A Puzzle About Inferential Strength and Probability

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ABSTRACT: *Inductive logic would be the logic of arguments that are not valid, but nevertheless justify belief in something like the way in which valid arguments would. Maybe we could describe it as the logic of "almost valid" arguments. There is a sort of transitivity to valid arguments. Valid arguments can be chained together to form arguments and such arguments are themselves valid. One wants to distinguish the "almost valid" arguments by noting that chains of "almost valid" arguments are weaker than the links that form them. But it is not clear that this is so. I have an apparent counterexample the claim. Though: as is typical in these sorts of situations, it is hard to tell where the problem lies.*

Introduction

Let an argument consist in premises and a conclusion. An argument is valid if and only if it is impossible that its premises be true and its conclusion false. A valid argument with no false premises provides a guarantee of its conclusion. Presumably, this fact provides the basis of an explanation of how sound arguments justify belief in their conclusion.

But there are arguments that justify belief in their conclusions without being valid - without being such that the truth of their premises would guarantee the truth of their conclusion. In such cases, if its premises are true, then it is very improbable that its conclusion be false. The highest degree of improbability is impossibility. This suggests a generalization of the notion of validity - i.e. strength.

The strength of an argument is the probability that its conclusion is true, given that its premises are true. Consider any argument from a propositions φ_1 (the premises) to a proposition φ_2 (the conclusion). Consider, that is, the argument $\varphi_1 \rightarrow \varphi_2$. Let '$STR(\varphi_1 \rightarrow \varphi_2)$' represent the strength of the argument $\varphi_1 \rightarrow \varphi_2$. I assume that the semantic value of 'STR' is a function that assigns some number *n* from the real interval [0,1] to arguments. For any argument, $\varphi_1 \rightarrow \varphi_2$: $STR(\varphi_1 \rightarrow \varphi_2) = 1$ if and only if $\varphi_1 \rightarrow \varphi_2$ is valid. For any argument, $\varphi_1 \rightarrow \varphi_2$: $STR(\varphi_1 \rightarrow \varphi_2) = 0$ if and only if it is impossible that φ_1 be true and φ_2 false. For any argument, $\varphi_1 \rightarrow \varphi_2$: $STR(\varphi_1 \rightarrow \varphi_2) = .5$ if and only if it is equiprobable that φ_2 be true and that φ_2 be false, given that φ_1 is true.[1]

Strengths and Probabilities

All this probability talk suggests that STR is a probability distribution (an assignment of probabilities). Indeed, I have half-way accepted this idea insofar as I have assumed that STR assigns elements of the interval [0,1] - with necessities represented by 1, equi-probabilities by .5, and impossibilities represented with 0.

Still: one might wonder how far we can use the apparatus of probability theory to model the strength of arguments. There is no reason to be confident that the use of terms like 'probable' and its

[1] It is worth mentioning that strengths need not be grasped cognitively. They are not meant to model degrees of confidence - though they might well have some interesting relationships to the degrees of confidence of ideal thinkers.

cognates requires that we model this domain with probability distributions. The term has a life in English outside of its use by mathematicians.

"Transitivity"

One of the nice things about valid arguments is that they can be assembled into chains. Valid entailment is "transitive".[2] Consider, for example:

STR((P & Q) & R) → (P & Q))	= 1
STR((P & Q) → Q)	= 1
STR((P & Q) & R) → Q))	= 1

Because of this, one can assemble increasingly long chains of valid inference without running an ever-increasing risk of error.

But things are apparently different when it comes to arguments with less-than-maximal strengths. The strength of an argument is a measure of its riskiness - of the risk that its premises are true and its conclusion is false. After all: if it is merely very probable that some conclusion is true (given the premises), then it is a small probable that it is false (given the premises). It is in the nature of risk to compound over "addition". The more risks you run, the higher the probability of failure. Therefore, supposing that two arguments have a degree of strength less than that possessed by valid arguments. Then: the result of chaining them together is an argument with a degree of strength less than either of the two.[3] Or so one might think.

[2] The scare quotes reflect the fact that I am being a bit sloppy about what sorts of objects are members of the domain of the inferential relation and which sorts of objects are members of the image of the inferential relation. The way I am presenting it, both collections contain propositions. But we might do well to take the domain of the inferential relation to contain sets of propositions (i.e. premise sets), whereas the image contains propositions (i.e. conclusions). If so: the inferential relation couldn't be transitive as it is normally understood. Instead, however, it could have a feature that might as well be transitivity.

[3] As long as they possess non-zero strengths! There is a least degree of strength. Accordingly, no argument have less strength than that.

Maybe Not

It is often mysterious which particular probabilities to assign to which outcomes - and so it can be difficult to reason about whether strengths are probabilities.[4] In a case at hand, that mystery is mitigated by the availability of an apparently reasonable probability distribution.

Consider a fair 6-sided dice. There are 6 equally probable rolls: {1, 2, 3, 4, 5, 6}. The probability of each possible roll is the same - i.e. 1/6. Thus, for example, the probability of rolling a 3 is 1/6. The probability of rolling an even number is 3/6 = 1/2. Etc.

Suppose the dice has been rolled.

Let T_1 = the roll was either a 1, 2, or 3.
Let T_2 = the roll was either a 2, 4, or 5.
Let T_3 = the roll was either a 1, 2.

Then:

$$\mathrm{PROB}(T_2 \text{ given } T_1) = 1/3$$
$$\mathrm{PROB}(T_3 \text{ given } T_2) = 1/3$$
$$\mathrm{PROB}(T_3 \text{ given } T_1) = 1$$

Suppose that for any argument $\varphi_1 \rightarrow \varphi_2$: $\mathrm{STR}(\varphi_1 \rightarrow \varphi_2) = \mathrm{PROB}(\varphi_2 \text{ given } \varphi_1)$

Then:

$$\mathrm{STR}(T_1 \rightarrow T_2) = \mathrm{PROB}(T_2 \text{ given } T_1) = 1/3$$
$$\mathrm{STR}(T_2 \rightarrow T_3) = \mathrm{PROB}(T_3 \text{ given } T_2) = 1/3$$
$$\mathrm{STR}(T_1 \rightarrow T_3) = \mathrm{PROB}(T_3 \text{ given } T_1) = 1$$

So, in this case: $\mathrm{STR}(T_1 \rightarrow T_2) < \mathrm{STR}(T_1 \rightarrow T_3)$

And: $\mathrm{STR}(T_2 \rightarrow T_3) < \mathrm{STR}(T_1 \rightarrow T_3)$

Whereas the arguments $T_1 \rightarrow T_2$ and $T_2 \rightarrow T_3$ possess less than maximal strength, the argument $T_1 \rightarrow T_3$ possess maximal strength. Given that the roll was either a 1 or 2 it is necessarily the case that the roll was either a 1, 2, or 3. So: it is impossible that T_1 should be true and T_3 should be false.

[4] One anxiety about using the resources of probability theory to give an account of the strength of arguments is that we often have very little idea which particular numbers from the real interval [0,1] ought to be assigned to which arguments. I often have some (weak, imprecise) sense of how various arguments ought to be ranked by strength - but very little idea what specific measure to assign. Which among the many possible probability distributions ought we select when assigning measures to arguments?

What gives?

Option 1: Strengths are probabilities, but they are not equivalent to these conditional probabilities. The strengths of the relevant arguments are a function of the probability distribution in which equal probabilities are assigned to each of the 6 rolls, but not equivalent to the conditional probabilities invoked above.

Option 2: Strengths are not probabilities - as understood according to the standard axioms.

Option 3: Strengths are probabilities, but which probabilities is not a function of the sort of probability distribution in which equal probabilities are assigned to each of the 6 rolls.

Option 4: ?

Way Forward?

Which feature of probability distributions allows for the sorts of inequalities that might exist between "chains" of conditional probabilities and their component links?

Are there other ways of chaining that get the right result - even assuming the identity of STR and PROB from above? I have assumed that a chain is equal to the first component of the first link together with the last component of the last link. But maybe this is the wrong way to think about it.