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Logical Discrimination

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Abstract. We discuss conditions under which the following 'truism' does indeed express a truth: the weaker a logic is in terms of what it proves, the stronger it is as a tool for registering distinctions amongst the formulas in its language.

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1. Introduction

Our topic is the idea that deductive strength varies inversely with discriminatory strength: the more a logic proves, the fewer distinctions (or discriminations) it registers. This is a thought often voiced, either in general terms, or with reference to a specific case. Here, for example, is what David Nelson had to say about the relationship between classical and intuitionistic logic:

> As we have suggested earlier, an argument favouring intuitionistic logic over the classical is the fact that the intuitionistic logic allows the classical distinctions in meaning and further ones besides. Classical logic is open to possible objection in that it identifies certain constructively distinct entities. Since we are speaking here of formal systems, we are interested in the general question of finding when one formal system allows distinctions among concepts which are not possible in another ([27], p. 215).

In a similar vein, Anderson and Belnap [1] write as follows when comparing the implicational fragments $\mathbf{T}_{\to}$, $\mathbf{E}_{\to}$, and $\mathbf{R}_{\to}$ (cited here in order of increasing deductive strength) of their logics of ticket entailment, entailment, and relevant implication; the initially mentioned "two systems" are the first and third just listed:

> These two systems, both intensional, exhibit two quite different ways of demolishing the theory of necessity enshrined in $\mathbf{E}_{\to}$: $\mathbf{R}_{\to}$ by making

stronger assumptions about identity or intersubstitutability (and hence having fewer propositional entities), and $\mathbf{T}_{\to}$ by making weaker assumptions (and hence having more distinct propositional entities). Modal systems, generally being weaker than their cousins, tend to make more distinctions; in $\mathbf{E}_{\to}$ we can distinguish A from $\Box A$, since, though the latter entails the former, the converse is neither true nor provable. As we saw in §5, adding $A \to .A \to A \to A$, i.e., $A \to \Box A$, to $\mathbf{E}_{\to}$ ruins this distinction and produces $\mathbf{R}_{\to}$: a stronger assumption produces fewer propositional entities ([1], p. 47).[1]

Sometimes the additional distinctions made available by passage to a weaker logic are thought of as making for an *embarras de richesses* when that logic is applied as the logic of a particular (typically, mathematical) theory. Troelstra and van Dalen [37] devotes a Section (3.7 of Chapter 1, entitled "Splitting of Notions") to replying to this objection – as it arises specifically in the passage from classical to intuitionistic logic – mainly by suggesting that in fact far fewer than the in-principle available distinct versions of what would in the classical case be alternative equivalent definitions of the same notion are of practical significance. In what follows we shall be concerned neither to sing the praises nor to lament the consequences of weakening a logic and thereby increasing the number of distinctions that have to be made as a result, contenting ourselves with an examination of the question of what background assumptions need to be in place in the general case for this "thereby" to be justified. We shall be concerned to see what these assumptions are, as well as to illustrate how, in cases in which they do not hold, a weaker logic may yet fail to support a greater number of distinctions. (See the discussion following Proposition 2.5 in this regard.) Alternatively put, strengthening a logic deductively need not, in such cases, result in collapsing any distinctions. We also consider the possibility, conversely, that a decrease (or increase) in discriminatory power need not signal a corresponding increase (or decrease, respectively) in deductive strength. While the particular distinctions that arise, to return to the previous example, in intuitionistic as opposed to classical mathematics – nonempty vs. inhabited, apartness vs. inequality (non-identity), etc. – might call for a logical discussion at the level of predicate logic, the general issue about discriminatory and deductive strength varying inversely can be illustrated without going beyond purely propositional logic. In the interest of simplicity, then, our general discussion as well as our illustrations are drawn from amongst propositional logics. The discussion presents a few elementary observations and examples, which might provide a stimulus for a general and systematic study of the topic, without itself pretending to constitute such a study.

[1]The passage continues with: "And of course further strengthening in the direction of the two-valued calculus produces a system which cannot tell the difference between Bizet's being French and Verdi's being Italian", rather lowering the tone since no formalization of "Bizet is French" and "Verdi is Italian" would render these two *logically* equivalent by the lights of classical logic.

Although we shall find the dictum that deductive and discriminatory strength vary inversely is not universally correct, it does hold up over a wide range of logics, so it is interesting to see the opposite presumption expressed in print. This is what we find in J. R. Lucas' discussion of a past-tense version of A. N. Prior's argument for the logical possibility of time without change (a version of which, purged of errors in an earlier formulation, appears in Prior [30]). Lucas [23], p. 10, writes: "Even within the austere framework of Lemmon's minimal tense logic K_t, we can distinguish between a dawn of creation in which the stars started in their courses the moment time began and a more leisurely inauguration in which they spent part of the morning doing nothing in unison". That is – presumably – we can distinguish the hypothesis that time had a beginning from the hypothesis that change had a beginning. Although what Lucas says may not seem especially clearly to amount to this, the present objection is different. It is to the confusion underlying any claim of the form "Even within the austere framework of Lemmon's minimal tense logic K_t, we can distinguish between X and Y." The *weaker* the logic, the *more* distinctions it allows, according to the by-and-large correct dictum enunciated above: so there is no "even" about it.[2]

We close this introduction with remarks on three related issues we shall not be further attending to. The first concerns the general theme of discrimination in logic, one aspect of which is the issue of more and less discriminating accounts of what a logic is. In Section 2 and 3, for example, we shall be concerned with logics as sets of formulas and logics as consequence relations.[3] It is well known that many distinct consequence relations on a given language induce (by taking the consequences of the empty set) the same logic-as-set-of-formulas,[4] and in this sense we may say that the 'consequence relations' account of what a logic is counts as *more discriminating* than the set-of-formulas account. Similarly, the use of generalized consequence relations in the style of Scott [32] (or more generally – see the preceding footnote – logics as sets of multiple-succedent sequents) is more discriminating still.[5] Another dimension of variation consists in how much attention is paid to rules: taking, for example, single-succedent sequents, we could say that two proof systems which render provable the same set of such sequents count as

[2]Setting aside the issue specifically about distinctions supported, teaching experience attests to the difficulty that students have with talk of one logic's being stronger than another, invariably intended by logicians, when no further qualification is added, to mean *deductively stronger*, but often suggesting the reverse to students, the stronger logic being taken to be the one making the more stringent demands in respect of what is provable. (Many examples of the customary usage alluded to here may be found in Mortensen and Burgess [26] and authors there quoted. The issue under discussion is whether for this or that purpose a stronger logic is better or worse than a weaker logic.)

[3]The latter could themselves be viewed as a special case of 'logics as sets of (single-succedent) sequents' – see the discussion after Proposition 3.2 below.

[4]In the terminology of Section 3 below, these are consequence relations which, though distinct, '1-agree'.

[5]See Gabbay [11], Theorem 13 on p. 8, Theorem 4 on p. 28, for example.

two systematizations of the same logic, or we could be more demanding and require for this that not only the same sequents should be provable but the same sequent-to-sequent rules should be derivable (= primitive or derived). Interesting as these issues are in their own right, they are not what we are talking about here. The discriminations we are concerned with are those made by a logic – however conceived – between formulas, not discriminations in respect of the individuation of logics themselves. This allows us derivatively to speak, for instance, of one logic making a finer discrimination between connectives than another, in the sense in which substructural logics support a distinction between, e.g., multiplicative and additive conjunction, which is collapsed in classical or intuitionistic logic – because this amounts to saying that the former logics discriminate, and the latter do not, between the *formulas* formed from two distinct propositional variables by compounding them on the one hand with the one connective and on the other with the other. (See pp. 15–17 of Paoli [28] for some discussion of the "suppression of distinctions" objection to the structural rules.)

Secondly, we are not directly concerned with semantically based measures of relative expressive power, such as closeness to functional completeness in logics determined by reducts of some single matrix, or the ability to distinguish between more frames (validating the logic) amongst normal modal logics interpreted by the Kripke semantics. ([14] gives one example of this kind of enterprise; note the title.) This is not to say that there are no connections between such issues and the more straightforwardly syntactical matter of discriminatory strength as understood here: just that we are not addressing any such connections here. Note that an 'inverse proportionality' between deductive strength and some such measures of expressive power is often remarked on – for example in Tennant [36] *à propos* of expressive power as the power to discriminate between non-isomorphic structures. There is also the matter of discrimination between elements within an individual structure, stylishly explored in Quine [31]. An algebraic incarnation of the latter theme arises with the (ternary) discriminator function t satisfying for arbitrary elements a, b, c: $t(a, b, c) = c$ if $a = b$, and $t(a, b, c) = a$ otherwise. Discriminator varieties – varieties generated by a class of algebras in which this function is a (fixed) term function – have turned out to have striking applications outside the realm of universal algebra: Burris ([6], esp. Section 5) shows how to 'reduce', in one reasonable sense of that word, an arbitrary first-order theory to an equational theory, in the context of such varieties.[6] No doubt there are further things that could go under the name of discriminatory strength from the point of view of interpretations of formal languages (and thus outside our present purview) but that should suffice by way of example.

Finally, a remark is in order on the measure of deductive strength we are employing, according to which one logic, S_1, is at least as strong as another, S_0, (resp., strictly stronger than S_0) when $S_0 \subseteq S_1$ (resp., $S_0 \subsetneq S_1$). Those formulations are suited to the logics-as-sets-of-formulas of the following section, while for Section

[6]See Bignall and Spinks [3] for some further developments and references.

3 the corresponding inclusions are between consequence relations. Though in the main cases we consider below the language does not change, the definition does not rule out the possibility that the language of S_0 should be properly included in that of S_1, in which case the issue of relative deductive strength becomes clouded by the possibility of a translation from the larger language to the smaller which allows for a faithful embedding of S_1 into S_0. Thus Łukasiewicz [24] argued that intuitionistic propositional logic should be regarded as a extension rather than – as is customarily maintained – a sublogic of classical propositional logic, because if the connectives of classical logic were taken as defined in terms of conjunction and negation, the similarly notated but now to be distinguished connectives of intuitionistic logic could be regarded as new non-classical primitives (somewhat in the style of modal logic). This alternative point of view was available because of Gödel's observation that the conjunction–negation fragments of classical and intuitionistic logic coincided, which is of course so on the 'set-of-formulas' conception of logics (favoured by Łukasiewicz), though not on the 'consequence relation' conception; however, other examples can be given of apparent reversals in comparative deductive strength attendant upon judicious definitional manoeuvres which do work equally well at the level of consequence relations. One such example is given a particularly crisp presentation in Béziau [2], where the puzzling nature of the general phenomenon is also emphasized. (Further discussion of the phenomenon, as well as of Béziau's specific example, appears in [18].) Here we simply set such matters to one side, taking relative deductive strength as given quite literal-mindedly by set-theoretic inclusion – no "re-notation" permitted.

2. Discrimination In Logics as Sets of Formulas

The idea that the stronger the logic (deductively), the more distinctions it collapses, voiced by the authors quoted in the preceding section, is conveniently formulated in general terms with the aid of the notion of synonymy in the sense of Smiley [34].[7] As also mentioned in that section, we confine ourselves to two (from amongst many possible) conceptions as to what constitutes a logic: the conception of logics as (certain) sets of formulas of a formal language, and the somewhat richer conception of logics as consequence relations on such a language. Working with the former conception, we say that formulas A and B are *synonymous* according to (or "in") a logic S (considered as a set of formulas from some language to which A and B belong) when for any formula $C(A)$ in which A occurs zero or more times as a subformula, and any formula $C(B)$ resulting from replacing zero or more such occurrences by B, we have $C(A) \in S$ if and only if $C(B) \in S$. (We can regard the 'context' $C(\cdot)$ as a formula $C(q)$ in which amongst others there occurs the propositional variable q, with $C(A)$, $C(B)$ the results of uniformly substituting A,

[7]In fact Smiley writes "synonymity", as do the authors of [10], explaining at p. 34 there the relation of this concept to the main concepts of abstract algebraic logic in the tradition alluded to in note 9 below.

B, respectively, for that variable.[8] Compare the notation with Δ below, in which the exhibited variables are the only ones allowed to occur.) In some formulations to follow, we render "$C(A) \in S$" in words by saying that $C(A)$ is provable in – or is a theorem of – the logic S. (Though we are confining ourselves to sentential logic for illustrative purposes here, a similar notion of synonymy could be given in an obvious way for expressions of arbitrary syntactic categories.) If we are thinking, as on the second conception of logic mentioned above, of a logic as a consequence relation, $\vdash$, then we say that A and B are *synonymous according to* $\vdash$ when, with the $C(\cdot)$ notation understood as above, $C_1(A), \ldots, C_n(A) \vdash C_{n+1}(A)$ if and only if $C_1(B), \ldots, C_n(B) \vdash C_{n+1}(B)$. Of course, as remarked in Section 1, many still richer conceptions of what should constitute a logic are possible, but our purposes will be served by considering only these two. In the present section, we stick with the first 'logics-as-sets-of-formulas' conception. In such a setting, the general idea that increasing deductive strength goes with reducing discriminatory power is embodied in (1) below, in which we denote, for logics S_0 and S_1, for simplicity presumed to have the same language, the relation of synonymy according to S_i by $\equiv_i$ (to avoid a proliferation of subscripts).

$$S_0 \subseteq S_1 \textit{ if and only if } \equiv_0 \subseteq \equiv_1 \tag{1}$$

Since the original idea is that increasing discriminatory power goes with decreasing logical strength, a formulation in terms of strictly increasing and decreasing discrimination and strength, respectively, may be found attractive:

$$S_0 \subsetneq S_1 \textit{ if and only if } \equiv_0 \subsetneq \equiv_1 \tag{2}$$

It is the "only if" direction of (2) that most directly encapsulates the dictum that the weaker a logic is deductively, the more discriminating it is amongst formulas, since it says that whenever one logic, here S_0, is strictly weaker (deductively) than another, S_1, then the former logic collapses strictly fewer distinctions between pairs of formulas than the latter, thus making finer discriminations between formulas. Arguably, in adding the converse, the biconditional formulation of (2) captures the idea that deductive and discriminatory strength vary inversely. (2) is a consequence of (1), but we shall concentrate on (1) itself, considering separately the possibility of counterexamples to its "if" and "only if" directions, and begin with some simple conditions which suffice to rule out such counterexamples. We follow a similar pattern in Section 3, except that there we take logics to be consequence relations rather than collections of formulas. In either case, we take the languages concerned to be based on a countable supply of propositional variables (sentence letters) amongst which are p, q, and r, with formulas generated from these by application of sentence connectives in the usual way. To avoid complications, when two logics are considered in the same breath (as with the S_0, S_1 of (1) and (2) above) we assume for the most part that they are logics in the same language.

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[8]The substitution of B for A, or better, *replacement* of A by B in the transition from $C(A)$ to $C(B)$ is of course itself required to be uniform.

Let Δ be a set of formulas in which the only propositional variables to appear are p and q, to emphasize which we write Δ as $\Delta(p, q)$, with $\Delta(A, B)$ as the result of substituting the formula A for every occurrence of p and B for every occurrence of q in the formulas in $\Delta(p, q)$. Adapting a usage of T. Prucnal and A. Wroński (see Czelakowski [8], [9]), we call logic S (in the set-of-formulas sense) *equivalential* if there is a set $\Delta(p, q)$ of formulas in the language of S with the property that A and B are synonymous according to S if and only if $\Delta(A, B) \subseteq S$. (*Cf.* also Porte [29], where the terminology of formula-definable congruences is used instead.)[9] The simplest example of such a $\Delta(p, q)$ would be $\{p \leftrightarrow q\}$, which shows, amongst many others, classical logic to be equivalential. In a purely implicational logic, such as ***BCI*** logic, which comprises all the consequences under the rule Modus Ponens of instances of the three schemas ***B***, ***C***, and ***I*** below, we obtain a similar effect by taking $\Delta(p, q)$ to be $\{p \to q, q \to p\}$.

B $\quad (B \to C) \to ((A \to B) \to (A \to C))$

C $\quad (A \to (B \to C)) \to (B \to (A \to C))$

I $\quad A \to A$

We shall return to this logic and some of its close relatives below. (These logics were intensively investigated by C. A. Meredith, to whom the combinator-derived labelling – "***BCI***" etc. – is also due. Discussion and extensive bibliographical references may be found in Hindley [12].) For the moment, we need to consider the following variation on this theme. Call logics, S_0 and S_1, presumed for simplicity to be in the same language, *similarly equivalential* if there is a set $\Delta(p, q)$ of formulas of that language with, for $i = 0, 1$, A and B are synonymous according to S_i if and only if $\Delta(A, B) \subseteq S_i$. Thus S_0 and S_1 are not just equivalential in that there is some set of formulas licensing the interreplaceability of arbitrary formulas A and B – i.e., the provability of appropriate substitution instances of which is necessary and sufficient for the synonymy of A and B, but it must be the *same* set for both logics. This relationship between S_0 and S_1 provides a simple and obvious sufficient condition for the "only if" direction of (1) above:

[9] In the original usage, it is logics as consequence relations rather than as sets of formulas, that are said to be equivalential. That usage has considerable currency in the literature on contemporary 'abstract algebraic logic' – [4], [8], [10], *q.v.* for the definition of "equivalential" as applied to consequence relations (or 'deductive systems' as this literature would have it). We have chosen to write "Δ" here to echo the choice made in Blok and Pigozzi [4] – but without their infix notation – for what they call a set of 'equivalence formulas' (though arguably 'congruence formulas' would be a more appropriate description). Though we make little direct contact with this tradition, there are some connections, especially as suggested in the following remark from Font *et al.* [10], p. 24: "One of the reasons why classical logic has its distinctive algebraic character lies precisely in the fact that logical equivalence and logical truth are reciprocally definable." (*Cf.* the proof of Proposition 2.3 below.) The remark just quoted could convey the misleading impression that the relation of logical equivalence – or more to the point, synonymy – associated with classical propositional logic is not thus associated with any other logic. If we take S as the set of classical tautologies in the language with, say, negation and implication as primitive connectives and S' as the set of formulas in this same language whose negations like in S, then S-synonymy and S'-synonymy coincide, even though $S \neq S'$.

Proposition 2.1. *For any similarly equivalential logics S_0 and S_1, $S_0 \subseteq S_1$ implies $\equiv_0 \subseteq \equiv_1$.*

Proof. Suppose S_0 and S_1 are similarly equivalential, with replacement-licensing formulas $\Delta(p, q)$, that $S_0 \subseteq S_1$, and that $A \equiv_0 B$. Since $A \equiv_0 B$, we have $\Delta(A, B) \subseteq S_0$, so since $S_0 \subseteq S_1$, $\Delta(A, B) \subseteq S_1$, and thus, finally $A \equiv_1 B$. □

Remark 2.2. As this proof shows, the requirement of being similarly equivalential is stronger than is actually called for (and was employed for the sake of a succinct formulation). If we let Δ_i be the set of replacement-licensing formulas for S_i ($i = 1, 2$), then all we have is that $\Delta_1 \subseteq \Delta_0$ – and not also the converse inclusion.

For the other direction of (1), we are also able to find a fairly simple sufficient condition, frequently satisfied in practice. Again some terminology is needed for its formulation. A logic S is *monothetic* if all its theorems are synonymous, i.e, if for all $A, B \in S$, we have $A \equiv_S B$. (The terminology is motivated by the consideration that for such logics there is, to within synonymy, only one theorem or 'thesis'.) Note that if the language of S has a binary connective $\to$ for which $\{p \to q, q \to p\}$ licenses replacements, and S is closed under Modus Ponens for this connective, then as long as every instance of the schema $\boldsymbol{K}$ is provable:

$\boldsymbol{K}$ $\quad A \to (B \to A)$,

S is monothetic. This applies to all the intermediate logics, intuitionistic and classical logic included, as well as to $\boldsymbol{BCK}$ logic, a pure implicational logic axiomatized as $\boldsymbol{BCI}$ logic was above, except putting $\boldsymbol{K}$ in place of $\boldsymbol{I}$ (all instances of which are now derivable). $\boldsymbol{BCI}$ logic itself, as well as $\boldsymbol{BCIW}$ logic, for which we add the contraction schema

$\boldsymbol{W}$ $\quad (A \to (A \to B)) \to (A \to B)$

are well known non-monothetic logics. (These last two are the implicational fragments, respectively, of Girard's linear logic and of the the relevant logic **R**, according to neither of which are the provable formulas $p \to p$ and $q \to q$ synonymous. See the discussion following Proposition 2.5 below.)

Proposition 2.3. *Let S_0 and S_1 be monothetic logics with $S_0 \cap S_1 \neq \varnothing$. Then $\equiv_0 \subseteq \equiv_1$ implies $S_0 \subseteq S_1$.*

Proof. Assuming S_0 and S_1 as described, choose $B \in S_0 \cap S_1$. Suppose that $\equiv_0 \subseteq \equiv_1$ and that $A \in S_0$, with a view to showing that $A \in S_1$. Since $A \in S_0$ and S_0 is monothetic, $A \equiv_0 B$, and so $A \equiv_1 B$. Since S_1 is also monothetic and $B \in S_1$, $A \in S_1$. □

We turn to the negative business for this section, with a counterexample – or family of counterexamples – to the "if" direction of (1) above. To describe the examples, we need to mention another schema, all instances of which are provable in $\boldsymbol{BCI}$ logic:

$\boldsymbol{B'}$ $\quad (A \to B) \to ((B \to C) \to (A \to C))$.

As with the other Meredith-style labelling, $\boldsymbol{BB'I}$ logic comprises the Modus Ponens consequences of all instances of the schemata named in the label. We use this convention without further comment for other cases as they arise, and further, write such things as "$\boldsymbol{BB'I} \subseteq \boldsymbol{BCI}$" to abbreviate the claim – just made – that $\boldsymbol{BB'I}$ logic is a sublogic of $\boldsymbol{BCI}$ logic. For a proof of the following, see Theorem 5.1 in Martin and Meyer [25], as well as the discussion in their introductory section:

Lemma 2.4. (E. Martin) *If for formulas A, B, we have $A \to B$ and $B \to A$ both provable in $\boldsymbol{BB'I}$ logic then A is the same formula as B.*

Proposition 2.5. *Let S be any logic with $\boldsymbol{I} \subseteq S \subseteq \boldsymbol{BB'I}$. Then the relation $\equiv_S$ is the relation of identity between formulas.*

Proof. Since $\equiv_S$ is reflexive for any S, we have only to show that for S between $\boldsymbol{I}$ and $\boldsymbol{BB'I}$, if $A \equiv_S B$ then $A = B$. Since $A \to A \in S$ for any $S \supseteq \boldsymbol{I}$, if $A \equiv_S B$ then $A \to B \in S$ and $B \to A \in S$. But we are also supposing that $S \subseteq \boldsymbol{BB'I}$, so each of $A \to B$ and $B \to A$ is also $\boldsymbol{BB'I}$-provable, implying by Lemma 2.4 that $A = B$. □

As a corollary to Proposition 2.5, then, we have that all logics between $\boldsymbol{I}$ and $\boldsymbol{BB'I}$ have the same synonymy relation, giving rise to a range of counterexamples to the "if" half of (1):

Example. (A range of examples, really.) If we take S_0 as $\boldsymbol{BB'I}$ logic and S_1 as any one of $\boldsymbol{I}$, $\boldsymbol{BI}$, $\boldsymbol{B'I}$, we have $\equiv_0 \subseteq \equiv_1$ while $S_0 \not\subseteq S_1$. (Alternatively, we can see these as counterexamples to the "only if" half of (2).)

Proposition 2.3 gave sufficient conditions which together ruled out this situation, namely (i) that each of S_0 and S_1 was monothetic, and (ii) that $S_0 \cap S_1 \neq \varnothing$. Clearly in the present instance condition (ii) is satisfied – indeed for the cases just listed, we have $S_1 \subseteq S_0$ – so it is condition (i) that fails. Like $\boldsymbol{BCI}$ logic, all of the logics here fail to be monothetic. (We can see that for all these logics, $\boldsymbol{BCI}$ included, $p \to p$ and $q \to q$ are both provable though the result of replacing the first occurrence of the former by the latter in the equally provable $(p \to p) \to (p \to p)$ is unprovable – an often-made observation with many interesting repercussions not germane to the present study.[10]) The example of S and S' at the end of note 9 also gave a counterexample to the "if" direction of (1), taking these as S_0 and S_1 respectively, or indeed vice versa. In this case, condition (i), the monotheticity condition, is satisfied and it is condition (ii) that fails: S and S' are disjoint.

Can we with equal ease illustrate how a failure of the sufficient condition in Proposition 2.1 can give rise to a counterexample to the "only if" half of (1)? The simplest cases in which strengthening a logic results in a loss of synonymies arise with a change of language, and so are not directly pertinent to the present enterprise since we have agreed to concentrate on comparisons amongst logics in

[10] *Cf.* Kabziński [21] and Section 4 of Humberstone [20].

the same language. For example, if we take the smallest modal logic,[11] or any of various non-normal modal logics such as Lemmon's S0.5, we have a proper extension of non-modal classical propositional logic in which classically equivalent formulas, synonymous in that logic, are no longer synonymous – indeed in which, as for the logics treated in Proposition 2.5, no two distinct formulas are synonymous. (See Porte [29].) Another well-known example is that of intuitionistic logic with 'strong negation', which we shall consider at the end of Section 3. Abiding by our 'same language' restriction on S_0 and S_1, one simple, if artificial, type of case arises as follows.

Example. Take again the language of (non-modal) classical propositional logic and S_0 as the empty set (certainly a subset of the set of formulas of this language, and answering to the most commonly proposed additional conditions on logics as sets of formulas – such as closure under Uniform Substitution[12]), with S_1 as classical logic. Although $S_0 \subseteq S_1$ we do not have $\equiv_0 \subseteq \equiv_1$, because every pair of formulas stand in the former relation while only formulas which are classically equivalent stand in the latter.

The above example is not very appealing because the empty set may not be regarded as a logic on the 'set-of-formulas' conception of logics (which does not say that *any* old set of formulas constitutes a logic), or is perhaps regarded only as an extreme and degenerate case of a logic. If one is interested in some 'atheorematic' logic such as the classical logic of conjunction and disjunction, one would normally pass to something like the consequence relation conception, noting that the set of pairs $\langle \Gamma, A \rangle$ standing in this relation is far from empty, even though the set of such pairs for which Γ is empty is itself empty.[13] Let us accordingly give a counterexample to the "only if" half of (1) not requiring $\varnothing$ to be acknowledged as a logic.

Example. Let the language have two connectives $\rightarrow$ and $\star$, say, of arities 2 and 1 respectively, and let S_0 consist all formulas of the form $\star A$, and S_1 of all all such formulas together with all formulas of the form $A \rightarrow A$. Then for any formulas A and B, $\star A \equiv_0 \star B$, though this is not so in the case of $\equiv_1$; for example $\star p$ is not synonymous with $\star q$ in S_1, because $\star p \rightarrow \star p \in S_1$ while $\star p \rightarrow \star q \notin S_1$.

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[11] We understand a modal logic here to be a set of formulas in the language of classical propositional logic with some functionally complete set of boolean primitives and one additional 1-ary connective $\Box$, containing all classical tautologies and closed under Modus Ponens and Uniform Substitution.

[12] All logics-as-sets-of-formulas we consider satisfy this condition, with the corresponding condition also satisfied for all logics-as-consequence relations in the following section.

[13] This is what we mean by an *atheorematic* consequence relation. Such consequence relations are called 'purely inferential' in Wójcicki [39] – except that Wójcicki tends to prefer formulations in terms of consequence operations rather than consequence relations.

3. Discrimination in Logics as Consequence Relations

We defined synonymy according to a consequence relation in Section 2, before putting this notion to one side in order to compare discriminatory and deductive strength in the simpler setting of logics as sets of formulas. We to take it up again here, to which end the following notation will convenient. For a consequence relation $\vdash$ ($\vdash_i$) we denote by $\equiv_\vdash$ ($\equiv_{\vdash_i}$, or for short $\equiv_i$) the relation of synonymy according to $\vdash$ ($\vdash_i$), and write $A \dashv\vdash B$ to mean "$A \vdash B$ and $B \vdash A$" ($A \dashv\vdash_i B$ to mean "$A \vdash_i B$ and $B \vdash_i A$"). As in Segerberg [33], we call a consequence relation $\vdash$ *congruential* when for all formulas A, B (in the language of $\vdash$) $A \dashv\vdash B$ implies $A \equiv_\vdash B$. (The converse implication holds for any $\vdash$. Thus a congruential consequence relation is one for which logical equivalence – the relation $\dashv\vdash$, that is – and synonymy coincide. Here we rely on the fact that the synonymy of A, B according to a consequence relation $\vdash$, as defined in Section 2, is equivalent to its being the case that for all contexts $C(\cdot)$, we have $C(A) \dashv\vdash C(B)$. Wójcicki [39] uses "self-extensional" for "congruential".)

Conceiving of logics as consequence relations rather than sets of formulas makes for the following modifications to (1) and (2):

$$\vdash_0 \subseteq \vdash_1 \textit{ if and only if } \equiv_0 \subseteq \equiv_1 \tag{3}$$

$$\vdash_0 \subsetneq \vdash_1 \textit{ if and only if } \equiv_0 \subsetneq \equiv_1 \tag{4}$$

Again, we concentrate on the first of these, and on the case in which $\vdash_0$ and $\vdash_1$ are consequence relations on the same language. Here is a very simple sufficient condition for the "only if" direction of (3):

Proposition 3.1. *If* $\vdash_1$ *is congruential and* $\vdash_0 \subseteq \vdash_1$, *then* $\equiv_0 \subseteq \equiv_1$.

Proof. Suppose that $\vdash_0 \subseteq \vdash_1$ for congruential $\vdash_1$, and that $A \equiv_0 B$. Since $A \equiv_0 B$, we have $A \dashv\vdash_0 B$, so since $\vdash_0 \subseteq \vdash_1$, $A \dashv\vdash_1 B$, whence by the congruentiality of $\vdash_1$, we get $A \equiv_1 B$. □

Proposition 3.1 is (nearly) a special case of the analogue for consequence relations of Proposition 2.1. Although the notion of an equivalential (set-of-formulas) logic was abstracted from the notion of an equivalential consequence relation, the latter turns out not to be the pertinent concept, and what we want instead is the concept of a consequence relations $\vdash$ with *sequent-definable synonymy*, by which we mean (*cf.* [29]) that there is a set $\Sigma(p, q)$ of pairs $\langle \Gamma, C \rangle$, all formulas occurring in which are constructed from only the variables p, q with the property that for all formulas A, B (in the language of $\vdash$) we have $\Sigma(A, B) \subseteq \vdash$ if and only if $A \equiv_\vdash B$. As in Section 2, we immediately pass to a relational version of this concept, saying that consequence relations $\vdash_0$ and $\vdash_1$ have *similarly sequent-definable congruences* if the same set $\Sigma(p, q)$ witnesses the sequent-definability of synonymy for $\vdash_0$ and $\vdash_1$. Then by a simple argument which replaces S_i in the proof of Proposition 2.1 by $\vdash_i$ and substitutions in the set of formulas $\Delta(p, q)$ by substitutions in the set of sequents $\Sigma(p, q)$, we obtain a proof of:

Proposition 3.2. *For any consequence relations* $\vdash_0$ *and* $\vdash_1$ *with similarly sequent-definable synonymies,* $\vdash_0 \subseteq \vdash_1$ *implies* $\equiv_0 \subseteq \equiv_1$.

The analogue of Remark 2.2 applies here too.

It may seem stretching things to use the term *sequents* for the ordered pairs $\langle\Gamma, C\rangle$, certain sets of which are consequence relations, since the the 'antecedent' of a sequent might typically be required to be a finite set, whereas these Γ will not all be finite. Indeed on many versions of what a sequent should be (e.g., for the sake of a convenient sequent-calculus), Γ wouldn't be a set (of formulas) at all but a multiset or a sequence. Nevertheless, the terminology is convenient and we ignore those objections to its use here. Let us further follow Blamey [5] in using $\succ$ as our sequent-separator – that is, we notate the sequent $\langle\Gamma, C\rangle$ more suggestively as $\Gamma \succ C$. We are now in a position to see Proposition 3.1 as close to being a special case of Proposition 3.2: a congruential consequence relation is one for which synonymy is defined by the set of sequents $\Sigma(p, q) = \{p \succ q,\ q \succ p\}$. "Close to being" a special case but not quite there, since Proposition 3.1 demands only that $\vdash_1$ be congruential, whereas the application just envisaged of Proposition 3.2 would appear to require the condition that both $\vdash_0$ and $\vdash_1$ be congruential (since they need to have similarly sequent-definable synonymies).[14]

We turn our attention to the provision of two counterexamples to the "if" direction of (3), each of which features a pair of consequence relations which, though distinct, yield the same synonymy relation. These examples, especially the second (appearing after Remark 3.12), are of some theoretical interest in their own right, and all four logics (playing the $\vdash_0$ and $\vdash_1$ roles in the two examples) are congruential, though that fact does not need to be exploited. The first example (immediately following Coro. 3.6 below) draw attention to a relation we shall call "1-agreement" between consequence relations, isolating which will assist in presenting the second example. After that discussion, we conclude with a counterexample (or two) to the "only if" direction of (3).

For the first of these examples, the language we use has only one connective, the 0-place connective (sentential constant) $\top$; we define $\vdash_0^\top$ to be the least consequence relation $\vdash$ on this language satisfying (5) for $\Gamma \neq \varnothing$, and $\vdash_1^\top$ to be the least consequence relation $\vdash$ on the language satisfying (5) for arbitrary Γ (equivalently, satisfying (5) for $\Gamma = \varnothing$):

$$\Gamma \vdash \top. \tag{5}$$

$\vdash_0^\top$ is a simplified version of idea of Roman Suszko's, described in note 7 of Smiley [34], and it is not hard to check that $\vdash_0^\top \subsetneq \vdash_1^\top$. Indeed, we will verify this twice over, the second proof following its statement below as Corollary 3.4. It is clear from the definitions that $\vdash_0^\top \subseteq \vdash_1^\top$; that the converse inclusion does not hold follows from the fact that $\varnothing \vdash_1 \top$ (again from the definition of $\vdash_1^\top$) while $\varnothing \nvdash_0^\top \top$. We can verify this latter fact syntactically by thinking of the above definition of $\vdash_0^\top$ as an inductive definition ("from below") of the class of pairs $\langle\Gamma, A\rangle$ standing in

[14] The author has the strong impression of missing an insight here.

this relation, which allows for a proof by induction on the length of a construction which would place $\langle\Gamma, A\rangle$ in $\vdash_0^\top$ only when $\Gamma \neq \varnothing$. (See Scott [32] for this type of argument; the characterization below in terms of valuations is also much inspired by Scott's work.)

An alternative to the above (quasi-)proof-theoretic argument, we can obtain the same conclusion by semantic reasoning, couched in terms of the notion of a consequence relation $\vdash$'s being *determined* by a class V of valuations (bivalent truth-value assignments to the formulas of the language of $\vdash$), a relation defined to hold between $\vdash$ and V just in case for all sets Γ of formulas of the language and all formulas A thereof: $\Gamma \vdash A$ if and only if for each $v \in V$, whenever $v(C) = \mathrm{T}$ for all $C \in \Gamma$, then $v(A) = \mathrm{T}$. (We use "T", "F", to denote the two truth-values; if $\vdash$ has been specified by means of a proof system, the "only if" and the "if" parts of this definition amount to the soundness and the completeness, respectively, of this system, with respect to V.) The easy proof of the following is left to the reader; the reference to valuations in both cases is to valuations for the (common) language of $\vdash_0^\top$ and $\vdash_1^\top$.

Proposition 3.3. *Let v_{F} be the unique valuation (for the language of $\vdash_0^\top$ and $\vdash_1^\top$) assigning the value* F *to every formula, and V be the class of all valuations (for this language) satisfying $v(\top) = \mathrm{T}$. Then*

(i) $\vdash_0^\top$ *is determined by $V \cup \{v_{\mathrm{F}}\}$*
(ii) $\vdash_1^\top$ *is determined by V.*

We now repeat the earlier syntactically argued assertion with its new semantic justification:

Corollary 3.4. $\vdash_0^\top \subsetneq \vdash_1^\top$.

Proof. That $\vdash_0^\top \subseteq \vdash_1^\top$ follows from Prop. 3.3 by a familiar Galois duality between consequence relations and classes of valuations, since $V \subseteq V \cup \{v_{\mathrm{F}}\}$; the failure of the converse inclusion (between the $\vdash_i^\top$) is illustrated by the fact that $\varnothing \vdash_1^\top \top$ while $\varnothing \nvdash_0^\top \top$. □

Remark 3.5. The formula $\top$, as it behaves according to $\vdash_0^\top$, is what is called in Humberstone [16], p. 59, a "mere follower": it follows from every formula and thus from every non-empty set of formulas – but not from the empty set of formulas. Note that so defined, only an atheorematic consequence relation can have a mere follower, and that any two mere followers are logically equivalent (each being a consequence of the other).

From Proposition 3.3 we may also infer (by an argument we leave to the reader) the following:

Corollary 3.6. *For all non-empty Γ and all formulas B, we have $\Gamma \vdash_0^\top B$ if and only if $\Gamma \vdash_1^\top B$.*

In particular, then, we have:

Example. For all formulas A, B, we have $A \dashv\vdash_0^\top B$ if and only if $A \dashv\vdash_1^\top B$. Since $\vdash_0^\top$ and $\vdash_1^\top$ are congruential, the induced synonymy relations $\equiv_0^\top$ and $\equiv_1^\top$ coincide, the fact that $\vdash_0^\top \subsetneq \vdash_1^\top$ notwithstanding, providing a counterexample to the "if" direction of (3), taking $\vdash_0$ and $\equiv_0$ (resp. $\vdash_1$ and $\equiv_1$) in (3) as $\vdash_1^\top$ and $\equiv_1^\top$ (resp. $\vdash_0^\top$ and $\equiv_0^\top$). Alternatively, we can see this as a counterexample to the "only if" direction of (4) – keeping the subscripts the same, this time. (In fact, the reference to congruentiality is not needed. See Remark 3.7 below.)

There is one aspect of the situation just reviewed we shall isolate for our second example. Say that consequence relations $\vdash$ and $\vdash'$ on the same language *n-agree* when for all formulas A and all sets of formulas Γ of cardinality $n \in \mathbb{N}$, we have $\Gamma \vdash A$ if and only if $\Gamma \vdash' A$. In this terminology Corollary 3.6 says that $\vdash_0$ and $\vdash_1$ n-agree for all $n \geq 1$. What actually matters for the above example though, is specifically that these consequence relations 1-agree:

Remark 3.7. Even if $\vdash$ and $\vdash'$ are not congruential, if $\vdash$ and $\vdash'$ 1-agree, then $\equiv_\vdash = \equiv_{\vdash'}$, since, adapting the characterization of congruentiality at the end of the opening paragraph of this section, A and B are synonymous according to a consequence relation just in case for all C, $C(A)$ and $C(B)$ are equivalent. But any 1-agreeing consequence relations also agree in respect of which formulas are synonymous – that is, have the same synonymy relation.

For our second example, included for its intrinsic interest, there is again only one connective in the language, and this time it is binary, and will be written – for reasons to become clear immediately – as "$\wedge$". Let $\vdash_0^\wedge$ and $\vdash_1^\wedge$ be the least consequence relations $\vdash$ on this language satisfying, for all formulas A and B and in the case of $\vdash_0^\wedge$, for all Γ of the form $\{C\}$ while in the case of $\vdash_1^\wedge$, for arbitrary Γ, the condition (6):

$$\Gamma \vdash A \wedge B \textit{ if and only if } \Gamma \vdash A \textit{ and } \Gamma \vdash B. \tag{6}$$

The consequence relations $\vdash_0^\wedge$ and $\vdash_1^\wedge$, or similarly related consequence relations with additional connectives present answering to their own conditions, are distinguished in Koslow [22] and Cleave [7]. $\vdash_1^\wedge$, is the restriction to the language with $\wedge$ of the consequence relations of intuitionistic or classical logic; it is called the logic of 'parametric' conjunction in [22], where essentially reasoning of Prop. 3.8 and Coro. 3.9 may be found (p. 129*f.*).[15]

Proposition 3.8. *Whenever* $\Gamma \vdash_0^\wedge A$*, we have* $C \vdash_0^\wedge A$ *for some* $C \in \Gamma$*.*

Corollary 3.9. $\vdash_0^\wedge \subsetneq \Gamma \vdash_1^\wedge$.

Proof. Clearly we have $p, q \vdash_1^\wedge p \wedge q$, since we may take Γ in (6) as $\{p, q\}$; but by Proposition 3.8 $p, q \nvdash_0^\wedge p \wedge q$, as otherwise we should have $p \vdash_0^\wedge p \wedge q$ or $q \vdash_0^\wedge p \wedge q$ (neither of which is even the case for $\vdash_1^\wedge$, of course). □

[15] In the case of Cleave [7], pp. 121*ff.* should be consulted. There are many problems with Cleave's discussion, and a few with Koslow's; see [15], esp. p. 478*f.* for these.

It was promised that the example involving the $\vdash_i^{\wedge}$ would be of some theoretical interest in its own right. There are two points of interest. A philosophical moral to be drawn is most easily seen if condition (6) is recast as a collection of sequent-to-sequent rules, in which case the weaker ($\vdash_0^{\wedge}$-defining) $\Gamma = \{C\}$ version of (6) emerges as follows, with semicolons separating the premiss-sequents from each other and "/" separating them from the conclusion-sequent:

(i) $C \succ A$; $C \succ B$ / $C \succ A \wedge B$. (ii) $C \succ A \wedge B$ / $C \succ A$. (iii) $C \succ A \wedge B$ / $C \succ B$.

The point of interest is that these rules already uniquely characterize $\wedge$ (to within logical equivalence),[16] even though they are weaker than the standard rules (i.e., the rules with the general set-variable "Γ" replacing C throughout – though it is easy to see that the rule (iii) would not be strengthened by this generalization). Thus it is not open to the intuitionist, for example, to complain that what is wrong with the classical rules governing negation is that they are 'stronger than needed' to characterize this connective uniquely, since the intuitionistically acceptable negation rules already suffice for uniqueness. To take such a line without further qualification would be to leave the intuitionist open to an objection to the intuitionistically accepted rules governing conjunction, since as just observed, these are also stronger than needed for unique characterization. In (the paper abstracted as) [13] it is suggested that the 'further qualification' needed will address the issue of rules being *fully general* in respect of side-formulas (so arbitrary Γ, rather than just C or more explicitly $\{C\}$, for instance), though what this comes to will naturally depend on exactly what form the sequents take – e.g., on whether multiple succedents are to be permitted. (Of course such sequents do not arise in the rules embodying conditions on consequence relations, but we are speaking of sequent-to-sequent rules for a notion of sequent that should be thought of as yet to be settled on, when issues of one logic *vs.* another are being aired.)

Philosophy of logic aside, the case of $\vdash_0^{\wedge}$ presents us with an interesting task in valuational semantics, namely that of informatively specifying a class of valuations which determines this consequence relation. To attack this problem, which will have dividends for our main business as well (see Coro. 3.11), we need some terminology and notation. If # is an n-ary connective with which some

[16]This is pointed out in Example 4.3(i) on p. 121 of [7]. (The parenthetical "to within logical equivalence" is an allusion to the possible contrast with unique characterization *to within synonymy*, on which see Humberstone [19], Sections 3 and 4.) The result of this is that if one party to a dispute about the logical powers of conjunction endorses only the $\vdash_0^{\wedge}$ conditions, while the other endorses the stronger $\vdash_1^{\wedge}$ conditions, they cannot agree to bury their differences by agreeing to adopt a logic with two connectives in place of $\wedge$ – $\wedge_0$ and $\wedge_1$, say – governed by rules embodiing the respective conditions: because even the weaker rules have the unique characterization property, the resulting combined logic then has the $\wedge_0$-conjunction of two formulas following from those formulas. The situation is just as with intuitionistic and classical negation, alluded to presently, in which the intuitionist would be ill advised indeed to concede the intelligibility of a connective governed by the rules for classical negation *alongside* and notationally distinguished from the favoured intuitionistic negation: the former's distinctive behaviour will then infect the latter, leaving no room, as Humberstone [13] concludes, for any such 'live and let live' attitude.

preassigned n-ary truth function f is associated, then we call a valuation $\#$-*boolean* if for all formulas $A_1, \ldots, A_n$, $v(\#(A_1 \ldots A_n)) = f(v(A_1), \ldots, v(A_n))$. Thus the class V featuring in Proposition 3.3 is the class of $\top$-boolean valuations, while the $\wedge$-boolean valuations (for a given language) are exactly those which assign the value T to formulas (of that language) $A \wedge B$ when they assign the value T to A and to B. A somewhat less frequently encountered notion is the following. (See [16].) For an arbitrary family of valuations, V we denote by $\sum V$ what we call the *disjunctive combination* of the valuations in V, defined to be the unique valuation u for which for all formulas A, $u(A) = \mathrm{T}$ if and only if there is some $v \in V$ with $V(A) = \mathrm{T}$. If $V = \{v_1, v_2\}$, we write $v_1 + v_2$ for $\sum V$. The dual – in the sense of poset duality, not Galois duality[17] – operation on valuations, *conjunctive combination* here denoted by $\prod V$ ($v_1 \cdot v_2$ in the binary case) is similarly defined but with "there is some" replaced by "for all"; these are fairly well known, being a bivalentized version of the notion of a supervaluation over V.[18] Their key logical significance is that the consequence relation determined by a class of valuations remains unaffected by adding conjunctive combinations of valuations to the determining class.[19] This, which is not so for generalized consequence relations, is due to the presence of a single formula on the *right* of the "$\vdash$". (The presence of at most one formula on the right, that is, rather than at least one, as in the preceding note.) In view of Proposition 3.8 above, which says that for the case of $\vdash_0^\wedge$ a consequence statement holds in virtue of a single formula from amongst those on the *left*, suggests the semantic characterization given in Proposition 3.10 below. It is well known that $\vdash_1^\wedge$ is easily seen to be determined by the class of all $\wedge$-boolean valuations; what we need for $\vdash_0^\wedge$ is the class of disjunctive combinations of such valuations, so we pause to observe that disjunctively combining $\wedge$-boolean valuations typically results in a valuation that is not $\wedge$-boolean (whereas the class of $\wedge$-boolean valuations is closed under *conjunctive* combination). We illustrate with the binary mode of combination.

Example. Let u and v be $\wedge$-boolean valuations satisfying: $u(p) = v(q) = \mathrm{T}$, $u(q) = v(p) = \mathrm{F}$. For their disjunctive combination $u + v$ we have $u + v(p) = \mathrm{T}$, since $u(p) = \mathrm{T}$, and also $u + v(q) = \mathrm{T}$ since $v(q) = \mathrm{T}$. However, $u + v(p \wedge q) = \mathrm{F}$, since neither u nor v verifies this conjunction, so $u + v$ is not $\wedge$-boolean.

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[17] In the terminology, though not the notation, of [16], $+$ is Galois dual to $\vee$ and $\cdot$ to $\wedge$. (Upward and downward pointing triangles are used in [16] to symbolize conjunctive and disjunctive combinations, large for the case of families of valuations and small in the case of the binary operation.)

[18] Incidentally, the standard 'gappy' version of what later became known as supervaluations appears already at the end of the second paragraph of §4 in Nelson [27].

[19] A special case is that of $V = \varnothing$, for which $\prod V$ is the valuation v_T assigning the value T to every formula. So any $\vdash$ determined by a class U of valuations is also determined by $U \cup \{v_\mathrm{T}\}$. For the same choice of V, $\sum V$ is the valuation v_F of Proposition 3.3, which taken together with Corollary 3.4 shows that, by contrast with the case of v_T, adding v_F to the determining class can change which consequence relation is determined. The explanation for this lies in the mandatory appearance of a formula on the right of the "$\vdash$" (or "$\succ$", at the level of individual sequents), as contrasted with the possible disappearance of all formulas from the left.

Having shown that we obtain a new class of valuations other than just that consisting of $\wedge$-boolean valuations when passing to arbitrary disjunctive combinations of such valuations, we proceed to our semantic characterization of $\vdash_0^\wedge$

Proposition 3.10. *The consequence relation $\vdash_0^\wedge$ is determined by the class of all valuations which are disjunctive combinations of families of $\wedge$-boolean valuations.*

Proof. We must show that $\Gamma \vdash_0^\wedge A$ if and only if every disjunctive combination of $\wedge$-boolean valuations which verifies each formula in Γ also verifies A. The "only if" direction is essentially a soundness proof for the system with, in addition to basic structural rules, the sequent-to-sequent rules (i), (ii), (iii), above), for which purpose it suffices to check that no disjunctive combination of $\wedge$-boolean valuations verifies all the left hand formulas without verifying the right-hand formula of any provable sequent. Since rules (ii) and (iii) are obviously equivalent (given the structural rules encoding the fact that our sequents are the elements of a consequence relation) to the zero-premiss rules $A \wedge B \succ A$ and $A \wedge B \succ B$ it is sufficient in their case to check that there are no countervaluations in the class w.r.t. which soundness is being shown, We consider the former by way of example. Suppose u is $\sum V$ for a family V of $\wedge$-boolean valuations, and $u(A \wedge B) = \mathrm{T}$. We must show that $u(A) = \mathrm{T}$. As $u(A \wedge B) = \mathrm{T}$ and $u = \sum V$, there is $v \in V$ with $v(A \wedge B) = \mathrm{T}$. But all valuations in V, v included, are $\wedge$-boolean, so $v(A) = \mathrm{T}$, and therefore $u(A) = \mathrm{T}$. We now check (i), showing that if there is a countervaluation to the conclusion sequent $C \succ A \wedge B$ of an application of this rule, then there is a countervaluation to one or another of the premiss-sequents $C \succ A$, $C \succ B$. So suppose that $u = \sum V$ for a collection V of $\wedge$-boolean valuations, and $u(C) = \mathrm{T}$ while $u(A \wedge B) = \mathrm{F}$. Then for some $v \in V$, we have $v(C) = \mathrm{T}$, but since $u(A \wedge B) = \mathrm{F}$, $v(A \wedge B) = \mathrm{F}$. As v is $\wedge$-boolean, either $v(A) = \mathrm{F}$ or $v(B) = \mathrm{F}$, so since v $(= \sum\{v\})$ is itself a disjunctive combination of $\wedge$-boolean valuations, it is either a countervaluation to $C \succ A$ or to $C \succ B$.

We turn to the "if" (completeness) direction of the claim. We must show then whenever $\Gamma \nvdash_0^\wedge A$, we can find a disjunctive combination of $\wedge$-boolean valuations verifying each formula in Γ but not A. For each $C \in \Gamma$ define the valuation v_C by setting $v_C(B) = \mathrm{T}$ iff $C \vdash_0^\wedge B$ for all formulas B. Note that v_C is guaranteed to be $\wedge$-boolean by the way $\vdash_0^\wedge$ was defined. Also observe that for each $C \in \Gamma$, we have $v_C(A) = \mathrm{F}$, since otherwise we should have $C \vdash_0^\wedge A$ and hence, by a defining property (variously called monotonicity, thinning, weakening,...) of consequence relations, $\Gamma \vdash_0^\wedge A$, contradicting our initial assumption. But together these facts imply that for $u = \sum\{v_C | C \in \Gamma\}$, u is a disjunctive combination of $\wedge$-boolean assigning T to every formula in Γ and F to A, as required. □

Corollary 3.11. *The consequence relations $\vdash_0^\wedge$ and $\vdash_1^\wedge$ 1-agree.*

Proof. Since $\vdash_0^\wedge \subseteq \vdash_1^\wedge$, we have only to show that for all formulas C, A, if $C \vdash_1^\wedge A$, then $C \vdash_0^\wedge A$. So, arguing contrapositively, suppose that $C \nvdash_0^\wedge A$. By Prop. 3.10 there is a valuation $u = \sum V$ with all $v \in V$ $\wedge$-boolean, with $u(C) = \mathrm{T}$ and

$u(A) = \mathrm{F}$. Thus for some $v \in V$, $v(A) = \mathrm{T}$ while $v(C) = \mathrm{F}$. But v is an $\wedge$-boolean valuation, so since $\vdash_1^\wedge$ is determined by the class of $\wedge$-boolean valuations, $C \nvdash_1^\wedge A$. □

Remark 3.12. Notice how this argument would have failed if we had tried to show that, for instance, if $C, D \vdash_0^\wedge A$ then $C, D \vdash_0^\wedge A$. In this case we have $u(C) = u(D) = \mathrm{T}$ while $u(A) = \mathrm{F}$, for $u = \sum V$ as above: but this allows $v \in V$ with $v(C) = \mathrm{T}$ and $v' \in V$ with $v'(D) = \mathrm{T}$, with no guarantee that $v = v'$ and so way to complete the argument – since as we saw in the Example preceding Prop. 3.10, $v + v'$ need not be $\wedge$-boolean. (We could have established Coro. 3.11 purely syntactically, but the semantic characterization seems illuminating.)

We have now assembled all the ingredients for the second of the counterexamples to be presented here to the "if" direction of (3).

Example. Although $\equiv_0^\wedge = \equiv_1^\wedge$, by Coro. 3.11 and Remark 3.7, $\vdash_0^\wedge \subsetneq \Gamma \vdash_1^\wedge$ (by Coro. 3.9).

We pause to notice that the $\vdash_i^\top$ and $\vdash_i^\wedge$ pairs (i = 0,1) with which we have illustrated the failure of the "if" direction of (3), are also convenient indicators of the falsity of a conjecture either to the effect that if consequence relations n-agree then they must m-agree whenever $m \leq n$ or to the effect that n-agreeing consequence relations must m-agree whenever $n \leq m$. Counterexamples to these conjectures are given respectively by the cases of the $\vdash_i^\top$, which 1-agree without 0-agreeing, and of the $\vdash_i^\wedge$, which 1-agree without 2-agreeing.

Remark 3.13. It should be noted, however, that the implication in the case of $m \leq n$: "$\vdash, \vdash'$ n-agree $\Rightarrow \vdash, \vdash'$ m-agree" holds under very weak conditions for all $m \geq 1$. The following additional condition secures the implication, for example: that for every formula A there is some formula $B \neq A$ such that $B \vdash A$, and likewise in the case of $\vdash'$. Alternatively, if one wished, one could secure the above implication, still with the $m \geq 1$ proviso in force, by a change in the definition of n-agreement, defining this relation to hold between $\vdash$ and $\vdash'$ just in case for all formulas $A_1, \ldots, A_n, B$: $A_1, \ldots, A_n \vdash B$ iff $A_1, \ldots, A_n \vdash B$. (This differs from the original definition because we can have $A_i = A_j$ when $i \neq j$.)

Before leaving the subject of 1-agreement altogether, we should take a moment to observe that despite its figuring in our examples distinct consequence relations with the same synonymy relation, 1-agreement is by no means a necessary condition for two consequence relations to coincide thus in respect of synonymy. The small reminder we include to that end, Proposition 3.14, requires the following concept. Let us call consequence relations on the same language $\vdash$ and $\vdash'$ *weakly dual* when for all formulas A, B, of that language, $A \vdash B$ if and only if $B \vdash' A$. Many consequence relations will weakly dual to any given consequence relation $\vdash$, and though they will 1-agree with each other, they will typically not 1-agree with $\vdash$ – the point of current interest.[20]

Proposition 3.14. *If $\vdash$ and $\vdash'$ are weakly dual, then they have the same synonymy relation.*

Proof. Weakly dual $\vdash$ and $\vdash'$, though not in general 1-agreeing, still 'agree' in respect of which formulas are equivalent to each other, and so, by the reasoning given in Remark 3.7, agree as to which pairs of formulas are synonymous. □

The counterexamples we have provided to the "if" direction of (3) have been of cases of differing consequence relations with the same same synonymy relations. We cannot similarly offer counterexamples to the "only if" direction of (3) – our final topic – in which $\vdash_0$ coincides with $\vdash_1$ while $\equiv_0$ and $\equiv_1$ differ, since $\equiv_i$ is fixed by $\vdash_i$. A somewhat artificial counterexample can be obtained by tinkering minimally with that given at the end of Section 2. We use the same language, with connectives $\to$ and $\star$, and define $\vdash_0$ and $\vdash_1$ as the least consequence relations $\vdash$ on this language such that (for the former) $\varnothing \vdash \star A$ for all A, and (for the latter) $\varnothing \vdash \star A$ as well as $\varnothing \vdash A \to A$ for all formulas A. The explanation given at the end of Section 2 as to why this is a counterexample applies here also, *mutatis mutandis.* Our final topic will be a more interesting 'naturally occurring' counterexample to the "only if" direction of (3).

Consider first the consequence relations of intuitionistic (propositional) logic, $\vdash_{IL}$, with any familiar set of primitive connectives, and of intuitionistic logic with strong negation $\vdash_{ILS}$, whose language contains a further 1-ary connective ('strong negation') written as "$-$", governed by principles which may be found in any discussion of the subject, such as Chapter 7, Section 2 of Gabbay [11].[21] (If the account specifies a logic in the set-of-formulas sense, by means of an axiomatization using Modus Ponens as the sole rule, then the consequence relation we are interested in relates Γ to A in the following familiar way: A stands at the end of a sequence of formulas each of which is either an axiom, an element of Γ, or

[20]From the definition of weak duality given here it is not hard to deduce the following. The smallest consequence relation weakly dual to a given $\vdash$ is the $\vdash'$ defined by: $\Gamma \vdash' A$ iff for some $B \in \Gamma$, $A \vdash B$. The largest consequence relation weakly dual to $\vdash$ is the $\vdash'$ defined by: $\Gamma \vdash' A$ iff for all B such that $C \vdash B$ for each $C \in \Gamma$, we have $B \vdash A$. This latter is essentially the notion of *the* dual of $\vdash$ offered by Wójcicki [38] (see also [35]) and §9.5 of Koslow [22], though there are slight differences. Koslow is discussing what calls implication relations rather consequence relations, which amounts to treating them as relations between finite but non-empty sets of formulas and individual formulas (subject otherwise to the usual defining conditions for consequence relations), while Wójcicki's definition is like ours except that what ours requires of Γ itself for $\Gamma \vdash' A$ to hold is instead required of some finite subset of Γ. With generalized consequence relations, of course, matters are much more straightforward since one can take the dual of such a relation just to be its converse. (See, e.g., Gabbay [11], p. 16.)

[21]Since we quoted Nelson [27] in our opening section, we should stress that the logic of strong negation presented in [27] is definitely *not* what we have in mind here (though it was earlier work by Nelson that inspired what we do have in mind), since that is not an extension of (even the implicational fragment of) intuitionistic logic.

follows from earlier formulas in the sequence by an application of Modus Ponens.) A well-known feature of $\vdash_{ILS}$ is that it is not congruential, since for example, writing "$\neg$" for (ordinary) intuitionistic negation, $\neg\neg\neg p$ and $\neg p$ are $\vdash_{ILS}$-equivalent (being $\vdash_{IL}$-equivalent), whereas $-\neg\neg\neg p$ and $-\neg p$ are not $\vdash_{ILS}$-equivalent.[22] The latter pair of formulas are $\vdash_{ILS}$-equivalent respectively to $\neg\neg p$ and p, which are not $\vdash_{ILS}$-equivalent, since they do not involve strong negation, are not $\vdash_{IL}$-equivalent and $\vdash_{ILS}$ is a conservative extension of $\vdash_{IL}$. Clearly, however, there is *something* non-conservative going on. We could say that the passage from intuitionistic logic to intuitionistic logic with strong negation fails to conserve synonymy – which should raise eyebrows amongst adherents of intuitionistic logic, the conservativity of the extension notwithstanding[23] – since evidently the 'strong negation'-free formulas $\neg p$ and $\neg\neg\neg p$ synonymous according to $\vdash_{IL}$ but not according to $\vdash_{ILS}$. This gives the counterexample we have in mind (and in fact could have presented in a suitably modified form in Section 2, as involving 'formula' logics):

Example. We have assembled the pieces for a counterexample to the following case of the "if" direction of (3):

$$\vdash_{IL} \subseteq \vdash_{ILS} \Rightarrow \equiv_{IL} \subseteq \equiv_{ILS},$$

since, as just observed, although $\vdash_{IL} \subseteq \vdash_{ILS}$, we have $\neg p \equiv_{IL} \neg\neg\neg p$ without $\neg p \equiv_{ILS} \neg\neg\neg p$.

This is, however, a 'two-language' example since strong negation is not a connective in the language of $\vdash_{IL}$, whereas we undertook to seek counterexamples without a change of language. One might think to get around this by considering in place of the consequence relation $\vdash_{IL}$ a variation which has strong negation in its language but enjoying no special logical behaviour, $\vdash_{IL(S)}$, we could call it, much as with $\Box$ in the smallest modal logic or $\bot$ in Minimal Logic (*Minimalkalkül*). For the counterexample, however, we should need $\neg p \equiv_{IL(S)} \neg\neg\neg p$, which is no longer the case since the two formulas involved here give non-equivalent results (relative to $\vdash_{IL(S)}$) when embedded in the scope of the strong negation connective: the very point we were exploiting concerning $\vdash_{ILS}$ (though with $\vdash_{IL(S)}$, the situation is more serious in that, as with several other logics we have considered, no two formulas are synonynmous). If there is a simple one-language counterexample in this vicinity to the claim that inclusion of consequence relations implies the corresponding inclusion of synonymy relations, we leave it for others to find.

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