Higher-Order Skolem's Paradoxes

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Abstract: Some analogous higher-order versions of Skolem's paradox will be introduced. The generalizability of two solutions for Skolem's paradox will be assessed: the course-book approach and Bays' one. Bays' solution to Skolem's paradox, unlike the course-book solution, can be generalized to solve the higher-order paradoxes without any implication about the possibility or order of a language in which mathematical practice is to be formalized.

1. Skolem's paradox

Let S be one of the well-known first-order axiomatizations of set theory (e.g. ZFC). Skolem's paradox (hereafter SP) is a seeming conflict between the *Downward Löwenheim-Skolem Theorem* about S and *Cantor's Theorem* within S:

> *Theorem (Downward Löwenheim-Skolem)*: Let Γ be a set of sentences in a language L of cardinality κ.[1] For all $\lambda > \kappa$, if Γ has a model of cardinality λ, then Γ has a model of cardinality μ, for all μ, $\kappa \leq \mu < \lambda$.
>
> *Theorem (Cantor)*: There are uncountable sets.

Suppose that S has a model. Since the language of S in standard formulations is countable, by the *Downward Löwenheim-Skolem Theorem*, S has a countable model, M. Now, by *Cantor's Theorem*, S proves that there is an uncountable set, hence there is an a in the universe of M such that a is uncountable; that is to say, a satisfies in M the open formula which defines uncountability in the language of S. Insofar as M is countable, there are only countably many o in the universe of M such that $o \in a$. It seems then that within M, a is countable. Therefore, a is countable from one perspective (within the model), uncountable from another (within the theory).

It is easily realizable that SP does not pose any logical contradiction. Nevertheless, it raises some philosophical issues. Is there any uncountable set *in a real sense*, that is to say, from no perspective? Does SP have anything to do with *the practice of mathematics*? The *course-book approach*'s answer to these questions is affirmative. According to this approach, SP provides an evidence for the deficiency and semantical inadequacy of first-order theories for formalizing mathematical practice around countability and uncountability. Actually, SP is not alone. Firs-order languages has shortcomings in formalizing many other concepts of ordinary mathematics, too; for example, finitude, well-ordering, well-founded-ness, powerset, etc.[2] In Shapiro's words, "[t]hese concepts form an important part of general mathematical practice, but they cannot be formulated in first-order languages. These concepts are clear and unambiguous as for instance when a mathematician asserts some set is finite; his listeners

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[1] Including, as usual, a countable set of variables, κ is at least countable.

[2] The course-book approach is mentioned and suggested vastly in familiar course-books of introductory mathematical logic, such as Mendelson (2015) and van Dalen (2013)

understand what he means." (1985, 722). He goes on "to suggest that nothing short of a language with second-order variables [with standard models] will do" (715).[3] The reason is clear. SP is caused by *Downward Löwenheim-Skolem Theorem* which is not valid for second-order languages (with standard models).

In the next section, we will introduce some forms of higher-order Skolem paradox (hereafter HOSP) and then argue that the course-book approach has its inadequacies regarding these paradoxes; namely, it must concede that there can be no language in which the mathematical practice (especially, set theory) could be formalized. Then in section 3 we show how the solution for SP suggested by Bays (2000) can be generalized to HOSPs without such unwanted consequence.

2. Higher-order Skolem paradoxes and the course-book approach

Let L be a language containing the first-order language with identity. Consider the following definition and theorem, both reported by Shapiro (1991, 147-8):

> *Definition.* (*Löwenheim number*) The Löwenheim number for L is the smallest cardinal κ such that for every formula φ of L, if φ is satisfiable, then it has a model with the cardinality at most κ.
>
> *Theorem.* (*Generalized Löwenheim*) If the collection of formulas of L is a set, then L has a Löwenheim number and the smallest extendible cardinal is an upper bound of it.

Now, an nth-order Skolem paradox can be formulated as follows.[4] Let Sn be an nth-order axiomatization of set theory which can prove that there are extendible cardinals. And let κ be its Löwenheim number. For a cardinal larger than κ, we have a proof for a sentence φ which says that there exists a set whose cardinality is larger than κ. By the *Generalized Löwenheim Theorem*, this sentence has a model, M, with the cardinality of at most κ. M satisfies "there exists a set whose cardinality is larger than κ," hence there is an a in the universe of M such that the size of a is larger than κ. While the cardinality of M is at most κ, there are at most κ objects o in the universe of M such that $o \in a$. It seems then that within M, a's size is at most κ. Therefore, a's size is at most κ from one perspective (within the model), larger than κ from another (within the theory).[5]

Reapplying the course-book approach to handle these nth-order paradoxes might seem to be appealing. Accordingly, the paradox could be solved by going to a higher-order language, but again an analogous higher-order paradox can be formulated for the higher-order language; and so on. Thus, the course-book approach to handle SP cannot be generalized to solve the parallel HOSPs, unless it is augmented by the claim that there is no unique language that the practice of mathematics (set theory, particularly) can be formalized within it. It might be so,

[3] Actually, it is doubtful to be correct to categorize Shapiro within the defenders of course-book approach. His project is to find the appropriate language for mathematical practice. Nevertheless, the course-book approach share something with Shapiro's thesis: Skolem's paradox is among evidences for going to higher-order languages.

[4] Our formulation of HOSPs appeals to the notion of *Löwenheim number*. Similarly, one can introduce other HOSPs by means of *Hanf number*, *set-Löwenheim number* and *set-Hanf number*, their definitions can be found in Shapiro (1991, 148). Here, we just focus on *Löwenheim* number, but the strategy can be reapplied for other numbers straightforwardly.

[5] Higher-order Skolem paradoxes is already mentioned by Hart (2000) based on considerations given by Hasenjaeger (1967).

but it seems to be superior if SP and its counterparts can be handled without conceding such radical claim about mathematical practice and its formalization.

In defense of the course-book approach, one might suggest that these new paradoxes are not as philosophically valuable as the original Skolem's paradox, for the large cardinals are not as much involved in mathematical practice as concepts like (un)countability. Second-order logic, though not apt for the *whole* practice of mathematics, is adequate for its *ordinary* part. This is a non-starter, however. Mathematical practice is not something stable and closed-end. It is extremely foreseeable that large cardinals will become more involved in mathematical practice than they are now. Furthermore, a valuable portion of the practice of set theory is already devoted to the study of large cardinals. An alternative approach seems to be attractive.

In the next section we pursue a treatment of SP suggested by Bays (2000)[6] and show how the strategy can be generalized to HOSPs. The virtue of this alternative approach is that it is neutral about the (im)possibility of axiomatization of mathematical practice.

3. An alternative treatment of higher-order Skolem paradoxes

Bays (2000) provides a solution for SP which appeals to an equivocation between model-theoretic and plain English interpretations of "$\exists x$ (x is uncountable)." We first summarize, with a bit simplification, his solution and then present how Bays' solution can be generalized to HOSPs.

a. Bays on SP

Let M be a countable model for a standard axiomatization of set theory (e.g. ZFC) and let $\Omega(x)$ be an articulation of "x is uncountable" in the language (of ZFC). Since M satisfies the axioms, there is an $m^* \in M$ such that $M \vDash \Omega[m^*/x]$. Bays, then, formulates SP as below: (hereafter *argument (A)*)

1. M is a countable model of ZFC.

2. $\Omega(x)$ says that "x is uncountable."

3. $M \vDash \Omega[m^*/x]$.

∴ 4. $\{x \mid x \in m^*\}$ is uncountable.

5. If M is countable and $m \in M$, so is $\{x \mid x \in m\}$.

∴ 6. $\{x \mid x \in m^*\}$ is countable. (Bays, 2000, 11)

Consider the ordinary English sentence "x is uncountable." This sentence is about the lack of a bijection between x and the natural numbers which could to be extracted as a sentence of ordinary mathematical English that contains only "equals," "is a member of," "not," "if. . . then," and "there is a set y, such that". By symbolizing these expressions with $=$, $\in$, $\neg$, $\rightarrow$, and $\exists$y, respectively, an ordinary English interpretation of Ω(x) arises. Bays denote this by $\Omega_E(x)$. On the other hand, $\Omega_M(x)$ gives $\Omega(x)$ a model-theoretic interpretation by means of M (Bays, 2000, 16).

[6] These materials also can be found in more recent works by Bays, namely 2007 and 2014. Here all references are to the original work, however.

It is not very hard to realize that the truth of the following conditional is integrated with the validity of argument (A):

$$\forall m \in M\, [\Omega_M (m) \Rightarrow \Omega_E (\{x \mid x \in m\})].$$

The reason, roughly, is that the most plausible reading of line 3 is based on the interpretation of $\Omega(x)$ as $\Omega_M(x)$ and the most plausible reading of line 4 is based on the interpretation of $\Omega(x)$ as $\Omega_E(x)$. Going from line 3 to line 4, thus, is contingent on presupposing such a connection between $\Omega_M(x)$ and $\Omega_E(x)$. (Bays, 2000, 16-8)

In order to block argument (A), all that is remained to be done is to show that the conditional is false. There are, as stated by Bays, at least two salient semantical differences between $\Omega_E(x)$ and $\Omega_M(x)$. Firstly, the semantics of $\Omega_E(x)$ and those of $\Omega_M(x)$ sometimes disagree about atomic formulas. For the semantics of $\Omega_E(x)$ interpret the symbol "$\in$" as a simple membership; whereas the semantics of $\Omega_M(x)$ interpret "$\in$" corresponding to the interpretation function for M. Secondly, for non-atomic formulas, there are other disagreements, too. While the semantics of $\Omega_E(x)$ interpret "$\exists x$" as "there is an x, such that", the semantics of $\Omega_M(x)$ interpret the expression "$\exists x$" corresponding to "there is an $x \in M$, such that". These asymmetries between $\Omega_M(x)$ and $\Omega_E(x)$ ensure that the conditional under consideration cannot be true. (Bays, 2000, 26-7)

In sum, according to Bays' formulation of SP, $\Omega_E(x)$ and $\Omega_M(x)$ are first-order formulas which neglecting their semantical difference leads to the paradox. The distinctive feature of Bays' solution is that it is silent with respect to the mathematical practice and its possibility of being formalized. Particularly, unlike the course-book approach, SP is not resolved by moving to the second-order language; all that is said is done in a first-order language.

b. Generalizing Bays' solution for HOSPs

Now, we generalize Bays' solution for HOSPs. In the following, we just talk about the second-order Skolem paradox. Treating other HOSPs is straightforwardly similar. Let κ be the Löwenheim number of a standard second-order axiomatization of set theory, S2. And let M be a model for S2 of cardinality at most κ, its existence is ensured by the *Generalized Löwenheim Theorem*. Now consider $\Psi(x)$ to be an articulation of "x is of cardinality larger than κ" in the language of S2. Since M satisfies S2, there is an $m^* \in M$ such that $M \vDash \Psi[m^*/x]$. We can now formulate the second-order Skolem paradox: (hereafter *argument (A*)*)

1. M is a model of S2 of cardinality at most κ.

2. $\Psi(x)$ says that "x is of cardinality larger than κ".

3. $M \vDash \Psi\, [m^*/x]$.

$\therefore$ 4. $\{x \mid x \in m^*\}$ is of cardinality larger than κ.

5. If M is of a cardinality at most κ, and $m \in M$ so is $\{x \mid x \in m\}$.

$\therefore$ 6. $\{x \mid x \in m^*\}$ is of cardinality at most κ.

Similar to $\Omega_E(x)$, $\Psi_E(x)$ can be introduced as a second-order ordinary English interpretation of $\Psi(x)$ which represents the lack of a one to one function from x into the smallest ordinal with the cardinality κ. Furthermore, like $\Omega_M(x)$, $\Psi_M(x)$ can be put to give a model-theoretic semantics to $\Psi(x)$.

Since the most natural justification of line 3 is to interpret $\Psi(x)$ as $\Psi_M(x)$ and the most natural justification of line 4 is to interpret $\Psi(x)$ as $\Psi_E(x)$, then the validity of argument (A*) will be dependent on the truth of the following conditional:

$$\forall m \in M\, [\Psi_M(m) \Rightarrow \Psi_E(\{x \mid x \in m\})].$$

It is left to argue that this conditional is false. Like Bays' original solution, the semantics of $\Psi_E(x)$ and $\Psi_M(x)$ might differ at least in two ways. First, the semantics of $\Psi_E(x)$ and those of $\Psi_M(x)$ may differ for atomic formulas, because the semantics of $\Psi_E(x)$ interpret the symbol "∈" as a simple membership. But, the semantics of $\Psi_M(x)$ interpret "∈" regarding the interpretation function for M. Second, there are other disparities in more complicated formulas. For second-order quantifiers, the semantics of $\Psi_E(x)$ interpret "$\exists x$" as "there is a set x, such that", whereas the semantics of $\Psi_M(x)$ interpret the expression "$\exists x$" corresponding to "there is a set $x \in M$, such that". These asymmetries between $\Psi_E(x)$ and $\Psi_M(x)$ guarantee that the conditional under consideration is not true.

Thus, we have seen how neglecting semantical disparities between $\Psi_E(x)$ and $\Psi_M(x)$ leads to the second-order paradox. So, as Bays did for SP we can conclude that the analogous second-order paradox neither have any possible consequence about the suitable order for the language by which we may formulate mathematical practice nor imply that there is not a unique language in which we can formulate the practice. Namely, unlike the course-book approach, the second-order paradox is not resolved by moving to a higher-order language; all that is said is done in a second-order language.

4. Conclusion

Higher-order Skolem paradoxes, as introduced here, are puzzling as much as the original Skolem's paradox. The course-book approach to solve the first-order paradox, however, is not generalizable to solve the higher-order paradoxes, unless one concedes that there is no language in which the practice of mathematics (especially set theory) can be formalized. This is not the end of the story, fortunately. Bays' solution to the original paradox has the power to be generalized to solve the higher-order paradoxes. Like Bays' original solution, the generalized one does not have any implication for the (im)possibility of a language in which the practice of mathematics may be formalized. This is a virtue for Bays' solution, and in effect for our generalization of it, that makes them to be superior to the course-book approach.

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