

Indistinguishability *right from the start* in standard quantum mechanics

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Abstract

We discuss a reconstruction of standard quantum mechanics assuming indistinguishability right from the start, by appealing to quasi-set theory. After recalling the fundamental aspects of the construction and introducing some improvements in the original formulation, we extract some conclusions for the interpretation of quantum theory.

Key words: Quantum Indistinguishability - Quasi-set theory

1 Introduction

The study of collections of quantum systems has given place to great debates in the philosophy of physics literature. The first remarkable thing that occurs when dealing with compound quantum systems, is that they can be prepared in entangled states [1, 2, 3, 4]. The second remarkable thing – and which is the subject of this work – is that quantum objects seem to be indistinguishable in a way that has no classical analogue [5, 6, 7, 8]. These two aspects of quantum mechanics should not be confused: indistinguishability has to be introduced as an independent axiom of quantum theory, and quantum objects can be prepared in symmetrized states without showing any entanglement at all [9].

The status of identity of quantum systems has been an issue almost since the conception of the theory. Many aspects of the quantum formalism suggest that quantum systems somehow lack identity. In a sense, they seem to be non-individuals. The positions in the literature defer with regard to the degree with which quantum systems depart from a classical notion of identity. Perhaps, the most radical position was that of E. Schrödinger, who claimed that quantum systems were *utterly indistinguishable* [6, 7]. The connections of quantum indistinguishability with the Principle of the Identity of Indiscernibles (PII) has been largely debated too, most

authors arguing that it is somehow violated in the quantum domain (see for example [10, 11]). But other voices appeared, claiming that elementary particles can be, at least, weakly discerned [12, 13, 14, 15], and tried to use that notion as an attempt to save the PII [16]. The status of identity in quantum systems has been also discussed in terms of ontologies based in bundles of properties (see for example [17]). The subject of quantum indistinguishability as a whole gave place to intense debates, and the literature about it is huge (see [8]; see also [18, 19, 20, 21] and references therein). It is important to remark that, for those who want to get stick to a classical ontology, Bohmian mechanics [22] offers an ontology of quantum objects for which – at the metaphysical level – there is no issue with identity: Bohmian particles can be considered individuals that can be identified by their *hidden* trajectories in space-time. But, as we explain in Section 5, even in Bohmian mechanics identity is hidden, and there is no empirical procedure that allows to identify (and re-identify) quantum systems. This is a remarkable feature of quantum phenomena, independently of any interpretation: under certain circumstances, there is no way to identify quantum systems of the same kind. Here, we call *empirical indistinguishability* to this peculiar feature of quantum theory.

Moreover, the symmetrization postulate [23, 24], closely related to the indistinguishability principle, can be used to explain remarkable physical processes. Among them, one can find the Pauli exclusion principle [25], the Bose-Einstein condensation [26], and all phenomena related to quantum statistics in general. Currently, quantum indistinguishability is considered as a resource, and exploited in quantum informational tasks (see for example [27, 28, 29]). In this sense, under the light of these developments and applications, the assumption of quantum indistinguishability is a very powerful feature of quantum theory, that has no classical analogue, and seems to be the right conceptual framework for the working physicist.

Independently of the interpretation chosen or the metaphysical commitments assumed, the symmetrization postulate and the incapability of distinguishing quantum systems must be reflected in the effective part of the theory (i.e., that part of the formalism that connects with experience). At this level, the symmetrization postulate plays a key role, in the sense that it is the mathematical procedure that physicists found in order to give place to predictions that describe quantum statistics correctly and, at the same time, reflect the fact that quantum systems cannot be discerned.

But the symmetrization postulate is implemented by appealing to a trick. First, quantum systems are labeled in order to create a tensor product Hilbert space, as if they were distinguishable. After that initial labeling, quantum states are symmetrized (or anti-symmetrized) in order to obtain the correct states. No trace of the initial labeling can be found in the final version of the formalism. While mathematically correct, this procedure seems to be rather artificial, because the initial labeling plays no real role in the empirical predictions of the theory. Thus, many authors posed the problem of finding a formulation of quantum theory in which indistinguishability is taken right from the start, eliminating the so called surplus mathematical structure of particle labeling [30, 31, 8]. One candidate for solving the “surplus structure” problem, is that of formulating quantum mechanics using the Fock-space formalism (FSF) [32, 33]. But, as observed in [8], the FSF also makes use of particle labeling and symmetrization in order to obtain the correct states. Among the attempts to solve this problem, one can find [34] and [35] (see also [36]).

In this work, we elaborate on the proposal presented in [34] and [35] (see also [37, 38, 39, 40]). We first review the fundamental features of the Fock-space formulation of quantum mechanics and the rudiments of quasi-set theory in Sections 2 and 3. Next, in Section 4, we present the quasi-sets reformulation of standard quantum mechanics in a new math fashion. In Section 5, after introducing the notion of empirical indistinguishability, we show in which sense the assumption of particle labelings and identities has a similar role to that of hidden variables, playing no real role in the empirical predictions of the theory. We argue that quantum indistinguishability is a positive feature of quantum systems that can be treated rigorously by

using the quasi-sets formalism. This, combined with the fact that quantum mechanics can be reformulated by assuming indistinguishability right from the start, favours an eliminativist approach with regard to the notion of classical identity in the interpretation of quantum theory. We draw our conclusions in Section 6.

2 Fock-space formalism

As is well known, the *FSF* can be used as an alternative mathematical framework for non-relativistic quantum mechanics [41]. In order to introduce the basic ideas, in this section we will expose the formalism in the way that is found in most physics textbooks (see for example [42, 41, 43]). A more technical introduction can be found in [44, 45]. In standard quantum mechanics, the Hamiltonian of n identical quantum systems with pairwise interaction, can be written as:

$$H_n = \sum_{i=1}^n \left[\left(-\frac{\hbar^2 \nabla_i^2}{2m} \right) + V_1(\mathbf{x}_i) + \sum_{i>j=1}^n V_2(\mathbf{x}_i, \mathbf{x}_j) \right] \quad (1)$$

where we have assumed that each quantum system is subject to an external potential $V_1(\mathbf{x}_i)$ and the pairwise interaction is represented by $V_2(\mathbf{x}_i, \mathbf{x}_j)$. The wave function

$$\Psi_n(\mathbf{x}_1, \dots, \mathbf{x}_n, t) \quad (2)$$

must be a solution of the Schrödinger's equation

$$H_n \Psi_n = i\hbar \frac{\partial}{\partial t} \Psi_n \quad (3)$$

The second quantization approach to QM has its roots in considering equation (3) as a *classical field equation*, and its solution $\Psi_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$ as a *classical field to be quantized*. This alternative view was originally adopted by Pascual Jordan [46, 47], one of the foundation fathers of quantum mechanics, and spread worldwide after the Dirac's paper [48]. Furthermore, it is a standard way of describing free fields in relativistic quantum mechanics (canonical quantization). The space in which these quantized fields operate is the Fock-space.

The standard Fock-space is built up from the one particle Hilbert spaces. Let \mathcal{H} be a separable Hilbert space and define:

$$\begin{aligned} \mathcal{H}^0 &= \mathbb{C} \\ \mathcal{H}^1 &= \mathcal{H} \\ \mathcal{H}^2 &= \mathcal{H} \otimes \mathcal{H} \\ &\vdots \\ \mathcal{H}^n &= \mathcal{H} \otimes \dots \otimes \mathcal{H} \end{aligned} \quad (4)$$

The Fock-space is thus constructed as the direct sum of n particle Hilbert spaces:

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n \quad (5)$$

When dealing with bosons or fermions, the symmetrization postulate (SP) must be imposed. In order to do so, let S_n be the group of permutations of the set $\{1, 2, 3, \dots, n\}$. Given any vector $\eta = \eta_1 \otimes \dots \otimes \eta_n \in \mathcal{H}^n$ and $P \in S_n$, let $P(\eta_1 \otimes \dots \otimes \eta_n) := \eta_{P(1)} \otimes \eta_{P(2)} \otimes \dots \otimes \eta_{P(n)}$. Next, define:

$$\sigma^n(\eta) = \frac{1}{n!} \sum_{P \in S_n} P(\eta_1 \otimes \dots \otimes \eta_n) \quad (6)$$

and:

$$\tau^n(\eta) = \frac{1}{n!} \sum_{P \in S_n} s^P P(\eta_1 \otimes \dots \otimes \eta_n) \quad (7)$$

where:

$$s^P = \begin{cases} 1 & \text{if } P \text{ is even,} \\ -1 & \text{if } P \text{ is odd.} \end{cases}$$

Calling

$$\mathcal{H}_\sigma^n = \{\sigma^n(\eta) \mid \eta \in \mathcal{H}^n\} \quad (8)$$

and:

$$\mathcal{H}_\tau^n = \{\tau^n(\eta) \mid \eta \in \mathcal{H}^n\} \quad (9)$$

we have the Fock-space

$$\mathcal{F}^+(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_\sigma^n \quad (10)$$

for Bosons and

$$\mathcal{F}^-(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_\tau^n \quad (11)$$

for Fermions. In what follows, in order to make the exposition clearer, we use the Dirac ket-bra notation to denote the vectors in $\mathcal{F}(\mathcal{H})$, $\mathcal{F}^+(\mathcal{H})$ and $\mathcal{F}^-(\mathcal{H})$. The standard second quantization procedure considers the one particle wave function $\psi(\mathbf{r}, t)$ and its hermitian conjugate $\psi(\mathbf{x}, t)^\dagger$ as operators acting on the Fock-space and satisfying [43]:

$$\begin{aligned} [\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)]_{\mp} &= 0 \\ [\psi(\mathbf{x}, t)^\dagger, \psi(\mathbf{x}', t)^\dagger]_{\mp} &= 0 \\ [\psi(\mathbf{x}, t), \psi(\mathbf{x}', t)^\dagger]_{\mp} &= \delta_{(\mathbf{x}-\mathbf{x}')} \end{aligned} \quad (12)$$

where $\delta(\mathbf{x} - \mathbf{x}')$ is the Dirac delta function. If A and B are operators, the brackets are defined by $[A, B]_{\mp} = AB \mp BA$ (where the “+” stands for Fermions and “−” for Bosons). The n particle wave function $\Psi_n(\mathbf{x}_1, \dots, \mathbf{x}_n)$ of the standard formulation is now written as

$$|\psi_n\rangle = (n!)^{-\frac{1}{2}} \int d^3x_1 \cdots \int d^3x_n \psi(\mathbf{x}_1)^\dagger \cdots \psi(\mathbf{x}_n)^\dagger |0\rangle \Psi_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \quad (13)$$

and it is an eigenvector (with eigenvalue n) of the particle number operator:

$$N := \int d^3x \psi(\mathbf{x})^\dagger \psi(\mathbf{x}). \quad (14)$$

We also have the relation

$$\Psi_n(\mathbf{x}_1, \dots, \mathbf{x}_n) = (n!)^{-\frac{1}{2}} \langle 0 | \psi(\mathbf{x}_1) \cdots \psi(\mathbf{x}_n) | \Psi_n \rangle \quad (15)$$

An arbitrary vector of the Fock-space will be a superposition of states with different particle number of the form

$$|\Psi\rangle = \sum_{n=0}^{\infty} \alpha_n |\Psi_n\rangle \quad (16)$$

and will not be in general an eigen-state of the particle number operator. Thus, according to the standard interpretation of QM, its particle number will be undetermined. This is very important, because in the presence of particle interactions, the states may evolve into an undefined particle state like (16) [43].

It is customary to make a Fourier decomposition of the operators $\psi(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)^\dagger$:

$$\psi(\mathbf{x}, t) = \sum_i u_i(\mathbf{x}) a_i(t) \quad (17a)$$

$$\psi(\mathbf{x}, t)^\dagger = \sum_i u_i(\mathbf{x}) a_i^\dagger(t) \quad (17b)$$

where the complex functions $u_i(\mathbf{x})$ are assumed to form a complete and orthonormal set. The operators $a_i^\dagger(t)$ and $a_i(t)$ acquire a simple interpretation when the Hamiltonian is time independent. In that case, it is useful to take the u_i 's as the eigen-functions of the stationary Schrödinger equation for one particle:

$$\left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{x}) \right) u_i(\mathbf{x}) = \epsilon_i u_i(\mathbf{x}) \quad (18)$$

By doing so, it is possible to write $a_i(t) = \exp\left(-\frac{i\epsilon_i t}{\hbar}\right) a_i$. The operators a_i^\dagger and a_i are called creation and annihilation operators and satisfy:

$$\begin{aligned} [a_i, a_j]_{\mp} &= \mathbf{0} \\ [a_i^\dagger, a_j^\dagger]_{\mp} &= \mathbf{0} \\ [a_i, a_j^\dagger]_{\mp} &= \delta_{ij} \end{aligned} \quad (19)$$

where the “−” and “+” signs stand for Bosons and Fermions, respectively.

3 Quasi-Set Theory

In Zermelo-Frankel (ZF) set theory, given an object a , we can always define the *singleton* $A = \{a\}$. Any other object b of the theory belongs to A , if and only if, it satisfies $b = a$. This simple task of identification can be carried out, due to the fact that identity is a primitive notion in the first order language in which the theory is formulated. Quasi-set theory (\mathfrak{Q} from now on) is a set-theoretical framework that can deal with collections of truly indiscernible objects [49, 50, 51, 8, 31]. In this context, “truly indiscernible” means that the theory is formulated in a first order language without identity (and then, \mathfrak{Q} is an example of a *non-reflexive logic*). Thus, the most simple identification tasks that are granted in ZF, are no longer available in \mathfrak{Q} . Instead of identity, an equivalence indistinguishability relation “ \equiv ” is postulated. The idea behind the axioms is that quasi-sets can be used to represent collections of quantum systems (possibly of the same kind, and thus, indiscernible). In this framework, indiscernibility does not implies identity: two elements of the theory might be indiscernible (in symbols $a \equiv b$), while not being identical. They fail to be identical in the following sense: the first order theory of identity is not valid for all objects of the theory. In the rest of this section, we discuss the basic formalism of quasi-set theory.

In order to mimic elementary particles, the objects of the theory are divided into two main groups: m -objects (these are ‘micro-objects’, intended to represent quantum systems) and M -objects (‘macro-objects’, introduced to represent classical objects). The m -objects are introduced as ur-elements in a standard way, while quasi-sets are collections of them (or collections of quasi-sets).

A derived notion of identity “ $=_E$ ” is introduced in the theory ($x =_E y$ is read ‘ x and y are extensionally identical’), iff they are both quasi-sets having the same elements (that is, $\forall z(z \in x \leftrightarrow z \in y)$), or they are both M -atoms and belong to the same quasi-sets (that is, $\forall z(x \in z \leftrightarrow y \in z)$). Due to the axioms of the theory, when applied to M -atoms (or collections of them), the relation of indiscernibility collapses into that of extensional identity and has the usual properties of the standard identity of ZFU (ZF with Urelemente). In this way, a copy of ZFU can then be constructed inside \mathfrak{Q} .

A quasi-cardinal can be assigned to quasi-sets, and it is intended to represent how many elements they have. Using the primitive notion of quasi-cardinal, an analogous of the axiom of weak extensionality is postulated in \mathfrak{Q} . It states that those quasi-sets that have the same quantity of elements of the same sort, are indistinguishable. It is important to remark that collections of m -atoms do not have an associated ordinal. The elements of a quasi-set formed by m -objects cannot be identified by names, nor counted, nor ordered. This is the reason why the notion of quasi-cardinal is postulated. In [52], the quasi-cardinal is treated as a derived notion (see also [38]). The problem of describing quantum systems with undefined particle number is discussed in [52, 53, 54].

In ZF, if $w \in x$, it is easy to show that $(x - \{w\}) \cup \{z\} = x$ if and only if $z = w$. The replacement of an element of a set by a different element, gives place to a different set. What happens if we try to make an analogous substitutions to a quasi-set formed by m -atoms? Let $[[z]]$ denote the quasi-set with quasi-cardinal 1 whose only element is an indistinguishable from z (this is called the strong singleton of z). Suppose that x is a finite quasi-set and that z is an m -atom such that $z \in x$. Given $w \equiv z$ and $w \notin x$, then, it is possible to show that $(x - [[z]]) \cup [[w]] \equiv x$. This can be clearly interpreted as follows: the permutation of indiscernible elements gives place to indiscernible collections.

Given x and y , it is possible to form $[x]$ and $[x, y]$, which are the collections of all indiscernibles from x and from x and y , respectively. Quasi-pairs can be built in the usual way as: $\langle x, y \rangle := [[x], [x, y]]$. A quasi-function f is a quasi-set formed by quasi-pairs in such a way that if $x \equiv z$ and $[[x], [x, y]]$ and $[[z], [z, w]]$ belong to f , we have $y \equiv w$.

4 The Ω -space

In this section we describe how to obtain a mathematical formalism based in Ω which is equivalent to the FSF described in Section 2. In this way, we provide a reformulation of standard quantum mechanics that uses quantum indistinguishability as a starting point, eliminating the surplus mathematical structure discussed in [33, 32]. We follow an analogous approach to the one given in [34, 35, 37, 38], but introducing technical improvements in the formulation. We use the axioms and definitions of quasi-set theory introduced in [55], with minor modifications.

Using the copy of ZFU in Ω , we start by considering a collection $\mathcal{E} = \{\epsilon_i\}_{i \in \mathbb{N}}$, where \mathbb{N} are the natural numbers. The ϵ_i 's are intended to represent outcomes of a maximal observable, and are thus distinguishable in the classical sense. This is the reason why we describe \mathcal{E} using the classical part of Ω . To fix ideas, the reader can consider its elements as the possible values of the energy of a single quantum system (assumed to have *discrete* spectrum). But it is important to keep in mind that the ϵ_i 's could be real numbers or just symbols used to distinguish the different outcomes of an experiment. The only important thing about \mathcal{E} is that it is a denumerable collection of distinguishable items (so we could have taken the natural numbers instead of \mathcal{E}). The constraint in the cardinality of \mathcal{E} implies the *separability* (i.e., that it admits a denumerable basis) of the Hilbert space that we construct, as the rigorous formulation of standard quantum mechanics requires. The fact that \mathcal{E} is denumerable and its elements distinguishable, also implies that it can be ordered. In the following, we choose a concrete order for its elements (given by $\epsilon_i < \epsilon_j$, whenever $i < j$).

We want to make sense of expressions such as “a quantum system has energy ϵ_i ” or “there are n quanta in the energy level ϵ_i ”, “there are n_i quanta in the energy level ϵ_i and n_j quanta in the energy level ϵ_j ”, and so on. For this aim, we use the non-classical part of Ω as follows. First, consider the quasi-set FIN_Ω formed by all possible finite and pure quasi-sets (with all the m -atoms of the same type). The existence of FIN_Ω can be granted as follows. Assume that there exists a quasi-set ω^λ whose quasi-cardinal is \aleph_0 (the smallest infinite cardinal number), representing the collection of all possible m -atoms of type λ . It can be considered as an infinite and abstract reservoir a type of quantum system collectively characterized by specifying their charge, mass, spin, etc., represented by λ . By applying the axiom of parts, consider $\mathcal{P}(\omega^\lambda)$, the quasi-set formed by its subsets. Now, apply the separation schema to obtain $FIN_\Omega := \{x \in \mathcal{P}(\omega^\lambda) \mid qcard(x) < \aleph_0\}$. Consider next the quasi-set \mathcal{F} formed by all possible quasi-functions f such that $f : \mathcal{E} \rightarrow FIN_\Omega$, and whenever $\langle \epsilon_{i_k}; x \rangle$ and $\langle \epsilon_{i_{k'}}; y \rangle$ belong to f and $k \neq k'$, then $x \cap y = \emptyset$. We also assume that the sum of the quasi-cardinals of the quasi-sets which appear in the image of each of these quasi-functions is finite. This means that $qcard(x) = 0$ for every x in the image of f , except for a finite number of elements of \mathcal{E} . Denote by \mathcal{E}_f to the collection of indexes for which f assigns an x such that $qcard(x) \neq 0$. There is no a priori order in \mathcal{E}_f , but its elements can be ordered, given that they belong to the classical part of Ω . Each f is a quasi-set formed by ordered pairs $\langle \epsilon_i; x \rangle$ with $\epsilon_i \in \mathcal{E}$ and $x \in FIN_\Omega$. Each order pair $\langle \epsilon_i; x \rangle$ represents the proposition “the quantum number ϵ_i has occupation number $qcard(x)$ ” or, equivalently, there are “ $qcard(x)$ quanta in the energy level ϵ_i ”. Thus, a quasi-function $f : \mathcal{E} \rightarrow FIN_\Omega$ can be interpreted as “there are $qcard(f(\epsilon_1))$ quanta in energy level ϵ_1 , $qcard(f(\epsilon_2))$ quanta in energy level ϵ_2 , $qcard(f(\epsilon_3))$ quanta in energy level ϵ_3 ...”, in such a way that $\sum_i qcard(f(\epsilon_i))$ is a finite number.

As discussed in Section 3, Ω is constructed in such a way that the permutation of indiscernible elements gives place to indiscernible collections. Given that these quasi-functions described above are constructed using the non-classical part of Ω , the permutation of quanta has no effect: the result of interchanging a quanta taken from a quasi-set associated to ϵ_i , with another one taken from the quasi-set associated to ϵ_j , gives place to the same quasi-function (see also

the discussion in [34]). In this way, the particle permutation operator of the standard formalism loses its meaning here. Each $f \in \mathcal{F}$ is characterized by the set \mathcal{E}_f and the “occupation numbers” associated to each $\epsilon_i \in \mathcal{E}_f$ (given by $qcard(f(\epsilon_i))$). There is no identification, nor ordering, nor labeling of quanta in this description, because there is no identity in the underlying logic of Ω for the m -atoms and their collections. In this way, a quasi-function $f \in \mathcal{F}$, faithfully represents a proposition such as “there are $qcard(f(\epsilon_1))$ quanta in energy level ϵ_1 , $qcard(f(\epsilon_2))$ quanta in energy level ϵ_2 , $qcard(f(\epsilon_3))$ quanta in energy level ϵ_3 ...”, without appealing to any labelling of the quanta involved.

Now, we proceed to associate a complex vector space structure to \mathcal{F} . This is a first important step if we aim to recover a formalism equivalent to that based in Fock spaces. In order to do that, let us recall a useful construction from algebra, [56, 57]. The idea of this construction is to assign an algebraic structure (a vector space, a commutative algebra, a non commutative algebra, etc.) to a given set, in such a way that it *generates* the desired structure. To fix ideas, let us consider first some examples (we work the examples over the complex numbers \mathbb{C} and using standard set theory, but notice that we can operate in a totally analogous way in Ω). Suppose that we have a set with one element, $S = \{x\}$, and we want to construct the commutative algebra A generated by S . Then, A should contain expressions such as “1”, “ $3ix$ ”, “ x^2 ”, “ $1 + ix - x^3$ ”, and so on. If A is a \mathbb{C} -algebra, then it must be equal to the polynomial algebra in one variable $\mathbb{C}[x]$. This construction is called the *free algebra generated by S* . The name “free”, comes from the fact that we are not imposing any *relation* among the generators in S . Another example that we can make is that of the free associative algebra B generated by S . If $S = \{x, y\}$, then $B = \mathbb{C}\langle x, y \rangle$ (the notation is standard). So, for example, $xy \neq yx$ in B . The last example that we want to address is that of a complex vector space V_S generated by $S = \{x, y\}$. In this case, in V we have expressions such as $(3 + 2i)x + 5iy$ or $x - y$, but the expressions x^2 or xy are not allowed. In fact, x and y become linearly independent vectors in V . Thus, if we have a set $S = \{x_i\}$, then we can construct the vector space generated by S by making formal linear combinations of its elements. By doing so, the elements of S become linearly independent. In order to construct the vector space V from the set S , let us denote it V_S , we can take the set of functions from S to \mathbb{C} ,

$$V_S = Fun(S, \mathbb{C}).$$

It is straightforward to check that V_S is a \mathbb{C} -vector space and the indicator functions $\iota(s)$ are the canonical basis for V_S . In fact, we can immerse S inside V_S , $\iota : S \rightarrow V_S$, by assigning to $s \in S$ the indicator function $\iota(s)$. Notice that with this construction we can obtain the commutative algebra A and the non-commutative algebra B . For example, given a set S , we can define the set of commutative monomials S' and the set of non-commutative monomials S'' . Then, A is the vector space generated by S' and B by S'' .

The above described algebraic techniques can be applied *mutatis mutandis* in Ω , in order to generate a complex vector space $V_{\mathcal{F}}$ using \mathcal{F} . Every function $f \in \mathcal{F}$ will have a copy in $V_{\mathcal{F}}$. This means that we will have a quasi-function $\iota : \mathcal{F} \rightarrow V_{\mathcal{F}}$, such that $\iota(f)$ is the copy of f in $V_{\mathcal{F}}$. Thus, given $f_1, f_2, \dots, f_n \in \mathcal{F}$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$, we can form expressions such as

$$\alpha_1 \iota(f_1) + \alpha_2 \iota(f_2) + \dots + \alpha_n \iota(f_n) \tag{20}$$

that can be interpreted as a linear combination of the quasi-functions f_i . Thus, we can formally express superposition states using Ω . We will denote the elements of $V_{\mathcal{F}}$ using Greek letters. Thus, any $\psi \in V_{\mathcal{F}}$ is a linear combination of the form (20).

The second step that we need to give in order to recover the FSF, is to endow the vector space $V_{\mathcal{F}}$ with a scalar product. This can be done in different ways (see for example [34]). Here, we notice that each $f \in \mathcal{F}$ has associated a collection of indexes \mathcal{E}_f . Given $f, g \in \mathcal{F}$, they can only differ in the content of the sets \mathcal{E}_f and \mathcal{E}_g , and in the number of quantum objects (the

occupation number) associated to each value of the observable. Thus, any physically meaningful scalar product between f and g , should depend only on \mathcal{E}_f , \mathcal{E}_g , and the respective occupation numbers. Taking into account that, for the case of indistinguishable quanta, the order of indexes is not relevant, there is no preferred order on \mathcal{E}_f . But it can be ordered, because its elements, representing outcomes of experiments, are distinguishable in the classical sense. We can exploit this to define a scalar product without appealing to a any kind of particle labeling. With this aim, let us first recall some useful notions of multilinear algebra. Let V be a vector space (possibly infinite-dimensional). Consider the Tensor Algebra $T(V)$, the Symmetric Algebra $S(V)$ and the Exterior Algebra $\Lambda(V)$ associated to V . The elements of the $T(V)$ are finite sums of non-commutative monomials $v_1 \otimes \dots \otimes v_n$ (called non-commutative polynomials), elements of $S(V)$ are finite sums of commutative monomials $v_1 \dots v_n$ (called polynomials), and elements of $\Lambda(V)$ are finite sums of skew-symmetric monomials $v_1 \wedge \dots \wedge v_n$ (called forms). For example, if v_1 is linearly independent from v_2 , then $v_1 \otimes v_2 - v_2 \otimes v_1 \neq 0$ in $T(V)$, $v_1 v_2 - v_2 v_1 = 0$ in $S(V)$ and $v_1 \wedge v_2 + v_2 \wedge v_1 = 0$ in $\Lambda(V)$. It is important to mention that we can combine constructions from algebra to achieve the same result. For example, if S is a set, we can take the vector space V_S generated by S , and then we can take the symmetric algebra $S(V_S)$. Alternatively, we can take the commutative algebra generated by S . It is a standard result from algebra that both constructions agree. Also, we can take the tensor algebra $T(V_S)$ and the non-commutative algebra generated by S . Again, both constructions agree. For more on these constructions, see in [56, Ch. III].

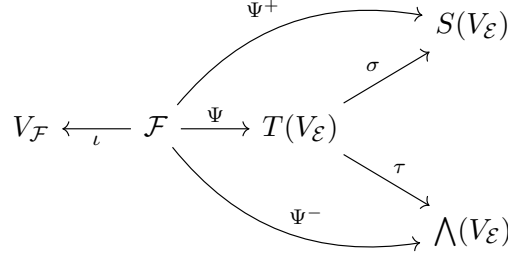
Notice the analogy between $T(V)$, $S(V)$ and $\Lambda(V)$ with respect to the spaces $\mathcal{F}(\mathcal{H})$, $\mathcal{F}_+(\mathcal{H})$ and $\mathcal{F}_-(\mathcal{H})$, introduced in section 2. But the analogy should not lead to confusion. In order to induce a scalar product in $\mathcal{V}_{\mathcal{F}}$, we will consider formal expressions formed by the distinguishable outcomes of an abstract observable, without appealing to any labeling of the quantum objects involved. Thus, consider the vector space $V_{\mathcal{E}}$ freely generated by \mathcal{E} (this means that the symbols ϵ_k and $\epsilon_{k'}$, are now considered linearly independent vectors whenever $k \neq k'$). First, notice that for each $f \in \mathcal{F}$, we have an element of $V_{\mathcal{F}}$ (denote this injection by $\iota : \mathcal{F} \rightarrow V_{\mathcal{F}}$). For each pair of these copies, we want to define a product. To do so, assign to each quasi-function $f \in \mathcal{F}$ a non-commutative monomial in $T(V_{\mathcal{E}})$ as follows. Specifically, given $f \in \mathcal{F}$, consider the elements of \mathcal{E}_f together with their respective multiplicities (given by $qcard(f(\epsilon_k))$, with $\epsilon_k \in \mathcal{E}_f$). Define the map $\Psi : \mathcal{F} \rightarrow T(V_{\mathcal{E}})$, that assigns to f the non-commutative monomial $\epsilon_{i_1} \otimes \epsilon_{i_2} \otimes \dots \otimes \epsilon_{i_m} \in T(V_{\mathcal{E}})$, where ϵ_i appears k times if $qcard(f(\epsilon_i)) = k$, and the order of the ϵ_{i_k} 's is chosen in such a way that ϵ_{i_k} appears to the left of ϵ_{i_j} , whenever $i_k < i_j$. For example, if for a given f we have $\mathcal{E}_f = \{\epsilon_2, \epsilon_4, \epsilon_5\}$, with $qcard(f(\epsilon_2)) = 3$, $qcard(f(\epsilon_3)) = 1$ and $qcard(f(\epsilon_5)) = 4$, and the order assigned is $\epsilon_2 < \epsilon_3 < \epsilon_5$, the monomial assigned to f would be: $\epsilon_2 \otimes \epsilon_2 \otimes \epsilon_2 \otimes \epsilon_3 \otimes \epsilon_5 \otimes \epsilon_5 \otimes \epsilon_5 \otimes \epsilon_5$.

An important aspect of our map $\Psi : \mathcal{F} \rightarrow T(V)$ is that it can be used to define a linear map $\tilde{\Psi} : V_{\mathcal{F}} \rightarrow T(V)$, preserving the structure of addition and scalar multiplication. Specifically, it sends the expression $\lambda_1 \iota(f_1) + \lambda_2 \iota(f_2) + \dots + \lambda_n \iota(f_n)$ to the non-commutative polynomial $\lambda_1 \tilde{\Psi}(\iota(f_1)) + \lambda_2 \tilde{\Psi}(\iota(f_2)) + \dots + \lambda_n \tilde{\Psi}(\iota(f_n)) \in T(V_{\mathcal{E}})$. The above maps can be depicted as:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\Psi} & T(V_{\mathcal{E}}) \\ \downarrow \iota & \nearrow \tilde{\Psi} & \\ V_{\mathcal{F}} & & \end{array}$$

Using these constructions, we can induce a scalar product on $V_{\mathcal{F}}$. Recall that if V has a scalar product, then $T(V)$, $S(V)$ and $\Lambda(V)$ also inherit a scalar product, [56]. Hence, instead

of taking $V_{\mathcal{E}}$ as the vector space freely generated by \mathcal{E} , we can take $V_{\mathcal{E}}$ as the vector space with scalar product freely generated by \mathcal{E} , that is, $\epsilon_k \perp \epsilon_{k'}$ if $k \neq k'$ and $\|\epsilon_k\| = 1$. Then, the map Ψ assigns to each quasi-function $f \in \mathcal{F}$ a monomial in $T(V_{\mathcal{E}})$. Also, recall that we always have canonical maps $\sigma : T(V_{\mathcal{E}}) \rightarrow S(V_{\mathcal{E}})$ and $\tau : T(V_{\mathcal{E}}) \rightarrow \Lambda(V_{\mathcal{E}})$ called symmetrization and anti-symmetrization respectively. Hence, we can assign to each f a polynomial or a form, by applying the compositions $\Psi^+ := \sigma \circ \Psi$ and $\Psi^- := \tau \circ \Psi$, respectively:



With these tools, we can define:

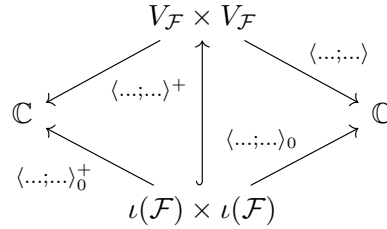
$$\langle \iota(f); \iota(g) \rangle_0 := \langle \Psi(f); \Psi(g) \rangle \quad \forall f, g \in \mathcal{F} \quad (21)$$

$$\langle \iota(f); \iota(g) \rangle_0^+ := \langle \Psi^+(f); \Psi^+(g) \rangle \quad \forall f, g \in \mathcal{F} \quad (22)$$

$$\langle \iota(f); \iota(g) \rangle_0^- := \langle \Psi^-(f); \Psi^-(g) \rangle \quad \forall f, g \in \mathcal{F} \quad (23)$$

where, in the right hand side of the above equations, we are using the scalar products induced in $T(V_{\mathcal{E}})$, $S(V_{\mathcal{E}})$ and $\Lambda(V_{\mathcal{E}})$, respectively, induced by the vector space with scalar product $V_{\mathcal{E}}$ freely generated by \mathcal{E} .

Clearly, 21 can be extended linearly to define a complex scalar product $\langle \dots; \dots \rangle : V_{\mathcal{F}} \times V_{\mathcal{F}} \rightarrow \mathbb{C}$. The completion of $V_{\mathcal{F}}$ with respect to this product gives place to a separable Hilbert space completely equivalent to $\mathcal{F}(\mathcal{H})$. The case of 22 is completely analogous to 21, and it can be used to define a scalar product $\langle \dots; \dots \rangle^+ : V_{\mathcal{F}} \times V_{\mathcal{F}} \rightarrow \mathbb{C}$, yielding a space completely equivalent to $\mathcal{F}^+(\mathcal{H})$. Some care must be taken with regard to 23: all the $f \in \mathcal{F}$ satisfying that $qcard(f(\epsilon_i)) \geq 2$, for at least one i , by construction, will have “null norm” in the following sense: $\langle \iota(f), \iota(f) \rangle^- = 0$. Thus, in order to recover a space equivalent to $\mathcal{F}^-(\mathcal{H})$, let us proceed as follows. First, extend 23 bilinearly to all $V_{\mathcal{F}}$. Next, take the quotient space $V_{\mathcal{F}} / \sim$, with respect to the equivalence relation $\psi \sim \phi$, iff $\langle \psi - \phi; \psi - \phi \rangle = 0$, for all $\psi, \phi \in V_{\mathcal{F}}$ (i.e., $\psi \sim \phi$ iff their difference has “null norm”). With regard to the equivalence classes, 23 can be used to define a complex scalar product $\langle \dots; \dots \rangle^- : V_{\mathcal{F}} / \sim \times V_{\mathcal{F}} / \sim \rightarrow \mathbb{C}$ on $V_{\mathcal{F}} / \sim$, whose closure is formally equivalent to $\mathcal{F}^-(\mathcal{H})$. For the non-symmetrized and symmetric products, we have the following diagram:



while for the anti-symmetric product, we have:

$$\begin{array}{ccc}
V_{\mathcal{F}}/\sim \times V_{\mathcal{F}}/\sim & & \\
\uparrow & \searrow \langle \dots; \dots \rangle^- & \\
\mathbb{C} & & \\
\downarrow & \nearrow \langle \dots; \dots \rangle_0^- & \\
\iota(\mathcal{F}) \times \iota(\mathcal{F}) & &
\end{array}$$

Thus, we have shown how to build, using quasi-set theory, spaces which are equivalent to $\mathcal{F}(\mathcal{H})$, $\mathcal{F}^+(\mathcal{H})$ and $\mathcal{F}^-(\mathcal{H})$. This is done by defining different scalar products on $V_{\mathcal{F}}$, in order to represent distinguishable quanta (understood as “quanta of different kinds”), Bosons or Fermions. The desired spaces are obtained by taking the closures with respect to the norms induced by the scalar products of the pre-Hilbert spaces $\langle V_{\mathcal{F}}, \langle \dots; \dots \rangle \rangle$, $\langle V_{\mathcal{F}}, \langle \dots; \dots \rangle^+ \rangle$ and $\langle V_{\mathcal{F}}/\sim, \langle \dots; \dots \rangle^- \rangle$, respectively. Once this point is reached, all constructions presented in Section 2 can be reproduced in $V_{\mathcal{F}}$ as usual, by choosing the right scalar products and defining suitable creation and annihilation operators (see [34]). In this way, we obtain a formulation of standard quantum mechanics by appealing to quasi-set theory, that incorporates indistinguishability right from the start.

We now describe the action of creation and annihilation operators for systems of Bosons and Fermions. Let us start describing Bosons. For this case, we describe the quasi-functions constructed above by explicitly indicating their support and occupation numbers. Thus, if $f \in \mathcal{F}$ and $\mathcal{E}_f = \{\epsilon_{i_1}, \epsilon_{i_2}, \epsilon_{i_3}, \dots, \epsilon_{i_n}\}$, we denote its copy in $V_{\mathcal{F}}$ by $\iota(f) := f_{n_{i_1} n_{i_2} n_{i_3} \dots n_{i_n}}$, where $n_{i_k} = \text{qcard}(f(\epsilon_{i_k}))$. It is important to remark again that the order of the indexes has no importance in $f_{n_{i_1} n_{i_2} n_{i_3} \dots n_{i_m}}$: it is just a notation that remind us the support of a given element of $V_{\mathcal{F}}$. We could have written $f_{n_{i_2} n_{i_1} n_{i_3} \dots n_{i_m}}$ instead of $f_{n_{i_1} n_{i_2} \epsilon_{i_3} \dots n_{i_m}}$, but the order has no physical meaning in the definition of these quasi-functions. We also write $f_{\emptyset} \in V_{\mathcal{F}}$ to denote the quasi-function whose $\mathcal{E}_f = \emptyset$ (or, equivalently, $\text{qcard}(f_{\emptyset}(\epsilon_i)) = 0$ for all i), which is intended to represent the ground state of the system under study (i.e., a state in which all occupation numbers are zero). Let us define the operator that creates a Boson in state ϵ_k by:

$$\mathbf{a}_{\epsilon_k}^{\dagger} f_{n_{i_1} n_{i_2} \dots n_{i_m}} = \sqrt{n_k + 1} f_{n_{i_1} n_{i_2} \dots (n_k+1) \dots n_{i_m}} \quad (24)$$

With the above normalization, its adjoint operator satisfies:

$$\mathbf{a}_{\epsilon_k} f_{n_{i_1} n_{i_2} \dots n_{i_m}} = \sqrt{n_k} f_{n_{i_1} n_{i_2} \dots (n_k-1) \dots n_{i_m}} \quad (25)$$

Now, consider the action of two creation operators (with $k < l$):

$$\mathbf{a}_{\epsilon_k}^{\dagger} \mathbf{a}_{\epsilon_l}^{\dagger} f_{n_{i_1} n_{i_2} \dots n_{i_m}} = \sqrt{n_k + 1} \sqrt{n_l + 1} f_{n_{i_1} n_{i_2} \dots (n_k+1) \dots (n_l+1) \dots n_{i_m}} \quad (26)$$

and

$$\mathbf{a}_{\epsilon_l}^{\dagger} \mathbf{a}_{\epsilon_k}^{\dagger} f_{n_{i_1} n_{i_2} \dots n_{i_m}} = \sqrt{n_k + 1} \sqrt{n_l + 1} f_{n_{i_1} n_{i_2} \dots (n_k+1) \dots (n_l+1) \dots n_{i_m}} \quad (27)$$

Subtracting equations 26 and 28, we obtain:

$$(\mathbf{a}_{\epsilon_k}^\dagger \mathbf{a}_{\epsilon_l}^\dagger - \mathbf{a}_{\epsilon_l}^\dagger \mathbf{a}_{\epsilon_k}^\dagger) f_{n_{i_1} n_{i_2} \dots n_{i_n}} = \quad (28)$$

$$= (\sqrt{n_k + 1} \sqrt{n_l + 1}) \left(f_{n_{i_1} n_{i_2} \dots (n_k+1) \dots (n_l+1) \dots n_{i_m}} - f_{n_{i_1} n_{i_2} \dots (n_k+1) \dots (n_l+1) \dots n_{i_m}} \right) = \mathbf{0} \quad (29)$$

A similar equation holds for $k = l$. Thus, since $f_{n_{i_1} n_{i_2} \dots n_{i_m}}$ is arbitrary, we obtain:

$$\mathbf{a}_{\epsilon_k}^\dagger \mathbf{a}_{\epsilon_l}^\dagger - \mathbf{a}_{\epsilon_l}^\dagger \mathbf{a}_{\epsilon_k}^\dagger = \mathbf{0} \quad (30)$$

Similarly, it is easy to obtain:

$$\mathbf{a}_{\epsilon_k} \mathbf{a}_{\epsilon_l} - \mathbf{a}_{\epsilon_l} \mathbf{a}_{\epsilon_k} = \mathbf{0} \quad (31)$$

Finally, let us compute the action of $\mathbf{a}_{\epsilon_k} \mathbf{a}_{\epsilon_l}^\dagger$:

$$\mathbf{a}_{\epsilon_k} \mathbf{a}_{\epsilon_l}^\dagger f_{n_{i_1} n_{i_2} \dots n_{i_n}} = \sqrt{n_k - 1} \sqrt{n_l + 1} f_{n_{i_1} n_{i_2} \dots (n_k-1) \dots (n_l+1) \dots n_{i_m}} \quad (32)$$

For $\mathbf{a}_{\epsilon_l}^\dagger \mathbf{a}_{\epsilon_k}$:

$$\mathbf{a}_{\epsilon_l}^\dagger \mathbf{a}_{\epsilon_k} f_{n_{i_1} n_{i_2} \dots n_{i_n}} = \sqrt{n_l + 1} \sqrt{n_k - 1} f_{n_{i_1} n_{i_2} \dots (n_k-1) \dots (n_l+1) \dots n_{i_m}} \quad (33)$$

Subtracting Equations and , we obtain:

$$\mathbf{a}_{\epsilon_k} \mathbf{a}_{\epsilon_l}^\dagger - \mathbf{a}_{\epsilon_l}^\dagger \mathbf{a}_{\epsilon_k} = \delta_{\epsilon_k \epsilon_l} \quad (34)$$

The above equations defined the desired commuting relations. The application of the change of basis operators [43] on the ground state f_\emptyset gives

$$f_{\mathbf{xy}} := \frac{1}{\sqrt{2!}} \Psi(\mathbf{y})^\dagger \Psi(\mathbf{x})^\dagger f_\emptyset = \frac{1}{\sqrt{2}} \sum_i \sum_j u_i(\mathbf{x}) u_j(\mathbf{y}) \mathbf{a}_{\epsilon_i}^\dagger \mathbf{a}_{\epsilon_i}^\dagger f_\emptyset = \quad (35)$$

$$= \frac{1}{\sqrt{2}} \sum_i \sum_j u_i(\mathbf{x}) u_j(\mathbf{y}) f_{\epsilon_i \epsilon_j} = \frac{1}{\sqrt{2}} \sum_{i < j} [u_i(\mathbf{x}) u_j(\mathbf{y}) + u_j(\mathbf{x}) u_i(\mathbf{y})] f_{\epsilon_i \epsilon_j} \quad (36)$$

Computing the scalar product between $f_{\epsilon_j \epsilon_k}$ and $f_{\mathbf{xy}}$

$$\langle f_{\epsilon_j \epsilon_k}; f_{\mathbf{xy}} \rangle^+ = \frac{1}{\sqrt{2}} [u_i(\mathbf{x}) u_j(\mathbf{y}) + u_j(\mathbf{x}) u_i(\mathbf{y})] \quad (37)$$

we obtain the right probability amplitudes for two particle Bosonic systems. A similar result holds for $f_{\mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_N} := \frac{1}{\sqrt{N!}} \Psi(\mathbf{x}_N) \dots \Psi(\mathbf{x}_1)^\dagger \Psi(\mathbf{x}_1)^\dagger f_\emptyset$

For Fermions, let us focus on the space $\langle V_{\mathcal{F}} / \sim, \langle \dots; \dots \rangle^- \rangle$. A function $f \in V_{\mathcal{F}} / \sim$ is characterized again by the set \mathcal{E}_f and its occupation numbers. But given that we are in the quotient space, only quasi-functions whose norm is non-null appear, and then, the occupation numbers are never greater than one. Thus, we can use a notation in which only the occupied levels are displayed. Thus, for $f \in V_{\mathcal{F}} / \sim$ such that $\mathcal{E}_f = \{\epsilon_{i_1}, \epsilon_{i_2}, \dots, \epsilon_{i_n}\}$, we use the notation $f_{\epsilon_{i_1}, \epsilon_{i_2}, \dots, \epsilon_{i_n}}$. As remarked above, while order of quantum systems has no meaning in f , the set \mathcal{E}_f can be ordered (and again, we use the order $\epsilon_i < \epsilon_j$, iff $i < j$). Given $\epsilon_k \notin \mathcal{E}_f$, define s_{f, ϵ_k} as the minimal number of permutations needed – starting from left to right – to add ϵ_k to the

sequence $\epsilon_{i_1} < \epsilon_{i_2} < \dots < \epsilon_{i_n}$ in such a way that the final result is ordered. As an example, suppose that $\mathcal{E}_f = \{\epsilon_3, \epsilon_5, \epsilon_7, \epsilon_8\}$, and we want to add ϵ_6 to the sequence $\epsilon_3 < \epsilon_5 < \epsilon_7 < \epsilon_8$, in such a way that the final result is ordered, using the minimal number of permutations. It is easy to check that, in this case, $s_{f, \epsilon_6} = 2$. If we were to add ϵ_1 , then $n_{f, \epsilon_1} = 0$. For ϵ_9 , we have $n_{f, \epsilon_9} = 4$, and so on. With these conventions in hand, for every $\epsilon_k \notin \mathcal{E}_{f, \epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}}$, define:

$$\mathbf{c}_{\epsilon_k}^\dagger f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}} = (-1)^{s_{f, \epsilon_k}} f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_k \dots \epsilon_{i_n}} \quad (38)$$

The interpretation of 39, is that we have created a quanta in level ϵ_k . Notice also that, in Equation 39, ϵ_k is placed in between ϵ_{i_1} and ϵ_{i_n} . If, for example, $\epsilon_k < \epsilon_{i_1}$, we should have placed ϵ_k at the beginning. But this is a meaningless detail of the notation which can also be find in the standard approach, has no effect in the definition of the operator, and should not lead to confusion. Naturally, whenever $\epsilon_k \in \mathcal{E}_{f, \epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}}$, adding a quanta to an occupied level, should result in a null-norm quasi-function. Since we are working in the quotient space $V_{\mathcal{F}} / \sim$, we simply set (for $\epsilon_k \in \mathcal{E}_{f, \epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}}$):

$$\mathbf{c}_{\epsilon_k}^\dagger f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}} = \mathbf{0} \quad (39)$$

where $\mathbf{0}$ is the neutral element for the sum in $V_{\mathcal{F}} / \sim$. Now, assuming first that $k < l$, by applying two creation operators, we obtain

$$\mathbf{c}_{\epsilon_k}^\dagger \mathbf{c}_{\epsilon_l}^\dagger f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}} = (-1)^{s_{f, \epsilon_k}} (-1)^{s_{f, \epsilon_l}} f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_k \dots \epsilon_l \dots \epsilon_{i_n}} \quad (40)$$

By reversing the order of application (but assuming again $i < k$), we have

$$\mathbf{c}_{\epsilon_l}^\dagger \mathbf{c}_{\epsilon_k}^\dagger f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}} = (-1)^{(s_{f, \epsilon_l} + 1)} (-1)^{s_{f, \epsilon_k}} f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_k \dots \epsilon_l \dots \epsilon_{i_n}} \quad (41)$$

Adding equations 40 and 42, we obtain:

$$(\mathbf{c}_{\epsilon_k}^\dagger \mathbf{c}_{\epsilon_l}^\dagger + \mathbf{c}_{\epsilon_l}^\dagger \mathbf{c}_{\epsilon_k}^\dagger) f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}} = \quad (42)$$

$$= (-1)^{n_{f, \epsilon_k}} (-1)^{n_{f, \epsilon_l}} f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_k \dots \epsilon_l \dots \epsilon_{i_n}} + (-1)^{(n_{f, \epsilon_l} + 1)} (-1)^{n_{f, \epsilon_k}} f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_k \dots \epsilon_l \dots \epsilon_{i_n}} = \mathbf{0} \quad (43)$$

Since $f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}}$ is arbitrary, we obtain:

$$\mathbf{c}_{\epsilon_k}^\dagger \mathbf{c}_{\epsilon_l}^\dagger + \mathbf{c}_{\epsilon_l}^\dagger \mathbf{c}_{\epsilon_k}^\dagger = \mathbf{0} \quad (44)$$

From the properties of $\mathbf{c}_{\epsilon_k}^\dagger$, it is possible to derive those of \mathbf{c}_{ϵ_k} as usual. Proceeding as usual, we obtain:

$$\mathbf{c}_{\epsilon_k} f_{\emptyset} = \mathbf{0} \quad (45)$$

and

$$\mathbf{c}_{\epsilon_k} \mathbf{c}_{\epsilon_l} + \mathbf{c}_{\epsilon_l} \mathbf{c}_{\epsilon_k} = \mathbf{0} \quad (46)$$

Let us now focus on the action of $\mathbf{c}_{\epsilon_k} \mathbf{c}_{\epsilon_l}^\dagger + \mathbf{c}_{\epsilon_l}^\dagger \mathbf{c}_{\epsilon_k}$. Assume first that $k < l$:

$$\mathbf{c}_{\epsilon_k} \mathbf{c}_{\epsilon_l}^\dagger f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}} = (-1)^{n_{f, \epsilon_k}} (-1)^{n_{f, \epsilon_l}} f_{\epsilon_{i_1} \epsilon_{i_2} \dots \bar{\epsilon}_k \dots \epsilon_l \dots \epsilon_{i_n}} \quad (47)$$

where the bar in $\bar{\epsilon}_k$ indicates that it has been removed from the list (if present before). On the other hand

$$\mathbf{c}_{\epsilon_l}^\dagger \mathbf{c}_{\epsilon_k} f_{\epsilon_{i_1} \epsilon_{i_2} \dots \epsilon_{i_n}} = (-1)^{(n_{f, \epsilon_l} - 1)} (-1)^{n_{f, \epsilon_k}} f_{\epsilon_{i_1} \epsilon_{i_2} \dots \bar{\epsilon}_k \dots \epsilon_l \dots \epsilon_{i_n}} \quad (48)$$

By adding Equations 47 and 48, we obtain a null result, in any case, for $k < l$. When $k = l$, $n_{f, \epsilon_k} = n_{f, \epsilon_l}$, and we obtain an identity, for every quasi-function. Thus, we conclude that

$$\mathbf{c}_{\epsilon_k} \mathbf{c}_{\epsilon_l}^\dagger + \mathbf{c}_{\epsilon_l}^\dagger \mathbf{c}_{\epsilon_k} = \delta_{kl} \quad (49)$$

The application of the change of basis operators on the ground state f_\emptyset gives

$$f_{\mathbf{xy}} := \frac{1}{\sqrt{2!}} \Psi(\mathbf{y}) \Psi(\mathbf{x}) f_\emptyset = \frac{1}{\sqrt{2}} \sum_i \sum_j u_i(\mathbf{x}) u_j(\mathbf{x}) \mathbf{c}_{\epsilon_i}^\dagger \mathbf{c}_{\epsilon_i}^\dagger f_\emptyset = \quad (50)$$

$$= \frac{1}{\sqrt{2}} \sum_i \sum_j u_i(\mathbf{x}) u_j(\mathbf{y}) f_{\epsilon_i \epsilon_j} = \frac{1}{\sqrt{2}} \sum_{i < j} [u_i(\mathbf{x}) u_j(\mathbf{y}) - u_j(\mathbf{x}) u_i(\mathbf{y})] f_{\epsilon_i \epsilon_j} \quad (51)$$

Computing the scalar product between $f_{\epsilon_j \epsilon_k}$ and $f_{\mathbf{xy}}$, we obtain

$$\langle f_{\epsilon_j \epsilon_k}; f_{\mathbf{xy}} \rangle^- = \frac{1}{\sqrt{2}} [u_i(\mathbf{x}) u_j(\mathbf{y}) - u_j(\mathbf{x}) u_i(\mathbf{y})] \quad (52)$$

which yields the right probability amplitudes for two particle Fermionic systems. Again, we obtain a similar result for $f_{\mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_N} := \frac{1}{\sqrt{N!}} \Psi(\mathbf{x}_N) \dots \Psi(\mathbf{x}_1)^\dagger \Psi(\mathbf{x}_1)^\dagger f_\emptyset$.

5 Implications of our construction

So far, we have seen how to write a version of the FSF that relies on quasi-set theory. As such, it assumes from the beginning that quantum objects are indistinguishable. In this section, we extract some conclusions of the above construction.

5.1 Empirical indistinguishability and ontological concealment

There are very concrete situations in which we can isolate quantum objects. A formidable example of this is given by electromagnetic traps. Using such devices, it is possible to isolate atoms [58], and even electrons [59] and positrons [60, 61]. The researchers that trapped a positron for the first time, called it *Priscilla*. Is this labeling of a quantum object physically meaningful? The manipulation of isolated quantum systems led some researchers to think that they can be indeed identified. But this is a hasty conclusion, as we explain below (see also the discussion presented in [5] and [62]).

The only thing that we can say for sure about a trapped atom or electron (and quantum objects involved in similar situations), is that there is a quantum object of a certain kind in the trap (or a collection of them, in case there are more than one). The sentences “there is a positron in the trap” and “Priscilla is in the trap” seem very similar in content. But the second one is much stronger than the first: it says that the particular positron named Priscilla has been trapped. It assumes that positrons can be identified and labeled. This is not assumed in the first one: even if we cannot identify positrons, the meaning of having just one of them in a trap is clear. One can say that one electron leaved a track in the cloud chamber, one photon provoked a click in the photon detector, a positron is trapped in a ion trap, and so on. But nothing grants that we can put physically meaningful labels to those quantum systems. From an operational point of

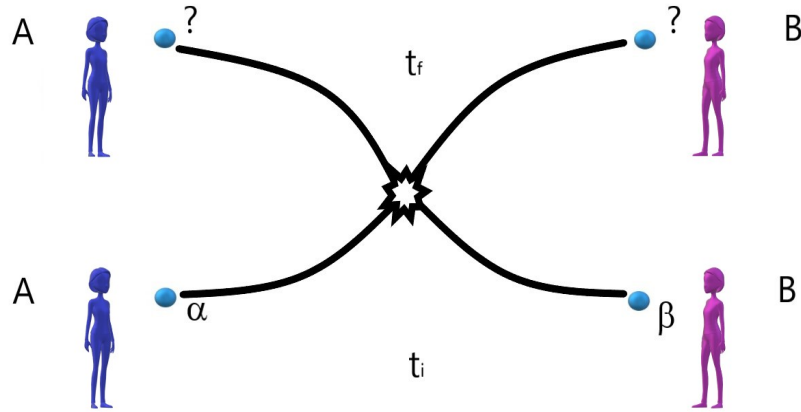


Figure 1: Two experimenters, A and B , assume that their electrons can be named α and β at time $t = t_i$. The electrons undergo an interaction process. At time $t = t_f$, the experimenters try to re-identify their electrons. Quantum mechanics predicts that there exists no physical mechanism whatsoever allowing for them to do this in a meaningful way.

view, there is no reasonable definition of particle labeling. Let us illustrate this with an example. Suppose that two experimenters, A and B have two electrons trapped in their labs. Suppose now that, by adopting the metaphysical standpoint that electrons are full individuals, they go on with it, and name them at time t_i . Experimenters A and B call their respective electrons α and β . The experimenters now make their electrons interact through a scattering process, and trap the outgoing electrons again on each lab at time t_f . If the experiment is designed in the right way, quantum theory *predicts* that there exists no physical mechanism -not even in principle- allowing each experimenter to know if they have recovered their previously possessed electrons after the scattering is performed. According to quantum theory, the identity of each electron is gone forever, in the sense that there is no way to tell whether α came back to A and β returned to B , or not. The question becomes meaningless – from an operational standpoint – because, according to quantum theory, it is physically impossible to identify which is which. Of course, metaphysics can always be put at work, and the experimenters can still assume that after the interaction, α is still α , β is still β , and that both electrons returned to their traps or they were switched, even if A and B don't know which option has taken place. The result of this discussion is that if we want to assume that quantum objects can be identified, in a way strong enough to allow naming them, this identification must be *hidden* (see also the discussion posed in [62]). If assumed (at the metaphysical level), the identity of quantum particles is as hidden as the hidden variables of Bohmian mechanics. An important remark is at stake here. It doesn't matter whether the electrons *actually* undergo a scattering process as the one described above or not. The very possibility of such an occurrence threatens their status as individuals. The *laws* governing the behavior of quantum systems render any assumption of particle identification a purely metaphysical claim, that cannot be subject to experimental control in a consistent way. The only meaningful assertions about quantum systems are of the form: “there is one positron in that trap and an electron outside of it”, “there are two electrons in an Helium atom” or “a quantum object is prepared in state ψ ”. From the perspective of physics, any identification becomes a matter of jargon, that can be useful in some cases (like “Priscila is in the trap”), but cannot be assumed to be valid in general.

Even if one assumes an ignorance interpretation about the identities of the electrons, this concealment is ontological in the following sense: if there would exist a sophisticated mechanism allowing for the quantum objects to be identified and re-identified in an empirically consistent way, quantum mechanics would be wrong. An analogous situation would be to design an experiment to actually detect simultaneously the position and momentum of a Bohmian particle. If this were possible, we should abandon quantum mechanics and replace it with a yet not known theory. The concealment of variables and identities of such hidden variable theories is, in this sense, ontological, due to the fact that it assumes that the world is such that there exists no physical mechanism allowing us to identify particles or detect trajectories. The world would be set as if there would exist a conspiracy in nature, that conceals the identities of the electrons from our experimental *control*. This situation is very different to a similar one with classical objects. We can make an analogous experiment for classical particles, and we may not know whether the particles were switched or not. But we – or some researchers from the future – could be able, in principle, to develop a very sophisticated mechanism capable of determining which is which. In the classical setting, there is no impediment in the laws of nature for knowing whether the particles were switched or not.

The situation described above with the electrons has no classical analogue. To see how weird this is, assume that two loving couples $C_1 = \{P_1, T_1\}$ and $C_2 = \{P_2, T_2\}$, share two twins, named T_1 and T_2 . Suppose that T_1 and T_2 are so similar, that C_1 and C_2 do not have any means to distinguish them physically. Now, suppose that T_1 and T_2 become involved in a strange form of interaction that hides their identities to P_1 and P_2 , the same as with electrons. When C_1 asks to the returned twin whether he is T_1 or not, he replies on the affirmative, and the same happens with P_2 . When asked about past memories, the twins answer that they don't remember anything, so there is no way for P_1 and P_2 to tell whether they were switched or not. Naturally, P_1 and P_2 want to recover their beloved couples. P_1 wants T_1 to come back home, and will not be willing to accept an indistinguishable from T_1 . In classical theories there always exists, no matter how difficult it is, a mechanism allowing P_1 and P_2 to re-identify their beloved ones, even if they can't do it in practice. The concealment is not ontological in this case. There is nothing in nature prohibiting the re-identification. It is just that it is very difficult for C_1 and C_2 to do it.

The situation is much worse than not knowing. By itself, concealment is not a problem for physics. There are many physical quantities which are not directly observable. For example, we don't observe an electric field. We just observe the action of it on some test particle that we put in a certain space region. But the subtle point here is that we can control, at least in principle, electric fields in such a way to make reasonable predictions in the lab. We usually postulate entities that we cannot directly observe, but we can *control* them in such a way that their effects can be expressed in empirically testable mathematical laws. Even if doing this is too difficult for many complex systems, we know that, according to the classical description of physics, there is no intrinsic concealment in nature, and that we could do it in principle. It may also happen that we are not able to do certain things at a stage of development of a theory, but we can hope for doing them in the future, without invalidating the theory. A paradigmatic example of this is the manipulation of a single atom. No one dreamed about that possibility in the early days of quantum mechanics, but according to the theory, there were nothing in nature preventing us to do so. It is perfectly possible to trap one atom or one electron using electromagnetic devices. The situation is very different for the case of indistinguishable particles: if there were a possibility of identifying (and re-identifying) in a consistent and verifiable way, quantum mechanics – in its current form – would be simply wrong.

The real problem with postulating hidden variables and identities in quantum mechanics is that, even if we assume hidden identities or trajectories, there is no physical experiment whatsoever (and it will never will if quantum mechanics is true), that allow us to re-identify quantum objects or to control the values of the hidden variables. In this sense, there is a very clear empirical feature of quantum phenomena that indicates that quantum systems cannot be

re-identified. We may call this feature the *principle of no re-identification*: if the experimenters adopt the metaphysical assumption that quantum objects can be labeled at a given moment, in the general case, there is no physical procedure allowing them to re-identify those systems in the future. It is important to stress that these limitations on the possibility of identifying consistently – from an empirical standpoint – are a common feature of *all* interpretations which are compatible with the predictions of quantum theory. And this is not just a metaphysical assumption: it is a constitutive physical feature of how the world is according to quantum theory. Remarkably enough, it is consistent with the laboratory observations up to now.

5.2 Lack of something?

The operational impossibility of re-identification described above is traditionally interpreted as a negative feature of quantum systems. Taking indistinguishability seriously from an ontological standpoint is usually criticized as a weird move, in favor of a classical ontology based in standard individuals. The “in” in “indistinguishability”, the “non” in “non-individuals” and the “weak” in “weakly discernible objects”, usually have a negative connotation. It is as if quantum objects would lack of something, because of the impossibility of naming or identifying them in a consistent way. The situation is usually presented as if being an individual in the classical sense would be “metaphysically” stronger than being a non-individual. But this picture is incorrect, if we stay close to the actual development of physics.

The theoretical and experimental research of the last decades, remarkably enhanced by the development of quantum technologies, indicates that, rather than a weakness, non-individuality is a positive feature of quantum systems. Put in simple words: non-individuals can do things that classical entities obeying the classical theory of identity, cannot do.

Indeed, quantum indistinguishability lies at the basis of many quantum technologies. for example, the Hong-Ou-Mandel [63] effect relies heavily in the fact the photons are prepared in a state in which they cannot be individuated by any means. The more indistinguishable the photons are prepared, more clearly the interference pick reveals itself in the laboratory. This device can be used to measure time differences with a very high precision. Recently, quantum indistinguishability has been identified as a resource for generating quantum entanglement [28, 29, 27], one of the main ingredients of quantum technologies. Boson sampling [64], is a very simple one-way quantum computer that is of big interest for testing fundamental features of quantum computing. It relies solely on indistinguishable photons (and remarkably, no entanglement seems to be involved). Recently, the notion of indistinguishability has been studied in connection with quantum contextuality [65, 66], showing its potential as an ontological principle. The main point that we want to make here is that the physical properties of quantum systems with regard to identity are behind many relevant developments in physics that have no analogue in the classical domain. The principle of indistinguishability – closely related to the principle of no re-identifications describe above – is one of the strongest features of quantum mechanics and is used by working physicists as the correct tool to describe nature.

Thus, if indistinguishability is so important for predicting new physical phenomena and for the development of technological devices, why not taking it seriously at the ontological level? The fact that we can always postulate hidden identities, should not lead to confusion: the impossibility of distinguishing predicted by quantum theory reveals a positive feature of quantum entities, and can be used as a resource in quantum information theory. To postulate the existence of weak discernible entities or hidden identities is of no use for the problems that the quantum physicists need to deal with. The explanatory power of these notions is usually empty (when not misleading) for the working physicist, and are dispensable.

Is it possible to make sense of non-individuals? Quasi-set theory shows that this is perfectly possible: we have a formal framework (at least one) that allows to speak about non-individuals in a rigorous way. This formalism gives a rigorous ground to the ideas that many physicists used in order to explain and predict new phenomena. Furthermore, the content of Section 4, shows

that it is even possible to *reformulate* quantum mechanics using quasi-set theory. In this way, we have exposed a formulation of quantum mechanics in which quantum objects are considered as non-individuals *right from the start*.

These developments do not imply that the ontologies based in individual entities should be rejected in quantum mechanics. But our work shows that the identity of quantum objects can be dispensed with (in the sense of being an *eliminable* feature). This eliminativist move works in different levels. First, it works in the operational level, given that no ontology based on individuals can make predictions in which quantum objects can be identified and re-identified in a satisfactory way. We have seen that this is not possible for quantum systems – provided that quantum mechanics is assumed to be correct. On the other hand, we have shown that it is possible – and useful – to consider non-individuality as a positive feature of quantum systems (and even as a *resource*), that can be formally described in a mathematics which relies in quasi-set theory.

6 Conclusions

In this work we have reviewed the approach to standard quantum mechanics based in quasi-set theory. We have shown how to recover a Fock-space formalism based in quasi-set theory, accomplishing the task of reformulating quantum mechanics using non-individuality right from the start.

We have also elaborated on *indistinguishability* from an operational standpoint, relating it to the limitations that quantum theory imposes on the attempts to identify and re-identify quantum objects (suggesting that a *no re-identification principle* is operating in the quantum domain). We have argued that this peculiar feature of quantum systems, which must be necessarily shared by all interpretations which are consistent with the predictions of quantum theory, leads to a sort of *ontological concealment* of identity in those interpretations which are based on individuals (such as Bohmian mechanics). We have argued that this, together with the formulation of quantum theory described in Section 4, makes the notion of identity eliminable, in the sense that it can be dispensed with. We have argued that, far from being a negative notion (“a lack of something”), non-individuality is a positive feature of quantum systems that can be rigorously formalized using quasi-set theory. It can also be used as a predictive tool and as a resource in quantum information theory, and it opens the door to a sound formulation of quantum mechanics based in non-individuals.

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