EXACTLY). That encoding provides the rationale for the Mathematical Object Picture; so such sentences should be expected to do no work in mathematical practice within that picture. (This is why, when first faced with such sentences, e.g., '0 is green', we're likely not to know what to make of them.) Thus our mathematical practice should, and does, involve only limited use of '$\vee$'-elimination, use confined to cases in which φ and ψ parse bivalent sentences; use of '$\supset$'-introduction and '$\exists$-elimination are also appropriately restricted. I can see no reason to insist that our practice involves acceptance of these rules in unrestricted forms. And when we restrict to ∞-bivalent sentences, our semantics reconstructs makes these rules sound. The Alternative theory need claim only that ∞-entailments involving ∞-bivalent sentences of $L^{x,\#}$ parse the inferences sanctioned by our mathematical practice; other ∞-entailments needn't reflect our inferential practices. Although '$\underline{P}(\bar{0})$' ∞-entails '$(\forall x)\underline{P}(x)$', we don't sanction the inference from '0 is green' to 'Everything is green'.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

Let φ be a number-theoretic sentence of $L^{(0,0),\#}$ iff:

> φ containing no occurences of members of S;
> all occurrences of $(\exists\nu)$ for ν of type-0 in φ are restricted by $\underline{N}(\nu)$;
> all occurrences of $(\exists\gamma)$ for γ of type-1 in φ are restricted by $(\forall\nu)(\gamma(\nu) \supset \underline{N}(\nu))$;
> and similarly for γ of type-$(0, 0)$.

For such a φ and any infinite model $\mathscr{A}$, $\mathscr{A} \models \varphi$ iff φ is ∞-valid. So our model-theory might be thought to reconstruct one central thesis of logicism: for number-theoretic sentences, truth is a sort of logical validity (or better, a sort of logical consequence of the assumption that there are infinitely many objects)! The significance of this will, of course, depend on how we understand the 'logic' in 'logicism', an issue which shall not be addressed here.

6.

In this section we'll take a more careful look at Observation 5; then we'll look at some technical questions about the logic introduced in Section 5. This section is primarily for logicians, and may be skipped by readers whose interest is "purely philosophical".

(1) The construction used in Observation 5 used a variable of type $(0, 0)$, even when φ contains no such variables. Is that required? In the presence of type-1 variables, the answer is "Yes". We'll construct $\varphi \in \infty\text{-Biv}(L^{1,\#})$ so that there is no $\psi \in \text{Sent}(L^{1,4}(\text{EXACTLY}))$ with φ ∞-equivalent to ψ. Let $S = \{\text{'}\underline{P}\text{'}, \text{'}\underline{Q}\text{'}, \text{'}\underline{R}\text{'}\}$; we construct φ so that for any infinite $\mathscr{A}$:

$$\mathscr{A} \models \varphi \text{ iff } \text{card}(\underline{P}^{\mathscr{A}}) \cdot \text{card}(\underline{Q}^{\mathscr{A}}) = \text{card}(\underline{R}^{\mathscr{A}}) < \aleph_0.$$

Let $\varphi'(\alpha)$ be:

$$(\forall\nu)(\alpha\nu \supset \underline{N}\nu) \;\&\; \alpha((\#\nu)\underline{P}\nu) \;\&\; (\forall\rho)(\alpha\rho \supset (\#\nu)\underline{P}\nu \leq \rho) \;\&\; \alpha((\#\nu)\underline{R}\nu) \;\&\; (\forall\rho)(\alpha\rho \supset \rho \leq (\#\nu)\underline{R}\nu) \;\&\; (\#\nu)\alpha\nu = (\#\nu)\underline{Q}\nu \;\&\; (\forall\nu_0)(\forall\nu_1)[(\alpha\nu_0 \;\&\; \alpha\nu_1 \;\&\; \nu_0 < \nu_1 \;\&\; \neg(\exists\nu)(\nu_0 < \nu \;\&\; \nu < \nu_1)) \supset (\#\nu)(\nu_0 \leq \nu \;\&\; \nu < \nu_1) = (\#\nu)\underline{P}\nu],$$

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

where α is a type-1 variable and all other indicated variables are of type-0 and are distinct from one another; let φ be

$$((\#\nu)\underline{Q}\nu = \bar{0} \ \& \ (\#\nu)\underline{R}\nu = \bar{0}) \vee (\exists\alpha)\varphi'(\alpha).$$

It's easy to see that φ is as desired. The key idea is that if $\mathscr{A}'$ is an expansion of $\mathscr{A}$ to a model for {'$\underline{P}$', '$\underline{Q}$', '$\underline{R}$', '$\underline{S}$'}, $\mathscr{A}, \varkappa \models \varphi'(\underline{S}/\alpha)$, and neither $\underline{Q}^{\mathscr{A}}$ nor $\underline{R}^{\mathscr{A}}$ is empty then $\underline{S}^{\mathscr{A}'}$ represents $\{i \cdot \mathrm{card}(\underline{P}^{\mathscr{A}}) : 1 \leqslant i \leqslant \mathrm{card}(\underline{Q}^{\mathscr{A}})\}$.

Suppose that ψ were ∞-equivalent to φ. For distinct type-1 variables $\alpha_0, \alpha_1, \alpha_2$ form ψ' from ψ by replacing '$\underline{P}$', '$\underline{Q}$', '$\underline{R}$' by $\alpha_0, \alpha_1, \alpha_2$ respectively. Then

$$(\exists\alpha_0)(\exists\alpha_1)(\exists\alpha_2)(\psi' \ \& \ (\text{EXACTLY } \mu_0\nu)\alpha_0\nu \ \& \\ (\text{EXACTLY } \mu_1\nu)\sigma_1\nu \ \& \ (\text{EXACTLY } \mu_2\nu)\alpha_2\nu)$$

defines ($\aleph_0$-defines, in the terminology of (Hodes 1988b)) multiplication on the finite cardinals within $L^{1,4}(\text{EXACTLY})$, where μ_0, μ_1, μ_2 are distinct type-2 variables; but no such formula exists by Theorem 2 of (Hodes 1988b). The essential point here is that the coding-down of type-2 to type-0 permits quantification of type-1 variables to act like quantification of type-3 variables; but nothing like quantification of type-3 variables is available in $L^{1,4}(\text{EXACTLY})$.

Conjecture 1: for each $\varphi \in \infty\text{-Biv}(L^{0,\#})$ there is a $\psi \in \mathrm{Sent}(L^{0,4}(\text{EXACTLY}))$ ∞-equivalent to φ. In two cases this conjecture may be easily seen to hold. (i) Let (Omega $\nu_0\nu_1)\theta$ be the obvious formula saying that, in the variables ν_0 and ν_1, θ defines an ordering of type ω. Suppose that there is a finite sequence $\theta_0, \ldots, \theta_{n-1}$ of formulae of $L^{0,\#}$ in each of which exactly the distinct variables ν_0 and ν_1 are free, and such that $\varphi[\neg\varphi]$ entails $\vee_{i<n}(\text{Omega } \nu_0\nu_1)\theta_i$. Then

$$\vee_{i<n}((\text{Omega } \nu_0\nu_1)\theta_i \ \& \ \varphi(\theta_i)) \\ [\&_{i<n}((\text{Omega } \nu_0\nu_1)\theta_i \supset \varphi(\theta_i))]$$

is as desired. (ii) If all members of S are at most 1-place, such a ψ may be constructed. The proof of this sheds no light on the general conjecture, and is left to the reader.

(2) We had to restrict Conjecture 1 to $\varphi \in \infty\text{-Biv}(L^{x,\#})$. To see why, introduce a further one-place predicate '$\underline{S}$'. Suppose that for $\psi \in \mathrm{Sent}(L^{0,4}(\text{EXACTLY}))$ we have for every infinite model $\mathscr{A}$ for {'$\underline{P}$', '$\underline{Q}$', '$\underline{R}$', '$\underline{S}$'}: $\mathscr{A} \models \neg\varphi'(\underline{S}/\alpha)$ iff $\mathscr{A} \models \psi$, where φ' is as in (1). Then:

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

$$\mathscr{A} \models \psi \;\&\; \neg((\#\nu)\underline{Q}\nu = \bar{0} \;\&\; (\#\nu)\underline{R}\nu = \bar{0})$$
$$\text{iff } \mathrm{card}(\underline{P}^{\mathscr{A}}) \cdot \mathrm{card}(\underline{Q}^{\mathscr{A}}) \neq \mathrm{card}(\underline{R}^{\mathscr{A}});$$

by existential quantifying-out the predicate-constants as done in (1), this can be turned into a $\aleph_0$-definition of multiplication in $L^{1,4}$(EXACTLY), violating Theorem 2 of (Hodes 1988b). Notice that if $\mathrm{card}(\underline{P}^{\mathscr{A}}) \cdot \mathrm{card}(\underline{Q}^{\mathscr{A}}) = \mathrm{card}(\underline{R}^{\mathscr{A}}) \neq 0$ then there are representors $\varkappa_0$ and $\varkappa_1$ for $\mathscr{A}$ so that $\mathscr{A}, \varkappa_0 \overset{\#}{\models} \varphi'(\underline{S}/\alpha)$ and $\mathscr{A}, \varkappa_1 \overset{\#}{\not\models} \varphi'(\underline{S}/\alpha)$; thus $\neg\varphi(\underline{S}/\alpha)$ isn't ∞-bivalent.

(3) For $\varphi \in \mathrm{Fml}(L^{x,\#})$ with free variables among $\nu_0, \ldots, \nu_{n-1}$, all of type-0, let $\varphi(\vec{\nu})$ define $V \subseteq \omega^n$ iff for every infinite model $\mathscr{A}$ and every $\vec{m} = \langle m_0, \ldots, m_{n-1}\rangle \in \omega^n$:

$$\text{if } \vec{m} \in V \text{ then } \mathscr{A} \models \varphi(\bar{m}_0, \ldots, \bar{m}_{n-1});$$
$$\text{if } \vec{m} \notin V \text{ then } \mathscr{A} \not\models \varphi(\bar{m}_0, \ldots, \bar{m}_{n-1}).$$

It's easy to see that if V is definable in $L^{x,\#}$ then V is defined by a pure formula of $L^{x,\#}$, that is, one based on $S = \{\ \}$; see (Hodes 1988b), Section 1. Using the construction from (1) it's easy to see that relations on ω definable in $L^{1,\#}$ are exactly those definable in second-order arithmetic. In fact the relations on ω definable in $L^{0,\#}$ are exactly those definable in Presburger arithmetic. Addition is defined by

$$\mu_1 = (\#\nu)(\nu < \mu_0 \vee (\mu_0 \leqslant \nu \;\&\; \nu < \mu_2)).$$

Suppose a formula φ of $L^{0,\#}(\{\ \})$ defines $R \subseteq \omega^k$. Consider the model $\mathscr{M}_2(\aleph_0)$ from (Hodes 1988b), Section 2, and let $\varkappa({}^2Q(n)) = n$ for all finite n. By treating '$\underline{N}(\nu)$' as '$\neg\perp$' we can translate φ to a formula φ' of L_2(EXACTLY) (see (Hodes 1988b, start of Section 2) so that

$$\mathscr{M}_2(\aleph_0), \varkappa \overset{\#}{\models} \varphi(\bar{m}_0, \ldots, \bar{m}_{n-1}) \text{ iff}$$
$$\mathscr{M}_2(\aleph_0) \models \varphi'(\underline{m}_0, \ldots, \underline{m}_{n-1});$$
$$\mathscr{M}_2(\aleph_0), \varkappa \overset{\#}{\not\models} \varphi(\bar{m}_0, \ldots, \bar{m}_{n-1}) \text{ iff}$$
$$\mathscr{M}_2(\aleph_0) \not\models \varphi'(\underline{m}_0, \ldots, \underline{m}_{n-1}).$$

Then apply Lemma 4 of (Hodes 1988b).

(4) The Skolem-number for $L^{x,\#}$ is the least aleph κ so that for any $\varphi \in \mathrm{Sent}(L^{x,\#})$:

> if φ is satisfiable then φ is true in some infinite model of cardinality $\leqslant\kappa$.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

If all predicate-constants in S are at most 1-place and all function-constants in S are at most 0-place then the Skolem-number of $L^{1,\#}$ is $\aleph_1$. It's at least $\aleph_1$, since for any infinite model $\mathscr{A}$, $\mathscr{A} \models (\exists \nu) \neg \underline{N}(\nu)$ iff $\mathrm{card}(\mathscr{A}) \geqslant \aleph_1$. Suppose $\zeta_0, \ldots, \zeta_{n-1}$ are the predicate-constants in S. Where $\mathscr{A}$ is a model for S and $\sigma \in {}^n2$, let $\sigma^{\mathscr{A}} = \cap_{i<n} P_i^{\sigma(i)}$, where $P^0 = \zeta^{\mathscr{A}}$ and $P^1 = |\mathscr{A}| - \zeta^{\mathscr{A}}$. Where $\mathscr{A}_0$ and $\mathscr{A}_1$ are such models let $\mathscr{A}_0 \simeq \mathscr{A}_1$ iff (i) for each $\sigma \in {}^n2$:

> for each individual-constant $\xi \in S$, $\xi^{\mathscr{A}_0} \in \sigma^{\mathscr{A}_0}$ iff $\xi^{\mathscr{A}_1} \in \sigma^{\mathscr{A}_1}$; either $\mathrm{card}(\sigma^{\mathscr{A}_0}) = \mathrm{card}(\sigma^{\mathscr{A}_1}) \leqslant \aleph_1$ or $\mathrm{card}(\sigma^{\mathscr{A}_0})$, $\mathrm{card}(\sigma^{\mathscr{A}_1}) > \aleph_1$;

and (ii) for any individual-constants $\xi, \xi' \in S$: $\xi^{\mathscr{A}_0} = \xi'^{\mathscr{A}_0}$ iff $\xi^{\mathscr{A}_1} = \xi'^{\mathscr{A}_1}$.

To show that the Skolem-number of $L^{1,\#}$ is at most $\aleph_1$ it suffices to show that if $\mathscr{A}_0 \simeq \mathscr{A}_1$ then

$$\text{for any } \varphi \in \mathrm{Sent}(L^{1,\#}): \mathscr{A}_0 \models \varphi \text{ iff } \mathscr{A}_1 \models \varphi.$$

This can be shown using Ehrenfeucht games. Conjecture 2: if all predicate- and function-constants in S are at most 1-place then the Skolem-number of $L^{1,\#}$ is $\aleph_1$.

(5) If S contains a 2-place element then the Skolem-number of $L^{1,\#}$ is that of full second-order logic. It's obvious that the first is at least the second. Suppose that '$\in$' $\notin S$, $\varphi \in \mathrm{Sent}(L^{1,\#})$, $\mathscr{A} \models \varphi$. Let θ be the conjunction of the axioms of extensionality, pairing, infinity sums, and the second-order sentence saying that ω is standard. Expand $\mathscr{A}$ to a model $\mathscr{A}'$ for $S \cup \{\text{'}\in\text{'}\}$ so that $\mathscr{A}' \models \theta$. Translate φ to $\varphi' \in \mathrm{Sent}(L^1(S \cup \{\text{'}\in\text{'}\}))$ by treating $\underline{N}(\tau)$ as "$\tau \in \omega$"; then $\mathscr{A}' \models \varphi'$. There is a model $\mathscr{B}'$ for $S \cup \{\text{'}\in\text{'}\}$ of cardinality less than or equal to the Skolem-number of full second-order logic, so that $\mathscr{B}' \models \theta \,\&\, \varphi'$; then contract $\mathscr{B}'$ to $\mathscr{B}$, a model for S; we have $\mathscr{B} \models \varphi$.

We'll now consider some questions of recursion-theoretic complexity. Let T be the truth-set for second-order arithmetic.

(6) It's easy to see that for any choice of S, T is 1-reducible to ∞-$\mathrm{Val}(L^{1,\#})$.

(7) If $S = \{\ \}$ then ∞-$\mathrm{Biv}(L^{x,\#}) = \mathrm{Sent}(L^{x,\#})$. If S contains a non-zero place predicate-constant then ∞-$\mathrm{Val}(L^{x,\#})$ is 1-reducible to ∞-$\mathrm{Biv}(L^{x,\#})$. The first claim requires an easy symmetry argument. For the second claim, use this fact. If ζ is a 1-place predicate-constant, for any $\varphi \in \mathrm{Sent}(L^{x,\#})$:

$$\varphi \text{ is } \infty\text{-valid iff } \varphi \vee \zeta(\bar{\infty}) \text{ is } \infty\text{-bivalent.}$$

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

(8) Suppose that all predicate-constants in S are at most 1-place and all function-constants in S are at most 0-place. Both ∞-Val$(L^{1,\#})$ and ∞-Biv$(L^{1,\#})$ are 1-reducible to T.

Without loss of generality, suppose that all predicate-constants in S are 1-place and all function-constants are 0-place. Let $S_0 = \{$'$\underline{N}$', '$<$', '$\underline{+}$'. '$\underline{\cdot}$', '$\underline{S}$', '$\underline{O}$'$\}$, disjoint from S. Let $\mathscr{A}$ be an ω-model for $S \cup S_0$ iff $(\underline{N}^{\mathscr{A}}; <^{\mathscr{A}}; \underline{+}^{\mathscr{A}}, \underline{\cdot}^{\mathscr{A}}, \underline{S}^{\mathscr{A}}, \underline{O}^{\mathscr{A}})$ is an ω-model of arithmetic. Let ω-Val = the set of second-order sentences based on $S \cup S_0$ true in all ω-models.

Lemma 1: ω-Val is 1-reducible to T. For this we'll need the following.

Lemma 2 ("splitting a formula into a $\underline{N}$-part and a non-$\underline{N}$-part"): Fix a copy S' of the set of predicate-constants in S, disjoint from S and S_0. Given a model $\mathscr{A}$ for $S \cup S_0$, let $\mathscr{A}'$ be the expansion of $\mathscr{A}$ to a model for $S \cup S' \cup S_0$ such that for any predicate-constant $\zeta \in S$ to which $\zeta' \in S'$ corresponds:

$$\zeta^{\mathscr{A}'} = \zeta^{\mathscr{A}} \cap \underline{N}^{\mathscr{A}};\ \zeta'^{\mathscr{A}'} = \zeta^{\mathscr{A}} - \underline{N}^{\mathscr{A}}.$$

Suppose $\varphi(\vec{\nu}, \vec{\alpha})$ is a formula of $L^1(S \cup S_0)$, $\vec{\nu}$ is $\nu_0, \ldots, \nu_{n-1}$, including all the type-0 variables free in φ, $\vec{\alpha}$ is $\alpha_0, \ldots, a_{m-1}$, including all type-1 variables free in φ, and $z \subseteq n$. Let $\vec{\nu}_z$ be the list of the ν_i for $i \in z$, in order; let $\vec{\nu}^z$ be the list of the ν_i for $i \notin z$, in order. Fix new distinct type-1 variables $\alpha_0^i, \ldots, \alpha_{m-1}^i$ for $i \in \{0, 1\}$. There is an $l_z = l \in \omega$ and formulae $\psi_{z,i}(\vec{\nu}_z, \vec{\alpha}^0) \in \mathrm{Fml}(L^1(S \cup S_0))$ and $\theta_{z,i}(\vec{\nu}^z, \vec{\alpha}^1) \in \mathrm{Fml1}(L^1(S' \cup S_0))$, with free variables among those listed, meeting these conditions: (i) each quantifier-prefix of the form $(\exists \nu)$ for ν of type-0 occurring in $\psi_{z,i}$ $[\theta_{z,i}]$ is restricted by $\underline{N}\nu$ $[\neg \underline{N}\nu]$; (ii) each quantifier-prefix of the form $(\exists \alpha)$ for α of type-1 occurring in $\psi_{z,i}$ $[\theta_{z,i}]$ is restricted by $(\forall \nu)(\alpha\nu \supset \underline{N}\nu)$ $[(\forall \nu)(\alpha\nu \supset \neg \underline{N}\nu)]$; (iii) for any ω-model $\mathscr{A}$ with $\vec{a} \in |\mathscr{A}|^n$, if

$$\text{for each } i < n\ a_i \in \underline{N}^{\mathscr{A}} \text{ iff } i \in z,$$

and $\vec{A} \in \mathrm{Power}(|\mathscr{A}|)^m$, then $\mathscr{A} \models \varphi(\vec{a}, \vec{A})$ iff for some $i \in l$

$$\begin{aligned} \mathscr{A}' \models \psi_{z,i}(\vec{a}_z, A_0 \cap\ & \underline{N}^{\mathscr{A}}, \ldots, A_{m-1} \cap \underline{N}^{\mathscr{A}})\ \& \\ \theta_{z,i}(\vec{a}^{\bar{z}}, A_0 -\ & \underline{N}^{\mathscr{A}}, \ldots, A_{m-1} - \underline{N}^{\mathscr{A}}). \end{aligned}$$

Lemma 2 follows by induction on the construction of φ; details are left to the reader.

To prove Lemma 1, suppose $\varphi \in \mathrm{Sent}(L^1(S \cup S_0))$. By Lemma 2

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

there is an l and ψ_i, θ_i for $i < l$, meeting restrictions (i) and (ii) from Lemma 1, with $(\varphi \equiv \vee_{i<l}(\psi_i \,\&\, \theta_i))$ true in all ω-models for $S \cup S' \cup S_0$. (Since φ is a sentence, $n = 0$.) We may effectively find an $k \in \omega$ and for each $i < l$ a set $x_i \subseteq k + 1$ such that for any ω-model $\mathscr{A}$:

$$\text{if } \mathrm{card}(|\mathscr{A}| - \underline{N}^{\mathscr{A}}) < k,\ \mathrm{card}(|\mathscr{A}| - \underline{N}^{\mathscr{A}}) \in x_i \text{ iff } \mathscr{A} \models \theta_i;$$
$$\text{if } \mathrm{card}(|\mathscr{A}| - \underline{N}^{\mathscr{A}}) \geqslant k \text{ then } k \in x_i \text{ iff } \mathscr{A} \models \theta_i.$$

If $\cup_{i<l} x_i \neq k + 1$ then $\vee_{i<l}(\psi_i \,\&\, \theta_i)$ is false in some ω-model; thus so is φ. Suppose that $\cup_{i<l} x_i = k + 1$. For each $i < k$ let θ'_i be a first-order sentence saying that there are exactly i-many non-$\underline{N}$s; let θ'_k by one saying that there are at least k many non-$\underline{N}$s. For each $j \leqslant k$ we may find a sentence ψ'_j meeting conditions like those (i) and (ii) impose on the $\psi_{z,i}$, so that

$$(\vee_{i<l}(\theta_i \,\&\, \psi_i)) \equiv ((\vee_{j\leqslant k}(\theta'_j \,\&\, \psi'_j)) \text{ is true in all } \omega\text{-models for } S \cup S' \cup S_0.$$

But by the "splitting" the left-hand side of this biconditional is ω-valid iff $\&_{j\leqslant k}\, \psi'_j$ is. The latter may be viewed as a sentence of second-order arithmetic, proving Lemma 1.

For α a type-1 variable, form $M(\alpha) \in \mathrm{Fml}(L^1(S_0))$ saying

$$(\forall \nu)(\alpha\nu \supset \underline{N}\nu) \,\&\, \alpha \text{ codes an } \omega\text{-model } \&\ (\forall \nu)(\text{“}\nu \in |\alpha|\text{”} \supset \underline{N}\nu).$$

(Keep in mind that we can express pairing of numbers in the vocabulary of arithmetic.) Given $\varphi \in \mathrm{Sent}(L^{1,\#})$ with α not occurring in φ, form $\varphi'(\alpha)$ by "translating" φ into $L^1(S \cup S_0)$, treating α as coding an ω-model of arithmetic; e.g., replace $\underline{N}(\nu)$ by "$\nu \in |\alpha|$"; replace $(\#\nu)\theta$ by a specification of the corresponding "member of $|\alpha|$". Then φ is ∞-bivalent iff:

$$(\forall \alpha)(\forall \alpha')([M(\alpha) \,\&\, M(\alpha')] \supset [\varphi'(\alpha) \equiv \varphi'(\alpha')]) \in \omega\text{-Val};$$

by Lemma 1 that question is 1-reducible to T. A similar reduction works for the ∞-validity of φ.

(8) continues to hold even if we allow 1-place function-constants into S; this is left to the reader.

(9) If there is at least one 2-place element of S then ∞-Val$(L^{1,\#})$, ∞-Biv$(L^{1,\#})$ and the set of validities of full second-order logic are 1-equivalent. This is left to the reader.

(10) If there is a 2-place predicate- or function-constant in S then ∞-Val$(L^{0,\#})$ is Π^1_1-complete. In this paragraph, let $\{e\} = \{n \in \omega$: the e'th

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

partial recursive function applied to n yields 1}. Let $u = \{e \in \omega : \{e\}$ is well-founded}. To show that W is 1-reducible to ∞-Val($L^{0,\#}$), the key idea is that $\{e\}$ is well-founded iff

$$(\exists \nu)(\nu \in \{e\} \ \& \ \underline{N}\nu) \supset (\exists \nu)(\nu \text{ is } \{e\}\text{-least so that } \underline{N}\nu)$$

is true in all ω-models of a suitable finite fragment of set-theory.

To show that ∞-Val($L^{0,\#}$) and ∞-Biv($L^{0,\#}$) are Π^1_1 we don't need the assumption about S. Let S_0 be as in (8). For $\varphi \in \mathrm{Sent}(L^{0,\#})$, "translate" φ to $\varphi' \in \mathrm{Sent}(L^1(S \cup S_0))$ so that $\varphi \equiv \varphi'$ is true in any ω-model for $S \cup S_0$; whether φ' is ω-valid is a Π^1_1 question (using the ω-completeness of ω-logic). Thus ∞-Val($L^{0,\#}$) is a Π^1_1-set. Using ideas from (8), ∞-Biv($L^{0,\#}$) can also be shown to be Π^1_1.

(11) If all members of S are individual constants then ∞-Val($L^{0,\#}$) is recursive. Use Lemma 4 of (Hodes 1988b); details are left to the reader. Conjecture 2: if members of S are 0- or 1-place predicate-constants or individual-constants then ∞-Val($L^{0,\#}$) remains recursive.

7.

Why did we restrict our attention to finite models? The difficulties faced in finite models by our analytic definitions of addition and multiplication have been mentioned already. More seriously, a finite model has no representors: if $\mathscr{A}$ is finite, although ^{2}EXACTLY is also finite, card($\mathscr{A}$) $<$ card(^{2}EXACTLY).

The Mathematical-Object picture of arithmetic, even if it is not inflated into a semantic theory, seems committed to the existence of infinitely many objects; for each natural number n, "There are at least n objects", though it can be expressed without use of mathematical vocabulary, is a consequence of some simple arithmetic truths, e.g., that there is no greatest number. So our restriction to infinite models appears to mirror a genuine presupposition of mathematical practice.

The Far-out response would be that arithmetic doesn't really presuppose an infinitude of objects, and it is coherent to suppose that there is a last natural number. We would consider fragments of representors, mapping each ${}^2Q(n)$ into $|\mathscr{A}|$ for $n <$ card($\mathscr{A}$) or for $1 \leqslant n \leqslant$ card($\mathscr{A}$). Of course relative to a finite model

$$(\forall \nu)(\underline{N}(\nu) \supset (\exists \rho)(\underline{N}(\rho) \ \& \ \nu \leqslant \rho \ \& \neg \rho \leqslant \nu))$$

is not true.

Since it involves a deep revision of mathematical practice, I shall

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

not pursue the Far-out response. But unlike those who accept the Mathematical-Object Theory, proponents of the Alternative theory cannot base their acceptance of an infinitude of objects on mathematics itself, e.g., on the infinitude of the natural numbers. Perhaps physics can assure us that, for example, there are infinitely many spatial volumes and temporal intervals. Or perhaps metaphysics can assure us that there are infinitely many non-actual possible objects.

The last suggestion is available only to the Individual-Possibilist; it doesn't help the Individual-Actualist, one who doesn't think that there are non-actual possible objects. But it does suggest a reconstrual of the commitment to the existence of infinitely many objects apparently carried by our mathematical practice. Perhaps some of the uses of existential quantification within the Mathematical-Object picture, e.g., in the assertion that there are at least n objects, should be construed as governed by a possibility operator. In particular, perhaps that picture is not committed to an actual, but merely to a potential, infinitude of objects: that for every natural number n there could be (or could have been) at least n objects. This would be compatible with there being a finite bound on the number of objects that actually exist. (Of course only for Individual-Actualists is this is a reconstrual of mathematics' apparent commitment to an infinitude of objects; the Individual-Possibilist would not construe the *prima facie* commitment to an infinitude of objects as a commitment to an infinitude of actual objects.)

Can the Individual-Actualist who also embraces the Alternative theory accept both arithmetic as we practice it and also the existence of only finitely many objects? Assuming only a potential infinitude of objects, can she make sense of the Mathematical-Object theory? The answer is sensitive to what we make of Individual-Actualism, a matter to which we shall return after presenting a modal version of the material from Section 2. (As in Section 2, we might do well to offer a three-valued semantics, not merely because our encoding semantics will be three-valued, but also because a three-valued model theory is particularly helpful for representing the concerns of the Individual-Actualist; see (Hodes 1986, 1987) for a detailed discussion. In the interest of simplicity, we will stick to a two-valued semantics.)

Form $L^{\Box,0}$ by adding '$\Box$' to the logical lexicon of L^0, functioning as a formula-forming operator on single formulae. To make life easier, we'll suppose that individual constants are our only function-constants in our vocabulary set S.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

A skeleton has the form $(W, R, A, \bar{A})$, where:

W and A are non-empty sets;
$R \subseteq W^2$; $\bar{A}: W \rightarrow \text{Power}(A)$;
$A = U_{w \in W} \bar{A}(w)$.

As usual R is the accessibility relation of this skeleton. A normal modal logic is determined by a class of skeletons. Of course the simplest non-trivial such logic is **S5**, determined by the class of all skeletons of the form $(W, W^2, A, \bar{A})$. **K** is the logic determined by the class of all skeletons; **T** is determined by the class of skeletons with reflexive accessibility relations. If we use '$\Box$' to represent necessity, we want our logic to be at least as strong as **T**; if we use '$\Box$' to represent real ("metaphysical") necessity, some think that our logic should be **S5**.

A modal structure $\mathscr{A}$ for S consists of a skeleton $(W, R, |\mathscr{A}|, \overline{|\mathscr{A}|})$ together with an assignment to elements of S as follows:

for any predicate-constant $\zeta \in S$,
  if $n > 0$, $\zeta^{\mathscr{A}}: W \times |\mathscr{A}|^n \rightarrow \{0, 1\}$;
  if $n = 0$, $\zeta^{\mathscr{A}}: W \rightarrow \{0, 1\}$;
for any individual-constant $\xi \in S$,
  either $\xi^{\mathscr{A}} \uparrow$ or $\xi^{\mathscr{A}} \in |\mathscr{A}|$.

Hereafter we write $\bar{\mathscr{A}}$ for $\overline{|\mathscr{A}|}$. Let $\mathscr{A}$ be total iff for all ξ as above, $\xi^{\mathscr{A}} \downarrow$. Let $\mathscr{A}$ be e-actualistic ('e' for 'extensionwise') iff for every n-place predicate-constant $\zeta \in S$ with $n > 0$, if $\langle a_1, \ldots, a_n \rangle \notin \bar{\mathscr{A}}(w)^n$ then $\zeta^{\mathscr{A}}(w, a_1, \ldots, a_n) = 0$. A modal model (hereafter just called 'a model') for S has the form $(\mathscr{A}, w)$ where $\mathscr{A}$ is a modal structure for S and $w \in W$; we call w "the actual world" for $(\mathscr{A}, w)$. Let $\hat{\mathscr{A}}(w) = \cup_{wRu} \bar{\mathscr{A}}(u)$. Relative to $(\mathscr{A}, w)$, members of $\bar{\mathscr{A}}(w)$ represent actual objects, and the Individual-Possibilist would say that the members of $\hat{\mathscr{A}}(w)$ represent possible objects. Let $|(\mathscr{A}, w)| = |\mathscr{A}|$. A model is total [$e$-actualistic] iff its structure is total [e-actualistic].

Let a skeleton $(W, R, A, \bar{A})$ be potentially infinite iff for every natural number n and every $w \in W$ there is a u so that wRu and $\text{card}(\bar{\mathscr{A}}(u)) \geqslant n$. Let a modal structure (model) be potentially infinite iff its skeleton (its structure) is so. Notice that an **S5**-skeleton is potentially infinite iff for every n there is a $w \in W$ with $\text{card}(\bar{\mathscr{A}}(w)) \geqslant n$. Where **L** is a normal modal logic (identified with a class of skeletons), let $p\infty$-**L** be the logic formed by restricting **L** to potentially-infinite skeletons.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

Given a model $(\mathscr{A}, w)$, for each $a \in |\mathscr{A}|$ introduce the new individual-constant $\underline{a}$, forming the language $L_{\mathscr{A}}^{\Box,0}$. We define $\mathrm{des}^{\mathscr{A}}$ on type-0 terms as usual. We define $\mathscr{A},w \models$ for sentences of $L_{\mathscr{A}}^{\Box,0}$ in the usual way, e.g. let:

$$\mathscr{A}, w \models \zeta(\tau_0, \ldots, \tau_{n-1}) \text{ iff}$$
$$\zeta^{\mathscr{A}}(w, \mathrm{des}^{\mathscr{A}}(\tau_0), \ldots, \mathrm{des}^{\mathscr{A}}(\tau_{n-1})) = 1;$$
$$\mathscr{A},w \models \tau = \sigma \text{ iff } \mathrm{des}^{\mathscr{A}}(\tau) \simeq \mathrm{des}^{\mathscr{A}}(\sigma).$$

Where ν is a type-0 variable, let:

$$\mathscr{A}, w \models (\exists\nu)\varphi \text{ iff for some } a \in \bar{\mathscr{A}}(w),\ \mathscr{A},w \models \varphi(\underline{a}/\nu);$$
$$\mathscr{A},w \models \Box\varphi \text{ iff for every } u \text{ with } wRu,\ \mathscr{A}, u \models \varphi.$$

Individual-Actualism is reflected in the actualistic clause just given for '$\exists$', which takes bound variables to range over objects that are "actual relative to" $(\mathscr{A}, w)$, i.e., over $\bar{\mathscr{A}}(w)$, and 'E' still parses 'exists', construed actualistically:

$$\mathscr{A}, w \models \exists(\tau)\ \&\ (\exists\nu)\mathsf{E}(\nu) \text{ iff } \mathrm{des}^{\mathscr{A}}(\tau) \in \bar{\mathscr{A}}(w);$$

the second conjunct is needed only to handle the case in which $\bar{\mathscr{A}}(w)$ is empty and $\mathrm{des}^{\mathscr{A}}(\tau)\uparrow$. We could enrich $L^{\Box,0}$ by introducing a new quantifier-expression '$\acute{\exists}$', binding variables of type-0, and letting:

$$\mathscr{A}, w \models (\acute{\exists}\nu)\varphi \text{ iff for some } a \in \hat{\mathscr{A}}(w)\ \mathscr{A}, w \models \varphi(\underline{a}/\nu).$$

A moderate Individual-Actualist maintains that, when it comes to first-order quantification, we can only quantify over actual objects, i.e., that bound variables of type-0 can range only over actual objects. On this view '$\acute{\exists}$' cannot represent a construction in any genuine language. This position is reflected in our model-theoretic semantics for $L^{\Box,0}$. But it is a somewhat unstable position, since it allows that free variables may range over, and that designators may designate, non-actual objects. A strict Individual-Actualist would reject even this as too possibilistic. A three-valued model-theoretic semantics reflecting this more strict position is presented and investigated in (Hodes 1986, 1987).

A lax Individual-Actualist might maintain merely that the central and most fundamental sort of first-order quantification is actualistic, i.e., over actual objects; so our "primary" notion of existence, represented by '$\exists$' under the model-theoretic semantics just presented, is actualistic. But he'd go on to claim that the possibilistic existential quantifier, represented by '$\acute{\exists}$', is legitimate, admitting it does not represent most

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

uses of quantifier-phrases, and it certainly is in no sense more basic than its actualistic cousin. This view rejects a only very strong sort of possibilism: that statements of the apparent form $(\exists \nu)\varphi$ are really of the form $(\acute{\exists}\nu)(\text{Actual}(\nu)\ \&\ \varphi)$, where 'Actual' represents a primitive predicate. Thus 'There is a non-actual object' is ambiguous: false under its most likely construal, though true under a less common construal of 'there is'.

I once thought that this lax view didn't merit the label 'Individual-Actualism'. Now I'm not so sure. Notice, for example, that as long as '$\acute{\exists}$' is not used within the scope of '$\Box$', the actuality operator '@' buys us '$\acute{\exists}$': we can take $(\acute{\exists}\nu)$ to abbreviate $\Diamond(\exists\nu)@$ within such contexts. Indeed, with the "world-travelling" operator introduced in (Hodes 1984), '$\acute{\exists}$' can be defined from '$\exists$'. It appears that even the moderate Individual-Actualist must deny that '@' can represent a construction in any genuine language.

In what way could an Individual-Actualist claim that a model $(\mathscr{A}, w)$ could represent the alethic underpinnings for a genuine language?

Relative to a model $(\mathscr{A}, w)$ based on the skeleton $(W, R, |\mathscr{A}|, \bar{\mathscr{A}})$, members of $\bar{\mathscr{A}}(w)$ represent objects, members of W represent "possible worlds", i.e., ways in which things could be or could have been, and of course $\mathscr{A}, w \models$ represents truth. But for an Individual-Actualist, relative to $(\mathscr{A}, w)$ members of $\bar{\mathscr{A}}(u) - \bar{\mathscr{A}}(w)$ don't represent anything. Rather they contribute to determining to what $(\mathscr{A}, w)$ bears $\models$ by virtue of the fact that relative to $(\mathscr{A}, u)$ they represent objects.

Our notion of an *e*-actualistic structure reflects Predicate-Actualism, a doctrine frequently associated with Individual-Actualism: that no predicate can apply to what doesn't exist. This is not a self-evident corollary of Individual-Actualism, unless the latter is construed in the strict way mentioned above. Suppose that necessarily all human beings are essential human; I can't see why an Individual-Actualist who "buys" the actuality operator couldn't also accept 'Necessarily every human being is actually human'. If we do not posit non-actual possible objects, we should not recognize "atomic facts" of which such objects are constituents. But an Individual-Actualist might not want to rest much weight on a metaphysics of facts; and even if she is quite happy with facts, determining what facts one is committed to can be a controversial matter. It's far from clear that the above sentence entails 'Necessarily for every human being there is an actual fact that he or she is human'. In any case, Predicate-Actualism is worth keeping in mind. Since our

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

semantics is to be kept two-valued, we've used the Falsehood Convention in our notion of being e-actualistic: we don't allow an n-place predicate ζ to apply to $\langle a_0, \ldots, a_{n-1}\rangle$ relative to $(\mathscr{A}, w)$ if that n-tuple does not belong to $\bar{\mathscr{A}}(w)$".

Form $L^{\Box,1}$ and $L^{\Box,(0,0)}$ by enriching L^1 and $L^{(0,0)}$ respectively by '$\Box$'. There are various kinds of second-order quantification available in the modal setting. Predicate-Actualists would likely want second-order quantification to be "actualistic". For $A \subseteq W \times |\mathscr{A}|$ $[A \subseteq W \times |\mathscr{A}|^2]$ let A be actualistic iff:

$$\text{for any } \langle w, a\rangle \in A, a \in \bar{\mathscr{A}}(w)$$
$$[\text{for any } \langle w, a, b\rangle \in A, a, b \in \bar{\mathscr{A}}(w)].$$

But we'll do best not to restrict the range of variables of types 1 and (0, 0) to actualistic values. For uniformity, we'll sometimes write '$A(w, a) = 1$' for '$\langle w, a\rangle \in A$'; similarly for '$A(w, a, b) = 1$'. For each $A \subseteq W \times |\mathscr{A}|$ $[A \subseteq W \times |\mathscr{A}|^2]$ introduce a new 1-place [2-place] predicate-constant $\underline{A}$. If ρ is a type-1 [type-(0, 0)] variable:

$$\mathscr{A}, w \models (\exists\rho)\varphi \text{ iff for some } A \subseteq W \times \bar{\mathscr{A}}(w)\ [A \subseteq W \times \bar{\mathscr{A}}(w)^2]$$
$$\mathscr{A}, w \models \varphi(\underline{A}/\rho).$$

8.

This section is a modal analog of Section 3. For 'x' replaceable by '0', '1' or '2', enrich $L^{x,4}$ to form $L^{\Box,x,4}$ by adding '$\Box$'. Our actualistic treatment of $(\exists\nu)$, for ν a variable of type 0, requires that cardinality-quantifiers, expressed by prefexes of the form (EXACTLY $\underline{n}\nu$), be equally actualistic. For any $n \in \omega$, any structure $\mathscr{A}$ with set W of worlds, and $w \in W$, let:

$${}^2Q(n, w) = \{A \subseteq \bar{\mathscr{A}}(w)\colon \mathrm{card}(A) = n\};$$
$${}^2Q(n) = \cup_{w\in W}\, {}^2Q(n, w);$$
$${}^2\mathrm{EXACTLY} = \{{}^2Q(n)\colon n \in \omega\}.$$

Notice how we have made sure that ${}^2Q(n)$ is actualistic. We define $\mathscr{A}, w \models$ for sentences of $L^{\Box,x,2}$ in the obvious way, with $\hat{\nu}\varphi^{\mathscr{A},w}$ defined actualistically for ν of type 0:

$$\hat{\nu}\varphi^{\mathscr{A},w} = \{a \in \bar{\mathscr{A}}(w)\colon\ \mathscr{A}, w \models \varphi(\underline{a}/\nu)\}.$$

So:

$$\mathscr{A}, w \models (\text{EXACTLY } \underline{n}\nu)\varphi \text{ iff } \mathrm{card}(\hat{\nu}\varphi^{(\mathscr{A},w)}) = n.$$

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

Since quantifiers do not exist (or fail to exist) at worlds, there is no sense in which prefexes of the form (EXACTLY $\underline{n}\mu$), for μ of type-2, could be actualistic. So let ${}^4Q(n)$ and ^{4}EXACTLY be defined as in Section 3.

As in Section 3, taking $\mathscr{D}_{2i} = {}^{2i}$EXACTLY for $i \in \{1, 2\}$, we call $L^{\Box,x,4}$ "$L^{\Box,x,4}$(EXACTLY)", adding '$\leqslant$' as a primitive when 'x' is replaced by '0'. With 'x' replaced by '1' or '(0, 0)' for variables μ and μ' of type-2 take $\mu \leqslant \mu'$ to abbreviate

$$\Diamond(\exists\alpha)(\exists\alpha')((\text{EXACTLY } \mu\nu)\alpha\nu \;\&\; (\text{EXACTLY } \mu')\alpha'\nu \;\&\; (\forall\nu)(\alpha\nu \supset \alpha'\nu)).$$

For any potentially infinite model $\mathscr{M}$ and any $n_0, n_1, n_2 \in \omega$:

$$\mathscr{M} \models \Box\text{Add}({}^2\underline{Q(n_0)}, {}^2\underline{Q(n_1)}, {}^2\underline{Q(n_2)}) \text{ iff } n_0 + n_1 = n_2;$$
$$\mathscr{M} \models \Box\text{Mult}({}^2\underline{Q(n_0)}, {}^2\underline{Q(n_1)}, {}^2\underline{Q(n_2)}) \text{ iff } n_0 \cdot n_1 = n_2.$$

Nonetheless we may also have $\mathscr{M} \models (\exists\mu)(\text{EXACTLY } \mu\nu)\mathsf{E}(\nu)$, i.e., $\mathscr{M}$ may satisfy "There are finitely many objects". We may now claim that $\Box\text{Add}(\mu_0, \mu_1, \mu_2)$ and $\Box\text{Mult}(\mu_0, \mu_1, \mu_2)$ (for μ_0, μ_1, μ_2 distinct variables of type 2) rather than Add(. . .) and Mult(. . .), provide "analytical" definitions of addition and multiplication.

Where $\varphi \in \text{Fml}(L^{\Box,x,4}(\text{EXACTLY}))$ has free variables among $\mu_0, \ldots, \mu_{k-1}$, all of type 2, let φ define $V \subseteq \omega^k$ iff for all potentially infinite models $\mathscr{M}$ and all $\vec{n} \in \omega^k$: $\vec{n} \in V$ iff $\mathscr{M} \models \varphi(\underline{n}_0, \ldots, \underline{n}_{k-1})$.

Theorem 2 of (Hodes 1988b) extends to the modal setting as follows. For any $V \subseteq \omega^k$. V is definable in $L^{\Box,1,4}$(EXACTLY) iff V is definable in Presburger arithmetic.

From right to left use $\Box$Add(. . .). From left to right it suffices, by Theorem 2 of (Hodes 1988b), to show that if V is definable in $L^{\Box,1,4}$(EXACTLY) then it's definable in $L^{1,4}$(EXACTLY); to show this, code any infinite non-modal model $\mathscr{B}$ by the obvious modal model $(\mathscr{A}, w)$ with $W = \{w\}$ and $|\mathscr{A}| = \bar{\mathscr{A}}(w) = |\mathscr{B}|$.

Curiously, if in our definition of "φ defines V" we had required that $\mathscr{M}$ be based on a skeleton $(W, R, \mathscr{A}, \bar{\mathscr{A}})$ with W infinite, then the relations on ω definable in $L^{\Box,1,4}$(EXACTLY) would be those definable in second-order arithmetic; in fact this holds even with variables of types 1 and (0, 0) restricted to actualistic values! For $A \subseteq W \times |\mathscr{A}|$ and $w \in W$ let $A_w = \{a: \langle w, a\rangle \in A\}$. The idea is that A encodes

$$\{n \in \omega: \text{for some } w \in W\; n = \text{card}(A_w \cap \bar{\mathscr{A}}(w))\};$$

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

furthermore any $B \subseteq \omega$ can be so encoded (by an actualistic A). We can then define multiplication as in Section 6, (1).

Observation 6: Every $\varphi \in \mathrm{Sent}(L^{\Box,(0,0),4}(\text{EXACTLY}))$ is $p\infty$-**S5**-equivalent to some $\psi \in \mathrm{Sent}(L^{\Box,(0,0)})$.

Note the argument from Observation 2.2 of (Hodes 1988a) would carry over quite directly if we were to add '$\acute{\exists}$'; without it we face extra work.

Fix a $p\infty$-**S5**-structure $\mathscr{A}$, with set W of "worlds". For $A \subseteq W \times |\mathscr{A}|$ let A be rigid iff for any $w, u \in W$ $A_w = A_u$. For a rigid A, let $A' = A_w$ for any $w \in W$. For $R \subseteq W \times |\mathscr{A}|^2$ let $R_w = \{\langle a, b\rangle : \langle w, a, b\rangle \in R\}$, and let R be rigid iff for any $w, u \in W$ $R_w = R_u$. Let R be a standard for $\mathscr{A}$ iff R is rigid and R' is a strict well-ordering of type ω. With such an R fixed, for $n < \omega$ let a_n be the n'th member of $\mathrm{Fld}(R)$ under R. For $a \in |\mathscr{A}|$ let $a^* = \{\langle w, a\rangle : w \in W\}$ be a's individual essence. For each subformula $\theta(\mu_1, \ldots, \mu_k)$ with type-2 variables among those listed, we want to construct a formula $\theta'(\alpha_1, \ldots, \alpha_k)$, with type-1 variables $\alpha_1, \ldots, \alpha_k$, so that for any $p\infty$-**S5**-model $(\mathscr{A}, w)$, any standard R for $\mathscr{A}$, any $n_1, \ldots, n_k < \omega$ and any assignment of values for the unindicated free variables in θ:

$$\mathscr{A}, w \models \theta(\underline{n}_1, \ldots, \underline{n}_k) \text{ iff } \mathscr{A}, w \models \theta'(\underline{a}^*_{n_1}, \ldots, \underline{a}^*_{n_k}).$$

(Without '$\acute{\exists}$' we can't directly quantify over $\{a_n : n < \omega\}$, the members of which are meant to code the values for variables of types 2 and 4; so instead we quantify over their individual essences.)

For type 1 variables α and β, type-$(0, 0)$ variable γ, and type-0 variable ν_0 there are formulae $\mathrm{Rgd}(\alpha)$, $\mathrm{Rgd}(\gamma)$, $\mathrm{Fin}(\alpha)$, $\leqslant(\alpha, \beta)$ and $\mathrm{Std}(\gamma)$ so that for any **S5**-model $(\mathscr{A}, w)$, $A, B \subseteq W \times |\mathscr{A}|$, $R \subseteq W \times |\mathscr{A}|^2$:

$\mathscr{A}, w \models \mathrm{Rgd}(\underline{A})$ iff A is rigid;
$\mathscr{A}, w \models \mathrm{Rgd}(\underline{R})$ iff R is rigid;
if A is rigid, then $\mathscr{A}, w \models \mathrm{Fin}(\underline{A})$ iff A' is finite;
if A and B are rigid, then $\mathscr{A}, w \models \leqslant(\underline{A}, \underline{B})$ iff $\mathrm{card}(A') \leqslant \mathrm{card}(B')$;
$\mathscr{A}, w \models \mathrm{Std}(\underline{R})$ iff R is a standard for $(\mathscr{A}, w)$.

$\mathrm{Rgd}(\alpha)$ is $\Box(\forall\nu)\Box(\alpha\nu \supset \Box\alpha\nu)$; $\mathrm{Rgd}(\gamma)$ is $\Box(\forall\nu)\Box(\forall\rho)\Box(\gamma\nu\rho \supset \Box\gamma\nu\rho)$. Form $\mathrm{Fin}(\alpha)$ and $\leqslant(\alpha, \beta)$ from their familiar non-modal versions by inserting '$\Box$' before each occurrence of '$\forall$' and '$\Diamond$' before each occurrence of '$\exists$', and restricting the initial existential quantification of a type-$(0, 0)$ variable by 'Rgd'. From these we may construct $\mathrm{Std}(\gamma)$. Let $\mathrm{Eqnum}(\alpha, \beta)$ be $\leqslant(\alpha, \beta)$ & $\leqslant(\beta, \alpha)$.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

Given $\theta \in \mathrm{Fml}(L_{\mathscr{A}}^{\Box,(0,0),4})$ with at most the type-0 variable ν free, and given a type-1 variable β, we want a formula $(\mathrm{Ext}\ \nu)(\beta, \theta)$ so that for any **S5**-model $(\mathscr{A}, w)$ and $B \subseteq W \times |\mathscr{A}|$:

$$\mathscr{A}, w \models (\mathrm{Ext}\ \nu)(\underline{B}, \theta) \text{ iff } B \text{ is rigid and } B' = \hat{\nu}\theta^{\mathscr{A},w};$$

For β' a distinct type-1 variable let $(\mathrm{Ext}\ \nu)(\beta, \theta)$ be:

$$\begin{aligned}&\mathrm{Rgd}(\beta)\ \&\ (\forall\nu)(\theta \equiv \beta\nu)\ \&\\ &(\forall\beta')([\mathrm{Rgd}(\beta')\ \&\ \neg(\exists\nu)\beta'\nu] \supset \Box(\forall\nu)(\beta'\nu \supset \neg\beta\nu)).\end{aligned}$$

For α a type-1 variable we want a formula $\mathrm{Numb}(\gamma, \alpha)$ so that for an **S5**-model $(\mathscr{A}, w)$ and R a code for $\mathscr{A}$, for any $A \subseteq W \times |\mathscr{A}|$:

$$\mathscr{A}, w \models \mathrm{Numb}(\underline{R}, \underline{A}) \text{ iff for some } a_n \in \mathrm{Fld}(R),\ A = a_n^*.$$

Let it be:

$$\begin{aligned}&\mathrm{Rgd}(\alpha)\ \&\ \Diamond(\exists\nu)(\alpha\nu\ \&\ \Box(\forall\nu')[\alpha\nu' \supset \nu' = \nu]\ \&\\ &\Diamond(\exists\nu')\gamma\nu'\nu]).\end{aligned}$$

We also want a formula $\mathrm{Section}(\gamma, \nu, \alpha)$, for ν a type-0 variable, so that for $(\mathscr{A}, w)$ and R as above and any $a \in |\mathscr{A}|$ and $A \subseteq W \times |\mathscr{A}|$:

$$\mathscr{A}, w \models \mathrm{Section}(\underline{R}, \underline{a}, \underline{A}) \text{ iff } A = \{b\colon \langle b, a\rangle \in R_w\}.$$

Suppose we're given $\varphi \in \mathrm{Sent}(L^{\Box,(0,0),4}(\text{EXACTLY}))$. To each type-2 or type-4 variable μ we associate a new type-1 variable α_μ. For each subformula $\theta(\mu_1, \ldots, \mu_k)$ of φ with free type-2 variables shown we want to construct $\theta'(\gamma, \alpha_{\mu_1}, \ldots, \alpha_{\mu_k})$ so that (with other free variables replaced by parameters in any type-appropriate way) for any $(\mathscr{A}, w)$ and R as above and any $n_1, \ldots, n_k < \omega$:

$$\mathscr{A}, w \models \theta(\underline{n}_1, \ldots, \underline{n}_k) \text{ iff } \mathscr{A}, w \models \theta'(\underline{R}, \underline{a}_{n_1}^*, \ldots, \underline{a}_{n_k}^*).$$

For μ of type 2 or 4, take $((\exists\mu)\theta)'$ to be $(\exists\alpha_\mu)(\mathrm{Numb}(\gamma, \alpha_\mu)\ \&\ \theta')$; for variables μ, μ', both of type 2 or both of type 4, take $(\mu \leqslant \mu')'$ to be:

$$\begin{aligned}&(\exists\beta)(\exists\beta')(\mathrm{Rgd}(\beta)\ \&\ \mathrm{Rgd}(\beta')\ \&\ \leqslant(\beta, \beta')\ \&\\ &\Diamond(\exists\rho)[\alpha_\mu(\rho)\ \&\ \mathrm{Section}(\gamma, \rho, \beta)]\ \&\\ &\Diamond(\exists\rho)[\alpha_{\mu'}(\rho)\ \&\ \mathrm{Section}(\gamma, \rho, \beta')]),\end{aligned}$$

for distinct type-1 variables β and β'. For μ of type 2 and ν of type 0 take $((\text{EXACTLY}\ \mu\nu)\theta)'$ to be

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

$$(\exists\alpha)(\exists\beta)((\text{Ext } \nu)(\alpha, \theta) \;\&\; \text{Rgd}(\beta) \;\&\; \text{Eqnum}(\alpha, \beta) \;\&\; \Diamond(\exists\rho)[\alpha_\mu(\rho) \;\&\; \text{Section}(\gamma, \rho, \beta)]),$$

for distinct new variables α and β of type 1 and ρ of type 0. For μ of type 4 and ρ of type 2, take $((\text{EXACTLY } \mu\rho)\theta)'$ to be:

$$(\exists\alpha)(\exists\beta)(\text{Rgd}(\alpha) \;\&\; \text{Rgd}(\beta) \;\&\; (\forall\alpha_\rho)[(\theta' \;\&\; \text{Numb}(\gamma, \alpha_\rho)) \equiv \Diamond(\exists\nu)(\alpha_\rho(\nu) \;\&\; \alpha\nu)] \;\&\; \Diamond(\exists\nu)[\alpha_\mu(\nu) \;\&\; \text{Section}(\gamma, \nu, \beta)]),$$

with α, β as above. Details are left to the reader. Take ψ to be $(\forall\gamma)(\text{Std}(\gamma) \supset \varphi'(\gamma))$.

Conjecture: if variables of type 1 and (0, 0) are restricted to actualistic values, this observation fails. Question: does this observation carry over to other familiar modal logics, e.g., **S4**?

9.

For 'x' replaceable as before, form $L^{\Box,x,\#}$ by enriching $L^{x,\#}$ with '$\Box$'. We'd like to find a way of encoding of $L^{\Box,x,2}(\text{EXACTLY})$ into $L^{\Box,x,\#}$. What shall a representor for a model $(\mathscr{A}, w)$ be? The most straightforward way to make our model-theoretic semantics model a modal Fregean version of the Mathematical-Object Theory would be to make sure that bound type-0 variables range over the range of representors. This suggests that we restrict our attention to structures $\mathscr{A}$ such that $\bigcap_{w\in W} \bar{\mathscr{A}}(w)$ is infinite, and take representors to be into that set. If we agree that the natural numbers are necessary existents, restricting our attention to such models would be completely reasonable.

If we want to accommodate the possibility of there being finitely many objects, such a restriction is unreasonable. But if $\bigcap_{w\in W} \bar{\mathscr{A}}(w)$ is finite we'll have to encode some members of ^{2}EXACTLY by members of $|\mathscr{A}| - \bar{\mathscr{A}}(w)$; relative to $(\mathscr{A}, w)$ such objects are not in the range of bound type-0 variables. This fact alone will block an encoding of $L^{\Box,x,2}(\text{EXACTLY})$ into $L^{\Box,x,\#}$.

Consider a structure $\mathscr{A}$ based on skeleton $\mathscr{S} = (W, R, |\mathscr{A}|, \bar{\mathscr{A}})$. For $a \in |\mathscr{A}|$, let a be safe in $\mathscr{S}$ (and in $\mathscr{A}$) iff for every $w \in W$ $a \in \bar{\mathscr{A}}(w) \cup \hat{\mathscr{A}}(w)$. So if $\mathscr{A}$ is an **S5**-structure all members of $|\mathscr{A}|$ are safe. Let $\mathscr{S}$, $\mathscr{A}$, and any model based on $\mathscr{A}$, support encoding iff there are infinitely many $a \in |\mathscr{A}|$ safe in $\mathscr{S}$. Let $\varkappa$ be a representor for $\mathscr{S}$ (and $\mathscr{A}$)

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

be a one-to-one function from ^{2}EXACTLY into $\{a\colon a$ is safe in $\mathcal{S}\}$. So a skeleton, or a structure, has a representor iff it supports encoding. Let a Frege-structure (Frege-model) have the form $(\mathcal{A}, \varkappa)$ $[(\mathcal{A}, \varkappa, w)]$ where $\varkappa$ is a representor for $\mathcal{A}$ [and $w \in W$].

For 'x' replaceable by '0', '$\cdot$', '1', or '(0, 0)' form $L^{\Box,x,\#}$ by adding '$\underline{N}$', '$\leqslant$' and '#' to the logical lexicon of $L^{\Box,x}$ under the usual formation rules. The definition of $\mathrm{des}^{\mathcal{A},\varkappa}$ and $A, \varkappa, w \overset{\#}{\models}$ involves relativizing the definition in Section 4 to w. In particular,

$$\mathrm{des}^{\mathcal{A},\varkappa,w}((\#\nu)\varphi) \simeq a \text{ iff for some } n < \omega \ \mathrm{card}(\hat{\nu}\varphi^{\mathcal{A},\varkappa,w}) \text{ and } a = \varkappa({}^2Q(n));$$

recall that $\hat{\nu}\varphi^{\mathcal{A},\varkappa,w}$ is the actualistic extension of φ for ν; thus '#' is actualistical as well.

For a model $\mathcal{M} = (\mathcal{A}, w)$ that supports encoding and $\varphi \in \mathrm{Sent}(L^{\Box,x,\#})$, let:

$$\mathcal{M} \models \varphi \text{ iff for every representor } \varkappa \text{ for } \mathcal{A}\ \mathcal{A}, \varkappa, w \models \varphi;$$
$$\mathcal{M} \dashv \varphi \text{ iff for every representor } \varkappa \text{ for } \mathcal{A}\ \mathcal{A}, \varkappa, w \not\models \varphi.$$

For $\Delta \cup \{\varphi\} \subseteq \mathrm{Sent}(L^{\Box,x,\#})$ and a model logic **L** let:

> Δ ***L**-entails φ iff for every $p\infty$-**L**-model $\mathcal{M}$ that supports encoding, if $\mathcal{M} \models \Delta$ then $\mathcal{M} \models \varphi$.

The definitions of ***L**-bivalence, ***L**-truthvaluelessness, etc., carry over from Section 5 in the obvious ways. By restricting attention to *e*-actualistic models for a model logic **L** we define *e*-actualistic ***L**-entailment, etc., in the obvious ways. Let:

> ***L**-Biv$(L^{\Box,x,\#})$ = the set of ***L**-bivalent sentences of $L^{\Box,x,\#}$;
> ***L**-Biv$^{ea}(L^{\Box,x,\#})$ = the set of *e*-actualistic ***L**-bivalent sentences of $L^{\Box,x,\#}$.

Note the following. For each $n \in \omega$, $E(\bar{n})$ is not ***K**-valid or even ***S5**-valid; nor is $(\exists\nu)\underline{N}(\nu)$. But $\mathsf{E}(\bar{n}) \vee \Diamond\mathsf{E}(\bar{n})$, and $(\exists\nu)\underline{N}(\nu) \vee \Diamond(\exists\nu)\underline{N}(\nu)$ are ***K**-valid; so $\Diamond E(\bar{n})$ and $\Diamond(\exists\nu)\underline{N}(\nu)$ are ***T**-valid. This is in keeping with the suggestion that many existential constructions within the Mathematical-Object picture should be understood as prefexed by the possibility operator. Hereafter, let's restrict our attention to **T**-structures and **T**-models; this restriction could be dropped at the cost of some notational complexity.

Even though our semantics does not treat #-terms as designators, it

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

permits us to reconstruct the non-semantic content of the doctrine that numerals are rigid designators: for any $n \in \omega$,

$$\Box(\forall x)(x = \bar{n} \supset \Box(x = \bar{n})) \text{ is } {}^*\underline{K}\text{-valid.}$$

The suggestion initiating the modal gambit was that certain existential statements to which the Mathematical-Object picture is committed (e.g., that there are at least n objects, for each $n < \aleph_0$), be understood as within the scope of an implicit possibility operator. Our semantics partly bears out that idea. For example, although $(\forall \nu)(\underline{N}(\nu) \supset (\exists \rho)(\underline{N}(\rho)\ \&\ \nu \leqslant \rho\ \&\ \neg \rho \leqslant \nu))$ is not even *$\mathbf{S5}$-valid, $\Box(\forall \nu)(\underline{N}(\nu) \supset \Diamond(\exists \rho)(\underline{N}(\rho)\ \&\ \nu \leqslant \rho\ \&\ \neg \rho \leqslant \nu))$ is *$\mathbf{T}$-valid, and expresses the "modal unboundedness" of the natural numbers.

But this idea does not lead to an encoding of $L^{\Box,x,2}$(EXACTLY) into $L^{\Box,x,\#}$. The problem is simple: relative to $(\mathscr{A}, \varkappa, w)$ for a type-0 variable ν, $\Diamond(\exists \nu)(\underline{N}(\nu)\ \&\ \ldots)$ "moves us" to other worlds u and then has us seek a witnessing member of $\mathrm{Rng}(\varkappa)$ in $\bar{\mathscr{A}}(u)$; we may find one, but then have no way "back" to w to see what hold for it there. For example, $(\exists \mu)(\text{EXACTLY } \mu\nu)\underline{P}(\nu)$, parsing 'There are finitely many Ps', appears not to be expressible in $L^{\Box,1,\#}$, even if we restrict ourselves to e-actualistic $\mathbf{S5}$-models.

Some of $L^{\Box,x,2}$(EXACTLY) can be encoded into $L^{\Box,x,\#}$. For $\varphi \in \mathrm{Sent}(L^{\Box,x,2}(\text{EXACTLY}))$, let φ be special iff each prefix of the form $(\exists \mu)$ occurring in φ, for μ of type 2, has scope of the form $\Box\psi\ \&\ \Diamond\theta_1\ \&\ \ldots\ \&\ \Diamond\theta_n\ \&\ \xi$, and ξ contains no occurrences of prefexes (EXACTLY $\mu\nu$) or of any predicate-constant.

Observation 7: There is a translation t from special sentences of $L^{\Box,x,2}$(EXACTLY) into $\mathrm{Sent}(L^{\Box,x,\#})$ so that for any special sentence φ of $L^{\Box,x,2}$(EXACTLY) and any $\mathbf{S5}$-model $\mathscr{M}$ that supports encoding: $\mathscr{M} \models \varphi$ iff $\mathscr{M} \models t(\varphi)$.

Form $t(\varphi)$ from φ by replacing each type-2 variable μ in φ with a new distinct type-0 variable ν_μ and replacing subformulae $(\exists \mu)(\Box\psi\ \&\ \Diamond\theta_1\ \&\ \ldots\ \&\ \Diamond\theta_n\ \&\ \xi)$ by $\Diamond(\exists \nu_\mu)(\underline{N}(\nu_\mu)\ \&\ \Box\psi\ \&\ \Diamond\theta_1\ \&\ \ldots\ \Diamond\theta_n\ \&\ \xi)$. Over an $\mathbf{S5}$-model, this preserves truth and falsity.

If we want more encoding, we're forced to compromise with Possibilism. I'll describe two ways to do this.

(1) We could form $L^{\cdot x,\#}$ by enriching $L^{\Box,x,\#}$ with '$\acute{\exists}$'. Then we can translate from $\mathrm{Sent}(L^{\Box,x,2}(\text{EXACTLY}))$ by replacing $(\exists \mu)\ldots$ by $(\acute{\exists} \nu_\mu)(\underline{N}(\nu_\mu)\ \&\ \ldots)$. Only a lax Individual-Actualist could be happy with this approach. But even this will not permit us to encode $L^{\Box,x,4}$(EX-

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

ACTLY) into $L^{\cdot,x,\#}$. This is because '#' is actualistic. To handle variables of type 4 we'd need to further enrich $L^{\cdot,x,\#}$, for example, by '$\acute{\#}$' under the clause:

$$\mathrm{des}^{\mathscr{A},\varkappa,w}((\acute{\#}\nu)\varphi) \simeq \varkappa({}^2Q(n)), \quad \text{where} \quad n = \mathrm{card}(\{a \in \hat{\mathscr{A}}(w): \mathscr{A}, \varkappa, w \overset{\#}{\models} \varphi(\underline{a}/\nu)\}).$$

So the compromise with Possibilism is substantial.

(2) We could enrich $L^{\Box,x,\#}$ with the quantifier '$\exists^*$' binding variables of type 1 and ranging over individual-essences. First we generalize the notion of an individual essence introduces in Section 8 to non-**S5** structures. Fix a structure $\mathscr{A}$ with set W of "worlds" and accessibility relation R. For $a \in |\mathscr{A}|$, let $a^* = \{\langle w, a\rangle: a \in \hat{\mathscr{A}}(w)\}$. (For **S5**-structures, this definition coincides that given in Section 8. By sticking with the latter, we'd spare ourselves some complication, but we would not have modal logics weaker than **S5** validate the principle that any individual essence is or could be (could have been) instanced.) For 'x' replaceable as usual, we enrich $L^{\Box,x}$ to $L^{\Box,x,*}$ by introducing the type-1 variables and '$\exists^*$' binding them and ranging over individual-essences, as follows: for α such a variable:

$$\mathscr{A}, w \models (\exists^*\alpha)\varphi \text{ iff for some } a \in \hat{\mathscr{A}}(w)\ \mathscr{A}, w \models \varphi(\underline{a}^*/\alpha).$$

Note: in **S5** we may regard $L^{\Box,1,*}$ $[L^{\Box,(0,0),*}]$ as a fragment of $L^{\Box,1}$ $[L^{\Box,(0,0)}]$, since over **S5**-structures we can express "α is an individual constant", for a type-1 variable α, by

$$\Box(\forall\nu)(\alpha\nu \supset \Box(\forall\rho)(\alpha\rho \supset \nu = \rho))\ \&\ \Diamond(\exists\nu)\alpha\nu.$$

For variables α, β of type-1, we adopt these abbreviations:

$$\underline{N}(\alpha) \text{ for } \Diamond(\exists\nu)(\alpha\nu\ \&\ \underline{N}(\nu));$$
$$\alpha \leqslant \beta \text{ for } \Diamond(\exists\nu)\Diamond(\exists\rho)(\alpha\nu\ \&\ \beta\rho\ \&\ \nu \leqslant \rho).$$

Though $(\forall\nu)(\underline{N}(\nu) \supset (\exists\rho)(\underline{N}(\rho)\ \&\ \nu \leqslant \rho\ \&\ \neg\rho \leqslant \nu))$ isn't even e-actualistically ***S5**-valid, $(\forall\alpha)(\underline{N}(\alpha) \supset (\exists\beta)\underline{N}(\beta)\ \&\ \alpha \leqslant \beta\ \&\ \neg\beta \leqslant \alpha))$ is ***T**-valid.

Observation 8. There is a translation t_3,

$$t_3\colon\ \mathrm{Sent}(L^{\Box,x,2}(\text{EXACTLY})) \to {}^*\mathbf{K}\text{-Biv}(L^{\Box,x,\#,*}),$$

so that for any model $\mathscr{M}$ that supports encoding and any $\varphi \in \mathrm{Sent}(L^{\Box,x,2}(\text{EXACTLY}))$: $\mathscr{M} \models \varphi$ iff $\mathscr{M} \models t_3(\varphi)$.

Given $\varphi \in \mathrm{Sent}(L^{\Box,x,2}(\text{EXACTLY}))$, for each variable μ of type 2 in φ

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

introduce a distinct new variable α_μ of type 1. Form φ' by replacing subformulae of φ as follows:

$$(\exists\mu)\theta \text{ by } (\exists\alpha_\mu)(\underline{N}(\alpha_\mu)\ \&\ \theta');$$
$$\mu \leqslant \mu' \text{ by } \alpha_\mu \leqslant \alpha_{\mu'};$$
$$(\text{EXACTLY } \mu\nu)\theta \text{ by } \alpha_\mu((\#\nu)\theta').$$

φ' is as desired.

Observation 9. There is a translation t_4,

$$t_4\colon \mathrm{Sent}(L^{\Box,(0,0),\#,*}) \rightarrow \mathrm{Sent}(L^{\Box,(0,0)}),$$

so that for any $\varphi \in {}^*\mathbf{K}\text{-Biv}^{ea}(L^{\Box,(0,0),*})$, φ is e-actualistically $^*\mathbf{S5}$-equivalent to $t_4(\varphi)$.

Given φ as above, since we're working in **S5** we may eliminate '$\exists^*$' in favor of '$\exists$'. Reletter to make sure every variable is bound at most once. As we did in proving Observation 5, transform φ into an e-actualistically **K**-equivalent sentence φ' in which #-terms only occur in equations; our restriction to e-actualistic models insures that for a predicate-constant ζ and #-term τ, $\zeta(\ldots,\tau,\ldots)$ and $(\exists\rho)(\rho = \tau\ \&\ \zeta(\ldots,\rho,\ldots))$ have the same truth-value (for a new variable ρ of type 0). The rest of the construction combines the arguments for Observations 5 and 6, and is left to the reader. The restriction to e-actualistic models in this previous observation is annoying; but I cannot see how to avoid it.

By (Hodes 1984c) Section 5 (14), even if we restrict ourselves to **S5**-models '$\exists^*$' does not give us '$\acute{\exists}$'; so one might argue that allowing quantification over individual essences is not as possibilistic as allowing a quantifier over possible objects. But, by p. 453, within **S5**, if we restrict ourselves to e-actualistic **S5**-models, '$\exists^*$' does give '$\acute{\exists}$'. So it's unclear that (2) has an actualistic advantage over (1).

Approach (2) also faces a "pre-metaphysical" problem. $L^{\Box,x,\#,*}$ permits two sorts what might be called quantification over numbers: $(\exists\nu)(\underline{N}(\nu)\ \&\ \ldots)$ and $(\exists^*\alpha)(\underline{N}(\alpha)\ \&\ \ldots)$. Granted, it seems that the latter does all the work; so perhaps the former should be set aside as not representing a construction used in mathematical practice. Still, if $L^{\Box,x,\#,*}$ models languages in which mathematics is practiced, it gives a surprising logico-syntactic analysis of such languages: it requires that apparently first-order quantification (binding variables of type 0) really be second-order (binding variables of type 1). Nothing in our inferential practice suggests this. The very idea is an ad hoc device to save the modal gambit from the laxest sort of Actualism. In conclusion: my view

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

of the modal gambit as expressed in my (1984a) article was too sanguine. It seems that only a lax Individual-Actualist can even get an Alternative model-theoretic semantics for $L^{\Box,0,\#}$ off the ground. Finally, it seems that even a lax Individual-Actualist can't encode $L^{\Box,0,4}$(EXACTLY) without allowing a non-actualistic "number-of"-operator. I tentatively conclude that an Individual-Actualist who accepts the Alternative theory does best to accept an actual infinitude.[19]

10.

I offer the model-theoretic semantics of Section 5 as philosophical therapy for those who, "gripped" by the Mathematical-Object picture, have gone on to swallow the Mathematical-Object Theory.[20] The point of an exercise in modelling is not always self-evident. So in this section I'll try to summarize the philosophical point of model-theoretic semantics.

Are there natural numbers? For example, does the number one "really" exist? Construed within the Mathematical-Object Picture (i.e., without reinterpretation) the answers to both questions are trivially "Yes". Indeed, our model-theory suggests "Yes, as a logical consequence of the existence of infinitely many objects". Provided we avoid the modal gambit, this does not violate Quine's plausible strictures, echoed by Dummett, against allowing that numbers exist, but only in a funny sense of 'exist'[21]:

> The only sense we have for 'exists' is that given by the existential quantifier in the sentences we ordinarily use . . . (3, p. 497)

Dummett goes on to say, however:

> if we have provided determinate truth-conditions for a certain existential statement, and, under those truth-conditions, the statement proves to be true, then there exists something satisfying the condition given in the statement, and that is an end of the matter.

Whether this is right depends on what the matter in question is. For one may reinterpret the question "Do numbers, e.g., the number one, exist?" metalinguistically, as "Do number-terms, e.g. '1', designate?" To somebody who accepts the Supposedly Self-Evident Principle from Section 1, this construal will seem no different from the preceding one. The model-theoretic semantics presented in Section 5 shows that it is at least formally coherent to reject that principle from right to left. In particular it is coherent to accept the literal truth of 'The number one

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

exists', but not of the semantic thesis that "the number one" designates something. In this case, the semantic and the existential questions are distinct.

Of course it is a mistake to formulate a semantic question as an existential one – though the mistake comes naturally if one's philosophical reflexes have been shaped by the principle just rejected. I think that in many cases it is a concern with the semantic question that motivates people to ask the otherwise trivial existential question, and that inclines those who do so to insert the scare-quoted 'really' as if to acknoweldge the infelicity of their formulation of the question.

To avoid such infelicity, let me put my point like this. There are two construals of Quine's slogan 'To be is to be the value of a variable'. Under the thin construal, Quine's point is Dummett's: the ontological commitments of a body of discourse are simply a matter of what existentially-quantified statements are asserted in that discourse under an optimal parsing. Under the thick construal, Quine's slogan tells us to read ontological commitments off of an optimal theory (or definition) of truth, or better, of satisfaction, for the language in which the discourse occurs; the values of variables are literally the values that variable-assignments, as relata of the satisfaction relation, assign to the variables. (Of course in this paper I formulated the model-theoretic definitions so as to avoid use of satisfaction and variable-assignments, in favor of truth plus the introduction of new constants. This was an expository convenience which I hope will not produce confusion: under the model-theoretic semantics presented here, the values of the type-0 variables are the designata of the terms of type-0.)

Now the contrast between existential and semantic questions may be described as "external" versus "internal"; but I can see no reason to rule out the "external" semantic question as illegitimate. In particular, asking it does not involve trying to speak and think "outside of all language". It does involve semantic ascent. Indeed, how we answer it depends on whether or not in so ascending we also extricate ourselves from the Mathematical-Object Picture. This is what the Alternative theorist prefers to do; the denial that, e.g., 'the number one' designates something, is made within a metalanguage representable as an $L^{x,4}$(EXACTLY) rather than as $L^{x,\#}$.

What advantage is there in this extrication from the Mathematical-Object Picture? Can we limn reality better within a language of the former form than within one of the latter form? In a limited way, yes.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

A sense-bearing language whose alethic underpinnings were modeled by $L^{x,4}$(EXACTLY) would be semantically uniform: lexical items of the same basic syntactic category would all do the same kind of semantic work. Our mathematical discourse is not semantically uniform; it can be modeled by uninterpreted languages of the form $L^{x,\#}(S)$, languages whose type-0 terms include #-terms as well as terms based on S. If we think within a semantically uniform metalanguage, we're not liable to certain metaphysical puzzlements and illusions, those which have fuelled philosophical controversy about mathematical objects.

But we cannot go beyond this to say, e.g., that the number one doesn't "really" exist: within a language representable as $L^{x,4}$(EXACTLY) a question like 'Does the number one exist?' can't even be formulated![22] Furthermore, if suitably understood, the Mathematical-Object Picure can be semantically innocuous, as it is mathematically innocuous. The Alternative theorist can introduce a non-robust disquotational sense of 'designates', allow that under it sentences like "1' designates 1' are literally true, and thereby imitate the Mathematical-Object Theorist. If not understood correctly, this move nicely disguises the distinction between thick and thin ontological commitment. Notice: if we want to conceive this model-theoretically, say by enriching $L^{x,\#}$ with meta-linguistic terms and a 2-place predicate '$\underline{D}$' to represent 'designates', then relative to a model $\mathscr{A}$ '$\underline{D}$', like '$\underline{N}$' and '$\leqslant$', would not have an extension. ('$\underline{D}$' would have one relative to each representor for $\mathscr{A}$; but these would be artifacts of the model.) See (Hodes 1990, section 8) for a detailed exposition. In non-model-theoretic terms: 'designates', in the extended use it is accorded by this move, doesn't stand for a relation. Does the availability of such an ersatz semantics lend weight to the Carnapian attitude that our external semantic question is merely a call for the establishment of a convention? Only in the trivial sense in which the meaning of our semantic vocabulary, like that of all vocabulary, is conventional. Whether or not we speak a semantically uniform metalanguage is, in a sense, a matter of convention. The important point then is that some conventions are less misleading than others.

The impression that the mere truth of 'The number one exists' insures that 'the number one' designated something dies hard. Certainly one needn't accept Frege's Context Principle under the Dummett and Wright reading (that all there is to the fact that a singular term designates is that it contribute uniformly to determine the truth-value of

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

sentences in which it occurs) to feel its pull. Dummett is quite aware of the strain put on the notion of reference when this principle is used to support the Fregean doctrine that numbers are objects designated by singular numerical expressions. He seems to defend the latter view, but only by qualifying it to the point of surrender:

> . . . reference may be ascribed to them only as a *façon de parler*. . . . their meaning cannot be construed after the realistic model, as determined by a relation of reference between them and external objects. (1973, p. 508)

I find the phrase 'after the realistic model' quite obscure; and talk of discerning "numbers as constituents of the external world" (1973, p. 505) doesn't help me. Perhaps by 'reference . . . ascribed only as a *façon de parler*', Dummett alludes to the sort of disquotational semantic ascent within a metalanguage of the form $L^{x,\#}$ described in the previous paragraph.

It would be somewhat in the spirit of Dummett's construal of the context principle to urge that an assignment of truth-values to sentences of $L^{x,\#}$ by the model-theoretic semantics from Section 5 would suffice to make #-terms designators. (This is not completely in that spirit, since such an assignment would not be total.) Let's try to picture this model-theoretically.

Let a Frege-model $(\mathscr{A}, \varkappa)$ be special iff

$$\text{for every } \varphi \in \mathrm{Sent}(L^{(0,0),\#})\ \mathscr{A}, \varkappa \overset{\#}{\models} \varphi \text{ iff } \mathscr{A} \models \varphi.$$

Of course our semantics in Section 4 was two-valued; so the existence of ∞-truth-valueless sentences, in even $L^{0,\#}$, prevent there from being special Frege-models. But by using a non-bivalent semantics from the start, as sketched in Appendix 2, we at least avoid this trivialization. We may then allow $\mathscr{A}$ to be a partial model, and we'd want to add a second clause to the definition of specialness:

$$\text{for every } \varphi \in \mathrm{Sent}(L^{(0,0),\#})\ \mathscr{A}, \varkappa \overset{\#}{\dashv} \varphi \text{ iff } \mathscr{A} \dashv \varphi.$$

Dummett says:

> Frege takes the possibility of giving an incontestably legitimate explanation . . . of the senses of sentences containing terms of a given kind that behave like proper names as a justification for regarding those terms as standing for objects.

In what seems to be a partial defense of this, he adds:

> If the stipulation is carried out in a logically unobjectionable manner, then no absurdity can arise from crediting those names with the property of standing for objects. . . . Numer-

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

ical terms, if explained in such a way, would, as it were, be said to have reference only by courtesy. (1981, p. 425–26)

For a special Frege-model $(\mathscr{A}, \varkappa)$, $\varkappa$ would be such a "courtesy" to $\mathscr{A}$. Under the semantics from Appendix 2, there are special Frege-models (total ones if S has no function-constants). But the condition that a given partial model have such a "courtesy" representor is very restrictive. For example, if ν is the only variable free in $\varphi \in \mathrm{Fml}(L^{(0,0)})$, we can't have $(\forall\nu)(\varphi \vee (\neg\varphi))$, $(\exists\nu)\varphi$ and $(\exists\nu)(\neg\varphi)$ all true in such a model. (Proof is left to the reader.) The moral: Dummett's courtesy can only be extended to the Alternative semantics for languages much weaker than those we speak, or under the assumption that the world is as uniform as such a model.

One way to stick to the Mathematical-Object Theory would be to maintain that numbers are literally products of linguistic practice, or its mental analog. A few pages before the *façon-de-parler* passage we find one in which Dummett might be advocating this position (though it is hard to be sure):

Pure abstract objects are no more than the reflections of certain linguistic expressions, expressions which behave, by simple formal criteria, in a manner analogous to proper names of objects. (1973, p. 505)

(Does 'reflection' carry the implication that pure abstract objects would not exist without such linguistic expressions that so behaved?) And numerous philosophers have said that numbers are produced by mental construction. (George Boolos has pointed out to me the analogy between the semantics offered in Section 5 and Dedekind's assertion that the mind constructs the natural numbers by abstracting away the differences between all ω-sequences. But the disanalogy is more salient: according to the Alternative theory, #-terms don't designate anything, including mental constructs.[23])

This sort of view is not absurd. Certain mildly abstract entities, e.g., clubs or nations, are products of complex social practices. Others, e.g., mathematical proofs under one construal of the phrase, are the products of mental activity. And it seems plausible to maintain that fictional characters in literature and legend are products of the mental activity and linguistic practices involved in the creation and persistence of literature and legend.

I'm disinclined to extend this thesis to the natural numbers; 'The number one did not exist one million years ago' seems literally untrue.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

(For what it's worth, the model-theoretic semantics of Section 5 suggests that it's not true: there are things that did exist one million years ago; let 'P' parse 'existed one million years ago'; note that for any infinite model $\mathscr{A}$, if $\mathscr{A} \models$ '$(\exists x)\underline{P}x$' then $\mathscr{A} \not\models$ '$\neg \underline{P}\bar{1}$'.) One could then retreat to possibilism, identifying numbers with permanently possible products of mathematical practice. Perhaps the only argument against both of these views is based on Ockham's razor: once we accept quantification over cardinality-quantifiers as legitimate, why posit objects that intrinsically encode them, especially when we can enjoy the comforts of the Mathematical-Object Picture without them? That we can has, I hope, also been demonstrated by this paper. Of course the Alternative theory doesn't really close the question of the ontological commitments of arithmetic; rather it refocuses the question where, I submit, it belongs – on the ontological commitments of higher-order languages.

APPENDIX 1

We'll consider a model-theoretic semantics that represents a non-Fregean Mathematical-Object theory, one that draws type-distinctions between objects. Introduce a countable set of new variables of type-**n** ('**n**' for 'numbers'). Where 'x' is replaceable by '0', '1', or '(0, 0)', form $L^{x,\mathbf{n}}$ from L^x by adding '$\leqslant$' and '$\#$' to the logical lexicon, with the formation rules:

> if ν is a variable of type 0 or type **n** and φ is a formula then $(\# \nu)\varphi$ is a term of type **n**;
> if ν and μ are variables of type **n** then $\nu \leqslant \mu$ is a formula.

For ν and μ of type **n** let $\nu = \mu$ be $\nu \leqslant \mu$ & $\mu \leqslant \nu$.

Let an **n**-model be of the form $(\mathscr{A}, R)$, where $\mathscr{A}$ is a model and R is a reflexive well-ordering of order-type ω. Form $L^{x,\mathbf{n}}_{\mathscr{A},R}$ as before, except also introducing a type **n** constant $\underline{e}$ for each $e \in \mathrm{Fld}(R)$. We define $\mathrm{des}^{\mathscr{A},R}$ and $\mathscr{A}, R \models$. The definitions are as usual, with these additional clauses:

> $\mathrm{des}^{\mathscr{A},R}((\# \nu)\varphi) \simeq n^R$, if $n = \mathrm{card}(\hat{\nu}\varphi^{\mathscr{A},R})$;
> $\mathscr{A}, R \models \underline{d} \leqslant \underline{e}$ iff dRe;

here n^R is the $n + 1$st element of $\mathrm{Fld}(R)$ under R, and if ν is a variable of type **n** then $\hat{\nu}\varphi^{\mathscr{A},R} = \{e \in \mathrm{Fld}(R)\colon \mathscr{A}, R \models \varphi(\underline{e}/\nu)\}$.

Relative to an **n**-model $(\mathscr{A}, R)$, members of $\mathrm{Fld}(R)$ represent natural

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

numbers; R itself may be thought of as coding the function assigning ${}^2Q(n)$ to n^R, which represents the standard representor. Notice that $|\mathscr{A}|$ and $\mathrm{Fld}(R)$ do not "interact"; they might as well be taken to be disjoint.[24]

As in Section 4, there are non-designating terms of type **n.** This may be avoided by using $\mathbf{n}^\infty$-models rather than **n**-models, i.e., by taking R to be a well-ordering of order-type $\omega + 1$ and adding the clause:

$$\mathrm{des}^{\mathscr{A},R}((\#\nu)\varphi) = \omega^R \text{ if } \hat{\nu}\varphi^{\mathscr{A},R} \text{ is infinite;}$$

here ω^R is the last element of $\mathrm{Fld}(R)$ under R.

We now make precise the sense in which $L^{x,\mathbf{n}}$ encodes $L^{x,4}(\textsc{exactly})$ $[L^{x,4}(\textsc{exactly}^\infty)]$.

Observation 10: There is a translation function u $[u^\infty]$,

$$u\colon \mathrm{Sent}(L^{x,4}(\textsc{exactly})) \to \mathrm{Sent}(L^{x,\mathbf{n}})$$
$$[u^\infty\colon \mathrm{Sent}(L^{x,4}(\textsc{exactly}^\infty)) \to \mathrm{Sent}(L^{x,\mathbf{n}})].$$

so that for any **n**-model [$\mathbf{n}^\infty$-model] $(\mathscr{A}, R)$ and any $\varphi \in \mathrm{Sent}(L^{x,4}(\textsc{exactly}))$ $[\mathrm{Sent}(L^{x,4}(\textsc{exactly}))]$:

$$\mathscr{A} \models \varphi \text{ iff } \mathscr{A}, R \models u(\varphi)\ [u^\infty(\varphi)].$$

This construction is like that from Section 4. However we now can reverse our translation.

Observation 11: There is a translation u^*,

$$u^*\colon \mathrm{Sent}(L^{x,\mathbf{n}}) \to \mathrm{Sent}(L^{x,4}(\textsc{exactly})),$$

so that for any **n**-model $(\mathscr{A}, R)$ and any $\varphi \in \mathrm{Sent}(L^{x,\mathbf{n}})$:

$$\mathscr{A}, R \models \varphi \text{ iff } \mathscr{A} \models u^*(\varphi).^{25}$$

Given $\varphi \in \mathrm{Sent}(L^{x,\mathbf{n}})$, find an **n**-equivalent φ' in which each occurrence of a $\#$-term τ occurs only in the context $\tau = \mu$, for a type-n variable μ. For example, replace $\tau = \sigma$ or $\sigma = \tau$ by $(\forall\mu)(\tau = \mu \equiv \sigma = \mu)$, where μ doesn't occur in τ or σ. In φ' replace each subformula $\mu = (\#\rho)\theta$ by $(\textsc{exactly}\ \mu\rho)\theta$, now making all type-**n** variables into type-2 variables. We may now have subformula of the form $(\textsc{exactly}\ \mu\rho)\theta$ in which ρ is of type-2; for each such, introduce a type-4 variable μ' corresponding to μ and replace that subformula by $(\exists\mu')((\textsc{exactly}\ \mu'\rho)\theta\ \&\ \mu = \mu')$. It's easy to see that this produces the required $u^*(\varphi)$. An analogous $u^{\infty *}$ may be constructed for $\mathbf{n}^\infty$-models and $L^{x,4}(\textsc{exactly}^\infty)$.

Observation 12. Where R is the reflexive closure of a well-ordering

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

of order-type $\omega + 1$, let R' be obtained by deleting ω^R from $\mathrm{Fld}(R)$. There is a translation s $[s^\infty]$ with

$$s\,[s^\infty]\colon\ \mathrm{Sent}(L^{x,\mathbf{n}}) \rightarrow \mathrm{Sent}(L^{x,\mathbf{n}}),$$

so that for every $\mathbf{n}^\infty$-model $(\mathscr{A}, R)$ and every $\varphi \in \mathrm{Sent}(L^{x,\mathbf{n}})$:

$$\mathscr{A}, R \models \varphi \text{ iff } \mathscr{A}, R' \models s(\varphi)$$
$$[\mathscr{A}, R' \models \varphi \text{ iff } \mathscr{A}, R \models s^\infty(\varphi)].$$

The construction of s^∞ is straightforward; the construction of s follows that used for Observation 1.

I am inclined to regard versions of the Mathematical-Object theory that draw type-distinctions between objects as *prima facie* unmotivated. Frege's type-distinctions, e.g., between objects and level-one concepts, rely on the uncontroversial fact that certain combinations of symbols, e.g., 'Frege Church' or 'is a philosopher is bald', are not sentences, and indeed can't even be construed as ungrammatical attempts at constructing sentences. One can't claim that it's equally obvious that '1 = Julius Caesar' is not a sentence. The thesis that the terms flanking the occurrence of '=' are designators only make such a claim less self-evident: if I can refer to two objects, why couldn't I say that they are identical? Those who accept such a type-distinction owe us some explanation of why they accept them; by itself, an appeal to "linguistic intuition" tells us nothing. (The model-theoretic semantics presented in Section 5 reconstructs some motivation for such a type-distinction, in so far as it characterizes what is anomalous about sentences like '1 = Julius Caesar'; but it also rejects such a distinction.)

Indeed Observations 10 and 11 together show that the picture of arithmetic discourse presented by the non-Fregean version of the Mathematical-Object picture (according to which the natural numbers constitute a logical type) can easily be reconstrued by the Alternative theory, translating from $L^{x,\mathbf{n}}$ to $L^{x,4}(\text{EXACTLY})$ or $L^{x,4}(\text{EXACTLY}^\infty)$, and back again. These facts justify the natural inclination to think that there is no mathematical reason for preferring $L^{x,\#}$ to $L^{x,\mathbf{n}}$, or vice versa. Of course such a cavalier attitude towards this choice is not available to those who accept the Mathematical-Object Theory.

APPENDIX 2

A partial model $\mathscr{A}$ for S consists of a set $|\mathscr{A}|$ and assignments as follows:

for an n-place predicate-constant $\zeta \in S$:

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

if $n > 0$, $\zeta^{\mathscr{A}}: \subseteq |\mathscr{A}|^n \to \{0, 1\}$;
if $n = 0$, either $\zeta^{\mathscr{A}} \uparrow$ or $\zeta^{\mathscr{A}} \in \{0, 1\}$;

for an n-place function-constant $\rho \in S$:

if $n < 0$, $\rho^{\mathscr{A}} : \subseteq |\mathscr{A}|^n \to |\mathscr{A}|$;
if $n = 0$, either $\rho^{\mathscr{A}} \uparrow$ or $\rho^{\mathscr{A}} \in |\mathscr{A}|$.

Where $\mathscr{A}$ is a partial model for S, extend $L^{(0,0)}$ to $L^{(0,0)}_{\mathscr{A}}$ as in Section 2. We define the partial designation function $\mathrm{des}^{\mathscr{A}}$ on closed terms of $L^{(0,0)}_{\mathscr{A}}$ as in Section 2. We simultaneously define $\mathscr{A} \models$ and $\mathscr{A} \dashv$ for sentences of $L^{(0,0)}_{\mathscr{A}}$ as follows:

$\mathscr{A} \dashv$ '$\perp$';
$\mathscr{A} \models \tau = \sigma$ iff $\mathrm{des}^{\mathscr{A}}(\tau) = \mathrm{des}^{\mathscr{A}}(\sigma)$;
$\mathscr{A} \dashv \tau = \sigma$ iff either $\mathrm{des}^{\mathscr{A}}(\tau) \downarrow$, $\mathrm{des}^{\mathscr{A}}(\sigma) \downarrow$, and
$\mathrm{des}^{\mathscr{A}}(\tau) \neq \mathrm{des}^{\mathscr{A}}(\sigma)$, or $\mathrm{des}^{\mathscr{A}}(\tau) \uparrow$ and $\mathrm{des}^{\mathscr{A}}(\sigma) \downarrow$ or
$\mathrm{des}^{\mathscr{A}}(\tau) \downarrow$ and $\mathrm{des}^{\mathscr{A}}(\sigma) \uparrow$;
$\mathscr{A} \models \zeta(\tau_0, \ldots, \tau_{n-1})$ iff $\zeta^{\mathscr{A}}(\mathrm{des}^{\mathscr{A}}(\tau_0), \ldots, \mathrm{des}^{\mathscr{A}}(\tau_{n-1})) = 1$;
$\mathscr{A} \dashv \zeta(\tau_0, \ldots, \tau_{n-1})$ iff $\zeta^{\mathscr{A}}(\mathrm{des}^{\mathscr{A}}(\tau_0), \ldots, \mathrm{des}^{\mathscr{A}}(\tau_{n-1})) = 0$;
for $A \subseteq |\mathscr{A}|$, let $\underline{A}^{\mathscr{A}}$ = the characteristic function for A on $|\mathscr{A}|$;
for $B \subseteq |\mathscr{A}|^2$, let $\underline{B}^{\mathscr{A}}$ = the characteristic function for B on $|\mathscr{A}|^2$;
$\mathscr{A} \models \psi \supset \psi$ iff either $\mathscr{A} \dashv \varphi$ or $\mathscr{A} \models \psi$;
$\mathscr{A} \dashv \varphi \supset \psi$ iff $\mathscr{A} \models \varphi$ and $\mathscr{A} \dashv \psi$;
$\mathscr{A} \models (\exists \nu)\varphi$ iff for some $a \in |\mathscr{A}|$ $\mathscr{A} \models \varphi(\underline{a}/\nu)$;
$\mathscr{A} \dashv (\exists \nu)\varphi$ iff for every $a \in |\mathscr{A}|$ $\mathscr{A} \dashv \varphi(\underline{a}/\nu)$;
where γ is a type-1 [type-(0, 0)] variable:
$\mathscr{A} \models (\exists \gamma)\varphi$ iff for some $A \subseteq |\mathscr{A}|$ $[A \subseteq |\mathscr{A}|^2]$ $\mathscr{A} \models \varphi(\underline{A}/\gamma)$;
$\mathscr{A} \dashv (\exists \gamma)\varphi$ iff for every $A \subseteq |\mathscr{A}|$ $[A \subseteq |\mathscr{A}|^2]$ $\mathscr{A} \dashv \varphi(\underline{A}/\gamma)$.

We adopt these further abbreviations:

$A \mid \varphi \quad : \mathscr{A} \not\models \varphi$ and $\mathscr{A} \not\dashv \varphi$;
$\mathscr{A} \models^{w} \varphi : \mathscr{A} \not\dashv \varphi$;

read the latter as "φ is weakly true in $\mathscr{A}$".

The semantics given above is, in (Hodes 1989), accorded to the logical lexicon $\mathrm{lex}_{1,s}$; notice that '$\supset$' is given the strong-Kleene truth-table. We introduce '$\neg$', '&', '$\vee$', '$\equiv$' and '$\forall$' by the standard abbreviations; thus '&' and '$\vee$' also have strong-Kleene truth-tables. Letting $\varphi \supseteq \psi$ abbreviate

$$(\varphi \supset \varphi) \ \& \ (\psi \supset \psi) \ \& \ (\varphi \supset \psi),$$

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

we introduce the weak-Kleene conditional '$\underline{\supset}$'. Note that '$\underline{\exists}$' is the quantifier that generalizes our truth-table for '$\vee$'. For a type-0 term τ, $E(\tau)$ still says "τ exists"; for any model $\mathscr{A}$:

$$\mathscr{A} \models E(\tau) \text{ iff } \mathrm{des}^{\mathscr{A}}(\tau)\downarrow;\ \mathscr{A} \dashv E(\tau) \text{ iff } \mathrm{des}^{\mathscr{A}}(\tau)\uparrow.$$

Let $(\underline{\exists}\nu)\varphi$ abbreviate $(\forall\nu)(\varphi \supset \varphi)\ \&\ (\exists\nu)\varphi$. For ν a variable of type-0, let φ be ν-bivalent in $\mathscr{A}$ iff:

$$\text{for every } a \in |\mathscr{A}|, \text{ either } \mathscr{A} \models \varphi(\nu/\underline{a}) \text{ or } \mathscr{A} \dashv \varphi(\nu/\underline{a}).$$

We then have the following:

$$\mathscr{A} \models (\underline{\exists}\nu)\varphi \text{ iff } \varphi \text{ is } \nu\text{-bivalent in } \mathscr{A} \text{ and for some } a \in |\mathscr{A}|,\ \mathscr{A} \models \varphi(\nu/\underline{a});$$
$$\mathscr{A} \dashv (\underline{\exists}\nu)\varphi \text{ iff for every } a \in |\mathscr{A}|,\ \mathscr{A} \dashv \varphi(\nu/\underline{a}).$$

Thus '$\underline{\exists}$' is the "weak" analog of the "strong" quantifier '$\exists$'.

Suppose that we have:

$$\mathscr{D}_2 \subseteq \mathrm{Power}(\{\langle A_0, A_1\rangle\colon\ A_0 \subseteq A_1 \subseteq |\mathscr{A}|\});$$
$$\mathscr{D}_4 \subseteq \mathrm{Power}(\{\langle A_0, A_1\rangle\colon\ A_0 \subseteq A_1 \subseteq \mathscr{D}_2\}).$$

Then we may expand $L^{(0,0),4}$ to $L^{(0,0),4}_{\mathscr{A},\mathscr{D}_2,\mathscr{D}_4}$ and define truth in $\mathscr{A}$, $\mathscr{D}_2$, $\mathscr{D}_4$ with these new clauses; where ν is a type-0 variable and $Q \in \mathscr{D}_2$:

$$\check{\nu}\varphi^{\mathscr{A},\mathscr{D}_2,\mathscr{D}_4} = \{a \in |\mathscr{A}|\colon \mathscr{A}, \mathscr{D}_2, \mathscr{D}_4 \models^{w} \varphi(\underline{a}/\nu)\};$$
$$\mathscr{A}, \mathscr{D}_2, \mathscr{D}_4 \models (Q\nu)\varphi \text{ iff } \langle \hat{\nu}\varphi^{\mathscr{A},\mathscr{D}_2,\mathscr{D}_4}, \check{\nu}\varphi^{\mathscr{A},\mathscr{D}_2,\mathscr{D}_4}\rangle \in Q;$$

similar clauses apply for ν a type-2 variables and $Q \in \mathscr{D}_4$. The basic point here is that we must attend to the "weak extension" of a formula at ν, as well as to its "strong extension".

We now face two possible choices for ^{2i}EXACTLY, between "weak" and "strong" quantifiers. For $n \in \omega$, the "strong" choice would be:

$${}^{2}Q(n) = \{\langle A_0, A_1\rangle\colon \text{ for some } A \subseteq |\mathscr{A}| \text{ with } \mathrm{card}(A) = n,\ A_0 \subseteq A \subseteq A_1\};$$

the "weak" choice would be:

$${}^{2}Q(n) = \{\langle A, A\rangle\colon\ A \subseteq |\mathscr{A}| \text{ and } \mathrm{card}(A) = n\}.$$

The corresponding choice must also be made for ${}^{4}Q(n)$. We gain generality with the strong choice, but this would make our life much harder when we come to the three-valued analog of Section 4. As in Section 3, we'll replace '$L^{x,4}$' by '$L^{x,4}$(EXACTLY)', with the corresponding other

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

notational changes. Suffice to say that the discussion in Section 3 may be extended in the natural way to these three-valued semantics. We may also develop a three-valued version of the material in Section 8; in so doing, we could introduce the notion of an *e*-actualist model without relying on the Falsehood Convention; see (Hodes 1987) for elaboration.

When $\mathscr{A}$ is a partial model, the definition of a representor for $\mathscr{A}$ carries over from Section 4 in the obvious way; similarly for the definition of a Frege-model. For a Frege-model $(\mathscr{A}, \varkappa)$, we might try defining $\mathrm{des}^{\mathscr{A},\varkappa}$ with this clause:

$$\mathrm{des}^{\mathscr{A},\varkappa}((\#\nu)\varphi) \simeq \varkappa(Q), \text{ where } \langle \hat{\nu}\varphi^{\mathscr{A},\varkappa}, \check{\nu}\varphi^{\mathscr{A},\varkappa}\rangle \in Q.$$

If we adopt the semantics for $L^{x,4}$(EXACTLY) under which variables of types 2 and 4 ranged over weak cardinality-quantifiers, the above definition would raise no problems; notice that it would make the following hold:

$$\text{if } \mathrm{des}^{\mathscr{A},\varkappa}((\#\nu)\varphi)\downarrow \text{ then } \varphi \text{ is } \nu\text{-bivalent in } (\mathscr{A}, \varkappa).$$

However if we adopt the semantics under which variables of types 2 and 4 ranged over strong cardinality-quantifiers, the above definition would fact this difficulty: there could be several $Q \in {}^{2i}\mathrm{EXACTLY}$ so that $\langle \hat{\nu}\varphi^{\mathscr{A},\varkappa}, \check{\nu}\varphi^{\mathscr{A},\varkappa}\rangle \in Q$.

This problem is not insurmountable. It would require that $\mathrm{des}^{\mathscr{A},\varkappa}$ accommodate multiple designation. One way would be to take its values be sets; where before we had $\mathrm{des}^{\mathscr{A},\varkappa}(\tau) = a$, now we'd have $\mathrm{des}^{\mathscr{A},\varkappa}(\tau) = \{a\}$; where before we had $\mathrm{des}^{\mathscr{A},\varkappa}(\tau)\uparrow$, now we'd have $\mathrm{des}^{\mathscr{A},\varkappa}(\tau) = \{\ \}$. Then our clause for '#' would be:

$$\mathrm{des}^{\mathscr{A},\varkappa}((\#\nu)\varphi) = \{\varkappa(Q) \colon \langle \hat{\nu}\varphi^{\mathscr{A},\varkappa}, \check{\nu}\varphi^{\mathscr{A},\varkappa}\rangle \in Q \quad \text{for some } Q \in {}^{2i}\mathrm{EXACTLY}\}.$$

We'd then have to change the definitions of $\overset{\#}{\models}$ and $\overset{\#}{\dashv}$ for atomic sentences. For example, we'd need:

$$\mathscr{A}, \varkappa \overset{\#}{\models} \tau = \sigma \text{ iff } \mathrm{des}^{\mathscr{A},\varkappa}(\tau) = \mathrm{des}^{\mathscr{A},\varkappa}(\sigma), \text{ which is a singleton;}$$
$$\mathscr{A}, \varkappa \overset{\#}{\dashv} \tau = \sigma \text{ iff } \mathrm{des}^{\mathscr{A},\varkappa}(\tau) \cap \mathrm{des}^{\mathscr{A},\varkappa}(\sigma) = \{\ \}.$$

In our final definition of truth and falsity in an infinite model $\mathscr{A}$, the clause for truth would be as in Section 5, but the clause for falsity would be:

$$\mathscr{A} \dashv \varphi \text{ iff for every representor } \varkappa \text{ for } \mathscr{A}\ \mathscr{A}, \varkappa \overset{\#}{\dashv} \varphi.$$

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

NOTES

[1] I read an ancestor of this paper at the December 1983 meeting of the Eastern Division of the American Philosophical Association. I'd like to thank the commentator Glen Helman for helpful comments; similar thanks go to John G. Bennett.

[2] The Alternative Theory presented in this paper was in my (1984a) called 'Coding-Fictionalism'. I've changed labels because the root 'fiction' has led to more misunderstanding than understanding. The narrowly philosophical content of this paper is presented from a slightly different angle, with focus on set theory rather than on arithmetic, in (Hodes 1990). There's more on set theory in my forthcoming article in *The Journal for Symbolic Logic*. This project has similarities to work of David Bostock; see his (1974) and my (1976).

[3] In a manuscript from the *Nachlass*, dated by the editors as from the last year of Frege's life, he wrote:

> I, for my part, never had any doubt that numerals must designate something in arithmetic, if such a discipline is to exist at all, and that it does is surely hard to deny. (5, p. 275)

Nonetheless, in a diary entry from the same period, Frege wrote:

> Indeed, when one has been occupied with these questions for a long time one comes to suspect that our way of using language is misleading, that number-words are not proper names of objects at all and words like 'number', 'square number' and the rest are not concept-words; and that consequently a sentence like 'Four is a square number' simply does not express that an object is subsumed under a concept and so just cannot be construed like the sentence 'Sirius is a fixed star'. But how then is it to be construed? (Frege 1979, p. 263)

The model-theory presented in Section 5 reflects an alternate construal.

[4] Dummett, and following him Wright, would delete this 'in part'. According to Wright, if we accept Frege's "arguments" for that thesis, arguments which consist in pointing out the syntactic "analogies between numerical expressions and paradigmatic singular terms elsewhere", then we must

> find the content of Frege's claim that numbers are objects in those very arguments. According to the latter course, the substance of the claim that the numbers are, if anything, Fregean objects, must then be simply that there are substantial analogies between the behavior of numerical expressions and that of paradigmatic singular terms in general; the existence of numbers as Fregean objects will be guaranteed by the presence of those analogies and the fact that certain appropriate contexts involving numerical expressions are true. (20, p. 12)

Keep in mind that the 'substantial analogies' to which Wright alludes are syntactic.

[5] I am not here endorsing the doctrine of "syntactic priority" that Dummett (1973) and Wright (1983, p. 57) attribute to Frege, according to which we may demarcate the class of singular terms of a language on purely syntactic grounds. I make no commitments as to how an initial class of paradigmatic singular terms (proper names of persons, places and events, pronouns used demonstratively or indexically, certain definite descriptions used referentially) in a logically imperfect language is to be demarcated. Of course in a

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

logically perfect language, type (indeed all of logical form) could be read off from superficial syntactic form; it's just in this that logical perfection would consist. Suffice it to say that such a class is expanded to include numerical singular terms only on the basis of broadly syntactic analogies with the paradigmatic singular terms. I say 'broadly' so as to include considerations of interanimation between sentences, as well as those concerning the construction of individual sentences.

[6] Dummett says that in *The Foundations* "The only absolute demand that Frege makes is that a sense should be provided for every identity-statement connecting any two proper names" (1981, p. 382), and in particular "it is allowed that . . . only certain predicates might be defined over directions, and the same would, by parity, apply to terms for numbers" (1981, p. 385). But by the time he wrote *The Basic Laws*, Frege clearly had adopted the principle that "every logical difference . . . must reflect a difference of logical type" (1981, p. 385), with the understanding that every difference of type is a difference within Frege's hierarchy of objects and functions.

[7] Of course their further typing in terms of what they called 'order' also had no counterpart in Frege's theory. I'll conform to contemporary misuse of 'order', using it to characterize Fregean typing into levels: so first-order quantifiers range over level zero, etc. As usual numerals are used as type-symbols according to this rule:

> '0' represents the type of objects;
> $(\sigma_1, \ldots, \sigma_n)$ represents the type of n-place relations between entities of type σ_1 in the first place and . . . and type σ_n in the n-th place;
> $n + 1$ abbreviates (n).

Note: for situations in which strict respect for the use-mention distinction would demand Quine's corner-quotes, I shall simply omit such quotes.

[8] Here Dummett has omitted mentioning the condition that sentences containing such an expression have truth-value; Wright puts it more carefully:

> When it has been established, by the sort of syntactic criteria sketched, that a given class of terms are functioning as singular terms, and when it has been verified that certain appropriate sentences containing them are, by ordinary criteria, true, then it follows that those terms do genuinely refer. (1983, p. 14)

[9] The suggestion that adjectival and singular-termlike occurrences of, e.g., 'four' are strictly homonymous is quite implausible. It asks us to ignore salient "interanimative" logico-syntactic facts, e.g., that 'The number of moons of Jupiter equals four' may be immediately inferred from 'There are exactly four moons of Jupiter', and vice-versa. The suggestion I seek to accommodate would make the use of 'four' as a singular term rather like use of adjectives as common nouns, e.g., in the transformation of 'John is black' to 'John is a black', where the latter could been seen as derived from 'John is a black man' by deletion. This issue is avoided in the model-theoretic semantics to be introduced in Sections 3 and 4, since languages of the form $L^{x,4}$(EXACTLY) represent only adjectival occurrences and those of the form $L^{x,*}$ represent only singular-term-like occurrences. The full range of constructions available in English would have to be represented by the union of a language of each of the above forms, endowed with a model-theoretic semantics that is the union of the one given in Section 3 with either that given in Section 4 or in Section 5.

[9a] Sally McConnell-Ginet pointed out the first two of these peculiarities, and that 'The

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

number four' is best viewed as an appositional construction, like 'the philosopher John Dewey', rather than as of the form [Det + NP + Adj]. Some English idioms do have the latter form (e.g. 'the attorney general'), with the adjective modifying the preceding noun-phrase. That is not the case in 'the number four', as pluralization shows: 'The numbers four' may be a cute archaisism applied, for example, to the perfect numbers known in antiquity; but it wouldn't do even if used to refer to inscriptions of '4' by someone who took them to be numbers.

[10] '$\downarrow$' means 'is defined', '$\uparrow$' means 'is undefined'. A sentence of the form $\ldots \simeq ---$ is true iff either $\ldots\downarrow$ and $---\downarrow$ and $\ldots = ---$ is true or both $\ldots\uparrow$ and $---\uparrow$; otherwise it is false. We work in standard set-theory; ω is the set of finite Von Neuman-ordinals. For a set x, card(x) is x's cardinality, which may be taken as an initial ordinal or as a Scott-cardinal; where x is finite, we'll identify it with a member of ω.

[11] Here '$\leqslant$' is an expression of type $(2, 2)$, and so it strictly speaking binds type-1 variables; thus where μ and μ' are variables of type-2, $\mu \leqslant \mu'$ abbreviates:

$$(\leqslant \gamma\gamma')((\text{EXACTLY } \mu\nu)\gamma\nu,\ (\text{EXACTLY } \mu'\nu)\gamma'\nu),$$

where γ and γ' are distinct type-1 variables and ν is a type-0 variable. So understood, our notation honors the Fregean requirement that type-2 expressions, here μ and μ', be "unsaturated".

[12] Insofar as Frege thought that arithmetic statements made in the context of the mathematical practice of his time had truth-values, he was committed to the existence of a standard representor. But in places he suggests that the mathematical practice of his time was defective in that it was not underpinned by a standard representor, and that part of his purpose was to fix one by stipulation.

[13] The use of 'Frege-model', and later of 'Frege-structure', in this paper should not be confused with the use of these phrases by Peter Aczel and others who work on extensions of the λ-calculus as a foundation for mathematics.

[14] Even after the mathematical community had assimilated Cantor's work, mathematicians used 'infinity' as if it were a univocal singular term. What can we make of this practice? This question should, I think, be an embarassment to the Mathematical-Object theory, which must, I think, accord it a referent. As we'll see, it's easily assimilated by the Alternative approach.

[15] Ostensibly, his desire to secure such non-identities motivated Frege in *The Foundations of Arithmetic* to identify numbers with certain extensions; but that move merely transfers the problem to extensions. When the question appeared in that form, in Section 10 of the *The Basic Laws of Arithmetic*, Frege says that he has introduced "only truth-values and courses-of-value as objects" (p. 47 of [6]), and stipulating that the former are to be, in effect, their own singletons. See p. 136–37 of (Hodes 1984a).

[16] Notice: I say 'encodes', not 'stands for'; it would be syntactically incoherent for a singular term to stand for a quantifier. Our classification of #-terms as singular is not compromised, since that is based on syntactic considerations. The inclination to think otherwise reflects an inability to imagine a singular term playing any semantic role other than that of designating an object. This is apparent in (Wright 1983); for example see Wright's initial characterization of what he calls 'reductionism' on p. 29 and p. 67.

[17] Here I speak of artifacts of our definition of truth and falsity in a model. Kaplan spoke of artifacts of models themselves: "When we construct a model of something, we must distinguish those features of the model which represent features of that which we model,

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

from those features which are intrinsic to the model and play no representational role" (Kaplan 1979), p. 216. My usage is connected with Kaplan's, since a definition of truth and falsity in a model is itself a model (in the engineering sense) for a definition of "real live" truth and falsity.

[18] Had we not directly defined $\models$ and $\dashv$ for $L^{x,\#}$, but instead offered t_0 $[t_0^{\infty}]$ as a rewrite-rule, then #-terms would have been introduced by contextual definitions in what Wright (his 1983, p. 68) calls the "the austere manner". Since this is not what we're doing, Wright's complaint, that such stipulations would fail to confer as semantic role on #-terms, do not apply. (In any case, I am dubious of Wright's distinction between conferring a mere use and conferring a semantic role on linguistic expressions.) I'm not sure that the model-theory offered here represents what Wright calls 'ontological reduction by analysis'; but if it is, it should answer his argument against it on p. 68–69: that any non-austere reading of equivalences like those between φ and $t_0(\varphi)$ for $\varphi \in \mathrm{Sent}(L^{x,4}(\text{EXACTLY}))$ would require us to construe the role of #-terms "as referential". (I'm inclined to think that remarks in my (1984a) would answer his argument from p. 31–36; but this issue evades model-theoretic representation.)

[19] Our approach to quantified modal model-theory has been, in essence, the one pioneered by S. Kripke, rather than that taken by A. Bressan. It's far from clear how a Bressanian model-theory bears on actual modal discourse. I think that a model-theory along Bressanian lines does have such application; but this would require a long story, some of whose details I have not worked out. Perhaps from this perspective, either Individual-Possibilism or the Actualist's modal gambit will appear more attractive than it has here.

[20] Of course a model-theoretic therapy can carry the risk of further philosophical illness. The standard model-theory for modal languages is particularly dangerous in this regard.

[21] Once again, the model-theoretic semantics of Section 9, unlike that of Section 5, violate this *prima facie* plausible thesis.

[22] In particular, the slogan 'Numbers are fictions' is misleading, for it asks us to simultaneously think within and without the Mathematical-object Picture. A similar difficulty might afflict uncontroversial attributions of fictionality to characters in literature, e.g., 'Hamlet is fictitious'. But it should be noted that the analogy between fictional characters and numbers, between a piece of fiction and the Mathematical-Object Picture, is far from perfect. Construed within the Mathematical-Object Picture (the only way it is intelligible), 'The number one exists' is true. Under one construal, 'Hamlet existed' is literally false, though it is fictionally true (i.e., true in Shakespeare's play). (That construal takes 'Hamlet' to designate a person, a Prince of Denmark; if it is construed as designating a fictional character, created by Shakespeare about four hundred years ago, 'Hamlet existed' is true, as is 'Hamlet exists'.)

[23] We can give a model-theoretic "picture" of Dedekind's view as follows. We'll use a three-valued semantics in place of that from Section 4; see the Appendix. Where $\mathscr{A}$ is a partial model, let a Frege-model $(\mathscr{B}, \varkappa)$ be a Dedekind-extention of $\mathscr{A}$ iff $|\mathscr{A}|$ and $\mathrm{Rng}(\varkappa)$ are disjoint, $|\mathscr{B}| = |\mathscr{A}| \cup \mathrm{Rng}(\varkappa)$, and for every $\varphi \in \mathrm{Sent}(L^{(0,0),\#})$:

$$\mathscr{B},\varkappa \overset{\#}{\models} \varphi \text{ iff } \mathscr{A} \models \varphi;\ \mathscr{B},\varkappa \overset{\#}{\dashv} \varphi \text{ iff } \mathscr{A} \dashv \varphi.$$

[24] Following the lead from Section 5, we could adopt these definitions, for $\varphi \in \mathrm{Sent}(L^{x,\mathbf{n}})$:

$$\mathscr{A} \models \varphi \text{ iff for every } R \text{ of order-type } \omega,\ \mathscr{A}, R \models \varphi;$$
$$\mathscr{A} \dashv \varphi \text{ iff for every } R \text{ of order-type } \omega,\ \mathscr{A}, R \not\models \varphi.$$

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

Problem: give an informative characterization of those models with Dedekind-extensions. This is not a three-valued semantics, because for any R and R', both of order-type ω, any model $\mathscr{A}$ and any $\varphi \in \mathrm{Sent}(L^{x,\mathbf{n}})$: $\mathscr{A}, R \models \varphi$ iff $\mathscr{A}.R' \models \varphi$. It's not hard to see that $L^{x,\mathbf{n}}$, under this semantics, is a notational variant of the language $L^{x,2^{\omega}}$(EXACTLY) under $\models_{\aleph_0}$, presented in my (1988a) and (1988b). In his (1974) Bostock appears to be considering a language of the latter form; but since he offers no model-theoretic semantics, it's hard to be sure.

[25] In effect, this observation is the collapsing theorem of my (1988a) applied to $\aleph_0$.

REFERENCES

Bostock, David: 1974, *Logic and Arithmetic, Volume I*, Oxford Press, New York and Oxford.

Burge, Tyler: 1980, 'Truth and Singular Terms', reprinted in Mark Platts (ed.), *Reference, Truth and Reality*, Routledge & Kegan Paul, London, Boston, and Henley.

Dummett, Michael: 1973, *Frege: The Philosophy of Language*, Duckworth, London.

Dummett, Michael: 1981, *The Interpretation of Frege's Philosophy*, Harvard University Press, Cambridge, Massachussetts.

Frege, Gottlob: 1964, *The Basic Laws of Arithmetic*, University of California Press, Berkeley.

Frege, Gottlob: 1979, *Posthumous Writings*, The University of Chicago Press, Chicago.

Furth, Montgomery: 1968, 'Two Types of Denotation', in N. Rescher (ed.), *Studies in Logical Theory*, The American Philosophical Quarterly Monograph Series, Oxford, pp. 9–39.

Grossman, Reinhardt: 1973, *Ontological Reduction*, Indiana University Press, Bloomington, Indiana.

Hodes, Harold: 1976, 'Review of D. Bostock's *Logic and Arithmetic, Volume I*', *The Journal of Philosophy* **73**(6), pp. 149–57.

Hodes, Harold: 1984a, 'Logicism and the Ontological Commitments of Arithmetic', *The Journal of Philosophy* **81**(3), pp. 123–49.

Hodes, Harold: 1984b, 'Axioms for Actuality', *The Journal of Philosophical Logic* **13**(1), pp. 27–34.

Hodes, Harold: 1984c, 'On Modal Logics Which Enrich First-Order S5', *The Journal of Philosophical Logic* **13**, pp. 423–54.

Hodes, Harold: 1986–1987, 'Individual-Actualism and Three-Valued Modal Logics', *The Journal of Philosophical Logic* **15**, pp. 369–401, and **16**, pp. 17–63.

Hodes, Harold: 1989, 'Three-Valued Logics: An Introduction, A Comparison of Various Logical Lexica, and Some Philosophical Remarks', *Annals of Pure and Applied Logic* **43**, pp. 99–145.

Hodes, Harold: 1988a, 'Cardinality Logics I: Inclusions Between Languages Based on "*exactly*"', *Annals of Pure and Applied Logic* **39**, pp. 199–238.

Hodes, Harold: 1988b, 'Cardinality Logics II: Definability', *The Journal for Symbolic Logic* **53**(3), pp. 765–84.

Hodes, Harold: 1990, 'Ontological Commitment, Thick and Thin', in George Boolos (ed.), *Meaning and Method: Essays in Honor of Hilary Putnam*, Cambridge University Press, Cambridge.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms

Hodes, Harold: (forthcoming), 'Where Do Sets Come From?', *The Journal for Symbolic Logic*.
Kaplan, David: 1979, 'How to Russell a Frege-Church', reprinted in Michael Loux (ed.), *The Possible and The Actual*, Cornell University Press, Ithaca, New York.
Wright, Crispin: 1983, *Frege's Conception of Numbers as Objects*, Aberdeen University Press, Aberdeen.

Sage School of Philosophy
Cornell University
Ithaca, NY 14853
U.S.A.

This content downloaded from
132.174.252.179 on Thu, 23 Mar 2023 00:32:00 UTC
All use subject to https://about.jstor.org/terms