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What Paradoxes Depend on

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Abstract This paper gives a definition of self-reference on the basis of the dependence relation given by Leitgeb (2005), and the dependence digraph by Beringer & Schindler (2015). Unlike the usual discussion about self-reference of paradoxes centering around Yablo's paradox and its variants, I focus on the paradoxes of finitary characteristic, which are given again by use of Leitgeb's dependence relation. They are called 'locally finite paradoxes', satisfying that any sentence in these paradoxes can depend on finitely many sentences. I prove that all locally finite paradoxes are self-referential in the sense that there is a directed cycle in their dependence digraphs. This paper also studies the 'circularity dependence' of paradoxes, which was introduced by Hsiung (2014). I prove that the locally finite paradoxes have circularity dependence in the sense that they are paradoxical only in the digraph containing a proper cycle. The proofs of the two results are based directly on König's infinity lemma. In contrast, this paper also shows that Yablo's paradox and its ∀∃-unwinding variant are non-self-referential, and neither McGee's paradox nor the ω-cycle liar has circularity dependence.

Keywords Circularity · Dependence · Paradox · Self-reference · Truth

1 Introduction

Provided a set of sentences represents a truth-theoretical paradox, is it self-referential or not? For those paradoxes involving in finite sentences, such as the

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Liar paradox, this can be easily answered. But there are also paradoxical sentences whose self-reference is still a controversial issue among contemporary truth theorists. One of such examples is the sentences in Yablo's paradox (Yablo (1993)). There had been lots of discussions around the problem whether the set of Yablo's sentences is self-referential: some people think that it is free of self-reference (see Sorensen (1998), Tennant (1995) and Bringsjord & Heuveln (2003)), while others insist that it is self-referential (see Priest (1997) and Beall (2001)). To a great extent, the point at issue is in what sense we use the word 'self-reference' when we talk about the paradoxical sentences and other pathological ones. As is suggested by Leitgeb (2002, p. 3), we has to put forward a 'clear-cut definition' for the self-reference before asserting some paradox is 'self-referential' or not, otherwise the relevant debates may be 'substantially flawed' because what we use might be an 'unclear and inadequate' notion of self-reference.

Literally, to say a sentence is self-referential is to say this sentence refers to or depends on itself. And so, in order to give the notion of self-reference, we must first define how a sentence depends on or refers to other ones. This is where we need a notion of the 'dependence (or reference) relation' over sentences. Furthermore, to incorporate the indirectly self-referential sentences into the extension of self-reference, we have to consider not only what a sentence depends on, but also what those sentences that the original sentence depends on depend on, and so on. Thus, for any arbitrary set of sentences, we need to provide a device to show how these sentences are related to each other according to the dependence relation. This device is the concept of so-called 'dependence digraph' or 'reference digraph'.

Provided that we work in a sentential language, it is possible to set up the dependence relation and the corresponding dependence digraph by an immediate way. Actually, the dependence relation of sentences is represented as 'sentence nets' and the dependence digraph is defined in terms of the syntactic constituents of sentences. To avoid digressing, I postpone the relevant discussions to the last section. For now, our working language is the standard language for truth and paradoxes, that is, the first-order language for arithmetic with a distinguished predicate T. The situation is different and more complex. Fortunately, we already have had the two necessary concepts for the first-order language. First, the dependence relation was advanced by Leitgeb in his paper 'What truth depends on' (Leitgeb (2005), p. 161). This concept is fundamental for studying the self-reference of paradoxes. I pay attribute to Leitgeb's ingenious idea about the dependence relation by the use of the title of the paper. Second, the dependence digraph was given by Beringer & Schindler (2015). This is a derived concept from Leitgeb's notion of dependence relation.

It is an easy task to give the definition of self-reference once we make clear the dependence relation and dependence digraph of sentences. Actually, Leitgeb (2005, p. 168) gave a formal definition of self-reference by use of his dependence relation. As we will see, Leitgeb's notion of self-reference is something like the notion of direct self-reference. This is too restricted because it throws these indirectly self-referential sentences out of the extension of self-reference. What Leitgeb missed is the concept of dependence digraph. [1] On the other hand, Beringer and Schindler

[1] This is not to reproach Leitgeb but to show a key concept in search of a more general notion of self-reference. After all, Leitgeb's main purpose of using the dependence relation, is not to study self-reference of paradoxes but to give an adequate definition of truth in the first-order language of arithmetic with the unitary predicate T. See Leitgeb (2005), pp. 171ff.

did not give a definition of self-reference yet, because their use of dependence digraph has a different goal.

In this paper, I will give a definition of self-reference on the basis of Leitgeb's dependence relation, and Beringer and Schindler's dependence digraph. After doing that, I will focus on a class of paradoxes of certain finite characteristics, namely the locally finite paradoxes.[2] Again, this is defined in terms of Leitgeb's dependence relation. Roughly speaking, locally finite paradoxes are those paradoxes consisting of only sentences which can depend on a finite set of sentences. The main target of this paper is to investigate the self-reference and related issues of locally finite paradoxes (together with some typical non-locally-finite paradoxes as a contrast). The main results I will establish are summarized in Table 1.

Paradoxes / Features	Test Group	Control Group	
	Locally finite paradoxes	Yablo's paradox and its variants	ω-cycle liar and McGee's paradox
Self-reference	✓	×	✓
Circularity-dependence	✓	✓	×

Table 1

The third row in Table 1 stands for the results about the self-reference of paradoxes. On one hand, all locally finite paradoxes are self-referential. More precisely, if a locally finite set of sentences is paradoxical, it is self-referential (Theorem 2). On the other hand, there are some non-locally-finite paradoxes that are not self-referential: Yablo's paradox and its variants are such examples, and at the same time, there are also some non-locally-finite but self-referential paradoxes such as the ω-cycle liar and McGee's paradox.

A point worth emphasizing is that the main object of our study is the locally finite paradoxes rather than Yablo's paradox or any variant of it. To give a formal definition of self-reference, we should first capture our intuition about those paradoxes whose self-reference is undisputed. As we will see, it is these locally finite paradoxes that include all the known paradoxes of finitary characteristics, whose self-reference is generally accepted by philosophers — no matter how they consider the self-reference of Yablo's paradox. In this sense, a reasonable definition of self-reference must, above all, pass the test of these paradoxes of finitary characteristics. That is why we take these paradoxes into the 'test group'. Of course, for an adequate definition of self-reference, this is only a necessary condition and other aspects must be considered. For instance, one may wonder whether there exist some examples of non-self-referential paradoxes according to the definition. The existence of such examples will make the notion of self-reference more interesting. At this point, the paradoxes in the control group play their role, and they are the candidates free of self-reference.

As far as I know, most current controversies about self-reference of paradoxes center around Yablo's paradox or its variants. However, Yablo's paradox is not so important as the locally finite paradoxes in the present work. When I prove that it

[2] The notion of local finiteness, in the form of digraph, was offered by Rabern et al. (2013, p. 754) in the context of sentential language. See the discussion of the last section. By the way, by a paradox I always mean a paradoxical set of sentences.

is not self-referential (Example 1), my main purpose is not to settle disputes about the self-reference of Yablo's paradox. Rather, Yablo's paradox is merely something like an element in the control group: it is a typical non-locally-finite paradox. My focus is still on the class of locally finite paradoxes. Let me stress again: if out target is to define a reasonable notion of self-reference, the test group, that is, the main object that we should study, is those undisputedly self-referential paradoxes. In this respect, Yablo's paradox and its variants are no more than members in the control group.

The fourth row of Table 1 is about the circularity dependence of paradoxes, which is a notion similar to the notion of self-reference but different from the latter. The notion of circularity dependence is introduced by Hsiung (2014, p. 35). And this notion is founded on the notion of paradoxicality in a digraph (Hsiung (2009a), p. 248). Roughly, when we say a paradox have circularity dependence, we mean it is paradoxical only in a digraph containing a proper cycle. The idea behind the notion of paradoxicality in a digraph is that paradoxes are conditionally contradictory. As we all know, paradoxical sentences lead to a contradiction, but unlike those contradictory sentences such as 'the snow is white and it is not white', they are not absolutely contradictory so that we could find some way to make them consistent with our cherished theories. Remember that in classical logic, a contradiction follows from applying Tarski's famous T-scheme (that is, $T\ulcorner A\urcorner$, iff A) to any paradoxical sentence. Now if we replace Tarski's T-scheme by Scheme (1), then whether a contradiction follows depends on what paradox is applied to Scheme (1) and what digraph is involved in this scheme.

$$T\ulcorner A\urcorner \text{ (holds) at } v, \text{ iff } A \text{ (holds) at } u, \tag{1}$$

where u and v are any points in the domain of a digraph such that u bears the binary relation of the digraph to v. It is at this point that we can use digraphs to show under what conditions a paradox actually generates a contradiction.[3] For instance, it can be proved that the Liar (sentence) is paradoxical in and only in the digraphs containing an odd cycle (Hsiung (2009a), p. 253).

Return to Table 1. The second main result I will prove about the locally finite paradoxes is that if a locally finite set of sentences is paradoxical in a digraph, then there is some proper cycle in this digraph (Theorem 5). Or briefly, all locally finite paradoxes have circularity dependence. We provide specific examples to illustrate this general result. Among the locally finite paradoxes, the simplest one is the Liar paradox. It has circularity dependence: as is just mentioned, the Liar is paradoxical only in those digraphs containing an odd cycle. More generally, for any positive number $n = 2^i(2j+1)$, the n-cycle liar (sometimes also called the n-liar) is paradoxical in a digraph, iff there is a cycle whose depth is indivisible by 2^{i+1} in this digraph (Hsiung (2014), p. 26). What is more, this can be further generalized to the Boolean paradoxes (Hsiung (2017)). From these example, we can see how their paradoxicality conditions are related to some certain circularity. The paradoxes in these examples, as elements in test group, are concrete examples of locally finite paradoxes. The second main result about locally finite paradoxes shows that all these paradoxes share a common property: the circularity dependence. On the other hand, there are non-locally-finite examples which have no

[3] Of course, this is merely a technical explanation, and for more philosophical motivations about paradoxicality in a digraph, please refer to Hsiung (2009a).

circularity dependence. The ω-cycle liar and McGee's paradox (but not Yablo's paradox or its variants) are such examples.

The structure of the paper is as follows. In Section 2, we first give two basic notions: one is Leitgeib's dependence relation, and the other is the notion of paradoxicality in a digraph. Then in Section 3, I generalize Leitgeib's notion of self-reference by use of his dependence relation, and then prove the self-reference of locally finite paradoxes. The circularity-dependence of locally finite paradoxes and related results will be proved in section 4. As we will see, both proofs are an application of König's infinity lemma. In section 5, we turn to discuss some typical non-locally-finite paradoxes. In the last section, we compare the present approach to the self-reference with the one in the context of a sentential language.

Graph-theoretical Preliminaries. A digraph (or a relational frame) is a pair $\langle G, R\rangle$, consisting of a non-empty set G, and a binary relation R on G.[4] In a digraph $\mathcal{G} = \langle W, R\rangle$, two points u and v in W are *adjacent* if either $u_i \, R \, u_{i+1}$ or $u_{i+1} \, R \, u_i$ holds. Let $u_0, u_1, \ldots, u_l$ be points in W. If u_i and u_{i+1} are adjacent for all $0 \leq i < l$, the sequence $\xi = u_0 \, u_1 \, \ldots \, u_l$ is a *walk* from u_0 to u_l in $\mathcal{G}$, u_0 and u_l are two *endpoints* of ξ, and l is the *length* of ξ.

(a) ξ is *closed*, if its two endpoints are equal, i.e. $u_0 = u_l$.
(b) ξ is *directed*, if $u_i \, R \, u_{i+1}$ for all $0 \leq i < l$.
(c) ξ is *a cycle*, if none of the points in ξ is repeated except that $u_0 = u_l$.

a *loop* is a directed cycle of length 1. An *odd cycle* is a cycle whose length is an odd number. A *directed acyclic digraph* (DAG) is a digraph which has no directed cycles (or equivalently, no closed directed walks, see Lemma 3).

Let $\xi = u_0 \, u_1 \, \ldots \, u_l$ be a walk in $\mathcal{G}$. *A sub-walk of* ξ is a walk from $u_a u_{a+1} \ldots u_b$ for some number a and b with $1 \leq a \leq b \leq l$. The *inverse* of ξ is the walk $u_l \, u_{l-1} \, \ldots \, u_0$, which is denote by ξ^-. Let $\zeta = v_0 v_1 \ldots v_m$ be another walk in $\mathcal{G}$. If $u_l = v_0$, we can define the *concatenation of* ξ *and* ζ (at u_l), denoted by $\xi^\frown\zeta$, to be $w_0 w_1 \ldots w_{l+m}$, where $w_i = u_i$ for $0 \leq i \leq l$ and $w_i = v_{i-l}$ for $l < i \leq l+m$.

A digraph is *connected*, if any two different points are connected by some walk. A *connected component* of a digraph is a sub-digraph of this digraph such that it is connected but any proper super-digraph of it is not connected. A digraph is *minimal reflexive*, if it consists of a single point that bears the binary relation R to itself. Clearly, all minimal reflexive digraphs are isomorphic, and so we can say *the* minimal reflexive digraph.

2 Dependence relation and Paradoxicality

Let $\mathscr{L}$ be the first-order language of the arithmetic, which includes S, $+$, $\cdot$ and $\mathbf{0}$ as its non-logical symbols. Let $\mathscr{L}^+$ be the language obtained from $\mathscr{L}$ by augmenting a distinguished unary predicate symbol T. Unless otherwise claimed, when we say a formula, we mean a formula of $\mathscr{L}^+$. We will also use $\mathscr{L}^+$ to denote the set of all sentences, and so by $A \in \mathscr{L}^+$, we mean A is a sentence of $\mathscr{L}^+$. The intended model of the language $\mathscr{L}$ is $\mathfrak{N} = \langle \mathbb{N}, ', +, \cdot, 0\rangle$, that is, the structure of natural numbers. Correspondingly, for $\mathscr{L}^+$, we will only consider those models of the form $\langle \mathfrak{N}, X\rangle$, where $X \subseteq \mathbb{N}$ is the extension of T. We can routinely define

[4] The digraph we define is actually the digraph without parallel directed edges. This restriction does not lose any generality for our purpose.

$\mathcal{V}_{\mathfrak{N},X}(A)$, i.e., the truth value of A in the model $\langle\mathfrak{N}, X\rangle$. Since the ground model $\mathfrak{N}$ is always fixed, we use $\mathcal{V}_X(A)$ instead of $\mathcal{V}_{\mathfrak{N},X}(A)$. When $\mathcal{V}_X(A) = \mathrm{T}(\mathrm{F})$, we will say A is true (false) for X. Sometimes, we also use $X \models A$ for $\mathcal{V}_X(A) = \mathrm{T}$. For brevity, we use $A \equiv B$ to denote that $A \leftrightarrow B$ is true for all $X \subseteq \mathbb{N}$.

For a sentence A, we use $\ulcorner A \urcorner$ for the Gödel's number of A, and $\overline{\ulcorner A \urcorner}$ for the corresponding numeral to the number $\ulcorner A \urcorner$. But, to avoid too many complications, we will often identify $\ulcorner A \urcorner$ with $\overline{\ulcorner A \urcorner}$, and identify a set Σ of sentences with the set of the Gödel's number of all sentences in Σ. For example, we will use $T\ulcorner A \urcorner$ instead of $T\left(\overline{\ulcorner A \urcorner}\right)$, and use $\mathcal{V}_\Sigma(A)$ instead of $\mathcal{V}_{\{\ulcorner B \urcorner \mid B \in \Sigma\}}(A)$. For any $n \geq 0$, define inductively $T^n\ulcorner A \urcorner$ as follows: $T^0\ulcorner A \urcorner = A$ and $T^{n+1}\ulcorner A \urcorner = T\ulcorner T^n\ulcorner A \urcorner \urcorner$ for $n \geq 0$.

Our method of constructing the paradoxes is the standard one via Gödel's diagonal lemma. For instance, by use of Gödel diagonalization, we can construct the Liar sentence λ, which satisfies the equivalence $\lambda \equiv \neg T\ulcorner \lambda \urcorner$. More generally, for any positive number n, we can construct a sentence λ_1^n in $\mathscr{L}^+$ such that $\lambda_1^n \equiv \neg T^n\ulcorner \lambda_1^n \urcorner$.

Leitgeb's dependence relation is defined as follows.

Definition 1 (Leitgeb (2005), p. 161) Let A be a sentence and Σ be a set of sentence. We define A *depends on* Σ, if for any $\Gamma_1, \Gamma_2 \subseteq \mathscr{L}^+$,

$$\Sigma \cap \Gamma_1 = \Sigma \cap \Gamma_2 \Rightarrow \mathcal{V}_{\Gamma_1}(A) = \mathcal{V}_{\Gamma_2}(A).$$

Informally, A depends on Σ, if and only if the truth value of A is determined by whether the sentences of Σ are present in the extension of the truth predicate. An equivalent definition is as follows: A depends on Σ: if for any $\Gamma \subseteq \mathscr{L}^+$, $\mathcal{V}_\Gamma(A) = \mathcal{V}_{\Sigma\cap\Gamma}(A)$ (Leitgeb (2005), 161).

Lemma 1 (Leitgeb (2005), p. 161) *For every sentence A of $\mathscr{L}^+$, we have:*

(a) *A depends on the set of all sentences of $\mathscr{L}^+$.*
(b) *If A depends on Σ and $\Sigma \subseteq \Gamma$, then A also depends on Γ.*
(c) *If A depends on both Σ and Γ, then A depends on $\Sigma \cap \Gamma$.*

When A depends on Σ, we will say Σ is a *dependence set* of A. We will use $\mathfrak{D}(A)$ to denote the family of all dependence sets of A. By Lemma 1, $\mathfrak{D}(A)$ is a filter and will be called the *dependence filter* of A. If $\mathfrak{D}(A)$ is a principal filter, then it contains a least set (about the set inclusion relation). In such a case, we will say A *essentially* depends on this least set. As Leitgeb has pointed out, not every sentence can essentially depends on a set. For instance, the sentence $\forall x \exists y(y > x \wedge T\ulcorner y = y \urcorner)$ depends on all cofinite subsets of the set $\{\mathbf{n} = \mathbf{n} \mid n \in \mathbb{N}\}$, but it does not depends on the empty set. Thus, this sentence does not essentially depends on any set.

Definition 2 (Beringer & Schindler (2015)) f is a *choice function*, if it is a function from $\mathscr{L}^+$ to the powerset of $\mathscr{L}^+$, such that $f(A) \in \mathfrak{D}(A)$ for every sentence A. For a choice function f, we define a binary relation $\prec_f$ on $\mathscr{L}^+$ as follows: for all sentences A and B, $A \prec_f B$, iff $B \in f(A)$. Let Σ be a set of sentence and let f be a choice function. We will say $\langle\Sigma, \prec_f\rangle$ is a *dependence digraph* of Σ.[5]

[5] In the digraph $\langle\Sigma, \prec_f\rangle$, $\prec_f$ is actually the restriction relation $\prec_f\rceil_{\Sigma\times\Sigma}$. This is always clear, and so the subscript is omitted.

The present study is also closely related to the revision theory of truth which has been mainly developed by Gupta & Belnap (1993). A basic notion of the revision theory is the revision sequence, which was originally defined for arbitrarily large length by Gupta (1982, p. 10) and Herzberger (1982, p. 68). But for the present purpose, we only need to consider the revision sequences of length ω.

Definition 3 (Gupta (1982); Herzberger (1982)) For a set Σ of sentences, define $\Sigma^r = \{A \in \mathscr{L}^+ \mid \Sigma \models A\}$. Define a sequence $\Sigma_0, \ldots, \Sigma_k, \ldots$ as follows: $\Sigma_0 = \Sigma$, and $\Sigma_{k+1} = \Sigma_k^r$ for all $k \geq 0$. This sequence is called the *revision sequence* starting from Σ.

We will generalize the notion of the revision sequence. To motivate the generalization, we recall that to say a set of sentences is paradoxical is to say there is no interpretation of T such that Tarski's scheme $T\ulcorner A \urcorner \leftrightarrow A$ holds for all A in this set. A precise definition is as follows.

Definition 4 A set Σ of sentences is *paradoxical*, if there is no Γ satisfying the condition: $\Gamma \cap \Sigma = \Gamma^r \cap \Sigma$. That is, there is no Γ such that for any $A \in \Sigma$, $\mathcal{V}_\Gamma(T\ulcorner A \urcorner) = \mathcal{V}_\Gamma(A)$.

From now on, we always use $\mathcal{G}$ to denote the digraph $\langle W, R \rangle$ unless otherwise claimed. See the end of section 1 for the definitions of the digraphs and related notions.

Definition 5 (Hsiung (2009a), pp. 243-244) Let Σ be a set of sentences. t: $W \to \mathscr{P}(\mathscr{L}^+)$ is a *revision mapping* for Σ in $\mathcal{G}$, if for all $u, v \in W$ satisfying $u\,R\,v$,

$$t(v) \cap \Sigma = t(u)^r \cap \Sigma \tag{2}$$

Σ is *paradoxical* in $\mathcal{G}$, if there is no revision mapping for Σ in $\mathcal{G}$.

When Σ is the set of all sentences, W is the set of natural numbers and R is the successor relation between natural numbers, a revision mapping t for Σ in $\mathcal{G}$ is a revision sequence starting from the set $t(0)$. And so the revision sequence is a special instance of the revision mapping. And the notion of being paradoxical in a digraph is also a generalization of being paradoxical. Actually, in the minimal reflexive frame, the equation (2) is collapsed to the equation of Definition 4, and so Σ is paradoxical, iff it is paradoxical in the minimal reflexive digraph. Note also that (2) is equivalent to

$$\text{for all } A \in \Sigma,\ \mathcal{V}_{t(v)}(T\ulcorner A \urcorner) = \mathcal{V}_{t(u)}(A).$$

And so the biconditional (2) is a formal representation of biconditional (1) in $\mathscr{L}^+$. Hence, when a set of sentences is paradoxical in a digraph, we can think that it is impossible to evaluate these sentences (without contradiction) in the digraph such that scheme (1) holds for all of these sentences.

Definition 6 (Hsiung (2009a), pp. 248, 254) Let Σ, Γ be two sets of sentences. Define $\Sigma \leq_P \Gamma$, if for any digraph $\mathcal{G}$, whenever Σ is paradoxical in $\mathcal{G}$, Γ is also paradoxical in $\mathcal{G}$. Define $\Sigma \equiv_P \Gamma$, if $\Sigma \leq_P \Gamma$ and $\Gamma \leq_P \Sigma$. Define $\Sigma <_P \Gamma$, if $\Sigma \leq_P \Gamma$ but $\Sigma \not\equiv_P \Gamma$.

Note that $\equiv_P$ is an equivalence relation. When $\Sigma \equiv_P \Gamma$, we will say Σ and Γ have the same *degree of paradoxicality*. When $\Sigma <_P \Gamma$, we say Σ has a (strictly) lower degree of paradoxicality than Γ.

3 Locally finite Paradoxes and Self-reference

Leitgeb gave a definition of self-reference in terms of his dependence relation. He defined that a sentence A is self-referential, if A belongs to any dependence set of it, i.e., for any set Σ of sentences, whenever A depends on Σ, A is an element of Σ (Leitgeb (2005), p. 168). On the basis of Leitget's definition, we now provide a more general notion of self-reference.

Definition 7 A sentence A is *self-referential*, if for all choice function f, there are finitely many sentences $A_1, \ldots, A_k$ such that $A \prec_f A_1 \prec_f \ldots \prec_f A_k \prec_f A$. In case $k = 0$ (that is, $A \prec_f A$) for all f, A is *directly self-referential*; otherwise, A is indirectly self-referential.

Leitget's self-reference is actually equivalent to the present definition of direct self-reference. To see this, only note that to say that for any set Σ, $\Sigma \in \mathfrak{D}(A)$ implies $A \in \Sigma$, is to say that for any choice function f, $A \in f(A)$, i.e., $A \prec_f A$.

Take the sentence λ_1^n as an example again. For $1 \le i < n$, let $\lambda_{i+1}^n = T\ulcorner\lambda_i^n\urcorner$. Then λ_{i+1}^n essentially depends on $\{\lambda_i^n\}$. We also have λ_1^n essentially depends on $\{\lambda_n^n\}$ since $\lambda_1^n \equiv \neg T\ulcorner\lambda_n^n\urcorner$. And so for any choice function f,

$$\lambda_1^n \prec_f \lambda_n^n \prec_f \ \ldots \prec_f \lambda_3^n \prec_f \lambda_2^n \prec_f \lambda_1^n.$$

When $n = 1$, we have $\lambda_1^n \prec_f \lambda_1^n$. It means that λ_1^1, i.e., the Liar sentence, is directly self-referential. Now suppose $n > 1$, we consider a choice function, say f_0, such that $f_0(\lambda_1^n) = \{\lambda_n^n\}$. Clearly, λ_1^n does not belong to the set $\{\lambda_n^n\}$. That means $\lambda_1^n \prec_{f_0} \lambda_1^n$ fails. It follows that when $n > 1$, λ_1^n is self-referential but not directly self-referential. In this case, λ_1^n is indirectly self-referential. See Figure 1, where the arrow stands for the relation $\prec_f$, and the points corresponding to the sentences λ_2^n, λ_3^n and so on are hollow since these sentences are merely auxiliary.

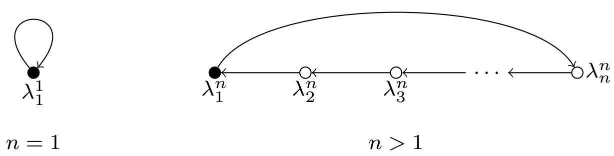

Fig. 1: Self-reference of λ_1^n

In the above example, we find it is more convenient to consider the set $\{\lambda_i^n \mid 1 \le i \le n\}$ rather than the single sentence λ_1^n. The set $\{\lambda_i^n \mid 1 \le i \le n\}$ will be called 'n-cycle liar', denoted by λ^n.[6] We now introduce the notion of self-reference for a set of sentences.

[6] The n-cycle liar is called the 'Liar cycle' in Leitgeb (2005, p. 164). It is also known as 'n-liar'. Generally, we can define α-liar for all ordinals α. Herzberger (1982, pp. 74-75) and Yablo (1985, p. 340). The case for $\alpha = \omega$ will be given in section 5. But we will use the term 'ω-cycle liar' rather than 'ω-liar', because the latter may cause confusing: for instance, the term 'ω-liar' sometimes is also used for Yablo's paradox. See Yablo (2004, p. 140).

Definition 8 A set of sentences is *self-referential*, iff each dependence digraph of it contains at least a closed directed walk. In particular, it is *directly self-referential*, iff each dependence digraph of it contains at least a loop.

The following notions are useful to study the self-reference of sets of sentences.

Definition 9 Let Σ be a set of sentences.

(a) Σ is *normal*, if every sentence of Σ essentially depends on some set.
(b) Σ is *locally finite*, if every sentence of Σ has a finite dependence set.

Lemma 2 (a) *If A depends on a finite set, then A essentially depends on a subset of this set.*
(b) *Any locally finite set of sentences is normal.*

Proof. (a) Suppose A depends on a finite set Σ. Let $\mathcal{D}$ be the set of all the subsets Γ of Σ such that A depends on Γ. By the hypothesis, $\Sigma \in \mathcal{D}$, and so $\mathcal{D}$ is non-empty. Let Γ_0 be the intersection of all the sets of $\mathcal{D}$. Since Σ is finite, $\mathcal{D}$ is also finite. By (c) of Lemma 1, A depends on Γ_0. Now for every set Γ, if A depends on Γ, then by (c) of Lemma 1 again, $\Gamma \cap \Sigma$ belongs to $\mathcal{D}$. It follows that $\Gamma_0 \subseteq \Gamma \cap \Sigma$, and $\Gamma_0 \subseteq \Gamma$. Consequently, A essentially depends on Γ_0.

(b) is immediate from (a). □

The following two graph-theoretical results are useful in studying the self-reference of a locally finite set of sentences.

Lemma 3 *A digraph contains a closed directed walk, iff it contains a directed cycle.*

The proof of this lemma is omitted, since we will late prove a similar but stronger result (see (b) of Lemma 5). By this lemma, we can see a set of sentences is self-referential, iff none of its dependence digraphs is a DAG. A basic property of a finite DAG is that its points can be linearly ordered such that whenever there is an edge from a point u to another v, then u comes after v in the ordering. The relevant notions are as follows.

Definition 10 Let $\mathcal{G} = \langle W, R\rangle$ be a digraph. A point u in W is a *sink* of $\mathcal{G}$, if for any v in W, $u\,R\,v$ always fails. A function $g : W \to \mathbb{N}$ is a *topological sorting* of $\mathcal{G}$, if for any u and v in W, whenever $u\,R\,v$, $g(v) < g(u)$.[7]

Lemma 4 (a) *Any finite DAG contains at least a sink.*
(b) *Any finite DAG has a topological sorting.*

Proof. (a) Let $\mathcal{G} = \langle W, R\rangle$ be a finite DAG. Assume no point of $\mathcal{G}$ is a sink, then we can find an infinite sequence of points in W, $u_0\, u_1\, u_2 \ldots$, such that $u_0\, R\, u_1\, R\, u_2\, R \ldots$. But W contains only finitely many points, there must be two repeated points in the above sequence. And so there is at least a closed directed walk in $\mathcal{G}$, but this is contradictory with the hypothesis that $\mathcal{G}$ is a DAG.

(b) Let $\mathcal{G}_0 = \mathcal{G}$, and by (a), we can find a sink in $\mathcal{G}_0$, and let it be u_0. Let $\mathcal{G}_1$ be the digraph obtained by deleting u_0 from $\mathcal{G}_0$. Then $\mathcal{G}_1$ is still a finite DAG. And

[7] The ordering we give here by the function g is the converse of the usual topological sorting of a digraph (see for instance, Cormen et al. (2009), p. 612). Such a ordering streamlines our induction in the proof of Theorem 1.

so, we can find a sink in $\mathcal{G}_1$, namely u_1. This process can be iterated indefinitely. In general, for any $k \geq 0$, let u_k be a sink of $\mathcal{G}_k$, and let $\mathcal{G}_{k+1}$ be the digraph obtained by deleting u_k from $\mathcal{G}_k$. Since W is finite, we can find a least number m such that $\mathcal{G}_{m+1}$ is empty. In that case, $W = \{u_i \mid 0 \leq i \leq m\}$. Note that for any $0 \leq i, j \leq m$, if $i \leq j$, then $u_i \, R \, u_j$ fails (otherwise u_i is not a sink of $\mathcal{G}_i$). Thus, if $u_i \, R \, u_j$, then $j < i$. Now define g such that $g(u_i) = i$ for $0 \leq i \leq m$. Then g is clearly a topological sorting of $\mathcal{G}$. □

Theorem 1 *If a finite set of sentences is paradoxical, then it is self-referential.*

Proof. Assume Σ is not self-referential, then there exists a dependence digraph of Σ, namely $\mathcal{G} = \langle \Sigma, \prec_f \rangle$, which is a DAG. We will show that Σ is not paradoxical.

By (b) of Lemma 4, $\mathcal{G}$ has a topological sorting, namely g. Let m be the greatest value of g. Define Σ'_k inductively as follows: $\Sigma'_0 = \Sigma$, and for any $k \geq 0$,

$$\Sigma'_{k+1} = \left\{ A \in \Sigma \mid \mathcal{V}_{\Sigma'_k}(A) = \mathrm{T} \right\}.$$

For brevity, $\Sigma'_k(A)$ is a shorthand for $\mathcal{V}_{\Sigma'_k}(A)$. We claim that for all $0 \leq l \leq m$ and for all $A \in \Sigma$, if $g(A) \leq l$, then for all $k \geq l$, $\Sigma'_{k+1}(A) = \Sigma'_k(A)$. If this claim is true, then in particular, we fix $l = m$, and then get for all $A \in \Sigma$, $\Sigma'_{k+1}(A) = \Sigma'_k(A)$ whenver $k \geq m$. That is to say, the sequence $\left\langle \Sigma'_k \mid k \in \mathbb{N} \right\rangle$ actually gets to a fixed point after stage m. It follows immediately that for every $A \in \Sigma$, $\Sigma'_{m+1}(T \ulcorner A \urcorner) = \Sigma'_{m+1}(A)$. Hence, Σ is not paradoxical.

We prove the above claim by induction on l. When $g(A) \leq 0$, A must be a sink in $\mathcal{G}$. Then by definition of the choice function, $\Sigma \cap f(A) = \varnothing$, otherwise $A \prec_f B$ for some $B \in \Sigma$, but this is impossible since A is a sink in $\mathcal{G}$. And so for all $k \geq 0$, $\Sigma'_{k+1} \cap f(A) = \Sigma'_k \cap f(A) = \varnothing$, and then by definition 1, $\Sigma'_{k+1}(A) = \Sigma'_k(A)$. Next, suppose $g(A) \leq l + 1$, then for any $B \in \Sigma$, provided $A \prec_f B$, $g(B) \leq l$. Then by inductive hypothesis, for all $k \geq l$, $\Sigma'_{k+1}(B) = \Sigma'_k(B)$. We notice $f(A) = \{B \mid A \prec_f B\}$, and so for all $B \in \Sigma \cap f(A)$ and for all $k \geq l$, $B \in \Sigma'_{k+2}$, iff $B \in \Sigma'_{k+1}$. That means for all $k \geq l$, $\Sigma'_{k+2} \cap f(A) = \Sigma'_{k+1} \cap f(A)$. Hence, by definition of dependence relation again, $\Sigma'_{k+2}(A) = \Sigma'_{k+1}(A)$ holds for all $k \geq l$. Therefore, for all $k \geq l+1$, $\Sigma'_{k+1}(A) = \Sigma'_k(A)$. As a result, for all $A \in \Sigma$, whenever $g(A) \leq l$, we can get for all $k \geq l$, $\Sigma'_{k+1}(A) = \Sigma'_k(A)$. The claim is proved. □

Recall that a tree is a partial order $\langle \mathcal{T}, < \rangle$ satisfying the following conditions: (i) $\mathcal{T}$ contains a unique minimal element (the root of $\mathcal{T}$); (ii) For any $\sigma \in \mathcal{T}$, the set $\{\sigma' \in \mathcal{T} \mid \sigma' < \sigma\}$ is a finite subset of $\mathcal{T}$ that is totally ordered by $<$. For brevity, $\{\sigma' \in \mathcal{T} \mid \sigma' < \sigma\}$ is denoted by $\mathrm{pd}_{\mathcal{T}}(\sigma)$. The height of σ is defined to be the size of $\mathrm{pd}_{\mathcal{T}}(\sigma)$. For $n \in \mathbb{N}$, the n-th level of $\mathcal{T}$ is the set of all $\sigma \in \mathcal{T}$ such that the height of σ is equal to n. σ' is an immediate successor of σ, if $\sigma < \sigma'$ and the height of σ' is the height of σ plus 1. A tree is finitely branching, if every point in the tree has only finitely many (possibly zero) immediate successors. A branch of a tree is a maximal totally ordered subset of $\mathcal{T}$. König's infinity lemma says that if a tree is infinite and finitely branching, then it has an infinite branch. For a proof of this lemma, see for instance Just & Weese (1997, p. 31).

Theorem 2 *If a locally finite set of sentences is paradoxical, it is self-referential.*

Proof. Let Σ be an infinite but locally finite set of sentences. Suppose it is not self-referential. We will use König's infinity lemma to prove Σ is not paradoxical.

By the supposition, there is a choice function f such that the dependence digraph $\mathcal{G} = \langle \Sigma, \prec_f \rangle$ is a DAG. Since Σ is locally finite, there is a choice function f' such that for all $A \in \Sigma$, $f'(A)$ is a finite set. Define a function f'' as follows: $f''(A) = f(A) \cap f'(A)$. By (c) of Lemma 1, f'' is also a choice function. And for all $A \in \Sigma$, $f''(A)$ is a finite set. Furthermore, $\langle \Sigma, \prec_{f''} \rangle$ is a DAG, since it is a sub-digraph of $\mathcal{G}$. Now by replacing f with f'', we can make a further supposition about the original f: for all $A \in \Sigma$, $f(A)$ is a finite set.

Let $\Sigma = \{A_k \mid k \in \mathbb{N}\}$. For brevity, we use Σ_k for the set $\{A_i \mid i < k\}$, and $\mathbb{N}_k$ for $\{i \in \mathbb{N} \mid i < k\}$. We now create a tree as follows. For $k \in \mathbb{N}$, let us say $\sigma : \mathbb{N}_k \to \{\mathrm{T}, \mathrm{F}\}$ is a k-sequence, if there exists a subset of Σ_k, namely Γ_k, such that for all $i < k$, (i) $\mathcal{V}_{\Gamma_k}(T\ulcorner A_i \urcorner) = \mathcal{V}_{\Gamma_k}(A_i)$ and (ii) $\sigma(i) = \mathrm{T}$, iff $A_i \in \Gamma_k$. k will be called the length of σ, denoted by $|\sigma|$. Let $\mathcal{T}$ be the set of all the k-sequences for $k \in \mathbb{N}$. Define a binary relation $<$ on $\mathcal{T}$ as follows: $\sigma < \sigma'$, iff $|\sigma| < |\sigma'|$, and for all $i < |\sigma|$, $\sigma(i) = \sigma'(i)$. It can be easily seen that $\langle \mathcal{T}, < \rangle$ is a tree with the 0-sequence (the empty sequence) as its root.

For any $k \geq 0$, by Theorem 1, there exists a set $\Gamma_k \subseteq \Sigma_k$, such that for all $i < k$, $\mathcal{V}_{\Gamma_k}(T\ulcorner A_i \urcorner) = \mathcal{V}_{\Gamma_k}(A_i)$. Define $\sigma_k : \mathbb{N}_k \to \{\mathrm{T}, \mathrm{F}\}$ such that $\sigma_k(i) = \mathrm{T}$, iff $A_i \in \Gamma_k$. Then σ_k is an element in the kth level of $\langle \mathcal{T}, < \rangle$. Thus, $\langle \mathcal{T}, < \rangle$ is an infinite tree. Furthermore, $\langle \mathcal{T}, < \rangle$ is finitely branching (it is actually a binary tree). Now by König's infinity lemma, the tree $\langle \mathcal{T}, < \rangle$ has an infinite branch. We put σ^*_k to be the restriction of this infinite branch up to the k-th level of the tree. Specifically, we can choose an infinite branch τ of $\langle \mathcal{T}, < \rangle$, and let $\sigma^*_k = \tau \restriction_{\mathbb{N}_k}$. Let $g = \bigcup_{k \geq 0} \sigma^*_k$, and let $\Gamma = \{A_k \mid g(k) = \mathrm{T}\}$. We prove that for all $k \geq 0$, $\mathcal{V}_\Gamma(T\ulcorner A_k \urcorner) = \mathcal{V}_\Gamma(A_k)$.

For any $k \geq 0$, by the supposition that $f(A_k)$ is a finite set, we can choose a number n_k such that $k < n_k$ and $f(A_k) \subseteq \Sigma_{n_k}$. In order to prove $\mathcal{V}_\Gamma(T\ulcorner A_k \urcorner) = \mathcal{V}_\Gamma(A_k)$, we notice that by definition of $\mathcal{V}_\Gamma$, $\mathcal{V}_\Gamma(T\ulcorner A_k \urcorner) = \mathrm{T}$, iff $A_k \in \Gamma$; by definition of Γ, $A_k \in \Gamma$, iff $g(k) = \mathrm{T}$; by definition of g and the fact that $k < n_k$, $g(k) = \mathrm{T}$, iff $\sigma^*_{n_k}(k) = \mathrm{T}$; by definition of σ^*_k, $\sigma^*_{n_k}(k) = \mathrm{T}$, iff $A_k \in \Gamma_{n_k}$; by definition of $\mathcal{V}_{\Gamma_{n_k}}$, $A_k \in \Gamma_{n_k}$, iff $\mathcal{V}_{\Gamma_{n_k}}(T\ulcorner A_k \urcorner) = \mathrm{T}$; and by condition for Γ_{n_k}, $\mathcal{V}_{\Gamma_{n_k}}(T\ulcorner A_k \urcorner) = \mathrm{T}$, iff $\mathcal{V}_{\Gamma_{n_k}}(A_k) = \mathrm{T}$. Thus it suffices to prove that $\mathcal{V}_{\Gamma_{n_k}}(A_k) = \mathrm{T}$, iff $\mathcal{V}_\Gamma(A_k) = \mathrm{T}$.

Note that Γ_{n_k} ($\subseteq \Sigma_{n_k}$) is the set corresponding to the n_k-sequence $\sigma^*_{n_k}$. Since A_k depends on $f(A_k)$, in order to show the last equivalence in the preceding paragraph, it is sufficient to prove $\Gamma_{n_k} \cap f(A_k) = \Gamma \cap f(A_k)$. First, suppose $A_i \in \Gamma_{n_k}$, then $\sigma^*_{n_k}(i) = \mathrm{T}$. Hence, $g(i) = \mathrm{T}$, and so $A_i \in \Gamma$. We thus obtain $\Gamma_{n_k} \subseteq \Gamma$. It follows $\Gamma_{n_k} \cap f(A_k) \subseteq \Gamma \cap f(A_k)$. And the converse is also true because $f(A_k)$ is already included in the set Σ_{n_k} by our choice of n_k: provided $A_i \in \Gamma \cap f(A_k)$, then $i < n_k$ and $g(i) = \mathrm{T}$, and so $\sigma^*_{n_k}(i) = \mathrm{T}$, and finally we have $A_i \in \Gamma_{n_k}$. □

4 Locally finite Paradoxes and Circularity-dependence

In this section, we will prove that the locally finite sets of sentences are paradoxical only in those frames containing a proper cycle. Thus, these locally finite paradoxes do depend on circularity in the following sense: they can generate some contradiction only if the digraphs by which we evaluate them satisfy some circularity condition.

Definition 11 (Hsiung (2014), p. 26) Let $\mathcal{G} = \langle G, R\rangle$ be a digraph. Define a mapping $d_{\mathcal{G}}$ from the set of all walks in $\mathcal{G}$ to the set of natural numbers as follows: for any world $u \in W$, $d_{\mathcal{G}}(u) = 0$; and for any walk $\xi = u_0\, u_1\, \ldots\, u_l\, u_{l+1}$ $(l \geq 0)$,

$$d_{\mathcal{G}}(\xi) = \begin{cases} d_{\mathcal{G}}\left(u_0\, u_1\, \ldots\, u_l\right) + 1, \text{ if } u_{l+1}\, R\, u_l; \\ d_{\mathcal{G}}\left(u_0\, u_1\, \ldots\, u_l\right) - 1, \text{ otherwise.} \end{cases}$$

$d_{\mathcal{G}}(\xi)$ will be called the *depth* of ξ in $\mathcal{G}$. We suppress the parameter $\mathcal{G}$ from $d_{\mathcal{G}}$ when no confusion arises. A cycle in a digraph is *improper*, if its depth is zero; otherwise it is *proper*.

Lemma 5 *Let $\mathcal{G} = \langle W, R\rangle$ be a digraph. Let ξ, ξ_0, and ξ_1 be walks in $\mathcal{G}$.*

(a) *If $\xi = \xi_0 \frown \xi_1$, then $d(\xi) = d(\xi_0) + d(\xi_1)$.*
(b) *$\mathcal{G}$ contains a proper cycle, iff it contains a closed walk of non-zero depth.*

Proof. Both (a) and the necessity of (b) are obvious, and we only prove the sufficiency of (b). It suffices to prove that for any closed walk ξ of non-zero depth in $\mathcal{G}$, ξ has a sub-walk which is a proper cycle. The proof is by mathematical induction on the length l of ξ. The case of $l = 1$ is apparently true, since a closed walk of length 1 is also a cycle.

For $l > 1$, suppose $\xi = u_0\, u_1\, \ldots\, u_l$. If all points in ξ are distinct except that $u_0 = u_l$, ξ is a cycle in itself. Otherwise, we can choose two points u_{i_0} and u_{j_0} in ξ such that $0 \leq i_0 < j_0 \leq l$, $u_{i_0} = u_{j_0}$, but either $i_0 \neq 0$ or $j_0 \neq l$. Consider the sub-walk from u_{i_0} to u_{j_0} of ξ. Let it be ξ_1. At the same time, consider the concatenation of the sub-walk from u_0 to u_{i_0} and the sub-walk from u_{j_0} to u_l at the point u_{i_0} (or u_{j_0}). Let it be ξ_2. Both ξ_1 and ξ_2 are closed and their lengths are less than l. At the same time, since $d(\xi_1) + d(\xi_2) = d(\xi)$ and $d(\xi) \neq 0$, either $d(\xi_1) \neq 0$ or $d(\xi_2) \neq 0$. So by inductive hypothesis, either ξ_1 or ξ_2 has a sub-walk which is a proper cycle. Such a cycle is also a sub-walk of ξ. □

Lemma 6 *Let $\mathcal{G} = \langle W, R\rangle$ be a digraph without any proper cycle.*

(a) *If ξ is a walk in $\mathcal{G}$, then $d(\xi^-) = -d(\xi)$.*
(b) *In $\mathcal{G}$, all walks with the same endpoints have the same depth.*

Proof. For (a), let $\xi = u_0\, u_1\, \ldots\, u_l$, then for any $0 \leq k < l$, either $u_k\, R\, u_{k+1}$ or $u_{k+1}\, R\, u_k$, but not both. From this, $d(\xi^-) = -d(\xi)$ can be easily proved by mathematical induction on the length of ξ. To prove (b), we fix arbitrarily two points u and v in W. Let ξ_0 and ξ_1 be two walks from u to v. Assume $d(\xi_0) \neq d(\xi_1)$, and consider the walk $\xi_0 \frown \xi_1^-$. By (a), $d\left(\xi_0 \frown \xi_1^-\right) = d\left(\xi_0\right) - d\left(\xi_1\right) \neq 0$. Thus, $\xi_0 \frown \xi_1^-$ is a closed walk of non-zero depth in $\mathcal{G}$. By (b) of Lemma 5, $\mathcal{G}$ contains a proper cycle, a contradiction. □

Theorem 3 *If a finite set of sentences is paradoxical in a digraph, then there is some proper cycle in this digraph.*

Proof. Let Σ be a finite set, and let $\mathcal{G} = \langle W, R\rangle$ is a digraph without any proper cycle. We show Σ is not paradoxical in $\mathcal{G}$. Note that Σ is paradoxical in $\mathcal{G}$, iff it is so in some connected component of $\mathcal{G}$. We can suppose that $\mathcal{G}$ is connected without loss of generality.

Define Σ'_k as in the proof of Theorem 1. Since Σ is a finite set, there exist numbers m and l such that $m < l$ and $\Sigma'_m = \Sigma'_l$. Put $p = l - m$. Now fix a point in $\mathcal{G}$, namely u_0. For any point u in $\mathcal{G}$, by (b) of Lemma 6, all walks from u_0 to u has the same depth. Let $d(u)$ be the depth of some (any) walk from u_0 to u. Now define a mapping $t : W \to \mathscr{P}\left(\mathscr{L}^+\right)$ as follows: if $d(u) \equiv k \,(\mathrm{mod}\ p)$ and $0 \leq k < p$, $t(u) = \Sigma'_{m+k}$. We show t is a revision mapping for Σ in $\mathcal{G}$, and so Σ is not paradoxical in $\mathcal{G}$.

Suppose u and v are two points in W satisfying $u\,R\,v$ and suppose $d(u) \equiv k\,(\mathrm{mod}\ p)$. Then $d(v) \equiv k + 1\,(\mathrm{mod}\ p)$. We consider two cases. In case $k + 1 < p$, then $t(u) = \Sigma'_{m+k}$ and $t(v) = \Sigma'_{m+k+1}$. From this, we can easily obtain $t(v) \cap \Sigma = t(u)^r \cap \Sigma$. Hence, t is a revision mapping for Σ in $\mathcal{G}$. In case $k + 1 = p$, then $t(u) = \Sigma'_{m+k} = \Sigma'_{l-1}$ and $t(v) = \Sigma'_m = \Sigma'_l$. We see this case can be similarly proved as the above case. □

Theorem 4 *If a set of sentences is paradoxical in a finite digraph, then there is some proper cycle in this digraph.*

Proof. Suppose Σ is an infinite set of sentences, and $\mathcal{G} = \langle W, R\rangle$ is a finite digraph without any proper cycle. We show Σ is not paradoxical in $\mathcal{G}$. The proof, like the one of Theorem 2, is an application of König's infinity lemma.

Let $\Sigma = \{A_k \mid k \in \mathbb{N}\}$. We still use Σ_k for the set $\{A_i \mid i < k\}$, and $\mathbb{N}_k$ for $\{i \in \mathbb{N} \mid i < k\}$. We now create a tree as follows. For $k \in \mathbb{N}$, let us say $\sigma : \mathbb{N}_k \to \mathscr{P}(W)$ is a k-sequence of worlds, if there exists a revision mapping t for Σ_k in $\mathcal{G}$ such that $u \in \sigma(i)$, iff $A_i \in t(u)$. Let $\mathcal{T}$ be the set of all the k-sequences of worlds for $k \in \mathbb{N}$. Define a binary relation $<$ on $\mathcal{T}$ as follows: $\sigma < \sigma'$, iff $|\sigma| < |\sigma'|$, and for all $i < |\sigma|$, $\sigma(i) = \sigma'(i)$. It can be easily verified that $\langle \mathcal{T}, <\rangle$ is a tree.

For any $k \geq 0$, by Theorem 3, there exists a revision mapping t_k for Σ_k in $\mathcal{G}$. Define $\sigma_k : \mathbb{N}_k \to \mathscr{P}(W)$ such that $\sigma_k(i) = \{u \in W \mid A_i \in t_k(u)\}$. Then σ_k is an element in the kth level of $\langle \mathcal{T}, <\rangle$. Thus, $\langle \mathcal{T}, <\rangle$ is an infinite tree. What is more, since W is a finite set, $\langle \mathcal{T}, <\rangle$ is finitely branching. This is the place where we use the finiteness of the digraph $\mathcal{G}$. Now by König's infinity lemma, the tree $\langle \mathcal{T}, <\rangle$ has an infinite branch. We put σ^*_k to be the restriction of this infinite branch up to the k-th level of the tree. Let $g = \bigcup_{k\geq 0} \sigma^*_k$, and define a mapping t^* on W as follows: $t^*(u) = \{A_k \mid u \in g(k)\}$. We prove that t^* is a revision mapping for Σ_k in $\mathcal{G}$.

Just as we do in the proof of Theorem 2, for any $k \geq 0$, we can choose a number n_k such that $k < n_k$ and $f(A_k) \subseteq \Sigma_{n_k}$. To prove t^* is a revision mapping for Σ_k in $\mathcal{G}$, we only need to prove that for any $A_k \in \Sigma$ and for any $u, v \in W$ with $u\,R\,v$, $A_k \in t^*(v)$, iff $t^*(u) \models A_k$. For this, first note that by definition of t^*, $A_k \in t^*(v)$, iff $v \in g(k)$; by definition of g and the fact that $k < n_k$, $v \in g(k)$, iff $v \in \sigma^*_{n_k}(k)$; by definition of $\sigma^*_{n_k}$, $v \in \sigma^*_{n_k}(k)$, iff $A_k \in t^*_{n_k}(v)$; and by definition of $t^*_{n_k}$, $A_k \in t^*_{n_k}(v)$, iff $t^*_{n_k}(u) \models A_k$. Now it suffices to prove that $t^*_{n_k}(u) \models A_k$, iff $t^*(u) \models A_k$.

Note that $t^*_{n_k}$ is a revision mapping for Σ_{n_k} in $\mathcal{G}$, which witnesses that $\sigma^*_{n_k}$ is a n_k-sequence of worlds. The last equivalence in the preceding paragraph holds because A_k depends on $f(A_k)$ and $t^*_{n_k}(u) \cap f(A_k) = t^*(u) \cap f(A_k)$. First, we can get $t^*_{n_k}(u) \cap f(A_k) \subseteq t^*(u) \cap f(A_k)$ from the easy fact that $t^*_{n_k}(u) \subseteq t^*(u)$. Conversely, suppose $A_i \in t^*(u) \cap f(A_k)$, then on one hand, by our choice of n_k, $i < n_k$, and on the other hand, by the definition of t^*, $u \in g(i)$. Thus, $u \in \sigma^*_{n_k}(i)$. It follows $A_i \in t^*_{n_k}(u)$. We obtain $t^*_{n_k}(u) \cap f(A_k) \supseteq t^*(u) \cap f(A_k)$. □

Theorem 5 *If a locally finite set of sentences is paradoxical in a digraph, then there is some proper cycle in this digraph.*

Proof. Suppose Σ is an infinite but locally finite set of sentences, and $\mathcal{G} = \langle W, R\rangle$ is a digraph without any proper cycle. We show Σ is not paradoxical in $\mathcal{G}$. Again, the proof is an application of König's infinity lemma.

We need to consider simultaneously two infinite parameters: one is the set Σ and the other is the digraph $\mathcal{G}$. Correspondingly, the tree we must create to apply König's infinity lemma is more complicated than the one in the proof of Theorem 4. First note that if the desired result holds for some (or any) countably infinite sub-digraph of $\mathcal{G}$, it must also hold for the whole $\mathcal{G}$. So we can suppose without loss of generality that $\mathcal{G}$ is countably infinite. Let $\Sigma = \{A_k \mid k \in \mathbb{N}\}$, and let $W = \{w_k \mid k \in \mathbb{N}\}$. For brevity, we use Σ_k for the set $\{A_i \mid i < k\}$, and W_k for the set $\{w_i \mid i < k\}$. Let $\mathcal{G}_k$ be the restriction of $\mathcal{G}$ to W_k. Now we construct a tree as follows. For $k \in \mathbb{N}$, let us say $\sigma : \mathbb{N}_k \to \mathscr{P}(W_k)$ is a k-sequence of worlds, if there exists a revision mapping t for Σ_k in $\mathcal{G}_k$ such that $u \in \sigma(i)$, iff $A_i \in t(u)$. Let $\mathcal{T}$ be the set of all the k-sequences of worlds for $k \in \mathbb{N}$. Define a binary relation $<$ on $\mathcal{T}$ as follows: $\sigma < \sigma'$, iff $|\sigma| < |\sigma'|$, and for all $i < |\sigma|$, $\sigma(i) = \sigma'(i) \cap W_{|\sigma|}$.

We verify that $\langle \mathcal{T}, <\rangle$ is a tree. First, it can be easily verified that $\mathcal{T}$ is (strictly) partially ordered by $<$, and the 0-sequence of worlds (i.e., empty sequence) is its the root. The details are omitted. Next, we must verify that for each $\sigma' \in \mathcal{T}$, $\mathrm{pd}_{\mathcal{T}}(\sigma')$ is a finite set which is totally ordered by $<$. To see this point, we first notice that whenever $\sigma < \sigma'$, $|\sigma| < |\sigma'|$. We also can easily see that there are at most 2^{k^2} function from $\mathbb{N}_k$ to $\mathscr{P}(W_k)$. Therefore, if $|\sigma'| = k$, then there are at most $\sum_{k=0}^{m} 2^{k^2}$ elements in the set $\mathrm{pd}_{\mathcal{T}}(\sigma')$. Thus, $\mathrm{pd}_{\mathcal{T}}(\sigma')$ is finite. Next, we suppose $\sigma_1, \sigma_2 < \sigma'$, we show either $\sigma_1 \leq \sigma_2$ or $\sigma_2 \leq \sigma_1$. Without loss of generality, we suppose $|\sigma_1| \leq |\sigma_2|$. For any $i < |\sigma_1|$, we have the following equations:

$$\begin{aligned}
\sigma_1(i) &= \sigma'(i) \cap W_{|\sigma_1|} && (\sigma_1 < \sigma') \\
&= \sigma'(i) \cap W_{|\sigma_2|} \cap W_{|\sigma_1|} && (W_{|\sigma_1|} \subseteq W_{|\sigma_2|}) \\
&= \sigma_2(i) \cap W_{|\sigma_1|} && (\sigma_2 < \sigma')
\end{aligned}$$

By the above result, if $|\sigma_1| = |\sigma_2|$, we have $\sigma_1 = \sigma_2$; if $|\sigma_1| < |\sigma_2|$, we have $\sigma_1 < \sigma_2$. As a result, we can conclude that $\langle \mathcal{T}, <\rangle$ is a tree.

For any $k \geq 0$, by Theorem 3, there exists a revision mapping t_k for Σ_k in $\mathcal{G}_k$. Define $\sigma_k : \mathbb{N}_k \to \mathscr{P}(W_k)$ such that $\sigma_k(i) = \{u \in W_k \mid A_i \in t_k(u)\}$. Then σ_k is an element in the kth level of $\langle \mathcal{T}, <\rangle$. Thus, $\langle \mathcal{T}, <\rangle$ is an infinite tree. What is more, since there are at most 2^{k^2} elements in the kth level of $\langle \mathcal{T}, <\rangle$, $\langle \mathcal{T}, <\rangle$ is finitely branching. Now by König's infinity lemma, the tree $\langle \mathcal{T}, <\rangle$ has an infinite branch. We put σ_k^* to be the restriction of this infinite branch to the k-th level of the tree. Let $g = \bigcup_{k \geq 0} \sigma_k^*$, and define a mapping t^* on W as follows: $t^*(u) = \{A_k \mid u \in g(k)\}$. We prove that for all t^* is a revision mapping for Σ in $\mathcal{G}$.

To prove t^* is a revision mapping for Σ in $\mathcal{G}$, we only need to prove that for any $A_k \in \Sigma$ and for any $w_l, w_m \in W$ with $w_l \, R \, w_m$, $A_k \in t^*(w_m)$, iff $t^*(w_l) \models A_k$. For this, first, just as we do in the proof of Theorem 2, we can choose a number n such that $l, m, k < n$ and $f(A_k) \subseteq \Sigma_n$. Note that n depends on not only k but also l and m, and so we write n (or more informative but cumbersome $n_{l,m,k}$) instead of n_k.

Now by definition of t^*, $A_k \in t^*(w_m)$, iff $w_m \in g(k)$; and by definition of g and the fact that $k, m < n$, the latter is equivalent to $w_m \in \sigma^*_n(k)$, which is equivalent to $A_k \in t^*_n(w_m)$ by definition of σ^*_n. And by definition of t^*_n and the fact that $l < n$, $A_k \in t^*_n(w_m)$, iff $t^*_n(w_l) \models A_k$. Finally, it suffices to prove that $t^*_n(w_l) \models A_k$, iff $t^*(w_l) \models A_k$.

Note that t^*_n is a revision mapping for Σ_n in $\mathcal{G}_n$, which witnesses that σ^*_n is a n-sequence of worlds. The last equivalence in the preceding paragraph holds because A_k depends on $f(A_k)$ and $t^*_n(w_l) \cap f(A_k) = t^*(w_l) \cap f(A_k)$. First, we can get $t^*_n(w_l) \cap f(A_k) \subseteq t^*(w_l) \cap f(A_k)$ from the easy fact that $t^*_n(w_l) \subseteq t^*(w_l)$. Conversely, suppose $A_i \in t^*(w_l) \cap f(A_k)$, then on one hand, by our choice of n, $i < n$, and on the other hand, by the definition of t^*, $w_l \in g(i)$. We now claim $w_l \in \sigma^*_n(i)$ (we must say this is not immediate). If we obtain this claim, then we know $A_i \in t^*_n(w_l)$, and so we can conclude $t^*_n(w_l) \cap f(A_k) \supseteq t^*(w_l) \cap f(A_k)$.

To verify the above claim, we first choose a number n' such that $i, n < n'$ and $w_l \in \sigma^*_{n'}(i)$. The existence of n' is guaranteed by the fact that $w_l \in g(i)$. By the definition of σ^*_k and the fact $n < n'$, we have $\sigma^*_n < \sigma^*_{n'}$. But $|\sigma^*_n| = n$, and then by the definition of the tree relation $<$, we know $\sigma^*_n(i) = \sigma^*_{n'}(i) \cap W_n$. But $l < n$, we know $w_l \in W_n$. Then $w_l \in \sigma^*_{n'}(i) \cap W_n$, that is, $w_l \in \sigma^*_n(i)$. □

Definition 12 (Hsiung (2014), p. 35) A set of sentences is said to have *circularity dependence*, if it is not paradoxical in any digraph unless this digraph contains some proper cycle. A set of sentences is said to have *digraph compactness*, if whenever it is paradoxical in a digraph, it must be paradoxical in some finite sub-digraph of this digraph.

Like the definition of self-reference, the above one of circularity-dependence is also used to formulate the informal notion of circularity about sentences. But, the two kinds of circularity have different meanings. As we all know, self-reference is essentially a semantical characteristic of sentences. And when we say a paradox is self-referential, it is the sentences of the paradox that are so. By contrast, when we say a paradox has circularity dependence, the object that has the feature of circularity is the digraphs in which this paradox is paradoxical. In other words, circularity-dependence is about digraphs rather than about sentences. It is a purely graph-theoretical notion. Circularity-dependence should not be confused with self-reference.

The following are immediate from Theorem 3 and 5.

Corollary 1 *Both the finite sets of sentences and the locally finite sets of sentences have the circularity dependence and the digraph compactness.*

5 Some Non Locally Finite Paradoxes

In this section, we first give some paradoxical examples which are not self-referential and then some ones which are paradoxical in a digraph without any proper cycle. Of course, all of them are non-locally-finite paradoxes.

First, we consider Yablo's paradox, which consists of countably infinite sentences $\nu(0)$, $\nu(1)$, ..., such that for all $n \in \mathbb{N}$,

$$\nu(n) \equiv \forall x \; (x > \mathbf{n} \rightarrow \neg T \ulcorner \nu(\dot{x}) \urcorner),$$

where $\mathbf{n}$ is the term $SS \ldots S\mathbf{0}$ (n occurrences of S), $x > \mathbf{n}$ is an abbreviation for the formula $\exists z\,(x = \mathbf{n}{+}Sz)$, and $\dot{x}$ is Feferman's dot notion which allows the scope of the quantifiers $\forall x$ covers the formula $\nu(x)$, even though $\nu(x)$ hides behind a closed term.[8] A variant of Yablo's paradox is as follows: $\nu'(0)$, $\nu'(1)$, …, such that for all $n \in \mathbb{N}$,

$$\nu'(n) \equiv \forall x\ \left(x > \mathbf{n} \to \exists y \left(y > x \wedge \neg T\ulcorner \nu'(\dot{y})\urcorner\right)\right).$$

We will use ν_n for $\nu(n)$ and ν'_n for $\nu'(n)$.

Example 1 Neither Yablo's paradox $\{\nu_n \mid n \in \mathbb{N}\}$ nor its variant $\left\{\nu'_n \mid n \in \mathbb{N}\right\}$ is self-referential.

Proof. It can be easily verified that for any $n \in \mathbb{N}$, ν_n essentially depends on the set $\{\nu_k \mid k > n\}$ (See Figure 2). Note that this fact also shows that Yablo's paradox is not locally finite. Now assume $\{\nu_n \mid n \in \mathbb{N}\}$ is self-referential, then consider the choice function f, which satisfies that $f(\nu_n) = \{\nu_k \mid k > n\}$ for all $n \in \mathbb{N}$. By definition 8, there is a closed directed walk in the dependence digraph $\langle\{\nu_n \mid n \in \mathbb{N}\}, \prec_f\rangle$. That is, for some numbers n_0, n_1, …, n_k, we have $n_0 = n_k$ and $\nu_{n_i} \prec_f \nu_{n_{i+1}}$ for $0 \leq i < k$. By the condition of f, $\nu_{n_{i+1}}$ must belong to the set $\{\nu_k \mid k > n_i\}$. Thus, for $0 \leq i < k$, we have $n_{i+1} > n_i$. Consequently, $n_k > n_0$, a contradiction!

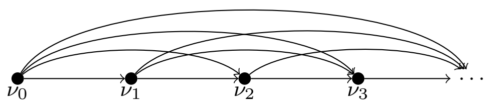

Fig. 2: Non-self-reference of Yablo's paradox

For the variant $\left\{\nu'_n \mid n \in \mathbb{N}\right\}$, first note that for any $n \in \mathbb{N}$, ν'_n depends on the set $\{\nu'_k \mid k > n+1\}$. Then we can consider the choice function f which assigns ν'_n to $\{\nu'_k \mid k > n+1\}$. The corresponding relation $\prec_f$ satisfies: if $\nu'_m \prec_f \nu'_n$, then $m < n$. As above, we can similarly prove $\left\{\nu'_n \mid n \in \mathbb{N}\right\}$ is non-self-referential. □

There are other variants of Yablo's paradox. For this, we notice that the definitional equivalences of $\left\{\nu'_n \mid n \in \mathbb{N}\right\}$ can be uniformly represented as follows:

$$\nu'(x) \equiv \forall y > x\, \exists z > y\, \neg T^{\ulcorner}\nu'\,(\dot{z})^{\urcorner}.$$

In this sense, The paradox $\left\{\nu'_n \mid n \in \mathbb{N}\right\}$ can be seen as a '$\forall\exists$-unwinding' variant of Yablo's paradox. Now for the quantifiers Q_1, …, Q_n, we can similarly give a $Q_1 \ldots Q_n$-unwinding variant of Yablo's paradox.[9] A basic fact about all the $Q_1 \ldots Q_n$-unwinding variants of Yablo's paradox is that only four kinds are not

[8] For more details about Feferman's dot notion, see for instance Halbach (2011, p. 32ff).

[9] The dual of Yablo's paradox, that is, the $\exists$-unwinding variant, was first given by Cook (2004, p. 771). And the dual of the $\forall\exists$-unwinding variant, i.e., the $\exists\forall$-unwinding variant, was put forward by Yablo (2004, p. 144). Yablo's paradox and the above two variants were generalized in Schlenker (2007a). The notion of unwinding was first formulated by Cook (2004, p. 770) and the present nomenclature '$Q_1 \ldots Q_n$-unwinding' comes from Cook (2014, p. 155).

logically equivalent to each other: the $\forall^n$-unwinding, $\exists^n$-unwinding, $\forall\exists$-unwinding, and $\exists\forall$-unwinding variants.[10] As above, it can be similarly proved that all these variants of Yablo's paradox are non-self-referential.[11]

Next, we turn to find the examples that can be paradoxical in a digraph without proper cycles. First note that Yablo's paradox (or its $\forall\exists$-unwinding variant) is not such an example, even though it is not locally finite. Actually, we can prove that Yablo's paradox even has the same degree of paradoxicality as the Liar paradox: Yablo's paradox is paradoxical in a digraph, iff the Liar is so in this digraph. But the Liar is paradoxical in a digraph, iff this digraph contains some odd cycles.[12] Thus, Yablo's paradox cannot be paradoxical in a digraph without proper cycles.

Another known non-locally-finite paradox is the ω-cycle liar. By Gödel's diagonal lemma, for $0 \leq \alpha \leq \omega$, we can construct sentences λ_α^ω such that $\lambda_0^\omega \equiv \neg T\ulcorner\lambda_\omega^\omega\urcorner$, $\lambda_{k+1}^\omega \equiv T\ulcorner\lambda_k^\omega\urcorner$ $(k \in \mathbb{N})$, and $\lambda_\omega^\omega \equiv \forall x T\ulcorner\lambda_{\dot{x}}^\omega\urcorner$. The set $\{\lambda_\alpha^\omega \mid 0 \leq \alpha \leq \omega\}$ is a paradox. It can be called the ω-cycle liar.[13] For any $k \in \mathbb{N}$, λ_{k+1}^ω essentially depends on the set $\{\lambda_k^\omega\}$. For this, only note that λ_{k+1}^ω does not depend on the empty set and for any $\Gamma \subseteq \mathscr{L}^+$,

$$\mathcal{V}_\Gamma(T\ulcorner\lambda_k^\omega\urcorner) = \mathcal{V}_{\Gamma\cap\{\lambda_k^\omega\}}(T\ulcorner\lambda_k^\omega\urcorner).$$

Similarly, λ_0^ω essentially depends on the set $\{\lambda_\omega^\omega\}$. It follows that for any choice function f, $\lambda_0^\omega \prec_f \lambda_\omega^\omega \prec_f \lambda_0^\omega$. That means any dependence digraph of the ω-cycle liar contains a closed directed walk. Consequently, the ω-cycle liar is a self-referential paradox. However, the ω-cycle liar is not locally finite. For this, we show the sentence λ_ω^ω essentially depends on the infinite set $\Sigma = \{\lambda_k^\omega \mid k \in \mathbb{N}\}$. Actually, since $\lambda_k^\omega \in \Gamma$ for all $k \in \mathbb{N}$ iff $\lambda_k^\omega \in \Gamma \cap \Sigma$ for all $k \in \mathbb{N}$, we can get $\mathcal{V}_\Gamma\left(T\ulcorner\lambda_k^\omega\urcorner\right) = \mathrm{T}$ for all $k \in \mathbb{N}$ iff $\mathcal{V}_{\Gamma\cap\Sigma}\left(T\ulcorner\lambda_k^\omega\urcorner\right) = \mathrm{T}$ for all $k \in \mathbb{N}$, that is,

$$\mathcal{V}_\Gamma\left(\forall x T\ulcorner\lambda_{\dot{x}}^\omega\urcorner\right) = \mathcal{V}_{\Gamma\cap\Sigma}\left(\forall x T\ulcorner\lambda_{\dot{x}}^\omega\urcorner\right),$$

and we get λ_ω^ω depends on Σ. It can be easily verified that λ_ω^ω does not depends on any proper subset of Σ. See Figure 3.

Definition 13 (Hsiung (2014), p. 27) A digraph $\langle G, R\rangle$ is *grounded*, iff for any point in the domain G, the depths of the walks starting from that point are bounded from above (or more formally, there is a number N, such that $d(\xi) \leq N$ holds for all walks ξ starting from that point). Otherwise, $\langle G, R\rangle$ is called *ungrounded*.

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[10] See Theorem 3.3.2 in Cook (2014).

[11] There are other variants of Yablo's paradox. For instance, Butler (2017) gave a recipe for constructing what he called 'infinitely non*-variants' of Yablo's paradox. There are even continuum-many such variants, whose formalized counterparts in $\mathscr{L}^+$ (if any) can be proved to be non-self-referential by the similar method we use in Example 1. By the way, Butler also asserted that some of these paradoxes are non-self-referential. But his criterion, proposed by Priest (1997), is whether a circular predicate is involved in the construction of a paradox.

[12] For details of the two results, see Hsiung (2013) and Hsiung (2009a) respectively.

[13] For any transfinite limit ordinal γ, we can similar set up the γ-cycle liar. See Herzberger (1982, pp. 74-75) and Yablo (1985, p. 340). Of course, only those γ-cycle liars consisting of countably sentences can be formalized in $\mathscr{L}^+$. Surprisingly, all of those transfinite γ-cycle liars have the same degree of paradoxicality. See Hsiung (2014, p. 36). Thus, all of these paradoxes are examples that can be paradoxical in a digraph without proper cycles.

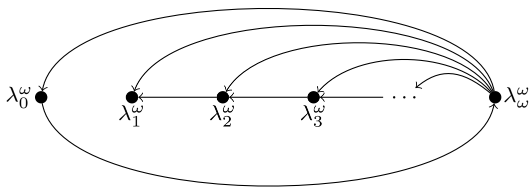

Fig. 3: Self-reference of the ω-cycle liar

Recall that a digraph $\langle G, R\rangle$ is well-founded, if there is no infinite sequence w_0, w_1, w_2, ... of elements of G such that $w_{k+1}Rx_k$ for every natural number k. The following lemma is obvious, and its proof is omitted.

Lemma 7 (a) *Any grounded digraph is well-founded.*
(b) *Any grounded digraph contains no proper cycle.*

Not all well-founded digraphs are grounded. For instance, let $\mathcal{G}$ be a frame whose domain contains three points u_0, u_1 and u_2, such that R satisfies only u_0Ru_1, u_0Ru_2 and u_2Ru_1. $\mathcal{G}$ is clearly well-founded but the lengths of the walks starting from u_0 have no upper bound. The converse of (b) of Lemma 7 is also not true. The set of natural numbers with the predecessor relation as its ordering is a ungrounded digraph which contains no proper cycle.

Theorem 6 (Hsiung (2014), p. 26) *The ω-cycle liar is paradoxical in a digraph, iff this digraph is ungrounded.*

By Theorem 6, the ω-cycle liar is paradoxical in the digraph whose domain is the set of natural numbers and whose ordering is the predecessor relation. As we just mentioned, this digraph contains no proper cycles, and so we find a paradox which can be paradoxical in a digraph without proper cycles. By the way, seeing Theorem 5, we may conjecture that if a set of sentences is paradoxical in a locally finite digraph, then there is some proper cycle in this digraph. But this is definitely wrong, because the digraph we just mentioned is a locally finite digraph in which ω-cycle liar is paradoxical, but this digraph, as we have known, contains no proper cycle at all.

Note that the indexes of the sentences in the ω-cycle liar are involved in the transfinite ordinal ω. But as we will see below, the use of the transfinite ordinal is not essential: labeling only by the natural numbers, we can construct a similar paradox with the same degree of paradoxicality. For this purpose, we now introduce two paradoxes. By Gödel's diagonal lemma, we can find a sentence μ_0 such that $\mu_0 \equiv \exists x \neg T^{S\dot{x}}\ulcorner\mu_0\urcorner$.[14] For any $k \in \mathbb{N}$, let $\mu_{k+1} = T\ulcorner\mu_k\urcorner$. It can be easily seen that the set $\{\mu_k \mid k \in \mathbb{N}\}$ is paradoxical. Let us call it 'McGee's paradox'. McGee's paradox is not locally finite, as μ_0 essentially depends on the set $\{\mu_k \mid k > 0\}$. And since μ_1 essentially depends on $\{\mu_0\}$, McGee's paradox is self-referential (see Figure 4). Similarly, let μ_0' be the sentence satisfying $\mu_0' \equiv \forall x \neg T^{S\dot{x}}\ulcorner\mu_0'\urcorner$ and call the set $\left\{\mu_k' \mid k \in \mathbb{N}\right\}$ 'the dual of McGee's paradox'.

[14] The sentence μ_0 was first introduced by McGee (1985, p. 400).

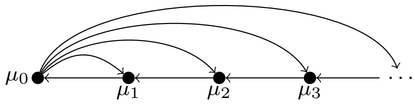

Fig. 4: Self-reference of McGee's paradox

Lemma 8 *The ω-cycle liar, McGee's paradox and its dual all have the same degree of paradoxicality.*

Proof. We first prove McGee's paradox is paradoxical in $\mathcal{G}$, iff its dual is so in $\mathcal{G}$. Suppose t is a revision mapping for McGee's paradox in $\mathcal{G}$, we define a mapping $t' : W \to \mathscr{P}(\mathscr{L}^+)$ as follows: $t'(u) = \{\mu'_k \mid \mu_k \notin t(u)\}$. We verify t' is a revision mapping for the dual of McGee's paradox in $\mathcal{G}$. Fix arbitrarily two points u and v with $u\,R\,v$. First notice $t'(v) \models T\ulcorner\mu'_k\urcorner$, iff $\mu'_k \in t'(v)$. Then by definition of t', $\mu'_k \in t'(v)$, iff $\mu_k \notin t(v)$ which is clearly equivalent to $t(v) \not\models T\ulcorner\mu_k\urcorner$. We get

$$t'(v) \models T\ulcorner\mu'_k\urcorner \iff t(v) \not\models T\ulcorner\mu_k\urcorner. \tag{3}$$

Next we prove

$$t'(u) \models \mu'_k \iff t(u) \not\models \mu_k. \tag{4}$$

Case 1: $k = i + 1$ ($i \geq 0$). Then as we prove (3), we can show $t'(u) \models T\ulcorner\mu'_i\urcorner$, iff $t(i) \not\models T\ulcorner\mu_i\urcorner$. Then, (4) follows immediately.

Case 2: $k = 0$. We notice $\mu_0 \equiv \exists x \neg T\ulcorner\mu_{\dot{x}}\urcorner$ and $\mu'_0 \equiv \forall x \neg T\ulcorner\mu'_{\dot{x}}\urcorner$. Then, we have $t'(u) \models \mu'_0$, iff for all $k \in \mathbb{N}$, $t'(u) \models \neg T\ulcorner\mu'_k\urcorner$. But note that $t'(u) \models \neg T\ulcorner\mu'_k\urcorner$, iff $\mu'_k \notin t'(u)$. Again, by definition of t', $\mu'_k \notin t'(u)$, iff $\mu_k \in t(u)$ which is equivalent to $t(u) \models T\ulcorner\mu_k\urcorner$. Thus, $t'(u) \models \mu'_0$, iff for all $k \in \mathbb{N}$, $t(u) \models T\ulcorner\mu_k\urcorner$, and we can obtain $t'(u) \models \mu'_0$, iff $t(u) \not\models \mu_0$. Now, (4) follows.

Since t is a revision mapping for McGee's paradox in $\mathcal{G}$, $t(v) \models T\ulcorner\mu_k\urcorner$, iff $t(u) \models \mu_k$. By (3) and (4), $t'(v) \models T\ulcorner\mu'_k\urcorner$, iff $t'(u) \models \mu'_k$. t' is a revision mapping for the dual of McGee's paradox in $\mathcal{G}$. In conclusion, we have shown that if McGee's paradox is non-paradoxical in $\mathcal{G}$, its dual is so in $\mathcal{G}$. By symmetry, the converse also holds.

Next we prove the ω-cycle liar is paradoxical in $\mathcal{G}$, iff $\{\mu_k \mid k \in \mathbb{N}\}$ is so in $\mathcal{G}$. Suppose t is a revision mapping for the ω-cycle liar in $\mathcal{G}$, define t' as follows:

$$t'(u) = \{\mu_0 \mid \lambda^\omega_\omega \notin t(u)\} \cup \{\mu_{k+1} \mid \lambda^\omega_k \in t(u), k \geq 0\}.$$

We show t' is a revision mapping for $\{\mu_k \mid k \in \mathbb{N}\}$ in $\mathcal{G}$. For any $u, v \in W$ with $u\,R\,v$, we need to verify that $t'(v) \models T\ulcorner\mu_k\urcorner$, iff $t'(u) \models \mu_k$. We consider two cases.

Case 1: $k = 0$. We notice $t'(v) \models T\ulcorner\mu_0\urcorner$, iff $\mu_0 \in t'(v)$. And by definition of t', $\mu_0 \in t'(v)$, iff $\lambda^\omega_\omega \notin t(v)$ which is equivalent to $t(v) \not\models T\ulcorner\lambda^\omega_\omega\urcorner$. At the same time, since $\mu_0 \equiv \exists x \neg T\ulcorner\mu_{\dot{x}}\urcorner$, we have $t'(u) \models \mu_0$, iff for some $n \geq 0$, $t'(u) \models \neg T\ulcorner\mu_{n+1}\urcorner$ (i.e., $\mu_{n+1} \notin t'(u)$). Now, by definition of t' again, $\mu_{n+1} \notin t'(u)$, iff $\lambda^\omega_n \notin t(u)$ which is equivalent to $t(u) \not\models T\ulcorner\lambda^\omega_n\urcorner$. Thus, we get $t'(u) \models \mu_0$, iff for some $n \geq 0$, $t(u) \not\models T\ulcorner\lambda^\omega_n\urcorner$. That is to say, $t'(u) \models \mu_0$, iff $t(u) \not\models \forall x T\ulcorner\lambda^\omega_{\dot{x}}\urcorner$. But $\lambda^\omega_\omega \equiv \forall x T\ulcorner\lambda^\omega_{\dot{x}}\urcorner$, and hence $t'(u) \models \mu_0$, iff $t(u) \not\models \lambda^\omega_\omega$. We also know $t(v) \models T\ulcorner\lambda^\omega_\omega\urcorner$, iff $t(u) \models \lambda^\omega_\omega$. To sum up, we obtain $t'(v) \models T\ulcorner\mu_0\urcorner$, iff $t'(u) \models \mu_0$.

Case 2: $k = i + 1$ ($i \geq 0$). On one hand, $t'(v) \models T\ulcorner\mu_{i+1}\urcorner$, iff $\mu_{i+1} \in t'(v)$. By definition of t', $\mu_{i+1} \in t'(v)$, iff $\lambda_i^\omega \in t(v)$ which is equivalent to $t(v) \models T\ulcorner\lambda_i^\omega\urcorner$. On the other hand, $t'(u) \models \mu_{i+1}$, iff $t'(u) \models T\ulcorner\mu_i\urcorner$ which is equivalent to $\mu_i \in t'(u)$. When $i = 0$, by definition of t', $\mu_i \in t'(u)$, iff $\lambda_\omega^\omega \notin t(u)$, which is equivalent to $t(u) \not\models \neg T\ulcorner\lambda_\omega^\omega\urcorner$. Since $\lambda_i^\omega \equiv T\ulcorner\lambda_\omega^\omega\urcorner$, we can deduce $\mu_i \in t'(u)$, iff $t(u) \models \lambda_i^\omega$. When $i = j + 1$ ($j \geq 0$), by definition of t' again, $\mu_i \in t'(u)$, iff $\lambda_j^\omega \in t(u)$ which is equivalent to $t(u) \models T\ulcorner\lambda_j^\omega\urcorner$. Now that $\lambda_i^\omega \equiv T\ulcorner\lambda_j^\omega\urcorner$, we get $\mu_i \in t'(u)$, iff $t(u) \models \lambda_i^\omega$. In either case, $t'(u) \models \mu_{i+1}$, iff $t(u) \models \lambda_i^\omega$. It follows immediately $t'(v) \models T\ulcorner\mu_{i+1}\urcorner$, iff $t'(u) \models \mu_{i+1}$.

In the above, we have proved that if the ω-cycle liar is non-paradoxical in $\mathcal{G}$, then $\{\mu_k \mid k \in \mathbb{N}\}$ is so in $\mathcal{G}$. Conversely, from a revision mapping t for $\{\mu_k \mid k \in \mathbb{N}\}$ in $\mathcal{G}$, we will find a revision mapping t' for the ω-cycle liar in $\mathcal{G}$. Define a mapping $t' : W \to \mathscr{P}(\mathscr{L}^+)$ as follows:

$$t'(u) = \{\lambda_\omega^\omega \mid \mu_0 \notin t(u)\} \cup \{\lambda_k^\omega \mid \mu_{k+1} \in t(u), k \geq 0\}.$$

For any $u, v \in W$ with $u R v$, it suffices to prove that for all $0 \leq \alpha \leq \omega$, $t'(v) \models T\ulcorner\lambda_\alpha^\omega\urcorner$, iff $t'(u) \models \lambda_\alpha^\omega$. We consider three cases.

Case 1: $\alpha = 0$. On one hand, $t'(v) \models T\ulcorner\lambda_0^\omega\urcorner$, iff $\lambda_0^\omega \in t'(v)$. By definition of t', the right side of the above biconditional is equivalent to $\mu_1 \in t(v)$. And so we can get $t'(v) \models T\ulcorner\lambda_0^\omega\urcorner$, iff $t(v) \models T\ulcorner\mu_1\urcorner$. On the other hand, $t'(u) \models \lambda_0^\omega$, iff $t'(u) \models \neg T\ulcorner\lambda_\omega^\omega\urcorner$, that is, $\lambda_\omega^\omega \notin t'(u)$. By definition of t' again, $\lambda_\omega^\omega \notin t'(u)$, iff $\mu_0 \in t(u)$ which is equivalent to $t(u) \models T\ulcorner\mu_0\urcorner$, i.e., $t(u) \models \mu_1$. We know $t(v) \models T\ulcorner\mu_1\urcorner$, iff $t(u) \models \mu_1$. It follows immediately $t'(v) \models T\ulcorner\lambda_0^\omega\urcorner$, iff $t'(u) \models \lambda_0^\omega$.

Case 2: $\alpha = i+1$ ($i \geq 0$). First, as above, we can easily see that $t'(v) \models T\ulcorner\lambda_{i+1}^\omega\urcorner$, iff $t(v) \models T\ulcorner\mu_{i+2}\urcorner$. Second, $t'(u) \models \lambda_{i+1}^\omega$, iff $t'(u) \models T\ulcorner\lambda_i^\omega\urcorner$, i.e., $\lambda_i^\omega \in t'(u)$, And by definition of t', $\lambda_i^\omega \in t'(u)$, iff $\mu_{i+1} \in t(u)$, that is, $t(u) \models \mu_{i+2}$. Now that $t(v) \models T\ulcorner\mu_{i+2}\urcorner$, iff $t(u) \models \mu_{i+2}$, it follows that $t'(v) \models T\ulcorner\lambda_{i+1}^\omega\urcorner$, iff $t'(u) \models \lambda_{i+1}^\omega$.

Case 3: $\alpha = \omega$. On one hand, $t'(v) \models T\ulcorner\lambda_\omega^\omega\urcorner$, iff $\lambda_\omega^\omega \in t'(v)$, which, by definition of t', is equivalent to $\mu_0 \notin t(v)$, that is, $t(v) \not\models T\ulcorner\mu_0\urcorner$. On the other hand, since $\lambda_\omega^\omega \equiv \forall x T\ulcorner\lambda_{\dot{x}}^\omega\urcorner$, we can see $t'(u) \models \lambda_\omega^\omega$, iff for all $n \geq 0$, $t'(u) \models T\ulcorner\lambda_n^\omega\urcorner$ (i.e., $\lambda_n^\omega \in t'(u)$). By definition of t' again, $\lambda_n^\omega \in t'(u)$, iff $\mu_{n+1} \in t(u)$, that is, $t(u) \models T\ulcorner\mu_{n+1}\urcorner$. Thus, we can get $t'(u) \models \lambda_\omega^\omega$, iff $t(u) \models \forall x T\ulcorner\mu_{S\dot{x}}\urcorner$, i.e., $t(u) \not\models \mu_0$. At last, note that $t(v) \models T\ulcorner\mu_0\urcorner$, iff $t(u) \models \mu_0$. We obtain that $t'(v) \models T\ulcorner\lambda_\omega^\omega\urcorner$, iff $t'(u) \models \lambda_\omega^\omega$. □

Theorem 7 *The language $\mathscr{L}^+$ (i.e., the set of sentences) is paradoxical in a digraph, iff this digraph is ungrounded.*

Proof. Suppose $\mathcal{G} = \langle W, R\rangle$ is a connected and grounded digraph. It suffices to prove that $\mathscr{L}^+$ is non-paradoxical in $\mathcal{G}$. By (a) of Lemma 7, we can find an R-minimal point in W. Let it be u_0. Then by (b) of Lemma 7 and (b) of Lemma 6, for any $u \in W$, we can define $d(u)$ as in the proof of Theorem 3. Since $\mathcal{G}$ is grounded, the set $\{d(u) \mid u \in W\}$ is bounded from above. Let N be the greatest number of this set.

Let $\langle \Sigma_k \mid k \in \mathbb{N}\rangle$ be the revision sequence starting from the empty set (or any other set of sentences). We define a mapping t from W to $\mathscr{P}(\mathscr{L}^+)$ as follows: $t(u) = \Sigma_{N-d(u)}$. t is well-defined, since for any $u \in W$, $d(u) \leq N$. We prove t is a revision mapping for $\mathscr{L}^+$ in $\mathcal{G}$. To verify this, we fix arbitrarily two points u and v such that $u R v$. Let $d(v) = k$, then $d(u) = k + 1$. And so, $t(u) = \Sigma_{N-k-1}$ and

$t(v) = \Sigma_{N-k}$. Clearly, by the definition of the revision sequence, for any $A \in \mathscr{L}^+$, we have $t(v) \models T\ulcorner A \urcorner$, iff $t(u) \models A$. □

Corollary 2 *The ω-cycle liar, McGee's paradox and its dual all have the highest degree of paradoxicality.*

6 Discussion

What we have done up to now is to investigate the self-reference and circularity-dependence of paradoxes in the first-order language $\mathscr{L}^+$. As is mentioned in the first section, there is a more immediate approach to the notion of self-reference, if we work in a sentential language. The main task of this section is to give an outline of such an approach and make a comparison between this approach and the present one.

Let $\mathscr{L}'$ be the sentential language, the symbols of which include countably infinite sentence names δ, λ, μ, ν (with and without subscripts), the connectives $\neg$, $\rightarrow$, $\vee$, $\wedge$, $\bigvee$, $\bigwedge$ and so on. The sentences of $\mathscr{L}'$ are formed as usual. For instance, we can use $\bigvee$ can combine infinitely many well-formed formulas to obtain a new formula. As usual, any truth-value assignment, that is, a function from the set of sentence names to the set $\{\mathrm{T}, \mathrm{F}\}$, can be extended uniquely to the set of sentences under the classical two-valued schema. For convenience, for a sentence A and an assignment σ, we still use $\sigma(A)$ to denote the truth-value of A under σ.

Now we can define a sentence net as a function from a set of sentence names to a set of sentences.[15] The sentence net provides a way to represent the pathological sentences without the use of diagonalization. For instance, the sentence net corresponding to the Liar paradox is the function $\mathbf{d}$ on $\{\lambda\}$ such that $\mathbf{d}(\lambda) = \neg\lambda$. This sentence net stipulates that λ refers to $\neg\lambda$. To Yablo's paradox, the corresponding sentence net is the function $\mathbf{d}$ on $\{\nu_n \mid n \in \mathbb{N}\}$ such that for any $n \in \mathbb{N}$, $\mathbf{d}(\nu_n) = \bigwedge_{k>n} \neg\nu_k$. And we can represent the $\forall\exists$-unwinding variant of Yablo's paradox as the sentence net $\mathbf{d}$ on $\{\nu_n \mid n \in \mathbb{N}\}$ such that for any $n \in \mathbb{N}$, $\mathbf{d}(\nu_n) = \bigwedge_{k>n} \bigvee_{i>k} \neg\nu_i$.

The sentence net is also an immediate way of showing to what a sentence refers. Let $\mathbf{d}$ be a sentence net. For any δ in the domain of $\mathbf{d}$, we can take $\mathbf{d}(\delta)$ to be the sentence to which δ refer. This determines how the sentences denoted by the sentence names depend on each other. Specifically, we can define a binary relation $R_{\mathbf{d}}$ on the domain of $\mathbf{d}$ as follows: for any δ, δ' in the domain of $\mathbf{d}$, $\delta R_{\mathbf{d}} \delta'$, iff δ' occurs as a syntactic constituent of $\mathbf{d}(\delta)$. Now the domain of $\mathbf{d}$ together with the relation $R_{\mathbf{d}}$ is defined to be 'the dependence digraph' of $\mathbf{d}$. What is more, we will say that $\mathbf{d}$ is self-referential, if its dependence digraph contains a directed walk. $\mathbf{d}$

[15] The notion of sentence net was first put forward independently by Bolander (2002) and Cook (2002). My presentation is based upon Bolander (2003, p. 89), Cook (2004, p. 767) and Rabern et al. (2013, p. 734). As Bolander himself pointed out (Bolander (2003), pp. 108-109), the notion of sentence net has some precursors such as Visser's 'stipulation list' in Visser (1989). It should be mentioned that Gupta & Belnap (1993, pp. 72ff.) also developed some of Visser's ideas about the stipulation lists.

is paradoxical, if there is a truth assignment σ such that for all δ in the domain of $\mathbf{d}$, $\sigma(\delta) = \sigma(\mathbf{d}(\delta))$.[16]

Most of the results we have obtained in the language $\mathscr{L}^+$ can be reformulated in $\mathscr{L}'$. For instance, it can be easily seen that the sentence net corresponding to the Liar is self-referential, but the ones to Yablo's paradox and its $\forall\exists$-unwinding variant are non-self-referential. They are all paradoxical. Let us say a sentence net is locally finite, if every point in its dependence digraph has a finite out-degree. Then we can show that if a sentence net with a finite domain is paradoxical, then it is self-referential; if a locally finite sentence net is paradoxical, then it is self-referential.[17] Finally, it is not hard to establish the corresponding results to Theorem 3, 4 and 5 in the context of the language $\mathscr{L}'$.

As far as the self-reference of pathological sentences, there are also important differences between the sentence-net approach and the present one. For this, we first consider the following sentence:

$$\text{sentence (5) is either true or untrue} \tag{5}$$

On one hand, the sentence net corresponding to sentence (5) is the function which maps δ to $\delta \vee \neg\delta$. This sentence net is clearly self-referential. On the other hand, In $\mathscr{L}^+$, sentence (5) can be represented as a formula A satisfying the condition $A \equiv T\ulcorner A\urcorner \vee \neg T\ulcorner A\urcorner$. Then what A refers to, being an instance of the excluded-middle principle, is always true under any extension of T. Hence, A depends on the empty set and so A is not self-referential.

The above example shows that if a sentence is self-referential by the definition of self-reference given in $\mathscr{L}'$, it is unnecessarily so according to the criterion of self-reference we set in $\mathscr{L}^+$. This difference comes from two different dependence digraphs which we use to determine the self-reference of a sentence. Given a sentence net, we can obtain the binary relation of the dependence digraph by checking whether a sentence name occurs as a syntactic constituent of the image of another sentence name. In this sense, the dependence digraph in $\mathscr{L}'$ can be determined 'syntactically'. For sentence (5), the dependence digraph of its sentence net has a reflexive binary relation just because δ is indeed a syntactic constituent of its image $\delta \vee \neg\delta$. However, the dependence relation in $\mathscr{L}^+$, as Leitgeib himself pointed out (Leitgeb (2005), p. 159), is a '*semantic*' dependence relation. The corresponding dependence digraph cannot be determined merely by the syntactical facts. This is exactly what we have seen in sentence (5).

Another important point we should point out is that some pathological sentences may not even have the corresponding sentence net in $\mathscr{L}'$ at all. Consider the following sentence:

$$\text{sentence `sentence (6) is true' is not true} \tag{6}$$

[16] The presentation of the three notions is based upon Bolander (2003, pp. 90-93). The notion of 'reference graph' given by Rabern et al. (2013, p. 737) is essentially the same as Bolander's 'dependency graph'.

[17] Unlike Bolander, Rabern et al. (2013) did not give a definition of self-reference, but studied what a dependence digraph is like if it supports a paradox. For instance, they proved that if a locally finite dependence digraph is acyclic, then it can not supports any paradox. This is actually equivalent to the statement I just mentioned in the text. A sentence-net version of Theorem 1 was also proved independently by Hsiung (2009b).

In $\mathscr{L}^+$, it can be represented as the sentence λ such that $\lambda \equiv \neg T^2\ulcorner\lambda\urcorner$. And by Definition 7, the sentence λ is indirectly self-referential. This captures the informal self-reference of sentence (6). However, any sentence net with a singleton domain is apparently either directly self-referential or not self-referential at all. In other words, there is no indirectly self-referential sentence net whose domain can be a singleton. Thus, it is impossible to use a sentence net with a singleton domain to formulate sentence (6) in $\mathscr{L}'$. In this sense, we can say that there is no way to formulate directly sentence (6) by use of the sentence net in $\mathscr{L}'$.

The reason why we cannot directly formulate a sentence by the sentence net is that the sentence involves in more than one iteration times of the truth predicate. In $\mathscr{L}'$, there is no occurrence of the truth predicate at all. This is its simplicity but this is also its shortage. The image of any sentence name under a sentence net is actually corresponding to a sentence of $\mathscr{L}^+$ in which there can be only one iteration of the truth predicate. For instance, the sentence net for sentence (5) is the function which maps δ to $\delta \vee \neg\delta$. Here $\delta \vee \neg\delta$ has a counterpart in $\mathscr{L}^+$, namely, $T\ulcorner A\urcorner \vee \neg T\ulcorner A\urcorner$. This feature determines that the sentence net has severely limited power in expressing the sentence involving in the iteration of the truth predicate. By the way, provided that sentence (5) were a sentence stating 'it is true that either sentence (5) or its negation holds', the corresponding sentence net would not be changed while the corresponding sentence in $\mathscr{L}^+$ would be a sentence A satisfying $T\ulcorner A \vee \neg A\urcorner$. From this point, we can also see that the expression power of $\mathscr{L}'$ is not so delicate as that of $\mathscr{L}^+$.

Reconsidering sentence (6), we find that we can introduce a new label for the sentence 'sentence (6) is true':

sentence (6) is true (7)

In this way, we transform sentence (6) into 'sentence (7) is not true' and so reduce the iteration time of truth predicate in (6) to one. Correspondingly, for sentence (6) and (7), we can introduce a sentence net on $\{\lambda_1, \lambda_2\}$, namely **d**, such that $\mathbf{d}(\lambda_1) = \neg\lambda_2$ and $\mathbf{d}(\lambda_2) = \lambda_1$. And of course, in the language $\mathscr{L}^+$ if we use λ' to denote $T\ulcorner\lambda\urcorner$, we obtain two sentences λ and λ' such that $\lambda \equiv \neg T\ulcorner\lambda'\urcorner$ and $\lambda' \equiv T\ulcorner\lambda\urcorner$. But we must emphasize that the sentence (6) alone is not equivalent to the set consisting of sentence (6) and (7). A substantial difference lies in, as is well known, that the sentence (6) by itself is not paradoxical, but the set consisting of it and its companion (7) is paradoxical. We can easily see this difference if working in the language $\mathscr{L}^+$. Actually, by Definition 4, the sentence λ such that $\lambda \equiv \neg T^2\ulcorner\lambda\urcorner$ is not paradoxical, but the set of λ and λ' is so. By contrast, as is just pointed out, sentence (6) by itself cannot be represented by a sentence net, much less to say that it is paradoxical in terms of $\mathscr{L}'$. Seeing this, we may say that the sentence net **d** that we just mention is not a faithful representation of sentence (6) alone. It is only corresponding to the set of sentence (6) and (7). The sentence net **d** does not yet capture the informal fact that sentence (6) alone is not paradoxical.

The iteration of the truth predicate is widespread in the construction of the pathological sentences. In some pathological sentences, the iteration of the truth predicate can be even infinite times. McGee's sentence is such an example. Informally, McGee's sentence is a sentence stating that not every result of prefixing the truth predicate to this sentence is true (McGee (1985), p. 400). In section 5, we have represented McGee's sentence as the sentence μ_0 such that $\mu_0 \equiv \exists x \neg T^{S\dot{x}}\ulcorner\mu_0\urcorner$.

Note that McGee's sentence by itself is not paradoxical. What is paradoxical is the set consisting of the sentence μ_0, μ_1 (equvalent to $T\ulcorner\mu_0\urcorner$), μ_2 (equvalent to $T\ulcorner\mu_1\urcorner$), ... This is what we called 'McGee's paradox'. In $\mathscr{L}'$, we can represent McGee's paradox as the function $\mathbf{d}$ on $\{\mu_n \mid n \in \mathbb{N}\}$ such that $\mathbf{d}(\mu_0) = \bigvee_{k>0} \neg\mu_k$, and for any $k \in \mathbb{N}$, $\mathbf{d}(\mu_{k+1}) = \mu_k$. However, there is still no sentence net corresponding to McGee's sentence alone.

To sum up, we have compared two definitions for the self-reference of sentences: one is set up by use of Leitgeib's dependence relation in $\mathscr{L}^+$, and the other is by use of the sentence net in $\mathscr{L}'$. We have found some differences when determining the self-reference of some non-paradoxical sentences according to the two definitions. The differences show that the method of investigating self-reference in the sentential language $\mathscr{L}'$ is on on hand more immediate than the one in the first-order language $\mathscr{L}^+$, on the other hand less delicate than the latter. But as far as the paradoxical sentences are concerned, we find that the self-reference is necessary to the locally finite paradoxes, no matter we consider the self-reference in $\mathscr{L}'$ or in $\mathscr{L}^+$. We also find that their paradoxicality are based upon some certain circularity conditions, and we can even compare the degrees of their paradoxicality according to the circularity conditions. From these observations, we can conclude that the locally finite paradoxes are a kind of simple but significant paradoxes, and it is their presence that reflects our naive thought that paradoxes are necessarily related to some kind of circularity.

In the end, we close our discussion by leaving two questions. Among the non-locally-finite paradoxes, we have found examples (such as Yablo's paradox and its $\forall\exists$-unwinding variant) that have circularity dependence but are free of self-reference, and we also have examples (such as the ω-cycle liar and McGee's paradox) that are self-referential but have no circularity dependence (see Table 1). This raises the following question: is there any paradox which neither is self-referential nor have circularity dependence? We think that the answer should be Yes. For this, consider again Yablo's paradox, which, as Cook (2004) had pointed out, can be obtained by 'unwinding' the Liar paradox. And the operation of unwinding may been taken as a procedure of eliminating self-reference of the Liar (see Sorensen (1998), Schlenker (2007a) and Schlenker (2007b)). Besides, as has been mentioned in Section 5, Yablo's paradox has the same degree of paradoxicality as the latter. That is, the operation of unwinding preserves the degree of paradoxicality of the Liar. And so, we conjecture that for any paradox, we can eliminate its self-reference while preserving its degree of paradoxicality.[18] If this conjecture were right, then by unwinding the ω-cycle liar and McGee's paradox, we would get examples to answer positively the above question.

As has been proved in Corollary 2, the ω-cycle liar and McGee's paradox have the highest degree of paradoxicality. The second question I would like to propose is whether there is a paradox with the lowest degree of paradoxicality. The significance of the existence of such a paradox is that the digraphs in which this paradox is paradoxical are exactly the ones with the weakest condition, such that any of paradoxes is paradoxical in these digraphs. That is, the condition is

[18] It seems that the method of proving the equiparadoxicality of Yablos paradox and the Liar in Hsiung (2013) can be somewhat generalized to the paradoxes with digraph compactness. And so it might not hard to prove that the unwinding preserves the degree of paradoxicality for the locally finite paradoxes. But the situation is different and difficult for the ω-cycle liar and McGee's paradox.

the one that is weakest for a sentence or a set of sentences to be paradoxical in a digraph. A candidate for the paradoxes with the lowest degree of paradoxicality, as is pointed out by Hsiung (2017), is the paradox whose primary periods are exactly the prime numbers.

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References

Beall, J. C. (2001). Is Yablo's paradox non-circular? *Analysis*, *61*(3), 176–187.

Beringer, T., & Schindler, T. (2015). Reference graphs and semantic paradox. Available at the website: https: www.academia.edu/19234872/Reference_Graphs_and_Semantic_Paradox.

Bolander, T. (2002). Restricted truth predicates in first-order logic. In T. Childers, & O. Majer (Eds.) *The LOGICA Yearbook 2002*, (pp. 41–55). Prague: Filosophia Press.

Bolander, T. (2003). *Logical Theories for Agent Introspection*. Ph.D. thesis, Technical University of Denmark.

Bringsjord, S., & Heuveln, B. V. (2003). The 'mental eye' defense of an infinitized version of Yablo's paradox. *Analysis*, *63*(1), 61–70.

Butler, J. M. (2017). An entirely non-self-referential Yabloesque paradox. *Synthese*. Online First: https://doi.org/10.1007/s11229-017-1443-7.

Cook, R. (2002). Patterns of paradox: Paradox and parity. In T. Childers, & O. Majer (Eds.) *The LOGICA Yearbook 2002*, (pp. 69–83). Prague: Filosophia Press.

Cook, R. T. (2004). Patterns of paradox. *Journal of Symbolic Logic*, *69*(3), 767–774.

Cook, R. T. (2014). *The Yablo Paradox : An Essay on Circularity*. Oxford: Oxford University Press.

Cormen, T. H., Leiserson, C. E., Rivest, R. L., & Stein, C. (2009). *Introduction to Algorithms (3rd ed.)*. Cambridge: MIT Press.

Gupta, A. (1982). Truth and paradox. *Journal of Philosophical Logic*, *11*(1), 1–60.

Gupta, A., & Belnap, N. (1993). *The Revision Theory of Truth*. Cambridge: MIT Press.

Halbach, V. (2011). *Axiomatic Theories of Truth*. Cambridge: Cambridge University Press.

Herzberger, H. G. (1982). Notes on naive semantics. *Journal of Philosophical Logic*, *11*(1), 61–102.

Hsiung, M. (2009a). Jump Liars and Jourdain's Card via the relativized T-scheme. *Studia Logica*, *91*(2), 239–271.

Hsiung, M. (2009b). Tarski's Theorem and Truth-theoretical Paradoxes. Dissertation (Chinese), Guangzhou: Sun Yet-Sen University. An English translation can be found in https://www.academia.edu/1555680/Tarski_s_Theorem_and_Truth-theoretic_Paradoxes.

Hsiung, M. (2013). Equiparadoxicality of Yablo's paradox and the Liar. *Journal of Logic, Language and Information*, *22*(1), 23–31.

Hsiung, M. (2014). Tarski's theorem and liar-like paradoxes. *Logic Journal of IGPL*, *22*(1), 24–38. (Corrigendum to Tarski's Theorem and Liar-like Paradoxes, 24 (2016), 2, 219).

Hsiung, M. (2017). Boolean paradoxes and revision periods. *Studia Logica*, *105*(5), 881–914.

Just, W., & Weese, M. (1997). *Discovering Modern Set Theory II: Set-Theoretic Tools for Every Mathematician*. Graduate Studies in Mathematics Volume 18. American Mathematical Society.

Leitgeb, H. (2002). What is a self-referential sentence? Critical remarks on the alleged (non-)circularity of Yablo's paradox. *Logique et Analyse*, *45*(177-178), 3–14.

Leitgeb, H. (2005). What truth depends on. *Journal of Philosophical Logic*, *34*(2), 155–192.

McGee, V. (1985). How truthlike can a predicate be? A negative result. *Journal of Philosophical Logic*, *14*(4), 399–410.

Priest, G. (1997). Yablo's paradox. *Analysis*, *57*(4), 236–242.

Rabern, L., Rabern, B., & Macauley, M. (2013). Dangerous reference graphs and semantic paradoxes. *Journal of Philosophical Logic*, *42*(5), 727–765.

Schlenker, P. (2007a). The elimination of self-reference: Generalized Yablo-series and the theory of truth. *Journal of Philosophical Logic*, *36*(3), 251–307.

Schlenker, P. (2007b). How to eliminate self-reference: A précis. *Synthese*, *158*(1), 127–138.

Sorensen, R. (1998). Yablo's paradox and kindred infinite liars. *Mind*, *107*(425), 137–155.

Tennant, N. (1995). On paradox without self-reference. *Analysis*, *55*(3), 199–207.

Visser, A. (1989). Semantics and the Liar paradox. In D. Gabbay, & F. Guenthner (Eds.) *Handbook of Philosophical Logic*, vol. IV, (pp. 617–706). Dordrecht: Reidel.

Yablo, S. (1985). Truth and reflection. *Journal of Philosophical Logic*, *14*(3), 297–349.

Yablo, S. (1993). Paradox without self-reference. *Analysis*, *53*(4), 251–252.

Yablo, S. (2004). Circularity and paradox. In T. Bolander, V. F. Hendricks, & S. A. Pedersen (Eds.) *Self-Reference*, (pp. 139–157). CSLI Publications.