An alternative to the Schwarzschild solution of GTR.

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Abstract.

The *Schwarzschild solution* (Schwarzschild, 1915/16) to Einstein's General Theory of Relativity (GTR) is accepted in theoretical physics as the *unique* solution to GTR for a central-mass system. In this paper I propose an alternative solution to GTR, and argue it is both logically consistent and empirically realistic as a theory of gravity. This solution is here called *K-gravity*. The introduction explains the basic concept. The central sections go through the technical detail, defining the basic solution for the geometric tensor, the Christoffel symbols, Ricci tensor, Ricci scalar, Einstein tensor, stress-energy tensor and density-pressure for the system. The density is integrated, and some consistency properties are demonstrated. A notable feature is the disappearance of the event horizon singularity, i.e. there are no black holes. So far this is for a single central mass. A generalization of the solution for multiple masses is then proposed. This is required to support K-gravity as a viable general interpretation of gravity. Then the question of empirical tests is discussed. It is argued that current observational data is almost but not quite sufficient to verify or falsify K-gravity. The Pioneer spacecraft trajectory data is of particular interest, as this is capable of providing a test; but the data (which originally showed anomalies that match K-gravity) is now uncertain. A new and very practical experiment is proposed to settle the matter. This would provide a novel test of GTR, and a novel test of the cause of the Pioneer anomalies. In conclusion, K-gravity has extensive ramifications for gravitational physics and for the philosophy of GTR and space-time.

An alternative to the Schwarzschild solution of GTR.

"The chief attraction of the theory lies in its logical completeness. If a single one of the conclusions drawn from it proves wrong, it must be given up; to modify it without destroying the whole structure seems to be impossible." Albert Einstein *(1919).*

1. Introduction.

The *Schwarzschild solution* (Schwarzschild, 1915/16) to Einstein's General Theory of Relativity (GTR) is accepted in theoretical physics as the unique solution to GTR for a central mass system (with no net charge or rotation). It is hard to imagine how its *uniqueness* could even be questioned. Being mathematically determined, even the slightest modification would seem to contradict GTR. Imposing a slightly different solution for the metric (e.g. to remove singularities) would seem to 'destroy the whole structure' as Einstein says. As a result, the implication of the existence of black holes in particular has been widely embraced as an inevitable consequence of GTR.[1] But in this paper I propose that there is an alternative solution, and argue that it is logically consistent and empirically realistic. The claim is that:

- *There is a central mass solution, distinct from the Schwarzschild solution, which is consistent with GTR, and leads to a realistic theory of gravity.*

This proposed alternative solution to GTR is here called *K-gravity*. This paper presents the basic mathematical solution for *K-gravity*, and argues that it represents a conceptually viable theory of gravity, within the general framework of GTR. This claim may at first glance seem impossible to theorists: the rest of this Introduction explains how it is possible. The mathematical analysis for the basic solution is given in the main sections. It is hoped this represents at least an interesting exercise for students, extending the usual exercises on the Schwarzschild solution, and some additional algebraic workings are given in an Appendix for this purpose.[2] However it raises the more controversial question of whether the basic solution can be extended to a full theory of gravity, which is discussed in Section 10; and whether it is empirically viable on current evidence, discussed in Section 11. Positive answers are argued for both of these, but more extended analysis is required to be conclusive. The

[1] Physicists have questioned the *universality* of the Schwarzschild solution, including its extrapolation to the black hole realm, and its combination with quantum field theory. But these are questions about the universality or completeness of GTR, not the uniqueness of the Schwarzschild solution, which to my knowledge has not been seriously questioned.

[2] Primary background to follow the main analysis is found in WEBREF Oas (2014) and WEBREF Vojinovic (2010). Will (2014) is a good review of experimental tests of GTR. Lehmkuhl (2008) is included as a pertinent discussion of the conceptual problems with the status of the stress-energy tensor. No attempt is made to cover the vast background of the subject, but it is important for students to realize that there are questions variously involving mathematics, theoretical physics, experimental physics, and philosophy. References include a small selection of examples of classic texts on: tensor calculus (Kay, 1976), differential geometry (Spivak 1979; Kobayashi, S. and K. Nomizu, 1963), theoretical physics (Landau and Lifshitz 1975; Misner, Thorne and Wheeler, 1973; Wald 1984), and philosophy of space-time (Torretti, 1983).

central aim here is to present the basic tensor calculus solution as the starting point, and show these are at least open questions.

K-gravity makes very small differences for ordinary gravity fields, in which empirical tests of GTR are done, and in which the value of the gravitational constant *G* or *GM* are measured. The Schwarzschild solution is the basis for all empirical tests of GTR that compare its predictions to the Newtonian theory. The sequel to this paper is the analysis of empirical data, to show in detail how K-gravity can be empirically tested against the Schwarzschild solution. Some key points about this are made in Section 11. The main goal here however is to introduce the concept, and show that it is a solution to GTR. It introduces a significant conceptual shift. One aspect is that it makes simplifies strong fields, removing the Schwarzschild event horizon (i.e. black holes). Since black holes are a purely theoretical extrapolation from the Schwarzschild solution, with no empirical evidence, and bizarre implications to physical intuition, this simplification represents a virtue. But this is really a secondary implication. Most essentially, K-gravity changes the interpretation of the stress-energy tensor, through which the metric of space-time is connected to the distribution of matter and energy by the Einstein equation.

The first step is to remove the mystery of how such an alternative solution is even possible. We start with the defining equation of GTR, the Einstein equation:

$$G_{\mu\nu} = R_{\mu\nu} + ½g_{\mu\nu}R = (8\pi G/c^4)T_{\mu\nu}$$

On the left hand side we have a *geometric construction,* defined from the space-time metric tensor, $g_{\mu\nu}$. It is $g_{\mu\nu}$ that represents the fundamental model of space-time as a differential manifold, and determines the motion of particles through Riemannian geometry and the geodesic principle. On the right hand side is the *stress-energy tensor.* This is normally said to be determined by the distribution of mass and energy in a physical system. The Einstein equation connects the two, meaning (geometric) properties of space-time are determined by (mechanical) properties of matter and energy.[3]

The Schwarzschild solution is for the simplest physical case, of a spherically symmetric central mass distribution. To derive it two types of assumption are required:[4]

(A) Symmetry assumptions: the mass distribution is radially symmetric and static (and we should add that it is of finite extent).

(B) Stress-energy tensor assumption: $T_{\mu\nu} = 0$ for empty space, i.e. all space outside the central mass.

These two assumptions do indeed determine the Schwarzschild solution uniquely from GTR. So if any alternative solution is possible within GTR, at least one must be dropped. The symmetry assumptions (A) are retained: they *define* the type of system being analyzed. It is the second assumption, (B), that we will reject. Explicitly:

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[3] But actually the exact nature of the relationship between the *metric tensor* and the *stress-energy tensor* and the *mass-energy distribution* is far from clear or obvious or deterministic; see Lehmkuhl (2008) for an interesting recent discussion.

[4] A third assumption is usually taken to be conformity in the limit with Newtonian gravity, but this is not really required to obtain the formal solution, but rather to solve the variables in the GTR solution to match the Newtonian solution. Also the static assumption is not strictly necessary.

(B*) In *K-gravity:* $T_{\mu\nu} \neq 0$ for empty space, and the stress-energy tensor for 'empty space' outside a central mass is not assumed to be zero.

Hence the novelty in *K-gravity* lies entirely in an alternative to the normal interpretation of the *stress-energy tensor* within GTR. The idea is that an *inertial mass M* produces an *extended gravitational mass density field* throughout the space around it. The form of this field follows a simple law. It corresponds to a non-zero pressure-density function, and corresponding non-zero $T_{\mu\nu}$ everywhere in space.

To realize why this does not contradict GTR, it must be recognized that the assignment of the *stress-energy tensor* is not part of the definition of GTR proper. It is something that is imported or interpreted from other physics. GTR postulates that there is a $g_{\mu\nu}$ characterizing *space-time*, and this has a precise interpretation within GTR – the primary concepts being from Riemannian geometry. GTR also assumes there is a $T_{\mu\nu}$ characterizing a *mass-energy distribution.* But the specification of $T_{\mu\nu}$ is not determined by GTR itself: it comes from other specialized branches of physics, i.e. theories of particles and fields, mass and energy. So what makes us assume that: $T_{\mu\nu} = 0$ for 'empty space'? It is our essentially classical intuition that mass is strictly *localized* within certain boundaries. It is this assumption that is questioned here, not the framework of GTR proper.

We will begin by viewing the *K-gravity* solution in two different ways. First it works as a purely conventional solution for a particular type of *extended* mass density distribution. Subsequently we propose a novel interpretation, modifying the usual stress-energy tensor law itself *for a central mass, M*. This is illustrated below.

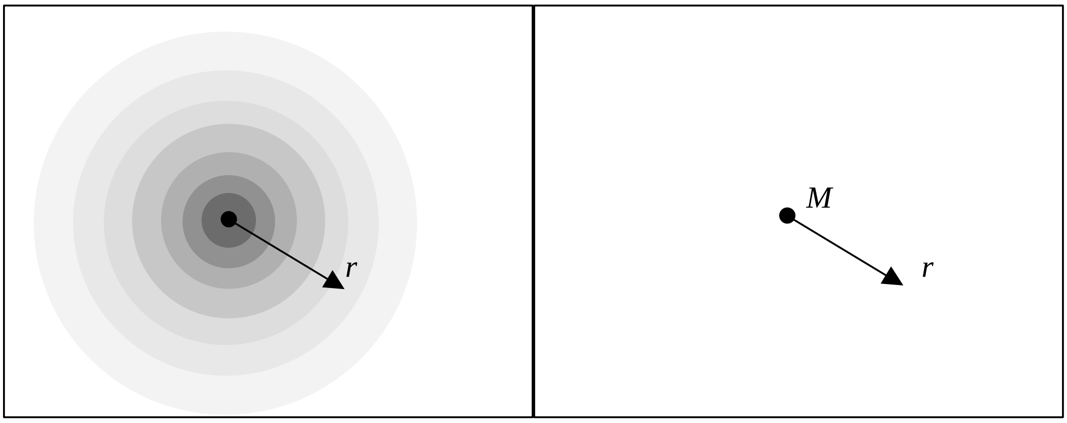

Figure 1(a). A conventional fluid with total mass M, smeared through space, thinning as r increases. Conventional GTR relates $g_{\mu\nu}(r)$ to the mass-energy density-pressure function, $\rho(r)$, via the tensor $T_{\mu\nu}(r)$. The K-gravity solution corresponds to a special case for $\rho(r)$.

Figure 1(b). A single inertial mass, M, produces a metric field $g_{\mu\nu}(r)$. The K-gravity solution for $g_{\mu\nu}(r)$ is the same as the conventional solution for the special case of the fluid $\rho(r)$. But in K-gravity, the central inertial mass M alone produces the smeared gravitational mass field $\rho(r)$.

In the first analysis, the metric tensor, $g_{\mu\nu}(r)$, for K-gravity corresponds to a *conventional* solution to GTR, for an appropriate mass-energy distribution, $\rho(r)$, and consequently $T_{\mu\nu}(r)$. This is a solution for an *infinitely extended* mass distribution, as in Figure 1(a). But since this mass distribution extends indefinitely outwards, this is *not a central mass solution,* and does not contradict the uniqueness of the Schwarzschild solution.

The second step, which gives K-gravity proper, is to take this extended mass distribution as *an alternative law for the stress-energy tensor, applying to any central inertial mass, M.* Taken this way, we are proposing that the *gravitational mass* of a particle is not concentrated simply at a central point or region (where the inertial mass appears to be). Instead it is extended throughout space, representing a natural law.

Gravitational mass thus becomes a *density field in space,* not a value at a point of inertial mass. This may be thought of as somewhat analogous to the quantisation of classical point particles. A classical particle with inertial mass *m* has a point-like location and definite trajectory. But this is treated in quantum mechanics as being 'smeared out through space', described by a wave function. The wave function obeys a universal law (e.g. Schrodinger or Dirac equation). This required a radical change in the conception of what physical particles are. Analogously, the proposal here is that the 'gravitational mass' of particles is also spread out through space, according to a universal law. This is a change in the conception of gravitational mass.[5]

This requires us to distinguish between *inertial mass,* which we conceive as the centralized mass of a localized body, and *gravitational mass,* which now becomes a *mass density function across space.* The conceptual distinction between inertial and gravitational mass was emphasized by Einstein, and played an important role in his thought. Inertial mass has a trajectory, and carries energy and momentum. It is what is accelerated by forces. 'Gravitational mass' is the *charge* for the gravitational field. Einstein recognized the conceptual importance of the fact that they are identified (in the Newtonian and GTR theories).[6] But the conceptual importance of identifying them also introduces the conceptual possibility of separating them – which is what K-gravity does. A total inertial mass *M* still corresponds to a total gravitational mass *M.* But while an inertial mass may be localized in a tiny region of space, in K-gravity its gravitational mass is proposed to be smeared out across all space.

From this point of view *K-gravity* represents an alternative to conventional GTR. The normal Schwarzschild solution is a mathematically consistent solution, but from the point of view of *K-gravity* it represents the wrong physical interpretation of mass, or $T_{\mu\nu}$. K-gravity holds that *gravitational mass is never concentrated in a finite central region, with* $T_{\mu\nu} = 0$ *elsewhere.*

The *K-gravity* solution is intimately related to the Schwarzschild solution:

[5] However the proposed 'smearing' of the gravitational mass does not correspond to conventional quantisation. I have proposed it corresponds to alternative 'geometric' model of quantum mechanics itself. But this is beyond the present paper.

[6] This is the big difference between the classical gravitational and electromagnetic forces: because the *inertial mass* also acts as the *gravitational charge* in the former, it has the special properties connected to the so-called *equivalence principle,* viz. all bodies experience identical acceleration in a gravitational field. The electric charge is distributed independently, and so affects bodies differently. K-gravity is also somewhat analogous to separating 'inertial mass' as the gravitational *charge* from 'gravitational mass' as a gravitational *field*, as we separate electric charge and electric field.

- *K-gravity for a central inertial mass is identical to Schwarzschild gravity for the same quantity of mass extended across space with an appropriate mass density function, $\rho(r)$.*

For K-gravity to make sense physically, the solution has to have some special properties, so that the law may be generalized. Two key properties are emphasized next as key to its initial plausibility.

The first is that it is very close to the Schwarzschild solution (necessary since the latter is very accurate for ordinary gravitational fields, up to solar-system scale). K-gravity has a similar looking metric form to the Schwarzschild solution – but the coordinate functions for $g_{\mu\nu}$ include higher-order terms in the dimensionless quantity: MG/c^2r. These higher order terms are very tiny in normal gravitational fields.

A second key property is that *the addition of two separate masses to form a larger single mass gives the same effect as the superposition of the two smaller masses taken sequentially*. This means we can apply K-gravity to fundamental mass particles, and obtain solutions for aggregate masses by superposition. The same form of the K-gravity law appears for aggregate masses of any size.

These two properties allow K-gravity to be a *prima facie* realistic theory. We now state the K-gravity solution for $g_{\mu\nu}$, and compare it to the Schwarzschild solution. The Schwarzschild solution is normally expressed in line metric form:

$$ds^2 = c^2 d\tau^2 = c^2dt^2/k^2 - k^2dr^2 - r^2d\theta^2 - r^2 sin^2\theta\, d\phi^2$$

with the characteristic term k (‘little k’) defined by:

$$k =_{df} 1/\surd(1\text{-}2MG/c^2r)$$

The K-gravity solution substitutes this term k with K (‘big K’), defined as:

$$K =_{df} exp(MG/c^2r)$$

The magnitude of K is extremely close to k in the normal gravity that we live in and experiment in. This is seen by expanding k and K as polynomial series. They differ only in 2nd-order and higher terms of the key dimensionless quantity: MG/c^2r. This is seen most simply by comparing $1/k^2$ with $1/K^2$. The former is just:

$$1/k^2 = 1\text{-}2MG/c^2r$$

The latter is the exponential series:

$$1/K^2 = 1 - 2MG/c^2r + (2MG/c^2r)^2(1/2!) - (2MG/c^2r)^3(1/3!)+ \ldots.$$

They differ in the second-order terms: $(2MG/c^2r)^2$ and higher. $1/K^2$ appears as an *analytic continuation of* $1/k^2$. For the direct comparison of k and K:

$$k = 1 + MG/c^2r + (3/2)(MG/c^2r)^2 + \ldots$$

$$K = 1 + MG/c^2r + (1/2)(MG/c^2r)^2 +\ldots$$

MG/c^2r is small in ordinary gravitational fields. Hence: $k \approx K + (MG/c^2r)^2$ for large r, and k is always slightly larger than K. Hence Schwarzschild gravity will predict slightly stronger effects (accelerations) than K-gravity, for the same source mass.

This substitution of K for k defines the K-gravity solution for $g_{\mu\nu}$. To verify its consistency, we will derive the stress-energy tensor $T_{\mu\nu}$, and consequently the pressure-density distribution required to produce it in conventional GTR. This lets us verify the solution is sensible in conventional GTR.

Intuitively, K-gravity corresponds to a smeared-out conventional mass M. Smearing a mass out beyond a given radial shell makes gravitational effects within that shell weaker. The calculation of mass-density required to produce $g_{\mu\nu}$ for K-gravity will show that $\rho(r)$ varies primarily with $1/r^4$ for large r. The series expansion in terms of $1/r$ is actually found to be:

$$\rho = (M^2G/4\pi r^4K^2) + (c^4/8\pi Gr^2K^2)((2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + \ldots)$$

The analysis of K-gravity below goes in the opposite direction to the derivation of the Schwarzschild solution: we start with the K-gravity solution for the metric, and work out the stress-energy tensor. All the tensor components in the Einstein field equation for the K-gravity solution will be derived. With the Schwarzschild solution, we start with stress-energy tensor, and derive the metric tensor. But here we do not derive the K-gravity metric: we just postulate it, and work backwards to verify its properties.

The key properties we initially want to verify are the two mentioned above, and we briefly review these next, to show the solution is *prima facie* sensible as an alternative theory of gravity. The first is that the solution is empirically realistic because it conforms closely to Schwarzschild gravity in normal gravity. The second is that the solution has appropriate symmetries to represent a *possible general law of gravity* (rather than just a particular solution for a special distribution of matter).

For the latter point, the most important symmetry is seen by considering the metric for an aggregate mass: $M = M_1+M_2$. This is generated by the term:

$$K(M_1+M_2) = exp((M_1+M_2)G/c^2r) = exp(M_1G/c^2r)exp(M_2G/c^2r)$$

This linear property corresponds to a basic superposition property in *K-gravity*:

- The effect of imposing an aggregate mass of: (M_1+M_2) on empty space (through: $K = exp((M_1+M_2)G/c^2r)$) is identical to imposing M_1 on empty space first (through: $K_1 = exp(M_1G/c^2r)$), and then imposing M_2 linearly on the resulting space (through: $K_{12} = K_1K_2 = exp(M_1G/c^2r)exp(M_2G/c^2r)$).

This and other nice properties of K underpin the fact that K-gravity is a meaningful physical theory. To extend the basic theory, we will subsequently introduce a more general superposition principle for the K-gravity metric, and show that it is at least *prima facie* coherent as a possible general theory.

For the first point, viz. the closeness of K-gravity to Schwarzschild gravity, the critical term is: MG/c^2r, which is about 10^{-8} for the strongest gravity we typically have available to experiment in - the gravity of the sun at roughly 1 AU.[7]

[7] There is of course a fundamental problem with gravitational experiments: we cannot produce large *g-fields* in the lab, or shield systems from gravity, as we can with EM fields, so we are constrained to using natural *g-fields* in the solar system as the only source of large fields. Experiments close to the sun (e.g. closely orbiting probes) to test stronger fields would be conflated by solar radiation, solar wind, oblateness, making it difficult to distinguish gravitational effects as we get closer. Laboratory experiments with weak gravity extended over long periods are used to determine G, but still to relative accuracy less than 10^{-6}, and cannot test effects of strong fields. There are now multiple experiments to try to test strong field limits, through observations of pulsars or 'blacks holes', but these are still unable to confirm the specific feature of a black hole event horizon.

Exercise: approximately what is (MG/c^2r) on Earth surface from Earth gravity?

- g-acceleration on Earth surface is approximately 10 m/s/s. Mean radius r is around 6,370 km = 6.37×10^6m. $c = 3\times10^8$. Thus: $10 = (M_{Earth}G/r_{Earth}^2)$, so: $(M_{Earth}G/c^2r_{Earth}) = 10r_{Earth}/c^2 = 6.37\times10^7/(3\times10^8)^2 = 7\times10^{-10} \approx 10^{-9}$.

Exercise: approximately what is (MG/c^2r) at Earth radius from Sun gravity?

- Estimate M_{Sun} is about 3.3×10^5 Earth masses and Earth-Sun orbit radius r_Θ is about $1.5*10^{11}$m. So: $(r_{Earth}/r_\Theta)(M_{Sun}/M_{Earth}) \approx 3.3\times10^5/23{,}500 \approx 14$. So: $M_{Sun}G/c^2r_\Theta \approx 14\times7\times10^{-10} \approx 9.9\times10^{-9}\approx10^{-8}$.

Although we typically think of the 'strength of gravity' as the acceleration field: MG/r^2, it is really the dimensionless field: MG/c^2r that is most important, and in this sense the sun's field is much stronger than any other in the solar system – even at the Earth's surface, where the sun's acceleration field is much weaker than Earth's.

Hence the terms k and K from the sun for inner planetary orbits differ typically by about: $(MG/c^2r)^2 \approx 10^{-16}$. But the key factor for differences in *accelerations* (including orbital precession) turns out to be: $K^2 \approx 1+2MG/c^2r$. Hence effects in the strongest currently testable domains (viz. accelerations by the sun at inner-planetary orbits) calculated with the Schwarzschild solution will be about $1+10^{-8}$ times larger than the effects calculated using K-gravity, for the same M/r. For the outer solar system the acceleration differences become weaker.

The 2014 CODATA-recommended value of the gravitational constant G alone has a relative uncertainty of 4.7×10^{-5}. (CODATA, 2014). This is about the uncertainty provided by laboratory-scale experiments (which cannot provide a test of K-gravity). This means there is at least the same error in measurements of solar or planetary masses.

But the relative uncertainty for $M_{sun}G$ (solar gravity is the key testing environment) is much better, claimed to be around 10^{-11}. This makes the value of $K \approx 1+10^{-8}$ to $1+10^{-9}$ for solar gravity in the solar system measurable with good precision. Thus we may expect that predicted differences between the two theories will be detectable. However it is not a matter of simply measuring K: what is needed is to compare measurements of acceleration *at different orbits.* A close analysis shows that the difference between the theories is around the limit of precision of data from present gravitational experiments to detect. But as far I know, available data is not yet accurate enough. I will argue that it requires a new experiment to test conclusively.

Thus it will be argued that the theory is theoretically consistent and empirically realistic enough to consider *K-gravity* theory *prima facie* as a realistic candidate for a viable physical theory. The interesting question for the experimentalist is then whether K-gravity is empirically viable. The observational evidence needs to be examined in detail. I analyzed this some time ago (2003; unpublished paper) for a number of gravitational phenomenon, including planetary orbits, the precession of the perihelion of Mercury, gravitational light deflection, gravitational red shift or time dilation, and the Pioneer 10 and 11 spacecraft trajectories. I found then that only the last is (potentially) sensitive enough to register the difference made by K-gravity. I think this is still the case, although in the meantime some gravitational experiments, in particular solar space probes, have improved the available data. I briefly mention this topic of empirical tests here, and return to it in the conclusion.

The first thing to realize is that the classic tests that confirm GTR over Newtonian gravity are not able to distinguish the Schwarzschild solution from K-gravity, because the *relativistic phenomena* of K-gravity are *qualitatively* identical to those of Schwarzschild gravity (in our weak gravity systems).[8] Bending of light, gravitational red shift, time dilation and orbital precession all work as in standard GTR. These phenomenon represent distinctive *mechanisms* in GTR that are absent from Newtonian theory, and hence they formed the primary points of comparison between those two theories. But there are no such *qualitative* differences (i.e. distinct causal mechanisms) separating Schwarzschild gravity and K-gravity. There are only fine quantitative differences (in accelerations, time dilation, etc), and these are at the current limits of experimental detection.

As a result, the data from the Pioneer spacecraft trajectories, which promised to give a sensitive quantitative measurement of gravitational acceleration, still represents the only measurement I am aware of that is potentially sensitive enough to test between the two theories. If the Pioneer data had unambiguously conformed to Schwarzschild GTR, this would contradict K-gravity. But instead, anomalies famously appeared in the data. I found (2003) that these anomalies are close to those predicted by K-gravity. But the waters are muddied, because (after many years searching) a conventional explanation to remove the anomalies was claimed around 2012; and the evidence is now unclear. This is a very interesting episode in gravitational physics, and is discussed briefly in the conclusion. A new experiment is proposed in the concluding section of this paper as the only way to decisively test the matter.

We now turn to the details of K-gravity. Sections 2-9 work through the tensor calculus and integration, and some algebraic workings are given in the Appendix, as I think this makes an interesting exercise for students. I have chosen to follow (Oas, 2014), as a convenient on-line derivation for students first tackling the Schwarzschild solution. Section 10 is more novel. It argues that the concept may be generalized to a real alternative theory of gravity, and derives the acceleration. Section 11 briefly summarizes the question of empirical tests, argues the empirical evidence is not yet sufficient to confirm or deny K-gravity, and proposes a new independent experimental test is in order.

[8] In strong gravitational fields like ‘black holes’ the theories diverge, but we cannot yet observe such fields in any detail – e.g. there is no direct evidence for the existence of the Schwarzschild event horizon yet.

2. Line-metric solution.

Solutions considered here are for the Einstein field equation of GTR:

$$G_{\mu\nu} = R_{\mu\nu} + ½g_{\mu\nu}R = (8\pi G/c^4)T_{\mu\nu}$$

We examine two solutions. First is the Schwarzschild solution, normally written in its line-element form with polar coordinates: [9]

The Schwarzschild Solution

Schwarzschild: $ds^2 = c^2d\tau^2 = c^2dt^2/k^2 - k^2dr^2 - r^2d\theta^2 - r^2sin^2\theta\, d\phi^2$ (1)

With k defined: $k =_{df} 1/\sqrt{(1\text{-}2MG/c^2r)} = (1\text{-}2MG/c^2r)^{-½}$

Schwarzschild solution metric tensor:

$$[g_{\mu\nu}] = \begin{pmatrix} c^2/k^2, & 0, & 0, & 0 \\ 0, & -k^2, & 0, & 0 \\ 0, & 0, & -r^2, & 0 \\ 0, & 0, & 0, & -r^2sin^2\theta \end{pmatrix}$$

The function *k* characterizes the solution. The *K-gravity solution* is defined by replacing *k* with *K*, defined as follows:

The K-gravity Solution

K-gravity: $c^2d\tau^2 = c^2dt^2/K^2 - K^2dr^2 - r^2d\theta^2 - r^2sin^2\theta\, d\phi^2$ (2)

With K defined: $K =_{df} exp(MG/c^2r)$

K-Gravity metric tensor:

$$[g_{\mu\nu}] = \begin{pmatrix} c^2/K^2, & 0, & 0, & 0 \\ 0, & -K^2, & 0, & 0 \\ 0, & 0, & -r^2, & 0 \\ 0, & 0, & 0, & -r^2sin^2\theta \end{pmatrix}$$

[9] I note that throughout I have used the full functional notation for quantities like MG/c^2r. In most textbooks this is compressed to *m/r*, usually justified by saying that we can set *G* and *c* conventionally to the value 1. However this suppresses the dimensional analysis, and the *logical construction,* and it is very confusing when we are trying to think our way through novel conceptual questions. Dimensional consistency of equations is the primary fundamental requirement of any theory: it is a requirement for a proposition to make sense *as a logical or semantic construction*. Setting: $c = 1$ does not make *c* a number, because it still has the physical dimensions of a *velocity.* We cannot write: $c = 1$ and $G =1$ and infer: $c = G$, because this cannot be a logical identity. E.g. it would let us formally infer (by substitution of identities) that: $MG/c^2r = M/cr$. But the first is a *dimensionless quantity,* the second is not, and such an equation is meaningless. Writing '*m/r'* instead 'MG/c^2r' is a pragmatic device for simplifying notation when working through a purely formal derivation. But when trying to comprehend meaning, especially in a novel context, which is the context for all students trying to conceptualize a new theory, it is essential to transparently see the logical construction of quantities. Students should be encouraged to think of equations of physics not as formal symbols but as *semantic constructions,* stating propositions about the world.

Both of these are 'Schwarzschild-type metrics', meaning essentially they are spherically symmetric and static. The essential functional dependence is on *r* alone, giving the simple Christoffel symbols in Section 4. Now of course, the Schwarzschild solution given in one coordinate representation can be transformed into another representation, where the coordinate functions look different. And this can cause some confusion about whether two different functional representations represent different *metrics,* or just coordinate transformations. To avoid any confusion, we can state that:

- *The K-gravity solution in (2) represents a metric in the same coordinate representation as the Schwarzschild solution in (1).*

In fact there is no coordinate transformation that would transform (2) to the form of (1). This would require transforming *r* to: $r' = f(r)$ in a way to make: $K(M,r) = k(M,f(r))$. This requires: $exp(A/r) = (1-2A/f(r))^{-½}$, for any positive *A* (which is a variable in *M*.) If we multiply *A* by *n*, this requires: $exp(A/nr) = (1-2A/nf(r))^{-½}$. And equally: $exp(A/(nr)) = (1-2A/f(nr))^{-½}$. Hence it requires: $(1-2A/nf(r))^{-½} = (1-2A/f(nr))^{-½}$, or: $f(nr) = nf(r)$. Hence *f* would have to be linear: $f(r)=Br$, for some constant *B.* But: $exp(A/r) \neq (1-2A/Br)^{-½}$. Hence it is impossible.

The metric (2) is not a coordinate transformation of (1), despite the appearance of a close similarity *in a certain range of values where* $r >> 2MG/c^2$. We will see the two metrics are fundamentally different: the Ricci scalars are different.

But first we note the similarity of the functions *k* and *K*.

Useful identities to compare k with K (3)

$1/k^2 = 1 - 2MG/c^2r$

$1/K^2 = 1 - 2MG/c^2r + (2MG/c^2r)^2(1/2!) - (2MG/c^2r)^3(1/3!)$

$k = 1 + MG/c^2r + (3/2)(MG/c^2r)^2 + (5/2)(MG/c^2r)^3 + (35/8)(MG/c^2r)^4 + \ldots$

$K = 1 + MG/c^2r + (1/2)(MG/c^2r)^2 + (1/6)(MG/c^2r)^3 + (1/24)(MG/c^2r)^4 + \ldots$

$k - K = (MG/c^2r)^2 + \ldots$ *(higher-order terms)*

$k/K = 1 + (MG/c^2r)^2 + \ldots$ *(higher-order terms)*

Comparing the series, we see *k* and *K* are close in value for large $r >> MG/c^2$. But they diverge sharply for small *r,* in the realm of $r = MG/c^2$.

We should ask first whether it makes sense to make this substitution of *K* for *k* and expect to get a sensible physical equation back. For a number of reasons it does.

- *K* is dimensionless, like *k.* So anywhere you can find *k* in a physical equation, it is logically consistent with the dimensional analysis to put *K* instead.
- *K* is very close to *k* except in large gravity fields. $1/k^2$ is just the first two terms in the polynomial expansion of $1/K^2$. They start to differ in the third term, by $2(MG/c^2r)^2$. Similarly, *k* and *K* start to differ in the third term, by $(MG/c^2r)^2$. This term is about 10^{-16} in solar gravity at 1 A.U.

- K is linear (separable) in M, meaning: $K(M+M') = K(M)K(M')$. This means that incrementally introducing mass to space with existing K by multiplying one field on the other: $K(M)K(M')$ gives the same result as applying the whole mass at once: $K(M+M')$. This is a strong symmetry of K which k does not have.
- $K(M,r)$ and $k(M,r)$ both have the same *scale symmetry*: $K(M,r) = K(aM,ar)$ and: $k(M,r) = k(aM,ar)$.

And there is an essential functional similarity between k and K:

- K differentiated by r is: $K' = dK/dr = -(MG/c^2r^2)K$.

  k differentiated by r is: $k' = dk/dr = -(MG/c^2r^2)k^3$.

It is these differentials that matter in calculating accelerations, and they have a similar functional pattern for k and K. The common term: $-(MG/c^2r^2)$ is responsible for the conformity to Newtonian gravity in the limit of large r, in both cases. The two differentials differ by: K/k^3, which is close to 1 for observable gravitational fields. This is essentially the difference in the accelerations (for a stationary observer at r) predicted by the two theories: more exactly, K-gravity is weaker by $1/K^2$ in proper acceleration.

To analyze the possibility of such a metric tensor in proper detail, we consider K-gravity (2) as a conventional solution to the Einstein equation. We work backwards, from the metric tensor (2) to derive the stress-energy tensor and mass-energy distribution that would be required to produce this. This requires a gravitational mass M distributed symmetrically in a field extending from the inertial center-of-mass. It can be thought of as a space-filling fluid, with a density law that generates the *K-gravity metric*.

To be precise, (2) tells us the *K-gravity* $g_{\mu\nu}$ tensor, and we will use it to derive $R_{\mu\nu}$ and R, and hence $G_{\mu\nu}$, and hence $T_{\mu\nu}$, through the GTR field equation. We find these are non-zero outside the central mass region (unlike the Schwarzschild solution). Einstein's equation relates $G_{\mu\nu}$ to $T_{\mu\nu}$, and this is related in the conventional theory of the stress-energy tensor to the pressure-density ρ. In this conventional analysis, this is a solution, *but it is not a central mass solution.*

But we will subsequently formulate the K-gravity theory within GTR as a *central mass solution* by defining a new rule for $T_{\mu\nu}$. We can think of this as proposing and alternative law for ρ and thus $T_{\mu\nu}$ for *a central mass*. The upshot is that K-gravity is a physically meaningful solution of GTR, *if we accept the hypothesis that 'gravitational mass' is naturally distributed around the center of inertial mass,* according to these new rules for ρ, or $T_{\mu\nu}$.

Finally we observe the linearity (separability) of K with respect to mass.

Linearity of K in Mass †[10] (4)

$k(M_1+M_2) \approx k(M_1)k(M_2)$	In weak gravity: $r >> 2MG/c^2$
$K(M_1+M_2) = K(M_1)K(M_2)$	Exact symmetry

[10] See Appendix for algebraic workings, indicated by the dagger.†

We find $k(2M)$ and $k(M)k(M)$ differ by a factor of approximately: $1+(2MG/c^2r)^2$ for large r. In solar gravity at Earth orbit, the difference is on the scale of $1+10^{-16}$, so very tiny. But the two functions are not identical, and it would be mathematically incoherent to formulate a theory that required an identity between these.

However $K(M_1+M_2) = K(M_1)K(M_2)$ is an identity. So in K-gravity, for a single central mass, the metric function K multiplies as a linear field in mass. E.g. the effect of introducing one larger mass: $M = M_1+M_2$ to a point in empty space is the same as having its two smaller masses introduced to the same point sequentially. This is what makes it possible to give the superposition principle for K-gravity in Section 10.

3. Generalized line metric functions.

We now follow the derivation of the Schwarzschild and the K-gravity solutions in parallel, formulated in a generalized static spherically symmetric metric. We adopt the usual spherically-symmetric metric in polar coordinates: (t,r,θ,ϕ), indexed by: μ,ν = (0,1,2,3), respectively, and the symmetry assumptions let us write this in the form:

Generalised Schwarzschild metric functions: U and V

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = U(r)dt^2 - V(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2 \qquad (5)$$

where U and V are spatial functions of r alone. This means:

$$g_{00} = U, \qquad g_{11} = -V, \qquad g_{22} = -r^2, \qquad g_{33} = -r^2\sin^2\theta \qquad (6)$$

K-gravity and *Schwarzschild gravity* are defined by alternative choices of U and V. The following identities are used repeatedly in the algebraic calculations, and grouped as Equations (7).

Identities for k, K, U, V, and their differentials. (7)

K-gravity	***Schwarzschild gravity***
$K = \exp(MG/c^2r)$	$k = (1-2\,MG/c^2r)^{-1/2}$
Definitions of U and V	
K-gravity:	***Schwarzschild gravity:***
$U = c^2/K^2 = c^2\exp(-2MG/c^2r)$	$U = c^2/k^2 = c^2(1-2MG/c^2r)$
$V = -K^2 = -\exp(2MG/c^2r)$	$V = -k^2 = -1/(1-2MG/c^2r)$
$U = -c^2/V$	$U = -c^2/V$
$V = -c^2/U$	$V = -c^2/U$
$UV = -c^2$	$UV = -c^2$
$U/V = -c^2/K^4$	$U/V = -c^2/k^4$
Derivatives of K and k by r	
$K = \exp(MG/c^2r)$	$k = (1-2\,MG/c^2r)^{-1/2}$

$K' = -(MG/c^2r^2)K$ $\qquad$ $k' = -(MG/c^2r^2)k^3$

$K^{2}{}' = -(2MG/c^2r^2)K^2$ $\qquad$ $k^{2}{}' = -(2MG/c^2r^2)k^4$

$K^{-1}{}' = (MG/c^2r^2)K^{-1}$ $\qquad$ $k^{-1}{}' = (MG/c^2r^2)k$

$K^{-2}{}' = (2MG/c^2r^2)K^{-2}$ $\qquad$ $k^{-2}{}' = (2MG/c^2r^2)$

Second derivatives by r

$K'' = (2MG/c^2r^3)K+(MG/c^2r^2)^2K$ $\qquad$ $k'' = (2MG/c^2r^3)k^3+(MG/c^2r^2)^2 3k^5$

Derivatives of U and V in terms of K and k:

$U' = -2c^2K'/K^3 = (2MG/r^2)/K^2$ $\qquad$ $U' = -2c^2k'/k^3 = 2MG/r^2$

$V' = -2KK' = (2MG/c^2r^2)K^2$ $\qquad$ $V' = -2kk' = (2MG/c^2r^2)k^4$

$U''= 4(MG/r^2K)^2 - (4MG/r^3K^2)$ $\qquad$ $U'' = -4MG/r^3$

$V''= -(4MGK^2/c^2r^3)(1 + MG/c^2r)$ $\qquad$ $V''= -(4MGk^4/c^2r^3)(1 + 2MGk^2/c^2r)$

Derivatives of U in terms of U:

$U' = (2MG/c^2r^2)U$ $\qquad$ $U' = 2MG/r^2$

$U'' = -(4MG/c^2r^3)(1-MG/c^2r)U$ $\qquad$ $U'' = -4MG/r^3$

$= 4(MG/c^2r^2)^2U - (4MG/c^2r^3)U$

Derivatives of V in terms of V:

$V' = -(2MG/c^2r^2)V$ $\qquad$ $V' = -(2MG/c^2r^2)V^2$

$V'' = 4(MG/c^2r^2)^2V + (4MG/c^2r^3)V$ $\qquad$ $V'' = 8(MG/c^2r^2)^2V^3 + 4(MG/c^2r^3)V^2$

$= (4MG/c^2r^3)(1 + MG/c^2r)V$ $\qquad$ $= (4MG/c^2r^3)(1 + 2VMG/c^2r)V^2$

4. Christoffel symbols.

Non-vanishing Christoffel symbols in terms of *U* and *V* are for a general Schwarzschild-type metric:[11]

Christoffel symbols written in U, V *(8)*

$\Gamma^0{}_{01}= \Gamma^0{}_{10}= U'/2U$

$\Gamma^1{}_{00}= U'/2V$ $\qquad$ $\Gamma^1{}_{11}= V'/2V$ $\qquad$ $\Gamma^1{}_{22}= -r/V$ $\qquad$ $\Gamma^1{}_{33}= -r\sin^2\theta/V$

$\Gamma^2{}_{12}= \Gamma^2{}_{21}= 1/r$ $\qquad$ $\Gamma^2{}_{33}= -\cos\theta\sin\theta$

$\Gamma^3{}_{13}= \Gamma^3{}_{31}= 1/r$ $\qquad$ $\Gamma^3{}_{23}= \Gamma^3{}_{32}= \cot\theta$

Others have no terms of U or V.

Substituting the coordinate functions for *U, V* we obtain the two different sets of functions, for K-gravity and Schwarzschild gravity:

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[11] See: Gary Oas, p. 3-4 for working.

Christoffel symbols in coordinate functions *(9)*

K-gravity:	***Schwarzschild gravity:***
$\Gamma^0_{01} = \Gamma^0_{10} = MG/c^2r^2$	$\Gamma^0_{01} = \Gamma^0_{10} = k^2MG/c^2r^2$
$\Gamma^1_{00} = -MG/r^2K^4$	$\Gamma^1_{00} = -MG/r^2k^2$
$\Gamma^1_{11} = MG/c^2r^2$	$\Gamma^1_{11} = k^2MG/c^2r^2$
$\Gamma^1_{22} = r/K^2$	$\Gamma^1_{22} = r/k^2$
$\Gamma^1_{33} = r\sin^2\theta/K^2$	$\Gamma^1_{33} = r\sin^2\theta/k^2$

5. Ricci tensors.

The Christoffel symbols determine the Ricci tensor, which has four non-zero terms.

Ricci Tensor written in U,V[12] *(10)*

$R_{00} = -U''/2V + U'V'/4V^2 + U'^2/4UV - U'/rV$

$R_{11} = U''/2U - U'^2/4U^2 - U'V'/4UV - V'/Vr$

$R_{22} = rU'/2UV + 1/V - rV'/2V^2 + 1$

$R_{33} = R_{22} \sin^2\theta$

Ricci tensor components written in coordinate variables [†] *(11)*

K-gravity:	***Schwarzschild gravity:***
$R_{00} = 2(MG/cr^2)^2/K^4$	$R_{00} = 0$
$R_{11} = -4MG/c^2r^3$	$R_{11} = 0$
$R_{22} = 1-1/K^2$	$R_{22} = 0$
$R_{33} = R_{22} \sin^2\theta$	$R_{33} = 0$
$R_{00}/R_{11} = -(MG/2r)/K^4$	

6. Ricci scalar.

Ricci Scalar written in U and V [13] *(12)*

$R = R^{\mu}{}_{\nu} = g^{\mu\nu}R_{\mu\nu}$

$\quad = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$

$\quad = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}\sin^2\theta\, R_{22}$

$\quad = R_{00}/U - R_{11}/V - R_{22}/r^2 - \sin^2\theta\, R_{22}/(r^2\sin^2\theta)$

$\quad = R_{00}/U - R_{11}/V - 2R_{22}/r^2$

Substituting for $R_{\mu\nu}$ and simplifying:[14]

$$R = -U''/UV + U'V'/2UV^2 + U'^2/2VU^2 - 2U'/rUV + 2V'/V^2r - 2/r^2(1+1/V) \qquad (13)$$

Then substitute for *U, V, U', V', U''* to obtain:

---

[12] See: Gary Oas, p. 5-7 for working, but error in: $R_{22} = rU'/2UV + 1/V - rV'/2V^2$ **– 1**.

[13] See Oas, p.6 for working.

[14] Oas, p.6 has: $R = -U''/UV + U'V'/2UV^2 + U'^2/2U^2V - 2U'/rUV + 2V'/rV^2 + (2/r^2)(1-1/V)$

Ricci Scalar in coordinate functions † *(14)*

R for K-Gravity

$R = (2/r^2K^2)(K^2 - 1 - (2MG/c^2r))$

$R \approx (4M^2G^2/c^4r^4K^2)$

R is positive and proportional to $1/r^4$ in its highest term.

(For Schwarzschild gravity, $R = 0$.)

7. Stress-Energy tensors.

Using the Einstein equation, we can now determine the $T_{\mu\nu}$ components directly. Only diagonal terms can be non-zero, and we obtain three independent equations as follows.[15]

Field Equations written in U, V *(15)*

$(8\pi G/c^4)T_{00} = R_{00} + \frac{1}{2}g_{00}R = UV'/rV^2 - (U/r^2)(1+1/V)$

$(8\pi G/c^4)T_{11} = R_{11} + \frac{1}{2}g_{11}R = -U'/rU - (V/r^2)(1+1/V)$

$(8\pi G/c^4)T_{22} = R_{22} + \frac{1}{2}g_{22}R$

$= (r/2V)(-U'/U + V'/V - rU''/U + rU'V'/2UV + rU'^2/2U^2)$

The fourth equation, for T_{33}, is equivalent to the third. In the Schwarzschild derivation these are set to zero, and this leads to the solutions: $V = k^2$ and $U = c^2/k^2$. We now use these equations to solve $T_{\mu\nu}$ for K-gravity. The solutions are given below in a number of different algebraic forms, including series in $1/r$, and approximations from above and below.

K-gravity Stress-Energy Tensor in coordinate functions †

T_{00} for K-Gravity *(16)*

$T_{00} = Mc^4/4\pi r^3K^4 + c^6/8\pi Gr^2K^4 - c^6/8\pi Gr^2K^2$

$= (Mc^4/4\pi r^3K^4) + (c^6/8\pi Gr^2K^4)(1-K^2)$

$= (c^6/8\pi Gr^2K^4)(1+(2MG/c^2r)- K^2)$

$= -(c^6/8\pi Gr^2K^4)((2MG/c^2r)^2/2!+(2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + \ldots)$

$= -(M^2Gc^2/4\pi r^4K^4)(1 + 2(2MG/c^2r)/3! + 2(2MG/c^2r)^2/4! \ldots)$

$= -(M^2Gc^2/4\pi r^4K^4) - (c^6/8\pi Gr^2K^4)((2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + \ldots)$

$-(M^2Gc^2/4\pi r^4K^2) \approx< T_{00} \approx< -(M^2Gc^2/4\pi r^4K^4)$ *for large r*

---

[15] See Oas, p.7, equations 8, 9, 10. Note Oas gives: (8) $(8\pi G/c^4)T_{00} = R_{00} + \frac{1}{2}g_{00}R = UV'/rV^2 + (U/r^2)(1-1/V)$ and: (9) $(8\pi G/c^4)T_{11} = R_{11} + \frac{1}{2}g_{11}R = -U'/rU + (V/r^2)(1-1/V)$.

T_{11} for K-Gravity (17)

$T_{11} = -T_{00}\,K^4/c^2 = T_{00}\,g_{11}/g_{00}$

$= -Mc^2/4\pi r^3 - (1-K^2)(c^4/8\pi Gr^2)$

$= (M^2G/4\pi r^4) + (c^4/8\pi Gr^2)((2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + \ldots)$

$(M^2G/4\pi r^4) \approx< T_{11} \approx< (M^2GK^2/4\pi r^4)$ *for large r*

T_{22} for K-Gravity (18)

$T_{22} = T_{11}\,r^2/K^4 = T_{11}\,g_{22}/g_{11}$

Note that:

$T_{11} = -T_{00}\,K^4/c^2 = T_{00}V/U = T_{00}\,g_{11}/g_{00}$

$T_{00} = -T_{11}c^2/K^4 = T_{11}U/V = T_{11}\,g_{00}/g_{11}$

$T_{22} = -T_{00}\,g_{22}/g_{00},\; T_{33} = -T_{00}\,g_{33}/g_{00}$

I.e.: $T_{\mu\mu} = T_{\nu\nu}\,g_{\mu\mu}/g_{\nu\nu}$ *(no summation)*

T_{00} *is negative, with its largest 1/r-term in:* $1/r^4$.

T_{11}, T_{22}, T_{33} *are positive, with largest 1/r-term in:* $1/r^4$.

8. Pressure-Density in K-gravity.

In K-gravity we conceive the gravitational mass, *M*, as if it were a fluid. We now calculate what the conventional distribution of this fluid would be. We can follow Vojinovic (2010) p.7. for a simple derivation.[16]

"The stress-energy tensor of a fluid element with density ρ, pressure p, and 4-velocity u^μ, is:

$T_{\mu\nu} = (\rho+p)u_\mu u_\nu + pg_{\mu\nu}$

We wish to describe the static fluid ($u_1 = u_2 = u_3 = 0$). So the stress-energy obtains the form:

$T_{00} = \rho u_0\,u_0 + p(u_0\,u_0 + g_{00}),\quad T_{11} = pg_{11},\quad T_{22} = pg_{22},\quad T_{33} = pg_{33}$

while other components vanish. Next the 4-velocity vector must be normalized, $u_\mu\,u_\nu\,g^{\mu\nu} = -1$, which means that $u_0\,u_0 = -g_{00}$."

Applying this to the K-gravity metric, this gives four equations:

K-Gravity: Pressure-Density Tensor Equations

$T_{00} = -\rho c^2/K^2,\quad T_{11} = pK^2,\quad T_{22} = pr^2,\quad T_{33} = -pg_{33}$ (19)

Or inversely:

$\rho = -T_{00}\,K^2/c^2,\quad p = T_{11}/K^2,\quad p = T_{22}/r^2$ (20)

[16] But note Vojinovic use the reverse definition of the metric signature, so we must reverse signs appropriately when we apply this.

For the Schwarzschild solution, these are all zero: $T_{\mu\nu} = 0$ so $p=0$ and $\rho = 0$.

We now calculate p and ρ for K-gravity. Since from Section 7: $T_{\mu\mu} = T_{\nu\nu} g_{\mu\mu}/g_{\nu\nu}$, there is really only one equation to solve, and: $p = \rho$. We will solve for ρ.

Substituting T_{00} from Equation (16) in the first equation above gives:

$$\rho = -Mc^2/4\pi r^3K^2 - c^4/8\pi Gr^2K^2 + c^4/8\pi Gr^2 \qquad (21)$$

$$= (c^4/8\pi Gr^2)(1 - 1/K^2 - 2GM/c^2rK^2)$$

$$= (c^4/8\pi Gr^2)(1 - 1/K^2k^2)$$

Or expanded as a series in $1/r$:

$$\rho = (M^2G/4\pi r^4) - (c^4/8\pi Gr^2)((2MG/c^2r)^3(2/3!) - (2MG/c^2r)^4(3/4!) + \ldots)$$

A first approximation (from below) for large r is:

$$\rho \approx< (M^2G/4\pi r^4) \qquad \textit{for large r} \qquad (22)$$

A second approximation (from above) for large r is:

$$\rho \approx> (M^2G/4\pi r^4K^2) \qquad \textit{for large r}$$

ρ is constrained between these two limits, and these are close when K is small. So we see that it essentially varies with: M^2/r^4 in the first approximation, for $r >> MG/c^2$, where $K \approx 1$. The first-order variation with M^2 may seem strange, because when we integrate ρ we will find the integral is proportional to M. But the integral is dependant on the behavior at small r, i.e. where $r < MG/c^2$, and higher-order terms in $1/r$ and M dominate.

Note K only becomes substantially larger than 1 in the region of the Schwarzschild (black hole) radius. E.g. at: $r = MG/c^2$, $K = exp(1) = e = 2.71828$. At $r = 2MG/c^2$, $K = exp(1/2) = \sqrt{e} = 1.64872$. When r becomes smaller than this, the value of ρ begins to diverge to infinity. As $r \rightarrow 0$, $\rho \rightarrow \infty$, and there is a central naked singularity. But we will see when we integrate for the mass that there is no conventional 'black hole' event horizon at any radius.

Dimensions

T_{00} has dimensions of: $MX^3T^{-4} \equiv Velocity^2 Energy/Volume$

T_{11} has dimensions of: $T_{00}/c^2 \equiv MX^{-1}T^{-2} \equiv$ *Energy per Volume* $\equiv$ *Force per Area*

ρ has dimensions of: $T_{00}/c^2 \equiv MX^{-1}T^{-2} \equiv$ *Energy/Volume* $\equiv$ *Force per Area*

p has dimensions of: $T_{11} \equiv MX^{-1}T^{-2} \equiv$ *Energy/Volume* $\equiv$ *Force per Area*

We may divide: ρ/c^2 to get *mass density* instead of *energy density*.

$\rho/c^2 \equiv MX^{-3} \equiv$ *Mass/Volume*

9. Integrating the mass-energy density.

We now show that the total mass-energy adds up to Mc^2, by integrating ρ over the spatial volume. This is required to match the Newtonian and Schwarzschild solutions in the limit. We first find the indefinite integral:

The mass-energy integral

$$I = \int (\rho)(4\pi r^2 dr) \qquad (25)$$
$$= \int (-Mc^2/4\pi r^3K^2 - c^4/8\pi Gr^2K^2 + c^4/8\pi Gr^2)(4\pi r^2 dr)$$
$$= \int (-Mc^2/rK^2 - c^4/2GK^2 + c^4/2G)dr$$

This has the exact solution:

The mass-energy integral solution

$$I = -rc^4/2GK^2 + rc^4/2G + E \qquad (26)$$
$$= (rc^4/2G)(1-1/K^2) + E$$

where E is an arbitrary constant of integration. To verify this calculate:

$$d/dr(rc^4/2GK^2) = c^4/2GK^2 + (-2rc^4/2GK^3)(dK/dr)$$
$$= c^4/2GK^2 + (-2rc^4/2GK^3)(-MG/c^2r^2)K$$
$$= c^4/2GK^2 + (Mc^2/rK^2)$$

And:

$$d/dr(rc^4/2G) = c^4/2G$$

Next we obtain the limit of I as: $r \rightarrow \infty$. We expand the solution in terms of r.

$$I = (rc^4/2G)(1-1/K^2) + E \qquad (27)$$
$$= (rc^4/2G)(1-1+2MG/c^2r -(2MG/c^2r)^2/2! + (2MG/c^2r)^3/3!) - \ldots) + E$$
$$= Mc^2 - M^2G/r + 2M^3G^2/3c^2r^2 - \ldots + E$$

As we limit $r \rightarrow \infty$ all terms in r disappear and only constant terms remain:

$$I_\infty = Mc^2 + E$$

We will set the constant E equal to 0,[17] so the indefinite integral is Mc^2 at $r = \infty$. Hence the indefinite integral is:

$$I = (rc^4/2G)(1-1/K^2) \qquad (28)$$

We next obtain the limit of I as $r \rightarrow 0$. To simplify, we can define: $r = \alpha 2MG/c^2$, i.e. r is defined as a multiple α of the fundamental distance: $2MG/c^2$. Thus: $\alpha \rightarrow 0$ as $r \rightarrow 0$, and: $\lim_{r\rightarrow 0} (I) = \lim_{\alpha\rightarrow 0} (I)$. Terms reduce to: $1/K^2 = exp(-2MG/c^2r) = exp(-1/\alpha)$, and: $rc^4/2G = \alpha Mc^2$. Substituting in I:

[17] For an otherwise empty universe: but remember in the real universe there is a lot of background energy that would really need to be included, and ultimately we must consider cosmological terms. However it is just the differentials of *I* that matter for the metric calculations in GTR.

$I = (Mc^2)(\alpha(1\text{-}exp(-1/\alpha)))$

We need the value of: $\alpha(1\text{-}exp(-1/\alpha))$ as: $\alpha \rightarrow 0$. This goes 0, because:

$exp(-1/\alpha) = 1/exp(1/\alpha)$ and: $exp(1/\alpha) \rightarrow \infty$ as: $\alpha \rightarrow 0$, so: $exp(-1/\alpha) \rightarrow 0$, so:

$\alpha(1\text{-}exp(-1/\alpha)) \rightarrow \alpha \rightarrow 0$. Hence:

$$I(0) = 0 \qquad \text{and:} \qquad I(\infty) = Mc^2 \qquad (29)$$

Hence the definite integral over the whole volume of space is:

The total mass-energy integral

$$I_{0 \text{ to } \infty} = {}_{r=0 \text{ to } \infty}\int (\rho)(4\pi r^2 dr) \qquad (30)$$

$$= [(rc^4/2G)(1\text{-}1/K^2)]_0^{\infty}$$

$$= Mc^2$$

The total mass-energy of the system is Mc^2.

Note the mass-energy within a radius r is:

$$I_{0 \text{ to } r} = {}_{0 \text{ to } \infty}\int (\rho)(4\pi r^2 dr)$$

$$= [(rc^4/2G)(1\text{-}1/K^2)]_0^{\,r}$$

$$= (rc^4/2G)(1\text{-}1/K^2)$$

$$= (rc^4/2G)((2MG/c^2r) - (2MG/c^2r)^2/2!+(2MG/c^2r)^3/3! - ...)$$

$$= (rc^4/2G)(2MG/c^2r)(1 - (2MG/c^2r)/2!+(2MG/c^2r)^2/3! - ...)$$

$$= Mc^2(1 - (2MG/c^2r)/2!+(2MG/c^2r)^2/3! - ...)$$

It is easily shown that the factor on the right is larger than $1/K^2$ and smaller than $1/K$ from at least: $r>10(MG/c^2)/3$, hence:

$$Mc^2/K > I_{0 \text{ to } r} > Mc^2/K^2 \qquad \text{for } r > (10/3)MG/c^2 \qquad (31)$$

For $r >> MG/c^2$, the total amount of gravitational mass outside the spherical shell of r is closely approximated by: M^2G/c^2r. Conversely $M(1–MG/c^2r)\approx M/K$ is approximately the gravitational mass within the sphere of radius r. The overall effect on proper acceleration at r is similar to a central mass M/K^2 in the Schwarzschild solution. We see the acceleration solution in the next section. The effect of the reduced mass within the shell combined with the additional mass outside the shell weakens the effective mass by M/K^2, not just M/K.

Two further simple results help confirm the physical consistency of this solution.

Black hole radius is consistent. Although the mass-density increases indefinitely as we approach the center, the Schwarzschild (black hole) radius r_s for the central mass within r is always smaller than r, so there is no conventional 'black hole' formed inside. The Schwarzschild radius is: $r_s = 2MG/c^2$. The mass within a radius r is: $M = (rc^2/2G)(1\text{-}1/K^2)$. Substituting for M we get: $r_s = (2G/c^2)(rc^2/2G)(1\text{-}1/K^2) = r(1\text{-}1/K^2)$, or: $r_s/r = (1\text{-}1/K^2) < 1$. Hence the mass distribution appears consistent, and no problems of singularities arise, except the central (naked) singularity, which appears as in conventional GTR.[18]

Pressure is consistent with a quasi-Newtonian force. Note if we differentiate the mass integral at r by r we get a force term, and this is exactly equal to: $dI/dr = 4\pi r^2 p$. Since $4\pi r^2$ is the surface area at r, this can be interpreted as meaning that the *total internal force of the mass distribution over the surface at r* generates the pressure term, p. Note (differentiate the series (27)) that this is like a gravitational self-attraction: $F = M^2G/r^2 - 4M^3G^2/3c^2r^3 + \ldots \approx M^2G/r^2$ for large r, as if the mass M was attracting to itself at a distance of r by a quasi-Newtonian force law. Note this begins to reduce at small r, e.g. at the point where: $r = 4MG/3c^2$, the Newtonian term cancels with higher order terms as: $M^2G/r^2 - 4M^3G^2/3c^2r^3 = 0$.

We have shown that a certain conventional mass-energy distribution ρ in GTR will generate the K-gravity metric solution, i.e. equation (2). For the physical model in K-gravity, we take the *inertial mass of a single fundamental particle* to be a mass M centered at $r=0$. This gives rise to a gravitational field exactly as if there is a gravitational mass-energy distributed symmetrically, as ρ. To deal with larger central masses, we need to be able to model them as aggregations of smaller masses. It is implicit in (2) that large spherically symmetric aggregations obey the same solution. However if we want this to represent a general law, we now have the problem of applying this to more complex systems of multiple masses, and for this we need an explicit *superposition principle*, to tell us how arrangements of multiple masses determine the metric tensor through K-gravity. We discuss this next.

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[18] I note the central singularity is removed in an extended theory for a finite closed universe, which takes global curvature into account, and where K may be modified by a small cosmological term related to $R_{universe}$, the radius of curvature for the universe, like: $K = exp((MG/c^2)(1/r + 1/\pi R_{universe}))$. The value as $r \rightarrow 0$ is $exp((MG/c^2\pi R_{universe})$. This is mentioned to emphasize that the central singularity should not be considered *physical*, or necessarily problematic, in a full treatment. But a proper treatment then also requires a cosmological model.

10. Superposition: K-gravity for multiple particles.

Throughout this section we now take $g_{\mu\nu}$ to be a generalised K-gravity metric, and $g_{\mu\nu}(M,r)$ to be the single mass solution (2). We have seen that the K-gravity metric for a single mass particle can work as a GTR solution. But to generalise K-gravity to a *general law of nature*, representing an alternative theory of gravity, we need a principle to determine the metric for multiple particles. The problem is that we cannot do this through superposition of the stress-energy tensor or mass density function directly, because they are not suitably linear.

Ordinarily, if we took two separate mass distributions in space, we could simply add their masses together to get a combined distribution. Now we do add the *inertial mass distributions* like this. But the K-gravity *density function* ρ cannot be added like this. It is immediately evident that $T_{\nu\mu}$ for the K-gravity central mass is not linear with mass. E.g. using the approximation: $T_{11} \approx M^2G/4\pi r^4$, which is accurate for large r, and defining: $M = M_1+M_2$, we see that:

$$
\begin{aligned}
T_{11}(M_1+M_2) &\approx (M_1+M_2)^2G/4\pi r^4 \\
&= (M_1{}^2+M_2{}^2+2M_1M_2)G/4\pi r^4 \\
&= M_1{}^2G/4\pi r^4+ M_2{}^2G/4\pi r^4+ 2M_1M_2G/4\pi r^4 \\
&\approx T_{11}(M_1) + T_{11}(M_2)+ 2M_1M_2G/4\pi r^4
\end{aligned}
$$

Hence: $T_{11}(M_1+M_2) > T_{11}(M_1) + T_{11}(M_2)$. E.g. When: $M_1 = M_2 = M/2$, we have: $T_{11}(M_1+M_2) \approx 2T_{11}(M_1) + 2T_{11}(M_2)$.

Consider doubling a central mass M, and define: ρ_2 as the density solution for the double mass $2M$, and ρ_1 for the single mass M. We do not find: $\rho_2 = 2\rho_1$. Instead, for large r: $\rho_2/2\rho_1 \approx ((2M)^2G/4\pi r^4)/2(M^2G/4\pi r^4) = 2$. This means that to get the superposition of two masses M at a point, we cannot take: $\rho_2 = 2\rho_1$ for the density: we would have to take: $\rho_2 \approx 4\rho_1$. The density function: $2\rho_1$ will appear at large r to be the density for a mass of approximately: $\sqrt{2}M$, not $2M$. But actually this is not the correct function for $\sqrt{2}M$ at all either: this is only approximately correct at large r. The ρ function changes its *shape* relative to M, and its spatial differentials and integrals change.

Now we might try to find another way of superposing multiple masses through their K-gravity density (or stress-energy tensor) functions. And I think there is a possible way this can be done – but it requires inventing a more detailed underlying model supporting ρ. For we do not have to assume that empty space has a zero ρ to get a *locally flat* metric. We can instead take empty space to have a large uniform ρ, representing a uniform background mass-energy field. In a finite closed universe, this results from the average of all the particle masses adding up over all the universe. Individual masses are then perturbations of this background field (the 'space-energy field'). It is *perturbations* – or differentials – on this field that cause effects on the metric. In fact in the context of K-gravity, we *have* to do this, because on the assumption of K-gravity, real space is filled with gravitational energy from all around the universe. I have found a certain type of underlying model can work.

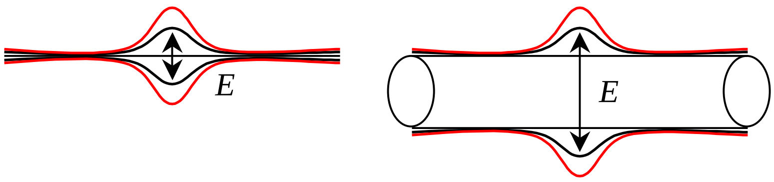

Figure 2. As an analogy, conceive a pipe of material which stretches by a perfect law of elasticity (if we insert energy localized at a point), so that stretching it at two places at once multiplies strains exponentially. But the size of the pipe makes a difference to the relative shapes of the strain functions. When we superpose two small strains in a large pipe, the result appears to be an approximately linear addition, although it is really exponential. Analogously, we have obtained the K-gravity stress-energy tensor as if it was a strain on a zero-radius pipe, but if we take a uniform background energy into account, a model for superposing through ρ can work.

But this takes us into deep water: it requires cosmology, and a quantum model for discrete particles. (And more radically, the model I have found to do this needs a multi-dimensional space of at least 6 dimensions.) So we will not explore this here.[19]

In the present treatment, we are only seeking to confirm that there is a realistic way to extend K-gravity metric to multiple masses, that makes a plausible theory, in the framework of GTR. To do this we propose a simple principle for *determining the K-gravity metric tensor directly from an arrangement of multiple static masses*, to show pragmatically how the theory may be generalized. The point is that we are not obtaining this superposition principle through an underlying superposition of stress-energy tensors, as GTR suggests: but the simple superposition principle below works for static systems. And as indicated above, I believe a fully general principle can be developed.

A superposition principle needs to produce certain symmetries, and needs to conform to Newtonian or Schwarzschild gravity in the limit. The most fundamental symmetry is that a metric for multiple particles should be independent of the composition of masses from smaller masses. Accelerations produced should also increase linearly with source masses. This symmetry arises already in the basic metric (2) from the linearity of K with respect to mass, i.e. $K(M_1+M_2) = K(M_1)K(M_2)$. (And we will see the linearity of accelerations below). This means that the K-gravity metric for two single stationary fundamental masses superimposed at the same point is the same as the metric for their total mass. We need similar properties for more general mass superpositions.

[19] See Lehmkuhl (2008) for an interesting discussion of the ontological relationship between the stress-energy tensor and the metric. Lehmkuhl argues that the latter is more fundamental.

We can start by thinking of inserting an additional mass M (into a static system) as *perturbing* an existing metric field. E.g. inserting a single mass changes the flat metric to the K-gravity metric. This is more obvious when we write the metric in local Cartesian (i.e. rectangular) coordinates with: (r,y,z) at the field point. Then we represent the original flat metric as $h_{\mu\nu}$:

Flat metric in local Cartesian coordinates.

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dr^2 - dy^2 - dz^2 \qquad (32)$$

$$[h_{\mu\nu}] = \begin{pmatrix} c^2, & 0, & 0, & 0 \\ 0, & -1, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix}$$

The K-gravity metric for a single mass then looks like this:

K-gravity metric in local Cartesian coordinates.

$$c^2 d\tau^2 = c^2 dt^2/K^2 - K^2 dr^2 - dy^2 - dz^2 \qquad (33)$$

$$[g_{\mu\nu}] = \begin{pmatrix} c^2/K^2, & 0, & 0, & 0 \\ 0, & -K^2, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix}$$

Thus the mass appears to perturb the flat metric elements by factors like: $\Delta_{00} = 1/K^2$ and $\Delta_{11} = K^2$ (the two perturbed elements), and: $\Delta_{22} = \Delta_{33} = 1$. But this is specific to this diagonalised coordinate representation only. We will generalize to two source masses, M_1 and M_2, at different points of space, to see the effect on the metric at an arbitrary field point, O.

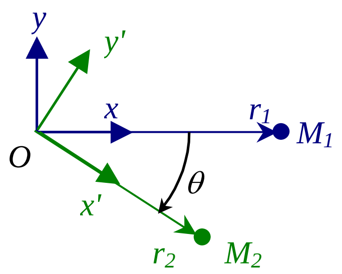

Figure 3. (The z-axis is coming out of the page.)

We now take the field point, O, to define the coordinate origin. Now we may define two different local Cartesian coordinates at this origin, rotated to align their *x-axes* to the two masses respectively. Specifically:

(1) The first coordinate system: (t,x,y,z) has the *x-axis* pointing towards M_1, and the *y-axis* in the plane of M_1, M_2 and O.

(2) The second coordinate system: (t',x',y',z') has the *x'-axis* pointing towards M_2, and the *y-axis* in the plane of M_1, M_2 and *O*.

Their time coordinate is common (there is no relative motion), so: $t = t'$. And the third spatial coordinate is also common, so: $z = z'$, i.e. *z* is orthogonal to plane of M_1, M_2 and *O*. So we may write the second coordinate system as: (t, x',y',z). The two coordinate systems are just rotations of the (x,y) coordinates to (x',y'), by an angle θ, which is the angle the two source masses appear at from the field point.

The K-gravity metric for M_1 by itself written in the (t,x,y,z) system is as $g_{\mu\nu}$ above. The only difference is that *K* is written in *x* instead of *r* (the radial vector from particle). We may define: $r_{(1,0)}$ as the *constant distance from M_1 to the field point*, the previous radial distance variable: *r* may be written instead as: $r_{(1)} = r_{(1,0)} - x$ giving:

$$K_1 = exp(M_1G/c^2(r_{10} - x)) = exp(M_1G/c^2(r_{(1)}))$$

Thus we will henceforth use: $r_{(n)}$ as the *radial distance variable* for the n^{th} mass. The coordinate transformation rule is: $g_{\mu\nu}' = g_{kl} (\partial x^k/\partial x'^\mu)(\partial x^l/\partial x'^\nu)$.

The Jacobian $(\partial x^k/\partial x'^\mu)$ ***for simple rotation*** *(34)*

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & \cos\theta, & -\sin\theta, & 0 \\ 0, & \sin\theta, & \cos\theta, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix}$$

We will now denote the coordinate representations of the tensors as: $[g_{(1)\mu\nu}]$ and: $[g_{(2)\mu\nu}]$ for the separate metrics from the two masses respectively. When we rotate $g_{\mu\nu}$ above to the (t, x',y',z) representation its components transform as follows:

K-Gravity metric tensor for M_1 in rotated Cartesian coordinates *(35)*

$$[g_{(1)\mu\nu}]' = \begin{pmatrix} c^2/K_1^2, & 0, & 0, & 0 \\ 0, & -1-\cos^2\theta(K_1^2-1), & -\cos\theta\sin\theta(K_1^2-1), & 0 \\ 0, & -\cos\theta\sin\theta(K_1^2-1), & -1-\sin^2\theta(K_1^2-1), & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix}$$

The representation of the metric $[g_{(2)\mu\nu}]$ for the second mass M_2 is diagonalised in these coordinates.

K-Gravity metric tensor for M_2 in local Cartesian coordinates *(36)*

$$[g_{(2)\mu\nu}]' = \begin{pmatrix} c^2/K_2^2, & 0, & 0, & 0 \\ 0, & -K_2^2, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix}$$

With K_2 defined:

$$K_2 = K(M_2,x') = exp(2M_1G/c^2(r_{(2)}+x'))$$

Now is there any simple way to obtain a combined metric tensor when these two masses are imposed simultaneously? Some simple considerations show that this cannot be done by any simple functional combination of the metric tensor coordinate functions. But there is a way using the gradient of K, that works for static systems at least.

We will now define a rule to get the metric $g_{(\Sigma N)}$ for N stationary masses, indexed by: $n = 1$ to N. We define a solution here only for *static* mass distributions, in the stationary rest frame of the collection of masses. To generalize it to allow moving masses will require writing the K_n's as *relativistic retarded potentials,* and introduces substantial complexity, and we will only give a static solution here.

The K field.

We first define the (scalar) *K field* as K defined over all the masses in space. It is just the product of all the individual $K(M_n,r_{(n)})$*'s* for the individual masses. At any field point K is defined:

$$K = K(M_1,r_{(1)})K(M_2,r_{(2)})\ldots K(M_N,r_{(N)}) \qquad (37)$$

The $r_{(n)}$ are the distances from the field point O to the masses M_n. We may write this:

$$K \equiv K(M_1/r_{(1)}+ \ldots +M_N/r_{(N)}) = exp((G/c^2)(\textstyle\sum_{n=1 \text{ to } N}(M_n/r_{(n)}))) \qquad (38)$$

This magnitude depends only the masses M_n and their distances: $r_{(n)} = |\mathbf{r}_{(n)}|$ from the field point. I.e. it is independent of the relative directions of the masses. But it is essential of course to remember when we differentiate that the $r_{(n)}$ are coordinate functions that do represent directions. We see this in the gradient of K.

The K-gradient field.

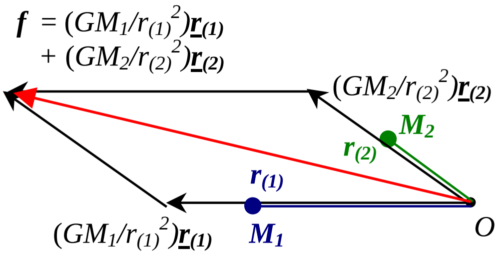

Figure 4. The vector sum ***f*** *of Newtonian acceleration fields, illustrated for two masses. The K-gradient field is closely related.*

Second we introduce the *gradient field of K,* which is defined in local rectangular coordinates for the empty space at a field point O as follows, with $i = 1,2,3$ and $\underline{\mathbf{x}}_i$ the basis vectors for coordinates: x^i.

$$\mathbf{K} = \mathbf{grad}(K) = (\partial K/\partial x^i)\underline{\mathbf{x}}_i \tag{39}$$

To differentiate note: $\mathbf{grad}(r_{(n)}) = \underline{\mathbf{r}}_{(n)}$ and: $\mathbf{grad}(M_n/r_{(n)}) = -(M_n/r_{(n)}{}^2)\underline{\mathbf{r}}_{(n)}$. So (summing over the masses *n*):

$$\mathbf{grad}(K) = \mathbf{grad}(exp((G/c^2)\sum(M_n/r_{(n)}))) \tag{40}$$

$$= K\,\mathbf{grad}((G/c^2)\sum(M_n/r_{(n)}))$$

$$= -(G/c^2)\sum((M_n/r_{(n)}{}^2)\underline{\mathbf{r}}_{(n)})K$$

The Newtonian acceleration field.

The K-gradient field above is just K/c^2 times the *Newtonian gravitational acceleration vector field*. This is just the vector sum of Newtonian accelerations, here called $\mathbf{f}$, defined as:

$$\mathbf{f} = -G\sum((M_n/r_{(n)}{}^2)\underline{\mathbf{r}}_{(n)}) \qquad \text{Newtonian acceleration field} \tag{41}$$

Thus:

$$\mathbf{grad}(K) = K\mathbf{f}/c^2 \tag{42}$$

($\mathbf{f}$ is commonly written as the gradient of the Newtonian potential field, ϕ.)

The magnitude of $\mathbf{f}$ may be written: $f = |\mathbf{f}| = (\mathbf{f}\cdot\mathbf{f})^{1/2}$. So f is a scalar field.

The direction of $\mathbf{f}$ may be written: $\underline{\mathbf{f}} = \mathbf{f}/f$.

The magnitude of $\mathbf{grad}(K)$ may be written: $grad(K) = |\mathbf{grad}(K)| = (\mathbf{grad}(K)\cdot\mathbf{grad}(K))^{1/2}$. So $grad(K)$ is a scalar field.

The scalar fields are thus also simply related:

$$grad(K) = Kf/c^2 \tag{43}$$

Note this is the key point where the special property of linearity of *K* comes into play. This solution for *grad(K)* cannot work for Schwarzschild gravity, because the Schwarzschild function *k* is not linear. It is impossible to give a superposition rule for Schwarzschild gravity like that for K-gravity.

Now we only need to work with one or other of $\mathbf{f}$ or $\mathbf{grad}(K)$ as they essentially interchangeable, and we will use $\mathbf{f}$ as it is simpler. The important point however is that the scalar field *K* already contains a 'picture' of $\mathbf{f}$ within itself. Note that the Schwarzschild function *k* cannot be meaningfully generalized to a scalar field for multiple masses like *K*, because it does not have the linear seperability property of *K*.

The rules for the metric tensor.

We can now state a set of rules to determine $g_{\mu\nu}$ for multiple source masses. We state this first *in a special local rectangular coordinate system, at the field point O, with* x^1 *chosen in the direction of* $\underline{\mathbf{f}}$. The $g_{\mu\nu}$ representation is diagonalised in this coordinate system. (Note if $\underline{\mathbf{f}} = 0$ we are at a local minima (of potential energy), and in this special case, we just set $g_{ij} = -\delta_{ij}$.) We will then give a more general rule for rotated coordinate systems, and show a basic consistency result that confirms our superposition rule is meaningful.

The *static assumption* means the off-diagonal terms with a time component are zero the *t-t* component g_{00} is simply:

$$g_{00} = c^2/K \qquad (44)$$

$$g_{01} = g_{02} = g_{03} = g_{10} = g_{20} = g_{30} = 0$$

In the special rectangular coordinates with $x = x^1$ chosen in the direction of $\boldsymbol{f}$ the metric tensor is defined like this:

$$\text{Coordinates:} \quad t=x^0 \quad x=x^1 \quad y=x^2 \quad z=x^3 \qquad (45)$$

$$[g_{\mu\nu}] = \begin{pmatrix} c^2/K^2, & 0, & 0, & 0 \\ 0, & -1-(\mathbf{f}.\underline{\mathbf{x}}_1/f)^2(K^2-1), & 0, & 0 \\ 0, & 0, & -1 & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix}$$

I.e. $g_{11} = -1-(\mathbf{f}.\underline{\mathbf{x}}_1/f)^2(K^2-1)$ and other spatial terms are: $g_{ij} = -\delta_{ij}$.

Now in fact: $(\mathbf{f}.\underline{\mathbf{x}}_1/f)^2$ just equals 1, because $\boldsymbol{f}$ is in the direction of $\underline{\mathbf{x}}_1$ so $\mathbf{f}.\underline{\mathbf{x}}_1 = f$. So we just get the same metric as before. But we put it in this functional form to introduce the form of the more general case next. First however we illustrate this case, and for this we need to summarize the calculation of acceleration in GTR.

Acceleration for a stationary test particle.

We will now use: U^μ for the velocity 4-vector and A^μ for the acceleration 4-vector. These are the differentials of the x^μ w.r.t. proper time, $d\tau$. Thus for a *stationary test particle* at our field point[20], which will be the key case for us to consider, $U^0 = dt/d\tau = c/\sqrt{g_{00}} = K$, and $U^i = dU^i/d\tau = 0$ for the spatial velocities. The general tensor relationship for acceleration is:[21]

$$\begin{aligned} A^\kappa &= U^\lambda \nabla_\lambda U^\kappa \\ &= U^\lambda(\partial U^\kappa/\partial x^\lambda + \Gamma^\kappa{}_{\lambda\mu}U^\mu) \end{aligned}$$

For the stationary particle, only $U^0 \neq 0$, and this simplifies to:

$$\begin{aligned} A^\kappa &= U^0(\partial U^\kappa/\partial x^0 + \Gamma^\kappa{}_{00}U^0) \\ &= (U^0)^2\Gamma^\kappa{}_{00} \qquad (46) \end{aligned}$$

For a Schwarzschild-type metric, the only non-vanishing Christoffel symbol is $\Gamma^1{}_{00}$. So the proper-time acceleration: $d^2x/d\tau^2$ of a stationary particle at a field point $\boldsymbol{O}$ is:

$$A^1 = (c^2/g_{00})\Gamma^1{}_{00} = (c^2/g_{00})(\partial g_{00}/\partial x)(1/2g_{11})$$

This is then equal to:

$$\begin{aligned} A^1 &= (c^2/g_{00})\Gamma^1{}_{00} = -(c^2K^2/c^2)(c^2\partial K^{-2}/\partial x)(1/2K^2) \\ &= -\tfrac{1}{2}c^2(\partial K^{-2}/\partial x) \end{aligned}$$

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[20] Stationary w.r.t. the static mass distribution.

[21] E.g. see *"The Schwarzschild solution in detail",* p.1.

In the case of the simple single-mass K, the differential is simply: $\partial K^{-2}/\partial x = 2MG/c^2r^2K^2$, and the result is: $A^1 = MG/r^2K^2$. (*c.f.* the Schwarzschild result is: $A^1 = MG/r^2$. Thus we see that the Schwarzschild acceleration is greater by a factor of K^2.) This is the acceleration with respect to proper time. The acceleration in real time *t for a stationary test particle with:* $dx/dt = 0$ is then: $a = d^2x/dt^2 = A^1(d\tau/dt)^2 = MG/r^2K^4$. (c.f. the Schwarzschild result is: $a = MG/r^2k^2$.)

However, in the more general case with multiple masses, we do not have a Schwarzschild-like metric, the Christoffel symbols Γ^{κ}_{00} other than Γ^{1}_{00} are not generally vanishing, and we have to go back to the more general equation (46).

However we now make use of our special assumption at the field point ***O***, that we have chosen $x = x^1$ in the direction of ***f***. The differentials of K tangent directions are then zero *at this point*, and *for this point* the Christoffel symbols Γ^{κ}_{00} do vanish except for Γ^{1}_{00}.

But of course the differential: $\partial K^{-2}/\partial x$ is now no longer simply: $2MG/c^2r^2K^2$. Rather it is given through the gradient function: $\partial K/\partial x = \mathbf{grad}(K).\underline{\mathbf{x}} = K\mathbf{f}.\underline{\mathbf{x}}/c^2$. We have: $\partial K^{-2}/\partial x = -2\mathbf{f}.\underline{\mathbf{x}}/c^2K^2$. Thus the result of calculating A^1 is more generally:

$$A^1 = (c^2/g_{00})\Gamma^{1}_{00} = -\tfrac{1}{2}c^2(\partial K^{-2}/\partial x) = \mathbf{f}.\underline{\mathbf{x}}/K^2 \qquad (47)$$

We can give a simple example to illustrate. Take a field-point ***O*** half-way between two masses of magnitude M and $2M$ respectively, at a distance r_0 from each.

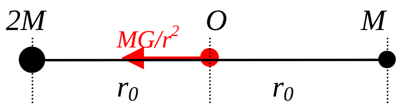

Figure 5. Field point halfway between two unequal masses.

The resultant Newtonian acceleration is towards the larger mass with: $f = MG/r^2$. The function K in x is:

$$K = exp((G/c^2)(2M/(r_0-x)+M/(r_0+x)) \qquad (48)$$

The magnitude at the field point, where $x = 0$, is simply: $K = exp((G/c^2)(3M/r_0)$. However notice the signs of the variable x are different in the two denominators in K, so when we differentiate we get:

$$\partial K/\partial x = ((G/c^2)(2M/(r_0-x)^2)- (G/c^2)(M/(r_0 +x)^2))K$$

The value at the field point where $x = 0$ is: $\partial K/\partial x_{|x=0} = (GM/c^2r_0)^2K = Kf/c^2$. And this is what gives the correct acceleration.

In fact the expression (48) is not a general expression for K either, because it does not show the general dependence on the other two coordinates, y and z. The general expression is rather given by defining radial distance variables:

$$r_{(1)} = \sqrt{((r_0+x)^2+y^2+z^2)}, \qquad r_{(2)} = \sqrt{((r_0-x)^2+y^2+z^2)}$$

and then writing:

$$K = exp((G/c^2)(M/r_{(1)} +2M/r_{(2)})) \qquad (49)$$

When we differentiate this w.r.t. *x, y* and *z* we get the same result.

The result is that *because we have chosen* ***x*** *in the direction* ***f*** *at the field point* ***O****, only the differential w.r.t. x is non-zero at that point.* This is why the matrix is diagonal *at the point* ***O*** in this coordinate system. However when we have multiple masses, the differentials in all directions are involved. And we are no longer writing general functions over the entire space: only special cases at the point ***O***. We have made a kind of recipe to determine the metric and acceleration at individual points, not a general metric function.

We now generalize the rule given above, for rectangular coordinates (x^i) rotated with respect to ***f*** in a plane of ***f*** by an angle θ. The space components are:

$$g_{ij} = -\delta_{ij} - (\mathbf{f}\bullet\underline{\mathbf{x}}_i/f)(\mathbf{f}\bullet\underline{\mathbf{x}}_j/f)(K^2-1) \qquad (50)$$

The dot product: $\mathbf{f}\bullet\underline{\mathbf{x}}_i$ gives the magnitude of ***f*** in the $\underline{\mathbf{x}}_i$ direction, and we may write this as: $\mathbf{f}\bullet\underline{\mathbf{x}}_i = f_i$. Now we may note a similarity in form with the rotated metric above (35). This is the essential point: we need to confirm that we get the same result if we use (50) to assign the components in a rotated frame as we get from rotating the original by the tensor transformation as in (35).

First consider the simple central mass case, where: $\mathbf{f}\bullet\underline{\mathbf{x}}_i = f$, and use (50) to assign components in a rotated frame. In the simple frame, $x = x^1$ is chosen in the direction ***f***, so: $\mathbf{f}\bullet\underline{\mathbf{x}}_1 = f$, and $\mathbf{f}\bullet\underline{\mathbf{x}}_2 = \mathbf{f}\bullet\underline{\mathbf{x}}_3 = 0$. Now suppose we rotate in the *x-y* plane by θ, as in the transformation (35). We now find that: $\mathbf{f}\bullet\underline{\mathbf{x}}_1 = f\cos\theta$ and: $\mathbf{f}\bullet\underline{\mathbf{x}}_2 = f\sin\theta$. This is simply the vector geometry of rotating the acceleration vector. Thus we find the components directly from (50) as:

$$g_{11} = -\delta_{11} - (f\cos\theta/f)(f\cos\theta/f)(K^2-1) = -1 - \cos^2\theta(K^2-1) \qquad (51)$$

$$g_{22} = -\delta_{22} - (f\sin\theta/f)(f\sin\theta/f)(K^2-1) = -1 - \sin^2\theta(K^2-1)$$

$$g_{12} = -\delta_{12} - (f\cos\theta/f)(f\sin\theta/f)(K^2-1) = -\cos\theta\sin\theta(K^2-1)$$

$$g_{21} = -\delta_{21} - (f\sin\theta/f)(f\cos\theta/f)(K^2-1) = -\cos\theta\sin\theta(K^2-1)$$

So we have the same result by applying (50) directly as we get by transforming the diagonalised metric to the rotated coordinate system (35). Now this works the same when $\mathbf{f}\bullet\underline{\mathbf{x}}_i = f_i < f$, i.e. generally, because f_i/f acts as a constant when we differentiate the g_{ij}.

Now we may write the rule (47) in a generalized matrix form similar to (35):

K-metric for a static system in rectangular coordinates *(49)*

$$\begin{pmatrix} c^2/K_1^2, & 0, & 0, & 0 \\ 0, & -1-(\mathbf{f}\bullet\underline{\mathbf{x}}_1/f)^2(K^2-1), & -(\mathbf{f}\bullet\underline{\mathbf{x}}_1/f)(\mathbf{f}\bullet\underline{\mathbf{x}}_2/f)(K^2-1), & -(\mathbf{f}\bullet\underline{\mathbf{x}}_1/f)(\mathbf{f}\bullet\underline{\mathbf{x}}_3/f)(K^2-1) \\ 0, & -(\mathbf{f}\bullet\underline{\mathbf{x}}_2/f)(\mathbf{f}\bullet\underline{\mathbf{x}}_1/f)(K^2-1), & -1-(\mathbf{f}\bullet\underline{\mathbf{x}}_2/f)^2(K^2-1), & -(\mathbf{f}\bullet\underline{\mathbf{x}}_2/f)(\mathbf{f}\bullet\underline{\mathbf{x}}_3/f)(K^2-1) \\ 0, & -(\mathbf{f}\bullet\underline{\mathbf{x}}_3/f)(\mathbf{f}\bullet\underline{\mathbf{x}}_1/f)(K^2-1), & -(\mathbf{f}\bullet\underline{\mathbf{x}}_3/f)(\mathbf{f}\bullet\underline{\mathbf{x}}_2/f)(K^2-1), & -1-(\mathbf{f}\bullet\underline{\mathbf{x}}_3/f)^2(K^2-1) \end{pmatrix}$$

Or in the slightly simpler notation:

K-metric for a static system in rectangular coordinates *(49)*

$$\begin{pmatrix} c^2/K_1^2, & 0, & 0, & 0 \\ 0, & -1-(f_1/f)^2(K^2-1), & -(f_1/f)(f_2/f)(K^2-1), & -(f_1/f)(f_3/f)(K^2-1) \\ 0, & -(f_2/f)(f_1/f)(K^2-1), & -1-(f_2/f)^2(K^2-1), & -(f_2/f)(f_3/f)(K^2-1) \\ 0, & -(f_3/f)(f_1/f)(K^2-1), & -(f_3/f)(f_2/f)(K^2-1), & -1-(f_3/f)^2(K^2-1) \end{pmatrix}$$

and this is consistent with general coordinate rotations. This is the metric represented in *orthogonal rectangular coordinates.* To get general coordinates we apply tensor transformations as usual.

The key is now to be able to write the function ***f*** in the coordinate system, and of course this gets messy as soon as there are multiple masses, and dynamic systems or trajectory functions are not analytically solvable in general, any more than in the ordinary GTR. But to conceptually verify the result, imagine that we define some arbitrary rectangular coordinates, with an arbitrary origin at ***O***, and write the metric as coordinate functions according to (49). Now to simplify the representation *at the point* ***O***, we rotate the coordinates so that *x* is in the direction of ***f*** *at* ***O***. Now $\mathbf{f}\bullet\underline{\mathbf{x}}_1 = f$, and $\mathbf{f}\bullet\underline{\mathbf{x}}_2 = \mathbf{f}\bullet\underline{\mathbf{x}}_3 = 0$. (The acceleration components tangential to ***f*** are zero.) So the metric now takes the diagonal form *at the point* ***O***. Then when we take the 4-acceleration, we get the same result we previously saw in (47): $A^1 = (c^2/g_{00})\Gamma^1{}_{00} = -½c^2(\partial K^{-2}/\partial x) = \mathbf{f}.\underline{\mathbf{x}}/K^2$. This is the total proper acceleration (since other components are zero). It conforms to the Newtonian and Schwarzschild accelerations for large *r*, within the factor K^2. Since the acceleration 4-vector is a tensor, transformations to any other coordinates will give the same total proper acceleration. Since we can do this for any point (except at mass points if we have point-mass singularities), the result is general.

Of course it is a further problem to generalize this for *dynamic systems.* The obvious concept is that source masses in motion be treated as retarded sources, like moving electric charges in electrodynamics. There seems no reason this will not be consistent for K-gravity. I note that it does avoid the important problem of 'self-gravitating fields', as produced by 'gravitons' in attempts to quantise GTR. Because gravitons are themselves energetic particles, they should also contribute to the stress-energy tensor – leading to infinite energies. But in K-gravity, there are no gravitons: *the K-field is the 'stress field' on the metric of space: it is not an energy source that also gravitates in turn.*

But perhaps the more immediate concern will be for the general status of covariance, or the tensorial nature of the stress-energy tensor. The Einstein equation is *relativistic* basically because the stress-energy tensor on the right hand side is (interpreted to be) relativistic. It is composed of multiple parts, for mass-energy, pressure energy, electrodynamic energy, etc. Now we have modified the *mass-energy* part of the tensor: does it still have this property? This is too complex to deal with, and we must leave the development of a full treatment at this point. It is unreasonable to expect to be able to answer such questions here: it took decades of (slow and generally unreliable) progress by a global community of leading physicists before the problem of motion in GTR was properly addressed - and it is still far from adequate.

> "However, the discovery [of a binary pulsar the astronomers wished to analyze through GTR] revealed an ugly truth about the "problem of motion". As Ehlers *et al.* pointed out in an influential 1976 paper [Ehlers 1976], the general relativistic problem of motion and radiation

was full of holes large enough to drive trucks through. They pointed out that most treatments of the problem used “delta functions” as a way to approximate the bodies in the system as point masses. As a consequence, the “self-field”, the gravitational field of the body evaluated at its own location, becomes infinite. While this is not a major issue in Newtonian gravity or classical electrodynamics, the non-linear nature of GR requires that this infinite self-field contribute to gravity. In the past, such infinities had been simply swept under the rug. Similarly, because gravitational energy itself produces gravity it thus acts as a source throughout spacetime. This means that, when calculating radiative fields, integrals for the multipole moments of the source that are so useful in treating radiation begin to diverge. These divergent integrals had also been routinely swept under the rug.” Will, 2014, p. 58.

It wasn’t until the 1990s-2000s that essential problems of motion in GTR were analyzed reliably. So we conclude the discussion of a superposition principle for K-gravity here. The aim has been to show that the basic solution (2) leads to a *prima facie* coherent theory, with a plausible generalization to multiple masses. The theoretical development so far, although naturally incomplete, is primarily meant to be sufficient to raise the question of its empirical accuracy.

11. Empirical tests.

> *"…they did notice that the Pioneers have been slowing down faster than predicted by Einstein's general theory of relativity. Some tiny extra force – equivalent to a ten-billionth of the gravity at Earth's surface – must be acting on the probes, braking their outward motion. …In 1994 Michael Martin Nieto of Los Alamos National Laboratory and his colleagues suggested that the anomaly was sign that relativity itself had to be modified."*
> Musser, 1998.

Once K-gravity is recognized a theoretical possibility consistent with GTR, and given its simplicity in removing the troublesome event horizon singularity, the leading question is naturally whether is empirically accurate. Since this question could actually be settled quite readily by experiment, this appears the best approach, rather than indulging in lengthy theoretical disputation. It has the advantage that it would provide a new test of GTR. To continue from Section 1, we can now note some further features of the empirical comparison of Schwarzschild gravity and K-gravity. I refer to (Will, 2014) as an authoritative general review of current experimental tests of GTR.

The conceptual starting point is that the predicted difference for low-velocity accelerations in weak gravity is K^2, seen in Section 10. For inner-planet orbits around the sun (1 AU), this is about $1+10^{-8}$. For Saturn orbit at around 10 AU, K^2 is down to about $1+10^{-9}$. Now for a stationary observer at a fixed orbit, adopting K-gravity (instead of Schwarzschild gravity) is practically the same as recalibrating the estimated magnitude of $M_{sun}G$ for the sun by the factor K^2 at that orbit. Testability then at first seems to depend upon whether acceleration measurements are made accurately enough to detect the difference between $M_{sun}G$ and $M_{sun}GK^2$. But measuring absolute accelerations at one orbit is no good: for these are what we use to *determine* $M_{sun}G$ in the first place. We must compare *accelerations*[22] *at two different orbits.* We can predict measurements *at a second orbit, using the recalibrated magnitude* $M_{sun}GK^2$ *established at an original orbit*, to test the differences between K-gravity predictions and the Schwarzschild predictions at the second orbit. This point is critical and needs a brief analysis.

$M_{sun}G$ is measured quite accurately, to a relative uncertainty better than 10^{-10} (and claimed to be around 10^{-11}.[23]) So it might seem a difference of K^2 could be immediately detected in absolute accelerations. But to repeat the point above, this is wrong. $M_{sun}G$ at a single orbit may be calculated from measuring *acceleration* (of orbiting bodies like planets or space probes), and then using the assumption of Schwarzschild gravity to infer $M_{sun}G$. If we assume K-gravity instead, we would just infer that $M_{sun}G$ is larger by K^2, using K for the orbit where we measured the acceleration. (Note because K-gravity is weaker, we infer a *larger* $M_{sun}G$ from the

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[22] The same applies to time dilation or red shift effects, but I will talk only of acceleration here. In fact red shift effects are verified to only about 10^{-6} (Will, 2014, p.13-15) and are not sensitive enough.

[23] Although inconsistencies in measurements give us reason to doubt more extreme values claimed. Researchers at the leading edge of a field of measurement naturally maximise claims of precision (they are competing to set new 'precision records'). But experimenters often do not take certain kinds of *systematic errors* into account when estimating *measurement uncertainty*, and leading results are often later overthrown. The measurement of MG for the Earth has the second best precision, at around 10^{-9}. But because of Earth's much weaker field, this is actually not ideal to test K-gravity.

same observed acceleration.) So $M_{sun}G$ inferred from Schwarzschild gravity from a single orbit will be consistent by definition with $M_{sun}GK^2$ inferred from K-gravity.

To use acceleration measurements, we need to measure accelerations at *two different orbits,* and compare their values. (The inference to a value of $M_{sun}G$ really a proxy for acceleration.) Suppose we first determine (proper) *accelerations:* $M_{sun}G/r_1^2$ and $M_{sun}G/r_2^2$ at two different orbits, using the assumption of Schwarzschild gravity to infer $M_{sun}G$. Their difference is:

$$\Delta_N = (M_{sun}G/r_1^2 - M_{sun}G/r_2^2) = (M_{sun}G)(1/r_1^2 - 1/r_2^2)$$

We can measure this accurately to the sum of relative uncertainties in the terms. This uncertainty involves the r terms as well as $M_{sun}G$. Let us define this uncertainty (to one standard error) as: $\pm\varepsilon M_{sun}G/r_1^2$. Now this depends on the measurements we have done at both orbits. If we do a very careful measurement at a primary orbit r_1 we may get a small error, but for a good comparison we need a similarly careful measurement at r_2 and as we will now see, we need r_2 to be in a suitable range to maximize the predicted acceleration differences.

On the assumption of K-gravity, we can *recalibrate* $M_{sun}G$ to the value: $M_{sun}GK_1^2$, at the primary orbit r_1, and then use this value for $M_{sun}G$ at r_2. The accelerations predicted by K-gravity will then be very close to: $M_{sun}GK_1^2/r_1^2K_1^2 = M_{sun}G/r_1^2$ and $M_{sun}GK_1^2/K_2^2r_2^2$. Their difference will then be predicted as:

$$\Delta_K = (M_{sun}G/r_1^2 - M_{sun}GK_1^2/K_2^2r_2^2) = (M_{sun}G)(1/r_1^2 - K_1^2/K_2^2r_2^2)$$

Expanding the K term, this is approximately:

$$\Delta_K \approx (M_{sun}G)(1/r_1^2 - 1/r_2^2 - (1/r_2^2)(2M_{sun}G/c^2)(1/r_1 - 1/r_2))$$

Then the absolute difference: $\Delta_K - \Delta_N$ is:

$$\Delta_K - \Delta_N = -(M_{sun}G)(1/r_2^2)(2M_{sun}G/c^2)(1/r_1 - 1/r_2)$$

This is the difference between the two theories for the accelerations predicted at r_2.

Define $\sigma = r_1/r_2$, so this is:

$$\Delta_K - \Delta_N = -(M_{sun}G\sigma^2/r_1^2)(2M_{sun}G/c^2r_1)(1-\sigma)$$

Now we need to compare this magnitude to the error term: $\pm\varepsilon M_{sun}G/r_1^2$.

Dividing gives:

$$(\Delta_K - \Delta_N)/(\varepsilon M_{sun}G/r_1^2) = -2\sigma^2(1-\sigma)(M_{sun}G/c^2r_1)(1/\varepsilon)$$

Effects become conclusively detectible when this is substantially greater than 1, let us say in the range: 10 to 100.[24] Let us set this to *±20* to define a *conclusively detectible limit,* so:

$$\sigma^2(1-\sigma)(M_{sun}G/c^2r_1) = \pm 10\varepsilon \qquad \textit{Conclusively detectible limit for } \varepsilon$$

Now we can put in approximate numbers, for $r_1 = 1\ AU$ as: $M_{sun}G/c^2r_1 \approx 10^{-8}$, so:

$$\varepsilon \approx \pm\sigma^2(1-\sigma)10^{-9}$$

[24] Theoretically 3-5 standard errors is sufficient; but the likelihood of systematic error through miscalculation of small effects, like radiation pressure, etc, means we really want to have a better precision to conclusively measure an effect.

This tells us the maximum limit of ε required at different choices of $\sigma = r_1/r_2$ to achieve a clear detection of the K-gravity effect.

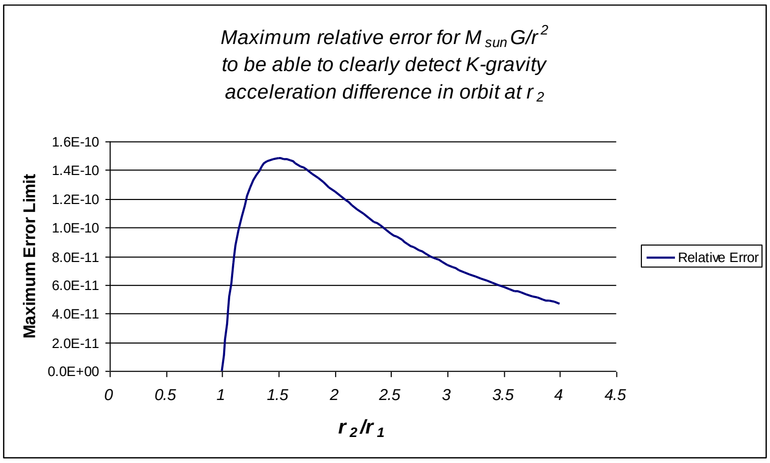

Figure 6. Graph of ε against r_2/r_1

This illustrates that a comparison of accelerations between two close orbits requires high precision or low measurement error, rising rapidly to a maximum at $r_2/r_1 = 1.5$, and then falling off again. (Or inversely: $r_1/r_2 = 1.5$). The most effective choice of the r_2 orbit is at 1.5 AU (or inversely 0.66 AU), assuming the primary orbit is at $r_1 = 1$ AU. A variation from about 1.3 – 2.5 may be good enough if measurement error is less than 10^{-10}. It requires that the relative error ε must be less than 10^{-10} *for the acceleration measured in for both orbits.* Note that this error must include compensation terms for forces other than the solar gravity component, e.g. solar radiation, solar wind, dust collisions, planetary tugs, EM forces, oblateness of the sun, etc.

Since the error claimed for $M_{sun}G$ is better than this value of 10^{-10}, and the term r can be measured with better accuracy for space probes, it is possible for this experimental test to be done with space probes. It is possible data for such a test *has* been collected. However I do not think this is the case.[25] Certainly no dedicated experiment or analysis has been deliberately done for this purpose. To do this would require high-precision measurements at two appropriate radial parameters, but without a theory to guide the experimental design, there is no way to guess this. I also note below that there is a better way to do the experiment than by measuring the sun's gravity directly through accelerations.

Measurements of planetary orbits, red-shift, Doppler effects or light delays are not accurate enough to test K-gravity: they have relative uncertainties of much more than 10^{-8}. The precession of the perihelion of Mercury is measured much less accurately (with relative uncertainty of about 10^{-3}; and as a proportion of the General Relativistic component about 10^{-2}), and cannot provide a test of K-gravity.

[25] Note also that multiple experiments are *averaged* to try to determine values for G or MG, and there have been perplexing inconsistencies in results from different experiments over the last few decades.

Thus we come back to the need for either a more precise direct measurement of gravitational accelerations, or another type of experiment that can be used to compare predicted trajectories accurately enough. The Pioneer spacecraft data potentially provided such an experiment, and as far I know, it is the only data of such precision available. There are two reasons it is more accurate. First, because it is taken over a long period of time, so small differences in acceleration can show up. And second, because it involves a free-fall trajectory over a large range of *r*, from about 10 AU (when the spacecraft left Saturn's orbit) to over 80 AU. The second point is most important. The functions *K* and *k* which determine accelerations *change shape over changes of r.* It is easier to detect the predicted anomalies in radial trajectories observed over a sufficient range of *r* than to detect differences in accelerations at two orbits directly.

This also leads to the most important realization when analyzing the effects on radial trajectories. Because K gravity is *weaker* than Schwarzschild gravity (for the same $M_{sun}G/r$), we intuitively expect it to predict that probes (like the Pioneers) traveling to large *r* will travel *faster.* "*The Pioneers have been slowing down faster than predicted",* and thus Musser reasons that "*some tiny extra force … must be acting on the probes, braking their outward motion."* This is true if the cause is non-gravitational. An attempt to explain this with a modified theory of gravity also intuitively also seems to need to predict *stronger* gravity. But in fact the opposite is the case.

Because $M_{sun}G/r$ is initially *calibrated* from the inner solar system (≈ 1 AU) on the assumption of Schwarzschild gravity, from the point of view of K-gravity this leads us to *underestimate* the magnitude of $M_{sun}G/r$ (by $1/K^2$). If K-gravity is correct, then we should *increase* the conventional magnitude of $M_{sun}G/r$ by this factor, i.e. K^2. In weak gravity, the differences between *K* and *k* are very small, and we will get almost the right acceleration predictions for K-gravity from the conventional Schwarzschild analysis - but by applying it with the larger value: $M_{sun}GK^2/r$ instead of $M_{sun}G/r$. This is what we saw in the analysis above.

If K-gravity is correct, we should notice the spacecraft slowing down faster than expected on the basis of the Schwarzschild solution. There should be an increasing delay in the expected position. This is what was first observed. Anomalies appeared on the scale of about a 16 seconds delay in the expected journey to around 80 AU by around 1998. In the only analysis done so far, I found a similar magnitude of difference (predicting about 12 - 18 seconds delay, sensitive to uncertainties in initial parameters), when I analyzed this in 2003. This result supported K-gravity.[26]

However the situation has subsequently become unclear, because after many years of searching for a mundane cause for the Pioneer anomalies, a number of research teams now claim (following Turyshev *et alia*, 2012) that a faint source of anisotropic heat radiation from the Pioneer spacecraft is responsible. If it could be proved that this correctly accounts for all effects, it would be important evidence supporting the fine-grained accuracy of Schwarzschild gravity. But there is too much uncertainty in the analysis for this to be considered in any way decisive as yet.

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[26] But this has not been independently checked, and it has significant complexity. I am not entirely confident of this result. As gravitational data has been significantly refined in the intervening 15 years, the evidence now has to be reevaluated in any case. But in any case, as argued next, the Pioneer data is not conclusive, and an independant experiment is needed.

The claim that the Pioneer anomalies are now explained will be taken by many physicists as immediate reason to dismiss any other possibility.[27] But there are substantial question marks around NASA's new explanation of the Pioneer anomalies, and the messy nature of this evidence means it cannot be taken as sufficiently conclusive to rule out K-gravity as an empirical theory without an experimental replication.

The problem is partly methodological. The NASA and other teams have been *searching for mundane causes* to explain away the Pioneer anomalies. They believe conventional physics and conventional GTR theory must be correct. But this can become a kind of *case-proving* approach, which can be a risky scientific procedure. The method is to search through possible causes, seeking to confirm a pre-conceived result – and then to *stop the search when a factor capable of supporting your case is found.* A lot of people are wrongly convicted of crimes by this kind of forensic methodology.

If such a tiny factor is able to be overlooked for 20 years, who knows if there are further tiny factors also overlooked? Tiny factors can be significant in this case, because the effects are amplified over a long period of time – and there are multiple possible effects to calculate, e.g. radiation and particle pressure from the sun, small planetary pulls, 'dark matter', heat anisotropy, dust collisions, so-called 'frame-dragging' effects, possible tiny EM forces - even the Hubble expansion of the universe – *and who knows what else?* Assuming they have indeed found a real effect in 'heat anisotropy', how thoroughly have they continued to search to see if there are still other effects involved?[28] Or indeed, how thoroughly have they incorporated all the other small effects?

And there must also be real uncertainty over the detail of their analysis. How much has *the expectation that they have found the culprit* affected the analysis, especially in terms of assigning parameters to critical values?[29] What is the real *uncertainty* in the analysis? As well as a bias towards seeking to reach a predetermined conclusion, the analysis is technically difficult. Data on the critical features of the spacecraft design was difficult to reassemble (due to poor NASA legacy IT), the model of heat-radiating properties is complex (no independent experimental confirmation), and a significant risk of modeling and calculation error must be considered possible.

This is mitigated by getting other teams to reproduce the analysis – but how independent are they really? Theorists are prone to copying each other's assumptions. The investigation is strongly driven by a conviction that conventional GTR theory *must be* correct - just as many astrophysicists are driven to keep searching for 'dark matter' from a conviction that GTR *must be* correct – and this is a real problem in

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[27] E.g. (Will 2014, p. 24).

[28] And if the current 'heat anisotropy' explanation had not been found, would the analysts not resort indefinitely to searching for further possibilities, until they found something that looks plausible, *and then stop*? Is it likely that they would ever accept failure to find a 'mundane physics' explanation as evidence for a *falsification of GTR?* This is a classic conundrum in philosophy of science – evident also in the search for 'dark matter'.

[29] In seeking to confirm a factor with uncertain magnitude as a cause, we tend to work backwards, and say: *what parameters would be required for this cause to explain the effect?* And then try to fit parameters to match our desired results.

scientific methodology – it shows up *repeatedly* in scientific history, as blindness to anomalous evidence in favor of established paradigms.

The upshot is that the NASA analysis of the Pioneer trajectories is vulnerable to too many possible delicate adjustments or uncertainties *precisely in the magnitude of anomalies predicted by K-gravity* to be a conclusive test. The simple fact is that the Pioneer situation was not sufficiently controlled to give a clear and indisputable result. So it cannot be taken as a conclusive experiment to decide between K-gravity and Schwarzschild gravity. Instead, it is in need of more conclusive experimental resolution. Single experiments of some kinds are conclusive: being bitten by a lion once proves sufficiently that lions bite. But single experiments of the Pioneer kind are not conclusive: experimental replication, with variations of conditions, is necessary.

And indeed this is the normal scientific solution: *replication of the experiment.* Replication is really the only way the cause of the Pioneer anomalies could be conclusively resolved.

However this normal solution is not feasible in this case, at least not as replication of the original experiment. It took 20-30 years to produce the original anomaly. To repeat the same trajectory with a new probe, designed without any heat anisotropy, is not realistic. We want a much simpler and faster experiment. But if we tried to speed it up, by sending a faster probe, or over a shorter distance, wouldn't this invalidate the replication? Well yes – *but it also depends on what the cause really is.* We do not have to replicate experiments *exactly:* rather, we want to replicate them to test for (suspected) *causes* of the phenomenon.

This is where having a specific alternative theory to test against is critical. A specific alternative theory lets us design variations of the original experiment, to enhance the anomalous effects on the assumption of the alternative cause.

So I propose an experiment to test K-gravity, and simultaneously try to replicate the phenomenon of the Pioneer anomalies, on the hypothesis that K-gravity is the cause. I think this can be done in a time-frame of around 3 years, by sending a simple probe in radial free-fall, from roughly Earth to Jupiter orbit, and tracking it precisely. The speed has to be optimized to amplify the effect predicted by K-gravity. It would only take about a three year journey at an optimized speed to obtain fairly good data to test K-gravity against Schwarzschild gravity; and the data will become quite conclusive over successive years. (And this is a more efficient and robust experiment than the previous idea of trying to directly measure acceleration in an orbit of 1.5 AU to a precision of around 10^{-11}.)

If the Pioneer anomalies really are due to K-gravity, this will reproduce the anomalous phenomena in a new setting, which would be strong evidence favor of K-gravity. If not, K-gravity will be falsified, and further weight will accrue to conventional GTR, and the 'mundane physics' explanation of the Pioneer anomalies. Without some *plausible alternative theory of the cause* to test against, we are at a loss to try to reproduce the anomalous phenomena, except by the impractical exercise of closely replicating the original experiment.

Why should we bother to do such an experiment to test K-gravity? Mainly because K-gravity identifies the domain of *the largest plausible error* that conventional GTR might have in weak gravity. Although there is a program for systematically testing GTR against a raft of alternatives, through the *parameterized* *post-Newtonian formalism,* this does not appear to include any variation resembling K-gravity.[30] There are a number of famous alternative theories to GTR (including Brans–Dicke theory, string theory, loop gravity, super-symmetry, etc) but they are *empirically untestable* against conventional GTR. And certainly none of the well-known alternatives predicts the Pioneer anomalies. So I think this would be a valuable novel test of both GTR and of the Pioneer anomalies at the same time.

On a philosophical note, this shows the importance of having a *specific alternative theories* to test current theories against. A specific alternative theory can tell us exactly where to look for differences, and the magnitude of effects to plan for in our measurements. It lets us design experiments – telling us how to adjust parameters (in this case initial radial speed and distance from the sun of the test probe) to amplify effects. Testing a theory like GTR *in a vacuum of an alternative theory* is very difficult. To test a good theory that is already verified to a fairly high precision, we need some theoretical idea of a domain where it is might go wrong. The idea of science exhaustively testing for anomalies *directly against nature* to falsify theories is a myth: we primarily test theories against other theories. The PPN testing program is well aware of this. The trouble is that they have made their own theoretical assumptions about the range of 'plausible alternatives', and K-gravity is not one of them.

[30] I cannot find any such test in (Will 2014) or elsewhere: but it is an open question whether there is effectively a test.

12 Conclusion.

The central sections of the paper analyze a novel solution to GTR, comprising a static system of a dispersed fluid. This should not be controversial, and hopefully is an interesting exercise for students, going beyond the usual Schwarzschild solution. It is the reinterpretation of this as the solution for a central mass, which involves reinterpreting the stress-energy tensor, that makes an alternative theory of gravity. This raises interesting conceptual questions, and I hope this is a valuable theoretical exercise, whether or not it turns out to be empirically correct. It has ramifications for the philosophy of GTR and space-time theory. I think we should recognize that the current deep problems in gravity and space-time theory ultimately come back to the fact that GTR does not cohere with quantum mechanics at a fundamental level. The stress-energy tensor should ultimately be a quantum construct (if you believe QM is fundamental) – but because 'gravitational mass' (or mass as the charge for the gravitational 'force') has no good quantum model, it remains an open question what its real treatment should be. The problem penetrates to GTR because the relation between the mass-energy-field distributions, the stress-energy tensor, metric tensor and the cosmological model, although constrained by the Einstein equation, is not clear. This is ultimately reflected in the fact that physics has a dualist ontology, with a mass-energy realm overlaid on a space-time realm.

To make a case for K-gravity, theoretically it has significant advantages in simplicity. Removing the event horizon singularity from GTR would be a big theoretical advantage. But its primary interest is empirical, because it is right on the cusp of current measurability and testability of GTR. A test of K-gravity against the Schwarzschild solution would set a new limit to the tested accuracy of the theory. It has value whether or not it is empirically true, because it provides insight into the conventional theory of gravity: for our knowledge includes not just our best current theory, but the possibilities for alternatives.

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Appendix. Extra Algebraic Workings.

This gives algebraic workings for some derivations not provided in the main text. These are indicated in the text by a dagger (†)

Equation 4.

$k(M_1+M_2) = 1/\surd(1-2(M_1+M_2)G/c^2r)$

$k(M_1)k(M_2) = 1/\surd((1-2M_1G/c^2r)(1-2M_2G/c^2r))$

$= 1/\surd(1-2(M_1+M_2)G/c^2r + M_1M_2(2G/c^2r)^2)$.

Hence: $k(M_1+M_2)^2/k(M_1)^2k(M_2)^2 = 1 + M_1M_2(2G/c^2r)^2/(k(M_1)^2k(M_2)^2)$

$\approx 1 + M_1M_2(2G/c^2r)^2$ for large r

So: $k(M_1)k(M_2) < k(M_1+M_2)$.

$K(M_1+M_2) = \exp((M_1+M_2)G/c^2r) = \exp(M_1G/c^2r)\exp(M_2G/c^2r) = K(M_1)K(M_2)$

Ricci Tensor. K-gravity:

$R_{00} = -U''/2V + U'V'/4V^2 + U'^2/4UV - U'/rV$

$-U''/2V = 2(MG/cr^2)^2/K^4 - (2MG/r^3)/K^4$

$U'V'/4V^2 = (MG/cr^2)^2/K^4$

$U'^2/4UV = -(MG/cr^2)^2/K^4$

$-U'/rV = 2MG/r^3/K^4$

$R_{00} = \text{sum} = 2(MG/cr^2)^2/K^4$

$R_{11} = U''/2U - U'^2/4U^2 - U'V'/4UV - V'/Vr$

$U''/2U = 2(MG/c^2r^2)^2 - (2MG/c^2r^3)$

$- U'^2/4U^2 = -(MG/c^2r^2)^2$

$- U'V'/4UV = -(MG/c^2r^2)^2$

$- V'/Vr = -2MG/c^2r^3$

$R_{11} = \text{sum} = 2(MG/c^2r^2)^2 - (2MG/c^2r^3) -(MG/c^2r^2)^2 - (MG/c^2r^2)^2 - 2MG/c^2r^3$

$= -4MG/c^2r^3$

$R_{22} = rU'/2UV + 1/V - rV'/2V^2 + 1$

$rU'/2UV = -MG/c^2rK^2$

$1/V = -1/K^2$

$- rV'/2V^2 = MG/c^2rK^2$

$1 = 1$

R_{22} = sum = $(1 - 1/K^2)$

$= (K^2-1)/K^2$

Ricci Tensor. Schwarzschild gravity:

$R_{00} = -U''/2V + U'V'/4V^2 + U'^2/4UV - U'/rV$

$-U''/2V = -2MG/r^3k^2$

$U'V'/4V^2 = (MG/cr^2)^2$

$U'^2/4UV = -(MG/cr^2)^2$

$-U'/rV = 2MG/r^3k^2$

R_{00} = sum = 0

$R_{11} = U''/2U – U'^2/4U^2 – U'V'/4UV – V'/Vr$

$U''/2U = -2MGk^2/c^2r^3$

$– U'^2/4U^2 = -(MG/c^2r^2)^2k^4$

$– U'V'/4UV = (MG/c^2r^2)^2k^4$

$– V'/Vr = +(2MGk^2/c^2r^3)$

R_{11} = sum = 0

$R_{22} = rU'/2UV + 1/V – rV'/2V^2 +1$

$rU'/2UV = -MG/c^2r$

$1/V = -1/k^2 = 2MG/c^2r -1$

$– rV'/2V^2 = -(MG/c^2r)$

$1 = 1$

R_{22} = sum = 0

Ricci Scalar:

Terms are:

$R_{00}/U = -U''/2VU + U'V'/4UV^2 + U'^2/4UUV - U'/rUV$

$-R_{11}/V = -U''/2UV + U'^2/4VU^2 + U'V'/4UVV + V'/VVr$

$-2R_{22}/r^2 = -U'/UVr - 2/Vr^2 + V'/V^2r - 2/r^2$

Sum:

$R = -U''/2UV -U''/2UV + U'V'/4UV^2 + U'^2/4UUV -U'/UVr + U'^2/4VU^2 - U'/rUV + U'V'/4UVV + V'/VVr - 2/Vr^2 + V'/V^2r - 2/r^2$

Simplifying:

$R = -U''/UV + U'V'/2UV^2 + U'^2/2VU^2 - 2U'/rUV + 2V'/V^2r - 2/r^2(1+1/V)$

Ricci Scalar. K-gravity:

$R = -U''/UV + U'V'/2UV^2 + U'^2/2U^2V - 2U'/rUV + 2V'/rV^2 - (2/r^2)(1+1/V)$

$-U''/UV = 4(MG/c^2r^2)^2/K^2 - (4MG/c^2r^3)/K^2$

$U'V'/2UV^2 = -2(MG/c^2r^2)^2/K^2$

$U'^2/2U^2V= -2(MG/c^2r^2)^2/K^2$

$-2U'/rUV = 4(MG/c^2r^3)/K^2$

$2V'/rV^2 = -4(MG/c^2r^3)/K^2$

$-(2/r^2)(1+1/V)= -(2/r^2)(1-1/K^2)$

$R = sum = -(2/r^2)(1-1/K^2) - (4MG/c^2r^3)/K^2$

$= (2/r^2)(1-1/K^2) - (4MG/c^2r^3)/K^2$

$= (2/r^2)(1-1/K^2 - (2MG/c^2r)/K^2)$

$= (2/r^2K^2)(K^2 - 1 - (2MG/c^2r))$

$= (2/r^2K^2)((2MG/c^2r)^2/2!+ (2MG/c^2r)^3/3! + ...)$

$R = (4M^2G^2/c^4r^4K^2) + (2/r^2K^2)(2MG/c^2r)^3/3! + ...)$

$R \approx (4M^2G^2/c^4r^4K^2)$ *for large r*

Ricci Scalar. Schwarzschild Gravity:

$R = -U''/UV + U'V'/2UV^2 + U'^2/2U^2V - 2U'/rUV + 2V'/rV^2 - (2/r^2)(1+1/V)$

$-U''/UV = -4(MG/c^2r^3)$

$U'V'/2UV^2 = 2(MG/c^2r^2)^2 k^4$

$U'^2/2U^2V= -2(MG/c^2r^2)^2 k^4$

$-2U'/rUV = 4(MG/c^2r^3)$

$2V'/rV^2 = -4MG/c^2r^3$

$-(2/r^2)(1+1/V)= -(2/r^2)(1-1/k^2) = -(2/r^2)(2MG/c^2r)= - 4MG/c^2r^3$

$R = sum = 0$

K-gravity Stress-Energy Tensor T_{00}

$(8\pi G/c^4)T_{00} = UV'/rV^2 - (U/r^2)(1+1/V)$

$UV'/rV^2 = (2MG/r^3)/K^4$

$-(U/r^2)(1+1/V) = -(c^2/r^2K^2)(1-1/K^2) = (c^2/r^2)(1/K^4-1/K^2)$

Sum: $(8\pi G/c^4)T_{00} = 2MG/r^3K^4 -(c^2/r^2K^4)(K^2-1)$

$T_{00} = Mc^4/4\pi r^3K^4 + c^6/8\pi Gr^2K^4 – c^6/8\pi Gr^2K^2$

As a polynomial in r:

$(8\pi G/c^4)T_{00} = (2MG/r^3)/K^4 + (c^2/r^2)(1/K^4-1/K^2)$

$= (c^2/r^2)(2MG/c^2r)/K^4 + (c^2/r^2)(1-K^2)/K^4$

$= (c^2/r^2)\,((2MG/c^2r + 1-K^2)/K^4$

$= (c^2/r^2K^4)\,((2MG/c^2r) + 1 – (1 + (2MG/c^2r) + (2MG/c^2r)^2/2! + (2MG/c^2r)^3/3! + ...))$

$= -(c^2/r^2K^4)((2MG/c^2r)^2/2! + (2MG/c^2r)^3/3! + ...)$

So:

$T_{00} = -(c^6/8\pi Gr^2K^4)((2MG/c^2r)^2/2! + (2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + ...)$

Factor out the term $(2MG/c^2r)^2/2!)$*:*

$T_{00} = -(c^6/8\pi Gr^2K^4)(2MG/c^2r)^2/2)(1 + 2(2MG/c^2r)/3! + 2(2MG/c^2r)^2/4! ...)$

$= -(M^2Gc^2/4\pi r^4K^4)(1 + 2(2MG/c^2r)/3! + 2(2MG/c^2r)^2/4! ...)$

Define $Z = (2MG/c^2r)$*. Call the term on the right Y:* $Y = (1 + 2Z/3! + 2Z^2/4! + 2Z^3/5! ...)$

Note: $YZ^2/2 = (Z^2/2 + Z^3/3! + Z^4/4! ...) = K^2-1-Z$

or: $Y = (Z^2/2 + Z^3/3! + Z^4/4! ...) = 2(K^2-1-Z)/Z^2$

Note that: $1 < Y < K^2$ *since each term of Y after 1 is positive but smaller than the corresponding term of* K^2*. Thus* T_{00} *falls in the interval:*

$-(M^2Gc^2/4\pi r^4K^4) >\approx T_{00} \approx> -(M^2Gc^2/4\pi r^4K^2)$

Or by expanding the series in r directly:

$T_{00} = -(M^2Gc^2/4\pi r^4K^4) - (c^6/8\pi Gr^2K^4)((2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + ...)$

K-gravity Stress-Energy Tensor: T_{11}

$(8\pi G/c^4)T_{11} = -U'/rU - (V/r^2)(1+1/V)$

$-U'/rU = -(2MG/c^2r^3)$

$-(V/r^2)(1+1/V)= (K^2-1)/r^2 = 2MG/c^2r^3 + (2MG/c^2r)^2/2!r^2 + (2MG/c^2r)^3/3!r^2 ...$

Sum: $(8\pi G/c^4)T_{11} = -(2MG/c^2r^3)+ (K^2-1)/r^2$

Comparing with the previous result:

$T_{11}/T_{00} = -K^4/c^2 = g_{11}/g_{00}$