Open problems in the development of a quantum mereology and their ontological implications

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Abstract

Mereology deals with the study of the relations between wholes and parts. In this work we will discuss different developments and open problems related to the formulation of a *quantum mereology*. In particular, we will discuss different advances in the development of formal systems aimed to describe the whole-parts relationship in the context of quantum theory.

Key words:Quantum Mereology; Quasiset theory; Quantum Indistinguishability; Undefined particle number; quantum non-separability

1 Introduction

Mereology deals with the study of the relations between wholes and parts (see for example [1, 2, 3]). In this work we will discuss different developments and open problems related to the formulation of a *quantum mereology* [4, 5]. In particular, we will discuss different advances in the development of formal systems aimed to describe the whole-parts relationship in the context of quantum theory [5, 6, 7, 8, 9]. Since macroscopic physical systems are assumed to be compounded by many quantum systems, understanding the challenges associated to the development of a quantum mereology is of relevance for the problem of explaining the emergence of a macroscopic classical reality out of a microscopic quantum realm. As our work will make clear, the notion of "component part" needs to be critically examined due to the peculiar features of quantum systems. This analysis has also implications for the discussion about whether it is possible to describe a macroscopical system – such as a measurement apparatus – as a simple collection of elementary particles, atoms, or molecules, interacting only through a unitary evolution. This point of view lies at the basis of the so-called *measurement problem*. Its very formulation implicitly assumes that quantum mechanics should be "universal" in that very specific sense and, as such, is an example of *reduction*. The assumption that everything can be reduced to a simple collection of interacting elementary entities, gathered as a simple whole, *partes extra partes*, is known to be very problematic. There is a lot of evidence that it is a naive assumption, even when trying to reduce thermodynamics to classical physics by appealing to statistical mechanics. As we will see below, the "*partes extra partes*" approach seems to be incompatible with the main features of quantum theory. Here is where the interaction between ontological problems and the concepts of practicing physicists play a key role, which cannot be disregarded.

When quantum systems are considered in aggregates, they can display several features that have no analog in classical physics. In this work we focus in three central characteristics of quantum mechanics which are, from a conceptual standpoint, different.

The first one is entanglement: the information of the whole cannot be recovered in terms of that of its parts. This feature gave place to multiple developments that allow to characterize, from a logical point of view, the problem of quantum non-separability [7, 9].

The second one, is that quantum systems of the same kind can be prepared in situations in which they become utterly indiscernible (see for example [10] and [11]). Quasiset theory is an example of a formal system that allows to capture the idea of collections of entities which are truly indiscernible [12]. We will revisit it together with Quaset theory, another formal system aimed to deal with the properties of quantum entities.

The third one is related to the fact that it is possible to prepare superpositions of states with a different particle number. This is the case, for example, of the coherent quantum states of the electromagnetic field, that have an undefined number of photons. This feature is particularly challenging for the developmnet of a quantum mereology, since it is difficult to capture it formally. We will revisit previous approaches that deal with this challenge [6, 13, 14]. It is also important to mention that it is also possible to represent an undefined number of components by appealing to the construction of a Fock-space using quasiset theory [15, 16].

In this work we will discuss the above three features under the light of the formal system introduced in [6] and the quantum logical approach presented in [7, 9, 17]. The work is organized as follows. In section 2 we analyze the main features of compound quantum systems. Next, in section 3, we formally describe compound quantum systems using a quantum logical approach that is based on the convex subsets of the set of quantum states. In section 4 we discuss examples of non-standard set-theoretical frameworks that are inspired in quantum theory. Even if they are not, strictly speaking, mereologies, it is instructive to discuss them here, because they can be used as a basis for the development of quantum mereologies in the future. In section 5 we analyze a logical system that can formally represent quantum systems with an undefined number of components. Finally, in section 6 we present some conclusions.

2 Compund quantum systems

Mereology studies the relationships between the whole and its parts. There exist different formal systems for studying this problem in a formal way. In its essence, it can be formulated as a mathematical theory about the whole and the parts. Our problem here is: how to formulate a *quantum mereology*? In order to attack it, we start first by revisiting the main features of the physics of composite quantum systems. Next, we will review some formal systems developed with the aim of capturing those features in a rigorous way. This is relevant for the discussion about possible ontologies for quantum theory, given that the formal description provides a precise characterization of certain concepts which, in natural language, can only be formulated in an intuitive way.

2.1 Quantum states

The (pure) state of a quantum system can be formally represented by a vector $|\psi\rangle$ in a separable Hilbert space $\mathcal{H}$. As such, linear combinations of pure states – if properly normalized to unity – give place to new states. If a source produces quantum systems in the state

$$|\psi\rangle = \alpha|a\rangle + \beta|b\rangle \tag{1}$$

then, if a measurement takes place, we would obtain the outcomes associated to $|a\rangle$ and $|b\rangle$ with probabilities $|\alpha|^2$ and $|\beta|^2$, respectively. Eqn. 1 describes what is known as a *superposition state*. More generally, states can be given a matrix representation, using the outer product:

$$\rho = |\psi\rangle\langle\psi| \tag{2}$$

In the matrix representation, the probabiities can be computed using the Born rule: $p_a = \mathrm{tr}(\rho|a\rangle\langle a|) = |\alpha|^2$ and $p_b = \mathrm{tr}(\rho|b\rangle\langle b|) = |\beta|^2$.

The state represented by Eqn. 1 (or, equivalently, Eqn. 2) should not be confused with the one represented by the matrix:

$$\rho_{inc} = |\alpha|^2|a\rangle\langle a| + |\beta|^2|b\rangle\langle b| \tag{3}$$

Alike ρ, the state ρ_{inc} is an incoherent mixture between $|a\rangle\langle a|$ and $|b\rangle\langle b|$. As such, it gives the same probabilities when we measure a and b, but will yield different probabilities if other experiments are performed. Thus, ρ and ρ_{inc} represent *different* physical states.

2.2 Entanglement

In case we have two *distinguishable* quantum systems involved (such as a proton and an electron, which can be distinguished, for example, by their charges and rest masses), if the first system is prepared in state $|a\rangle$ and the other is prepared independently in state $|b\rangle$, the state of the compound system can be represented by:

$$|\psi\rangle = |a\rangle \otimes |b\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \tag{4}$$

The fact that states of compound systems must be constructed using the tensor product, is an independent postulate in standard quantum mechanics. According to the rules of quantum theory, it is also possible to form a superposition

$$|\psi\rangle = \alpha|a\rangle \otimes |b\rangle + \beta|a'\rangle \otimes |b'\rangle$$

with a and b different from a' and b'. Such states can be prepared using suitable interactions, as is the case when a pair of photons is emitted after a laser beam targets certain non-linear crystals. It turns out that such a superposition cannot written as a product state (i.e., a state of the form $|\psi_1\rangle \otimes |\psi_2\rangle$). When this is the case, it is said that the system is *entangled*. Numerous experiments suggest that the information about an entangled state cannot be recovered out of the information contained in its parts. This fact gives place to the following slogan:

Slogan 1. THE WHOLE IS NOT EQUAL TO THE SUM OF ITS PARTS.

Most of the extant literature addressing the problem of developing a quantum mereology, focus solely in the study of the features of quantum theory related to entanglement.

2.3 Indistinguishability

But entanglement is no the end of the story. If the quantum systems considered are of the same class, as is the case, for example, when we have two electrons or two photons, we must use the *symmetrization postulate*, and the states must be symmetrized, according to whether the systems belong to two (and only two) classes:
Bosons:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|a\rangle \otimes |b\rangle + |b\rangle \otimes |a\rangle \tag{5}$$

Fermions:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|a\rangle \otimes |b\rangle - |b\rangle \otimes |a\rangle \tag{6}$$

At the moment of writing this work, there are no other known types of quantum systems in nature. All empirical evidence suggests that there exist only two general classes of elementary quantum systems. The symmetrization postulate has a lot of consequences. Among them, the most known is, perhaps, the Pauli exclusion principle [18]. But Bose Einstein condensates have become very relevant as well. The symmetrization postulate entails that

Slogan 2. Quantum systems of a same kind are indistinguishable.

Of course, the slogan must be taken carefully. If an electron on earth is created in a completely independent way from an electron in a distant star (i.e., an electron coming from a space-like separated region), no non-trivial correlations can take place among them. There is no wave function overlap nor entanglement among them, and, for all practical purposes, they can be considered as distinguishable [10]. In other words, considering them as distinguishable, works as a reasonable approximation, since all correlations originated in the symmetrization of the quantum state are negligible. But the "for all practical purposes" should not lead to confusion, due to the following fact: quantum indistinguishability is operating anyway, because, should the electrons undergo an interaction process (some causal event connecting them), a physical situation might take place in which there exists no operational way to determine which one is which. This is a purely quantum feature, in the sense that it has no classical analogue. Two classical particles can be, in principle, always distinguished by a suitably designed experiment. There will always exist an empirical procedure allowing us to tell which is which, even if, under certain circumstances, it may be extremely difficult to do so. On the contrary, quantum mechanics forbids the definition of an operationally well behaved way of distinguishing. In this sense, from the ontological standpoint, quantum systems cannot be considered as individuals in the usual sense. They lack "identity cards", given that there exist situations in which there is no way to retrieve their identities. Taking this into account, one could say that any identity or labeling attributed to a quantum system is mock or fake. Of course, one can *imagine* that quantum systems have hidden identities, to which one cannot access (as is the case, for example, in some versions of Bohmian mechanics). But, if quantum theory is correct, there exist situations in which this assumption has no correlate with any empirical procedure. This is a distinctive feature of quantum theory, given that, even if nothing prevents us to assume that quantum systems are individuals, they can be prepared in states in which any labels attributed to them must be essentially hidden. Due to the above reasons, the standard formulation of quantum theory assumes that quantum systems are indistinguishable in an deep ontological sense. This is the intuition that guides working physicists when dealing with quantum systems of the same kind. As we will see below, this intuition can be formulated in a rigorous way using a non-standard set theoretical framework.

Before we continue, it is very important to make the following remark:

Remark 1. *Entanglement and indistinguishability are not the same thing and should not be confused.*

Equations 5 and 6 look like entangled states. But no genuine entanglement can be present due solely to the symmetrization of a state. As an example, for Fermions, if we want two have non-null entanglement, we need to create a superposition of two Slater determinants (and equation 6 only contains one). The technicalities behind this fact are far beyond the scope of this article. For our purposes here though, it is important to keep in mind the following:

Remark 2. *Indistinguishability is, besides superposition and entanglement, an independent and crucial physical feature of quantum systems.*

The assumption that quantum systems can be in situations of utter indiscernibility, leads naturally to the derivation of the so called *quantum statistics*. It can be explained in terms of counting: if one has two Bosons and two states, if they were distinguishable, there would exist four possibilities. But, in the properly quantum regime, there is no sense in saying that Boson A is in state 1, and Boson B is in state 2. Since they are indistinguishable, those classical alternatives become one whenever the correlations originated in the symmetrization of the state cannot be neglected. Therefore, for two Bosons and two states, there are only three alternative configurations (and not four, as in the classical case). Similarly, due to the Pauli exclusion principle, one cannot place two Fermions in the same state. Therefore, for the case of two Fermions and two states, there is only one physically meaningful configuration in the quantum regime.

2.4 Undefined particle number

The reader who is not familiar with quantum theory might feel a little bit exhausted. Not only the whole is not equal to the sum of its parts but, besides that, some component parts can be indistinguishable in a deep ontological sense. Yet, there is still another crucial physical feature that plays its role when we deal with compound quantum systems: it is possible to prepare quantum systems in a superposition of different particle number states. If we take quantum physics literally, this implies that the number of components can be undefined in an ontological sense. To illustrate this, consider first a state of two Bosons:

$$|2\rangle = \frac{1}{\sqrt{2}}(|a\rangle \otimes |b\rangle + |b\rangle \otimes |a\rangle \tag{7}$$

(notice the symmetrization). Next, consider a state formed of three Bosons:

$$\begin{aligned}|3\rangle = \frac{1}{\sqrt{6}}(&|a\rangle \otimes |b\rangle \otimes |c\rangle + |b\rangle \otimes |a\rangle \otimes |c\rangle + |a\rangle \otimes |c\rangle \otimes |b\rangle + \\ &|c\rangle \otimes |b\rangle \otimes |a\rangle + |c\rangle \otimes |a\rangle \otimes |b\rangle + |b\rangle \otimes |c\rangle \otimes |a\rangle)\end{aligned}$$

Now, according to the rules of quantum theory, nothing prevents us of forming a coherent superposition between the above states:

$$|\psi\rangle = \alpha|2\rangle + \beta|3\rangle \tag{8}$$

If, given a system prepared in the above state, we perform a particle number measurement, we will detect two particles with probability $|\alpha|^2$, and three particles with probability $|\beta|^2$. Following the standard interpretation of quantum theory, a system prepared in a state such as the one given by Eqn. 8, has no defined number of components. An important example is that of a coherent state:

$$|\alpha\rangle = \exp^{\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{9}$$

It is important to recall here the difference between a coherent state and a mixture:

$$|\alpha\rangle\langle\alpha| \neq \exp^{\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} |n\rangle\langle n| \tag{10}$$

Also, squeezed states [19] play a crucial role in quantum information tasks (see [20] for details):

$$|S\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \tanh r^n \frac{\sqrt{(2n)!}}{2^n n!} |2n\rangle \tag{11}$$

The above states are prepared in nowadays laboratories on a daily basis. Out of the considerations of this section, we add a new slogan:

Slogan 3. THE NUMBER OF COMPONENTS CAN BE UNDEFINED.

2.5 Conclusions of the first part

When quantum systems with multiple components are considered, there might be – at least – three crucial physical features at play:

- *Entanglement*: the system contains more information than that of its parts.
- Qantum systems of the same kind are *indistinguishable* (they can enter situations for which the notions of labeling, re-identification and individuation lack of any well defined physical meaning).
- The *number* of components can be *undefined*.

Most studies about quantum mereology focus in the phenomenon of entanglement. In some previous works, besides entanglement, indistinguishability and undefined particle number have also been considered. In what follows, we will review some of these developments, with the aim of illustrating the challenges that appear in the development of a quantum mereology.

3 A formal description of the relation between the whole and its parts

Let us first consider what happens with the propositions associated to classical systems in the compound case. For a classical system, if the subsystems are prepared in states $s_1 = (p_1, q_1)$ and $s_2 = (p_2, q_2)$, then, the state of the compound system can be described by the pair (s_1, s_2). In this sense, the *information* of the compound system is just *the sum of that if its parts.*

Any testable proposition about a physical system has the form: "the value of the physical quantity A lies in the interval Δ". It turns out that, for classical systems, the set of testable propositions forms a distributive lattice with regard to the classical logic connectives "$\wedge$", "$\vee$" and "$\neg$". To fix ideas, one can represent those propositions as the *measurable subsets* of the phase space (according to the Lebesgue measure). In this representation, the connectives "$\wedge$", "$\vee$" and "$\neg$", are represented by the set theoretical intersection "$\cap$", the union "$\cap$", and the set

$$\begin{tikzcd} & \mathcal{L}_{\mathcal{CM}_1} \times \mathcal{L}_{\mathcal{CM}_2} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathcal{L}_{\mathcal{CM}_1} & & \mathcal{L}_{\mathcal{CM}_2} \end{tikzcd}$$

Figure 1: The different maps between $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_1 \times \mathcal{L}_2$, and $\mathcal{L}$. π_1 and π_2 are the canonical projections.

theoretical complement "$(...)^c$", respectively. The diagram displayed in Figure 3 illustrates the relation between the lattice of propositions of the compound system and those of its subsystems. There exists a map – which is the canonical projection associated to the Cartesian product – that establishes a one to one correspondence between states.

But the testable propositions associated to quantum systems have a very different structure. To begin with, the standard way of describing quantum systems is essentially probabilistic. Accordingly, in most circumstances, information about a quantum system is expressed in probabilistic terms. Second, besides those properties which can be considered "classical" (such as charge, intrinsic spin and rest mass), we can also make assertions such as: "The value of the observable A lies in the interval Δ". In the quantum formalism, these testable propositions are represented by closed subspaces $P_A(\Delta)$ (or, equivalently, by their associated orthogonal projections). The mathematical representatives of the propositions of quantum systems form an algebraic structure known as *orthomodular lattice* which, alike its classical counterpart, is not distributive. Here we call $\mathcal{L}_{v\mathcal{N}}$ to that lattice. In this case, the logical connectives "$\wedge$", "$\vee$" and "$\neg$" are represented by the closed linear subspaces intersection "$\cap$", direct sum "$\oplus$", and orthogonal complement "$(...)^{\perp}$", respectively.

3.1 Improper mixtures

Can we find maps as in Figure 3 for quantum systems? The situation is depicted in Figure 3.1. Can we find maps ξ_1 and ξ_2 connecting states of the compound system with those of its subsystems? In order to answer that question, we notice that quantum states can be *mixed*. In order to illustrate that concept, let us consider a compound quantum system. To simplify the description, let us assume that its subsystems are distinguishable (i.e., they are not quantum systems of the same kind). Assume that the compound system is in the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle).$$

Its associated density operator is given by

$$\rho = |\psi\rangle\langle\psi| \in \mathcal{L}_{v\mathcal{N}}$$

Then, taking the partial trace over the second subsystem, we obtain that the state of the first subsystem is

$$\rho_1 = \mathrm{tr}_2(\rho) = \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|).$$

But ρ_1 is not a pure state, since it is written as a non-trivial convex combination of $|+\rangle\langle+|$ and $|-\rangle\langle-|$. Then, it is not possible to describe it as an element of $\mathcal{L}_{v\mathcal{N}_1}$. This is a clear expression of the fact that the whole is not the sum of its parts. It is therefore not possible to use partial traces to map the properties of the compound quantum system to the properties of its subsystems, given that states of the subsystems can be mixed, and therefore, they do not have representatives as elements of the lattice of properties. Let us dig a little bit on this feature

$$
\begin{tikzcd}
 & \mathcal{L}_{v\mathcal{N}} \arrow[dl, "\xi_1?"'] \arrow[dr, "\xi_2?"] & \\
\mathcal{L}_{v\mathcal{N}_1} & & \mathcal{L}_{v\mathcal{N}_2}
\end{tikzcd}
$$

Figure 2: The different maps between $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_1 \times \mathcal{L}_2$, and $\mathcal{L}$. π_1 and π_2 are the canonical projections.

of the quantum formalism. Actual properties are defined as those which are associated with testable propositions whose probability of occurrence equals one. If we restrict to pure states, the conjunction of all actual properties defines the state of the system (see for example [21]) in the following sense:

$$\{s\} = \bigwedge \{X \in \mathcal{L}_{v\mathcal{N}} \mid \text{X es actual}\} \tag{12}$$

The above Equation provides the connection between states and properties: a (pure) state can be considered as the collection of all properties that are true at a given time. But, since states of subsystems are, in most cases, unavoidably mixed, it is no longer possible to use Equation 12 to establish a connection between states ad properties. This fact precludes the possibility of finding a correct way to connect the states of the whole with the states of its parts (at the level of the lattice of propositions associated to a physical system). A possible solution is to extend the notion of proposition associated to a quantum system to probabilistic assertions, such as: "The system has property E with probability p". Thus, one can define something like:

$$C_E(p) = \{\rho \in \mathcal{C} \mid \mathrm{tr}(\rho E) = p\} \tag{13}$$

It is possible to show that if E is a quantum effect, $C_E(p)$ is a convex subset of $\mathcal{C}$ [9]. Thus, consider the following set (that defines the collection of all possible density operators representing quantum states):

$$\mathcal{C} = \{\rho : \rho = \rho^{\dagger};\ \ \rho \geq 0;\ \ tr(\rho) = 1\}. \tag{14}$$

The faces of $\mathcal{C}$ form a lattice (in a canonical way) which is isomorphic to the orthomodular lattice of projection operators (see for example [22]). As an extension, in [9], it was proposed to consider a lattice formed by convex subsets of $\mathcal{C}$. The operations are defined as follows.

Definition 1. $\mathcal{L}_{\mathcal{C}} := \{C \subseteq \mathcal{C} \,|\, \text{is a convex subset of } \mathcal{C}\}$

In order endow this set with a lattice structure, we introduce the following operations in $\mathcal{L}_{\mathcal{C}}$ (where $conv(A)$ describes the convex hull associated to a given set A):

Definition 2. For all $C, C_1, C_2 \in \mathcal{L}_{\mathcal{C}}$

$\wedge$ $C_1 \wedge C_2 := C_1 \cap C_2$

$\vee$ $C_1 \vee C_2 := conv(C_1, C_2)$. It is convex and it is contained in $\mathcal{C}$. contained

$\neg$ $\neg C := C^{\perp} \cap \mathcal{C}$

$\longrightarrow$ $C_1 \longrightarrow C_2 := C_1 \subseteq C_2$

The new lattice has the following properties:

$$
\begin{tikzcd}
& \mathcal{L}_{\mathcal{C}} \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] \arrow[d, shift left, "\tau"] & \\
\mathcal{L}_{\mathcal{C}_1} & \mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, shift left, "\Lambda"] & \mathcal{L}_{\mathcal{C}_2}
\end{tikzcd}
$$

Figure 3: The different maps between $\mathcal{L}_{\mathcal{C}_1}$, $\mathcal{L}_{\mathcal{C}_2}$, $\mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2}$, and $\mathcal{L}_{\mathcal{C}}$. π_1 and π_2 are the canonical projections.

- Contraposition and non-contradiction hold:

$$C_1 \longrightarrow C_2 \Longrightarrow \neg C_2 \longrightarrow \neg C_1$$

$$C \wedge (\neg C) = \mathbf{0}$$

- Double negation does no longer holds. Therefore, $\mathcal{L}_{\mathcal{C}}$ is not an ortholattice.
- $\mathcal{L}_{\mathcal{C}}$ is a lattice that contains all convex subsets of the space of quantum states.
- It includes a copy $\mathcal{L}_{v\mathcal{N}}$ (its elements represented as faces of the convex set of quantum states).
- All possible quantum states are propositions of the form $\{\rho\} \in \mathcal{L}_{\mathcal{C}}$ (because each point of $\mathcal{C}$ is a quantum state).

By appealing to the extended lattices described above, the relationship between a quantum whole and its parts can be expressed as described in Figure 3. Using the extended lattices, it is possible to define different canonical maps between $\mathcal{L}_{\mathcal{C}_1}$, $\mathcal{L}_{\mathcal{C}_2}$, $\mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2}$, and $\mathcal{L}_{\mathcal{C}}$. The impossibility of reducing the information of the whole to that of its parts can be expressed in formal terms by the Equation [23, 9, 24]:

$$\Lambda \circ \tau \neq \tau \circ \Lambda \tag{15}$$

The above equation should be clear by itself: it says that going from the whole to the parts, and then going from the parts to the whole, will not convey the same information as the reversed operation. In short: going down and going up, is not the same as going up and going down.

The mathematical structures described in this section allow for a formal description of the connection between the properties of the whole and the properties of its parts. In the next section, we focus on the description of collections of quantum entities from the point of view of set theoretical structures.

4 Non-standard set theories

When we deal with physical (or mathematical) objects, it is natural to describe them in terms of collections. Therefore, it is natural to assume that a mereology must have a built-in way of dealing with collections of entities or parts. But collections of quantum entities must be handled with care. To fix ideas, one can think in the collection of natural numbers, or the collection of planets in the solar system. In those examples, the components of the collection have well defined identities: each planet or number can be unambiguously distinguished from others. Also, the membership relationship is sharply defined: the number two is either is even or not. Therefore, it belongs or not to the collection of even numbers. There is no third possibility. As explained above, quantum objects can enter situations in which they can be considered as truly indistinguishable. Therefore, they way in which electrons form a collection, should not be the same as in the examples of planets and numbers – at least, from a formal standpoint.

Even the notion of membership has been questioned in the quantum domain: up to which point can we clearly define a membership relation for the collection of the electrons with spin up in a given material? The formal systems described below aim to capture these quantum features in a formal way, providing a rigorous formulation for the notion of a collection of quantum entities. Of course, these formal systems can be considered as the basis of a non-standard mathematics, inspired in quantum theory.

4.1 Quasisets and Quasets

In order to formally capture the peculiar features of quantum systems, two different set theoretical frameworks were proposed: QST (Quaset theory) and $\mathfrak{Q}$ (Quasiset theory). Though both theories are inspired in quantum theory and they share some structural similarities, they are quite different with regard to the way in which they depart from ZF.

QST was developed by María Luisa Dalla Chiara and Giuliano Toraldo di Francia [25]. It is formulated in a first order language with identity and its main characteristic is that it relies on an additional primitive membership notion in its non-logical axiomatic layer: "$\not\in$". The axioms are set in such a way that $x \not\in y$ can be interpreted as "*x is not a member of y with certainty*". This new connective, alike ZF, is independent of "$\in$". The following well formed formula of QST *is not valid*: $x \not\in y \longleftrightarrow \neg(x \in y)$. The direct implication of this fact is the existence of undetermined membership relations. If a given quaset y admits elements whose membership status is undetermined, then, there exists an element x such that $[\neg(x \in y) \wedge \neg(x \not\in y)]$. This is the main feature that distinguishes QST from ZF. The proponents of this formal framework suggested that this feature could be related to the notions of quantum uncertainty and superposition states.

On the other hand, Quasiset theory ($\mathfrak{Q}$ from now on) was proposed by Décio Krause [26, 27], and subsequently developed by other authors [12, 28, 29]. Alike QST, its first order metalanguage does not includes identity as a primitive. It aims to describe collections of truly indiscernible objects, for which the notion of identity cannot be applied. Therefore, it is not possible to write formulas such as $x = y$ for some elements of the theory. For the so called $m-$atoms, we can only appeal to an indiscernibility relation "$\equiv$". Two $m-$atoms can be indiscernible, but still count as two[1]. In this way, a formal description of the idea of collections of objects that are truly indiscernible (non-individuals) is obtained. It is a non-standard set theory (a variant of the Zermelo-Frenkel system), but it is not a mereology *strictu sensu*. Notice that the membership relationship "$\in$" has a different meaning than that of "being a part of..." (which we will denote by "$\sqsubset$" below). If x, y represent quantum entities of the same kind (i.e., if they are m-atoms of the same kind), then, the formula "$x \equiv y$" is interpreted as *x is indiscernible from y*. Thus, even if they are identical in all their attributes, they are not the same. In this sense, indiscernibility is weaker than identity, but the notions coincide when applied to entities that belong to a certain *classical domain*. Intuitively, if x, y are variables meant to represent electrons, then, the formula $x = y$ is not well formed, while $x \equiv y$ expresses the fact that those entities are indiscernible. In this way, one can consider them as different "*solo numero*" (i.e., only numerically discernible). As a consequence, the Principle of Identity of Indiscernibles is challenged.

$\mathfrak{Q}$ provides a formal framework to deal with collections of indiscernible items without the usual tricks of confining them into equivalence classes or non-rigid (deformable) structures: in quasi-set, theory one can speak and treat those items without those standard tricks and with logical precision. In what follows, we describe the axioms of these non-standard set theories with some detail.

[1]Notice that, in a standard set theory, if $x = y$, then, there are no two objects, but one.

4.2 Brief description of the non-logical layer of QST and $\mathfrak{Q}$

In this section, we briefly describe the elementary basis of Quaset and Quasisets. The non-logical language of QST includes the following primitives:

1. A monadic predicate (ur-object or ur-element) "O".
2. Three binary predicates: positive membership "$\in$", negative membership "$\notin$", and the inclusion relationship "$\subseteq$".
3. A unary functional symbol: the quasicardinal "*qcard*".
4. A binary functional symbol: quasets intersection "$\sqcap$".

Definición 4.1. A quaset is something that is not a ur-object:

$$Q(x) := \neg O(x)$$

Axioma 4.1. If something has an element, then, it is a quaset:

$$\forall x \quad \forall y \;\; (x \in y \longrightarrow Q(y))$$

Axioma 4.2. If we know with certainty that something does not belongs to a quaset, then, it is not true that it belongs with certainty to the given quaset. But the converse is not granted in general:

$$\forall x \; \forall y \; (x \notin y \to \neg(x \in y))$$

The previous axiom implies that not all instances of the principle $((x \in y) \vee (x \notin y))$ are true, and then, there exists the possibility of undetermined membership relations. On the other hand, the notion of "*certainly knowing*" can be interpreted as "*belongs with certainty*" – if one wishes to avoid an epistemic interpretation of the theory. As is well known, quantum mechanics admits both, ontological and epistemic interpretations of the uncertainty principle.

Axioma 4.3. The inclusion relationship "$(\subseteq)$" is a partial order (i.e., it is a reflexive, symmetric and transitive relationship).

The symbol $\subseteq$ has an *intensional* meaning but, in general, it lacks an extensional one. Also, "$x \subseteq y$" can be understood as "concept x implies concept y".

Axioma 4.4. Quasets inclusion implies extensional inclusion.

$$\forall x \quad \forall y \;\; (x \subseteq y \longrightarrow \forall z \; ((z \in x \longrightarrow z \in y) \wedge (z \notin y \longrightarrow z \notin x))$$

In QST, conjunction (or intersection) is a primitive notion. This is denoted by the symbol $\sqcap$ and satisfies:

Axioma 4.5. "$\sqcap$" represents the weak conjunction of quasets. It coincides with the usual intersection when restricted to sets:

$$\forall_Q x \;\; \forall_Q y \;\; ((x \sqcap y \subseteq x \;\wedge\; x \sqcap y \subseteq y) \wedge (Z(x) \wedge Z(y) \to x \sqcap y = x \cap y))$$

Notice that the axioms of quasets do not demand that there exist proper quasets (i.e., quasets which are not sets). From an intuitive point of view, the qextension of a proper quaset does not represents a suitable semantic counterpart for the usual notion of extension. As an example, we mention that the qextension de un quaset – even with a cuasicardinal greater than zero– could be empty.

Let us now focus on $\mathfrak{Q}$. The primitive non-logical symbols are a binary predicate '$\equiv$', which stands for 'indiscernible', or 'indistinguishable'; three unary predicates 'Z', 'm' and 'M', which have the following intuitive interpretation: $Z(x)$ says that x is a *set*, really a copy of an entity of ZFU; $m(x)$ says that x is a m-atom, which is intended to represent quantum systems (be they considered quantized particles or field excitations); $M(x)$ says that x is an atom of ZFU. The theory still has a derived notion of *extensional identity*, symbolized by '$=_E$', which is defined only for M-atoms that belong to the same quasi-sets or quasi-sets having the same elements. Let us call $\mathfrak{Q}$ to a first order theory whose primitive vocabulary contains, beyond the vocabulary of standard first order logic without identity (propositional connectives, quantifiers, etc.), the following specific symbols: (1) three unary predicates m, M, Z, (2) two binary predicates $\in$ and $\equiv$, (3) one unary functional symbol qc. Notice again that identity is not part of the primitive vocabulary, and that the only terms in the language are variables and expressions of the form $qc(x)$, where x is an individual variable, and not a general term.[2] The intuitive meaning of the primitive symbols is given as follows:

(i) $x \equiv y$ (x is indiscernible from y)

(ii) $m(x)$ (x is a "micro-object", or an m-atom)

(iii) $M(x)$ (x is a "macro-object" or an M-atom)

(iv) $Z(x)$ (x is a "set" – a copy of a ZFU set)

(v) $qc(x)$ (the quasi-cardinal of x)

The underlying logic of $\mathfrak{Q}$ is a *non-reflexive* one, a logic where the standard theory of identity is limited in some way; in our case, by restricting the application of the very predicate of identity; see also [12, 30].

5 The $ZF^{\star}$ system and the problem of describing an undefined number of components

Both, QST and $\mathfrak{Q}$, are set-theoretical frameworks. This means that they are meant to describe *collections* of certain entities that have quantum features. The formal analysis of collections is important for the goal of developing a quantum mereology, but it is not enough. QST and $\mathfrak{Q}$ lack of an explicit definition of one of the key features of any mereology, namely, the notion of *part*. A propper quantum mereology was introduced in [31]. The proposed system has several interesting properties: it can describe, at the same time, indistinguishability and an undefined number of components. As a by-product, it allows to describe superpositions as a form of logical undecidability. From now on, we refer to the system introduced in [31] as ZF^{*}. We give here a brief description focused in the ontological aspetcs of the problem, and refer the reader to [31] for a detailed description of its axiomatics.

[2] This restriction prevents that, for example, $qc(qc(x))$ end up being a term.

5.1 Undefined number of components as logical undecidability

Let us first introduce the primitive symbols of ZF^*:

- First order logic symbols with "$\in$" (membership).
- "$\mathcal{C}(\dots)$" (in such a way that "$\mathcal{C}(X)$" reads "X is a set").
- "$\sqsubset$", in such a way that "$X \sqsubset Y$" reads "X is a part of Y".
- The theory contains *sets* and *physical things*. The latter are not sets, that is, they are not *collections*.

A definition of indistinguishability can be introduced as follows:

$$\alpha \equiv \beta := \alpha \sqsubset \beta \wedge \beta \sqsubset \alpha \tag{16}$$

Alternatively, it could be introduced following the same strategy as in $\mathfrak{Q}$ (i.e., based in a relationship such as "$\equiv$"). Using the above, it is possible to give the following definitions. Given a well formed formula F, the notion of a collection of entities satisfying that formula can be defined as:

$$\exists\{x \,|\, F(x)\} := (\exists y)(\forall x)(x \in y \longleftrightarrow F(x)) \tag{17}$$

We will also need the notion of a *Cantorian entity*:

$$Cant(\alpha) := \exists\{\beta \,|\, \beta \sqsubset \alpha\} \tag{18}$$

The above equation says that a given entity is Cantorian whenever we can describe its parts as a well behaved collection. In case that a given entity is Cantorian, we can formally assign to it a number of components, as the cardinal associated to the collection of all its parts:

$$\sharp(\alpha) := \sharp(\{\beta \,|\, \beta \sqsubset \alpha\}) \tag{19}$$

If α satisfies that $\neg Cant(\alpha)$, then, there is no way to assign a cardinal to it. Therefore, as a consequence of the axioms of the theory, non-Cantorian entities might exist. For them, the number of components will be undefined. The reader can already notice that here, undefinitness, is associated to the logical impossibility of granting the existence of an object in a logical system. In a certain sense, one can represent something which is assumed to actually happen in the real world (i.e., the existence of physical systems for which the number of components is not defined, such as those described by Eqns. (9) and 11) as a limitation of a formal system.

5.2 Undefined things

As a byproduct of the above discussion, we can obtain a more general result. If α is such that $\neg Cant(\alpha)$, given the formula $F(x)$, it is not possible – using the axioms of ZF^* – to grant the existence of the set:

$$\alpha_F = \{\beta \sqsubset \alpha \,|\, F(\beta)\} \tag{20}$$

As a consequence, we cannot, in general, describe the components that satisfy certain properties as a well behaved collection. This gives place to undefined properties in the following sense: if α non-Cantorian, we cannot assert that its parts have the property defined by $F(x)$, neither that it doesn't has it. This type indeterminancy in the posession of a property can be used to describe the situation originated when a quantum system is prepared in a superposition state, such as:

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \tag{21}$$

According to the standard formulation of quantum theory, we cannot tell whether the system has spin up or down, because that property is undefined previous to measurement. The $ZF^\star$ system can thus give a formal description of one of the most important concepts of quantum theory, namely, superposition states.

How to understand undefined things? How to formalize the idea that something is undefined (in a stronmg ontological sense)? The message of ZF^* can be summarized as follows:

$$\text{INDETERMINANCY} \Longleftrightarrow \text{UNDECIDABILITY}$$

We think that the above double implication opens a very interesting line of philosophical inquiry. Up to which point can we relate the notions of quantum indeterminancy and logical undecidability? The deepest intuitions that guide quantum physicists rely on the idea that there is an intrinsic randomness in nature. Notice that this guiding idea underlies, for example, the whole industry of building commercial quantum random number generators (i.e., of using quantum systems as sources of true randomness [32]). Due to the assumption of true randomness, it is not possible to ascribe results of observations to properties possessed by the system previous to measurement, because the results of future experiments are simply not defined before their concrete instantiation. In that sense, those properties are undefined. In this way, logical undecidability – or indefiniteness – could be used as a conceptual tool to give a formal representation of the intuitions of working physicists. The ZF^* system is a first step in this direction. We end this section by noticing that this type of analysis is motivated by questions that go well beyond the philosophy of physics: a deeper understanding of the connections between randomness, undefinitness and logical undecidability, could have direct implications in physics and mathematics, in the fields of random numbers generation and randomness certification.

6 Conclusions

As we have shown, the study of the relationship between the whole and its parts poses special challenges for the philosophy of quantum theory. When quantum systems are presented in aggregates, we can find non-separability, indistinguishability and undefined number of components. While most studies focus in the problem of non-separability, we have discussed here different formal systems which allow to describe these notions in a way that help to overcome the ambiguities of natural language. It follows that it is possible to give a rigorous formal representation of many of the intuitive notions associated to quantum physics. The notions of indistinguishability, the imposibility of reducing the whole to its parts, and the possibility of having an undefined number of components, can be captured in a formal way.

The formalization of the peculiar features that quantum systems display when considered in aggregates is relevant for a better understanding of the connections between the macroscopic and microscopic levels of reality. Our analysis suggests that the picture of a macroscopic object as a simple "partes extra partes" aggregate of small localized entities is highly misleading. Having this into account is crucial in the analysis of the so-called measurement problem, given that it assumes implicitly that a measurement apparatus can be described as a simple aggregate of elementary systems evolving unitarily. This "reductionist" point of view has been shown to be problematic in many other areas of research, and cannot be naively implemented in the quantum domain either. The analysis presented in this work reinforces the need of overcoming the measurement problem with a different non-reductionist strategy, adopting a critical re-examination of the notion of "part" in the quantum domain.

We have also discussed the connection between quantum inteterminancy and logical undecidability. In other words, we have suggested that logical undecidability could be used as a tool to formalize many of the the intuitions that underlie quantum theory. In particular, we have pointed out the relevance of this ideas in the context quantum random number generation and certification. The formal treatment of the intuitions underlying quantum indeterminancy could be of help for a deeper understanding of the concept of *true randomness*.

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