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ORIGINAL ARTICLE

Truth-value relations and logical relations

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Abstract

After some generalities about connections between functions and relations in Sections 1 and 2 recalls the possibility of taking the semantic values of n-ary Boolean connectives as n-ary relations among truth-values rather than as n-ary truth functions. Section 3, the bulk of the paper, looks at correlates of these truth-value relations as applied to formulas, and explores in a preliminary way how their properties are related to the properties of "logical relations" among formulas such as equivalence, implication (entailment) and contrariety (logical incompatibility), concentrating for illustrative purposes on binary logical relations such as those just listed. To avoid an excess of footnotes, some points have been deferred to an Appendix as "Longer Notes".

KEYWORDS

exhaustification, logical relations, truth functions

1 | INTRODUCTION

Our interest here will be on relations among truth-values and relations holding among sentences (or formulas) in virtue of their truth-values. In particular, the concern will be with such relations as are systematically associated with truth functions. The setting will be bivalent, with the two truth-values taken as T and F. A *valuation* for a language is any mapping assigning one of these values to each formula of the language.[1]

[1]All object languages considered here are sentential, with their formulas constructed by suitable iterated application of primitive connectives (varying from language to language) to a denumerable set, Π, of sentence letters (or propositional variables) $p_1, p_2, \ldots$ (usually written as $p, q, \ldots$). Lest the restriction to bivalent valuations be thought unduly restrictive, recall that every consequence relation $\vdash$ on such a language – and not just those with a two-element (strongly) characteristic matrix – is *determined by* a class $\mathcal{V}$ of such valuations, meaning by this that it is of the form $\vDash_{\mathcal{V}}$, defined thus: for any set $\Gamma \cup \{A\}$ of formulas, $\Gamma \vDash_{\mathcal{V}} A$ iff for no $v \in \mathcal{V}$, do we have $v(C) = T$ for each $C \in \Gamma$, while $v(A) = F$. We use the customary notational abbreviations in connection with consequence relations, "$A, B \vDash_{\mathcal{V}} C$" for "$\{A, B\} \vDash_{\mathcal{V}} C$", "$\vDash_{\mathcal{V}} A$ for $\emptyset \vDash_{\mathcal{V}} A$", and so forth. When $\vDash_{\mathcal{V}}$ is being thought of as a relation of logical consequence – for this or that logic – one typically expects it to respect substitutions, a property secured by requiring that for any $\Pi_o \subseteq \Pi$ there is some $v \in \mathcal{V}$ such that $v(p_i) = T$ iff $p_i \in \Pi_0$. The informal use, here, of "respecting substitutions" is made precise in the definition of substitution-invariance for consequence relations in note 22 below. This condition is satisfied whenever no constraint is imposed on the treatment of sentence letters by the valuations in $\mathcal{V}$, but only on compound formulas, as with the main choices of interest below, the most prominent of which is the class of all Boolean valuations.

A *Boolean connective* is one for which there is a conventionally associated truth function used to interpret it in the semantics of classical logic ($\wedge$, $\vee$, $\neg$, $\rightarrow$, etc.). A *Boolean valuation* v for such a language is one for which that interpretation is in force; for example, if $\rightarrow$ is among the language's (binary) connectives, we require that for all formulas A, B of the language $v(A \rightarrow B) = T$ except when $v(A) = T$ and $v(B) = F$. Returning to the second sentence of the opening paragraph above and its talk of relations associated with truth functions, the next point to notice is that there is no single systematic way of associating such a relation with a given truth function, but rather two such ways. This twofold contrast applies more generally than just to the case of truth functions themselves, so the present section pauses to locate it in this more general setting. Doing so will involve us in rehearsing some considerations that will be familiar to most readers, but it is worth taking a moment to get the discussion off on the right foot.

Since we are concentrating on connections between functions and relations, and initially on two ways, in particular, of associating a function with a relation, the cleanest way to proceed will be to regard functions as *sui generis* rather than by identifying them with functional relations, where, as usual, an $(n+1)$-ary relation R on a set S is a subset of S^{n+1}, and is *functional* just in case for all $x_1,\ldots,x_n \in S$ there is exactly one $y \in S$ for which $\langle x_1,\ldots,x_n,y\rangle \in R$.[2] In that case the corresponding function, $\mathsf{Fun}(R)$, we may call it, is the function f mapping any $x_1,\ldots,x_n \in S$ to that unique y. Although the correspondence between functions and functional relations is a bijection, distinguishing them allows us to respect the usual "arity" talk: applying $\mathsf{Fun}(\cdot)$ turns an $(n+1)$-ary relation into an n-ary function, when the relation is a functional relation (and is undefined otherwise). Travelling in the reverse direction, we have the operation $\mathsf{Rel}(\cdot)$, say, which turns any n-ary (or n -place) function f – this time without restriction – into an $(n+1)$-ary relation $\mathsf{Rel}(f) = \{\langle x_1,\ldots,x_n,y\rangle \mid f(x_1,\ldots,x_n) = y\}$. Rel, in other words, is the operation that trades in a function for its graph. (On the corresponding linguistic move, see Longer Note A.)

$$\mathsf{Rel}(\mathsf{Fun}(R)) = R \text{ whenever } \mathsf{Fun}(R) \text{ is defined.} \tag{1.1}$$

$$\mathsf{Fun}(\mathsf{Rel}(f)) = f \text{ for any function } f \text{ whatever.} \tag{1.2}$$

Thus, to the extent that Fun's being a partial operation – defined only on functional relations – permits it, Fun and Rel are each other's inverses. Rel is the first of the two ways to be distinguished here of associating a relation with a given function.[3]

The second way of effecting such an association will be arity-preserving rather than arity-reducing.[4] To introduce it, we start with the corresponding reverse – relation-to-function – transition, recalling that for $R \subseteq S^n$, the *characteristic function* of R, most familiar from the case in which $n = 1$,[5] $\chi(R)$, often (as here) written as χ_R, is defined by:

$$\chi_R(x_1,\ldots,x_n) = \begin{cases} T & \text{if } \langle x_1, \ldots, x_n\rangle \in R \\ F & \text{if } \langle x_1, \ldots, x_n\rangle \notin R \end{cases}$$

Since for current purposes it is most convenient to have a single underlying set S to supply arguments and values for our functions and the relata for our relations, to accommodate $\chi(\cdot)$ we

[2]When convenient below we will, as usual, write "$Rx_1\ldots x_n$" – or, when the R part of the label is cluttered with superscripts or subscripts, "$R(x_1,\ldots,x_n)$" – in place of "$\langle x_1,\ldots,x_n\rangle \in R$".

[3]Of course what we are calling the *operation* Rel is also a function (and Fun a partial function), but for convenience we are reserving the term "function" in the main discussion – with one exception: in note 6 – for those mapping, for some n, n elements of our single basic type S, to elements of S. Here, to minimize complications, we acquiesce in the customary representation of a function's taking n arguments with its taking a single argument, that being an ordered n-tuple of elements, despite the inadequacy of this as a general account. The issue here is explained toward the end of footnote 13 in Humberstone (1993). – as well, no doubt, as elsewhere.

[4]Another commonly encountered arity-preserving way of associating relations with functions is described in Longer Note B.

[5]Here we presume an account of ordered n-tuples which identifies $\langle x\rangle$ with x.

make the presumption that this underlying set contains the truth values T and F. (In fact for Section 2 and much that follows that, it won't contain anything else – see the opening sentence of the present section.) Again, in the interests of simplicity we may take n-ary functions under discussion as having domain S^n and codomain S, and, concentrating on the case of $n = 1$, recall a familiar ambiguity of the notation $f^{-1}(\cdot)$. As a temporary expedient we introduce two notations, $f^{\|-1\|}(\cdot)$ and $f^{|-1|}(\cdot)$ to do the disambiguating.

Given $f : S \to S$, $f^{\|-1\|}$ is the relation $\{\langle b,a\rangle \mid f(a) = b\}$ – not in general a functional relation – and $f^{\|-1\|}(b)$ is the set of all a to which b stands in this relation, that is, the set of all $a \in S$ mapped by f to b.[6] On the other hand, for the case in which $f^{\|-1\|}$ *is* a functional relation, then $f^{|-1|}(\cdot)$ is instead $\mathsf{Fun}(f^{\|-1\|})$, mapping b to the unique a for which $f(a) = b$.[7] From now on, the inverse image notation "f^{-1}" will always be understood in the first of these two ways, as meaning "$f^{\|-1\|}$". Note that this makes sense without further ado, by contrast with the "$f^{|-1|}$" case – when f is an n-ary function with $n > 1$: $f^{-1}(b) = \{\langle a_1,\ldots,a_n\rangle \mid f(a_1,\ldots,a_n) = b\}$.

We are now in a position to undo the action of χ, and trade in a characteristic function for the relation whose characteristic function it was. Where f is the function concerned, its de-characteristic relation, $\overleftarrow{\chi}(f)$, is defined thus:

$$\overleftarrow{\chi}(f) = f^{-1}(T),$$

and one easily checks that we have the following, where, in (1.4), we call a function *truth-valued* when it only takes values in the set $\{T, F\}$:[8]

$$\overleftarrow{\chi}(\chi(R)) = R \text{ for any relation } R \text{ whatever} \tag{1.3}$$

$$\chi(\overleftarrow{\chi}(f)) = f \text{ for any truth-valued function } f \tag{1.4}$$

The cases in which f is not a truth-valued function are "don't care" cases not arising in the way $\chi(R)$ (alias χ_R) was specified above. Even in the case in which the range of f is completely disjoint from $\{T, F\}$, the "worst" that can happen is that $f^{-1}(T)$ is empty; there is no need to insist that $\overleftarrow{\chi}(f)$ should be other than $\emptyset$ – for example that it should be *undefined* – in such cases.[9]

2 | TRUTH FUNCTIONS AND TRUTH-VALUE RELATIONS

To pick up again on the opening sentence of Section 1: we have now finished describing two ways of associating a relation with a given function – as well as of making an association in the converse direction. The first way, via Rel, passed from a function to its graph, and the second

[6]The notation here reminds us that an $(n+1)$-ary relation on S is also "informationally equivalent" to a function from S to $\wp(S^n)$, though we will not explicitly dwell on the operations taking us to and from in this equivalence. Likewise with the transition between functions from S^n to S and functions from S to functions from S^{n-1} to S (the iterated use of which is associated with Schönfinkel and Curry).

[7]A more general version of the introducing $f^{|-1|}(\cdot)$ would turn it into a partial function, defined only for those b for which $f^{\|-1\|}(b)$ has a unit set as value, with $f^{\|-1\|}(b)$ then being the sole element of that set. Issues in the vicinity of the ambiguity currently under discussion are touched on in the text Makinson (2020)., in the paragraph spanning pp. 87 and 88, and in the final paragraph of the coloured box on p. 91.

[8]We are not saying that $\{T, F\}$ is the range of the function – the constant true function would satisfy the current condition, for example – nor that $\{T, F\}$ is its codomain, since we are taking all 1-ary (total) functions to have the same domain and codomain: namely a single fixed universe of discourse, S. Note also that truth-valued functions include truth functions, the latter arising when S is just $\{T, F\}$.

[9]In the original version of this material, all relations and functions were "sorted" in the sense of having prescribed basic types of their relata, arguments, and values, and what is here called $\overleftarrow{\chi}$ was a partial operation (like Fun in (1.2)), defined only for truth-valued functions as arguments. The present treatment was suggested to me by Katalin Bimbó as a simplifying improvement (one of several, in fact).

way, via $\overleftarrow{\chi}$, from a function – at least, a characteristic function – to its de-characteristic relation. Since every truth function can be viewed as the characteristic function of some relation, this association of a relation with an antecedently given function will always be available now that we turn our attention specifically to truth functions. And in this setting, it is the second ("de-characteristic") association, figuring in (1.3)–(1.4), that will be the focus of our attention. $\overleftarrow{\chi}$ delivers, given an n-ary truth function – a function from $\{T, F\}^n$ to $\{T, F\}$ – an n-ary *truth-value relation*, as we shall put it: a subset of $\{T, F\}^n$. (A comment on this terminology will be made in Longer Note C in the Appendix.)

Example 2.1. By way of example, consider the ternary "majority" truth function f, mapping $\langle T, T, T\rangle$, $\langle T, T, F\rangle$, $\langle T, F, T\rangle$, $\langle F, T, T\rangle$ to T and the remaining triples to F. Then $\overleftarrow{\chi}(f)$ is simply the set of triples just explicitly listed (the "mapped to T" cases). ◀

As with the term *truth function* itself, which is sometimes used for linguistic expressions – either for connectives interpreted via truth functions proper, or for formulas with such connectives as their main connectives[10] – so the phrase *truth-value relation* may be given a secondary usage for linguistic expressions interpreted via (esp. de-characteristic) truth-value relations, such as the relations $R^{\#}_{v}$ defined in Section 3 below. Indeed, historically, the latter use of the phrase "truth-value relation" is historically the primary use, given its appearance in the title and the body of the article by Welding (1976). The situation is complicated by the fact that Welding focuses his attention on *propositions* – however, exactly, these are conceived[11] – rather than linguistic expressions (formulas or sentences) themselves, as with our $R^{\#}_{v}$, and also by the undeniable murkiness of much of the discussion in Welding (1976). That discussion nevertheless introduces the relational perspective by essentially considering at the level of propositions, the ($\overleftarrow{\chi}$-obtained) truth-value relation associated with any truth function. Welding contrasts this, not, as above, with the truth-value relation obtained from the given truth function by the application of Rel, but with the truth function itself, which he sees it as somehow usurping:

> *If, for instance, a conjunction is conceived to be a relation, it is not, logically speaking, correct to say that the conjunction of* A *and* B *is true: we should rather assume that the conjunction holds between the truth both of* A *and* B. *(…) We should observe that the assumption that logical connectives are (non-functional) truth-value relations does not involve any difference in logical operations on them. It is not self-contradictory to say that sometimes logicians do precisely know how to operate on something and yet fail to know what precisely it is they have been operating on.*[12]

As we saw, though, there is no competition between giving a semantic description using truth functions and giving one in terms of truth-value relations. After all, (1.3 and 1.4) make the functional and relational perspectives intertranslatable and equally legitimate. Peter Simons argues a version of this case for the incarnation of the relations concerned as relations among statements, sentences or propositions – the differences among which, he remarks Simons (1982, p. 209)., are not to the point here – very reasonably concluding (p. 211) as follows, after

[10]Sometimes leading to confusions, as is observed in Dale (1982); on which, see also Humberstone (2014, p. 22*ff*).

[11]Two candidate explications arise in passing in what follows: one treatment (alluded to in passing in Longer Note G) construes propositions as certain equivalence classes of sentences or formulas; and a version of the propositions-as-sets-of-points in a model arises in the discussion after Example 3.8; *c.f.* also the role of propositions as sets of verifying valuations in some antecedently given class of valuations, implicit in the definition of $R^{\#}_{\mathcal{V}}(A, B)$ in Section 3. Welding does not have anything as clearly articulated as any of these notions in mind, and is in any case working in a more informal setting than would make them available.

[12]Welding (1976, p. 160).

mentioning the functional treatment and (what was called above) the **Rel**-obtained relational treatment:

> *On the other hand it is possible also to consider the truth functions to correspond to certain relations among propositions, or again, among truth-values. The latter alternative has been suggested by Dr Steen Olaf Welding, though provided we keep clear as to the general priority of relations over functions, there seems to be no reason to accepting Dr Welding's criticisms of the functional view of truth functions, which can, as we have demonstrated, be defended provided it is cleared of confusion. In the case of truth functions, it so happens that we have arrived at the functional rather than the relational view first, and have perhaps found it natural to proceed that way. But the other way is equally acceptable: for instance we can define a relation between propositions k such that p k q iff both p and q are true.*[13]

The theoretical interchangeability of the functional and the ("de-characteristic") relational perspective does not mean that both perspectives are always equally helpful in practice.

An illustration of how the latter perspective can sometimes make things more evident will be given after Remark 3.6 below, for which we need some background earlier in Section 3. It will involve the ($\overleftarrow{\chi}$-obtained rather than **Rel**-obtained) truth-value relations associated with two-place truth functions, which are especially pleasant to deal with because of their easy visualisability, contrasting in this respect with Example 2.1above. We can represent the ordered pairs of truth-values belonging to the relation in question by means of an arrow going from the one to the other. Figure 1 contains such arrow diagrams for four binary truth-value relations, identified in the figure caption by the truth functions with which they are associated. The third diagram in that figure represents approximately what has been called the relation of "material conjunction":[14]

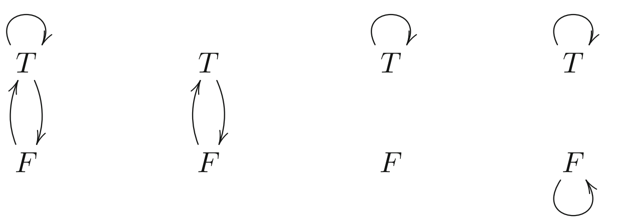

FIGURE 1 Truth-value relations for inclusive disjunction, exclusive disjunction, conjunction, material equivalence

[13]Simons chooses the letter "k" because of the use of a capital *K* in Welding's own remarks which take off from material by Stefan Körner using this letter (presumably inspired by Łukasiewicz-style Polish notation). The early reference, in the Simons passage quoted here, to "the general priority of relations over functions" sounds like something set-theoretically and category-theoretically oriented thinkers might disagree about. And formulations such as "criticisms of the functional view of truth functions" do not seem optimal, for evident reasons.

[14]This phrase comes from the opening page of Woods (1967), in discussion of some publications by Everett Nelson from the 1930s. (References to the Everett papers can be found in Woods' critique: Woods (1967, 1969). The current application would only be approximate, because, like Welding's – as noted above after Example 2.1 – Woods' discussion is conducted in terms of propositions. The extensional or "material" relations between them would be those holding in virtue of the present truth-value relations holding between their respective truth-values. On our more linguistically oriented treatment, as in the following section, of material conjunction relative to a Boolean valuation *v*, would be as the binary relation $\overleftarrow{\chi}(\wedge)$ between the results of applying $v(\cdot)$ to the conjuncts – the sentences or formulas themselves – just as, in the 1-ary case of ¬, for example, we have the 1-ary relation (alias property) of *falsity*, (relative to the given *v*), $\overleftarrow{\chi}(\neg)$ as predicated of the formula filling the blank in $v(\cdot)$.

Recall that the binary truth function indirectly represented in such relational diagrams is the characteristic function of the relation explicitly depicted by the presence of an arrow from x to y ($x,y \in \{T,F\}$) when the function involved maps $\langle x,y \rangle$ to T. The cases in which a pair is mapped to F are indicated by the absence of a corresponding arrow. So, because a truth-table for the inclusive disjunction, exclusive disjunction and conjunction truth functions has respectively three, two and one, occurrence of T in its main column, the first three diagrams in Figure 1 have arrows appearing in them respectively three times, twice and once. The fact that the second and fourth diagrams are complementary digraphs reflects the Boolean complementation between exclusive disjunction and material equivalence. Figure 2 depicts four more cases, so this leaves the reader to draw corresponding diagrams for the remaining eight binary truth functions, to help get a feel for the ($\overleftarrow{\chi}$-derived) relational perspective on truth functions.

3 | FROM TRUTH-VALUE RELATIONS TO LOGICAL RELATIONS

Truth-value relations in the sense of our discussion, as relations among truth-values, are not, as was mentioned in the preceding section, the only things called by that name: relations among linguistic expressions – and here our attention will be on the formulas of sentential languages – as well as relations among propositions, have been called truth-value relations when they obtain in virtue of the truth-value relations in the narrower sense among the truth-values of the expressions or propositions concerned. Relations of this kind were introduced directly in 3.34 of Humberstone (2011)., rather than via the truth-value relations proper, with the binary relations between formulas of some language defined (p. 504) in terms of a class $\mathcal{V}$ of valuations and for that language and a binary connective # of the language. The case of n-ary # for n in general is clear enough, and we stick with the binary case for illustrative purposes, superscripting the # and subscripting the references to valuations, so as to highlight the binary relations that emerge when these parameters are fixed; we do not require in the general setting that # be interpreted by a truth function over $\mathcal{V}$:

1. $R^{\#}_{v}(A,B)$ iff $v(A \# B) = T$; and
2. $R^{\#}_{\mathcal{V}}(A,B)$ iff for all $v \in \mathcal{V}$: $R^{\#}_{v}(A,B)$.

Remarks 3.1. (i) The comment about there being no need for # to be interpreted by a truth function over $\mathcal{V}$ means that we do not require there to be some $g:\{T,F\}^2 \to \{T,F\}$ such that for all $v \in \mathcal{V}$ and all formulas A, B, $v(A \# B) = g(v(A),v(B))$. When, however, such a g does exist, as in the case of all Boolean connectives # over any class of Boolean valuations, we denote the g in question with the aid of boldface: **#**. In this case, one could, in turn, spell out the right-hand side of the above definition of $R^{\#}_{v}(A,B)$ in familiar functional terms by replacing it with $v(A)\,\mathbf{\#}\,v(B) = T$, or alternatively, using the truth-value relational form, spell it out instead by writing "$\overleftarrow{\chi}(v(A),v(B))$". (In some degenerate cases, there

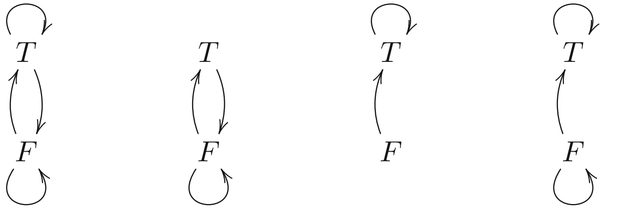

FIGURE 2 Relations for constant true, nand, second projection, material implication

may be more than one g a given $\mathcal{V}$ provides, for which it is correct to say that # is interpreted by g over $\mathcal{V}$, in which case # can be taken indifferently as any one such candidate.[15]) Humberstone (2011). puts matters in terms of the *association* of a truth function with a connective over $\mathcal{V}$ rather than the *interpretation* of a connective by a truth function over $\mathcal{V}$, but here that would be confusing in view of the talk of associating a relation with a function, the two modes of doing which were contrasted in Section 1.[16]

(ii) One might also be inclined to trade in valuations-as-functions for valuations-as-relations, as in Priest (2008, §8.4a), for example, or, more elaborately, in Priest (2014). But that would be a different kind of move, replacing a 1-place function with a 2-place relation, rather than, as here, an n-ary function with an n-ary relation – as with the **Rel**-invoking relational treatment of truth functions distinguished in Section 2 from the present $\overleftarrow{\chi}$-invoking version. ◀

Someone wanting to prioritize the relational perspective may prefer to have something less roundabout than the current "$\overleftarrow{\chi}_{\#}$" (or "$\overleftarrow{\chi}(\#)$"), along the lines of "#" for denoting the relevant binary truth-value relation rather than the truth function from the start. Then in the inductive definition of truth on a Boolean valuation, one would explain that for any A, B: $v(A \# B) = T$ iff $\#(v(A), v(B))$, and the r.h.s. here could be used in place of the r.h.s. of the definition of $R^{\#}_{v}$ above.[17] We return to illustrating the occasional greater suggestiveness of the current ($\overleftarrow{\chi}$-based) relational perspective below – in the paragraph following Remark 3.6 – and for that illustration we need to consider logical relations among formulas. For the case of binary logical relations – equivalence, implication, contrariety, subcontrariety, and so forth (to invoke some very traditional terminology in these last two cases) – it is the definition of $R^{\#}_{\mathcal{V}}(A, B)$ above rather than that of $R^{\#}_{v}(A, B)$ that is relevant.

Lemmon's account of the (binary) logical relations makes use of some of this terminology, but produces a far from traditional list of what those relations are, when applied to the case of classical logic, as it is in Lemmon (1965, pp. 69–71). Now reverting from the truth-relational use of "#" mooted in the preceding paragraph to its originally proposed use for the binary truth function interpreting a candidate truth-functional connective, over the class of all Boolean valuations taken as $\mathcal{V}$, the binary logical relations are precisely the 16 relations $R^{\#}_{\mathcal{V}}(\cdot,\cdot)$ as # ranges over pairwise non-equivalent connectives interpreted (over $\mathcal{V}$) by the truth function #. Here it does not matter whether we think of # as a primitive or as a defined connective of the language, as long as the primitives are functionally complete. Thus the logical relation of equivalence is the relation $R^{\leftrightarrow}_{\mathcal{V}}$, subcontrariety is the relation $R^{\vee}_{\mathcal{V}}$, and so on.[18]

[15]For example, suppose v is the unique Boolean valuation for the language with primitive connectives $\rightarrow$ and $v(p_i) = T$ for all sentence letters p_i. Then on v (or over $\mathcal{V} = \{v\}$), $\rightarrow$ can equally correctly be said to be interpreted by $\rightarrow$, $\wedge$, $\vee$, and so forth. – any binary truth function mapping $\langle T, T\rangle$ to T.

[16]This connection with the relations $R^{\#}_{v}(\cdot,\cdot)$ used in defining the basic binary Lemmon-style logical relations $R^{\#}_{\mathcal{V}}(\cdot,\cdot)$ provided the present author's point of entry into the realm of truth-value relations, rather than encounters with Welding (1976). and Simons (1982)., and similar idiosyncratic concerns have no doubt been operative in their re-discovery by others. I learned from Jean-Yves Béziau in mid-2020, for example, that he had, again independently, noticed the one-to-one correspondence between (in particular, the binary) truth functions and the associated (as it is put here, "de-characteristic" binary) truth-value relations, and was considering writing up an account of such relations in terms of a hexagon of opposition in the style of Blanché (1969).

[17]Despite our talk of the relational perspective, there is an important respect in which these truth-value relations are intuitively "not genuinely relational": see Longer Note D in the Appendix.

[18]See Humberstone (2013)., in the terminology of which, the Lemmon logical relations are just the beginning of a classification of what are there called the (binary) coercive logical relations from the perspective of classical logic, and see Humberstone (2020). (or, more perfunctorily, Humberstone (2019, Appendix 2).) for a discussion attempting to transcend the specifically classical case. For more on this, see Longer Note E. A referee asks to hear a bit more about the differences between Lemmon's use of the terminology for various logical relations and the more traditional usage, alluded to at the start of this paragraph; Longer Note F has been added to meet this request in a way that keeps the present discussion self-contained, as well as supplying a somewhat different slant from what is offered in the material just cited.

As already mentioned, the local $R^{\#}_{v}$ relations and the global $R^{\#}_{\mathcal{V}}$ relations were introduced into the discussion in Humberstone (2011, 3.34). in order to throw light on the curious fact that many writers on logical matters mix up, even in the same breath, talk of a something like conjunction as being symmetric, on the one hand, and being associative, on the other. Others may say that $\wedge$ is commutative and that $\rightarrow$ is transitive, and so on.[19] Thus, they have not cleared their thoughts as to whether it is the language of binary relations or the language of 2-place functions (binary operations) that is appropriate. The suggestion made in that discussion is that, with some definite class of valuations in mind, they pass from thinking of the connective #, a function mapping (in the binary case) a pair of formulas to a formula, to thinking of the relation $R^{\#}_{\mathcal{V}}$ (where $\mathcal{V}$ is the class in question) or – less typically – to one of the local relations $R^{\#}_{v}$ (where v is an element of the class in question).

Now in general – that is, for arbitrary $\mathcal{V}$ – it is one thing to ask whether for every $v \in \mathcal{V}$ the relation $R^{\#}_{v}$ is, for example, transitive, and another to ask whether the relation $R^{\#}_{\mathcal{V}}$ is transitive. Where $\vDash_{\mathcal{V}}$ is the consequence relation determined by $\mathcal{V}$ in the sense explained in note 1, the former asks whether

$$\text{For all } A, B, C : A \# B, B \# C \vDash_{\mathcal{V}} A \# C \tag{1a}$$

whereas the latter asks whether

$$\text{For all } A, B, C : \text{If } \vDash_{\mathcal{V}} A \# B \text{ and } \vDash_{\mathcal{V}} B \# C \text{ then } \vDash_{\mathcal{V}} A \# C. \tag{1b}$$

The former (local) transitivity claim is easily seen to imply the latter (global) transitivity claim.

Remark 3.2. The implication just noted arises because the intersection of transitive relations is transitive, and

$$R^{\#}_{\mathcal{V}} = \bigcap_{v \in \mathcal{V}} R^{\#}_{v},$$

for any (not just binary) # of the language for which $\mathcal{V}$ is a set of valuations. A more general version of the relevant point can be given using the notion of a universal Horn sentence (defined below, though no doubt familiar to most readers): for any condition on a relation expressed by such a sentence, if the condition is satisfied by a family of relations, it is satisfied by their intersection. (Relatedly, $R^{\#}_{\mathcal{V}}(A, B)$ iff $u(A \# B) = T$, where u is what is called in Humberstone (2011)., e.g., p. 60, the "conjunctive combination" of the valuations in $\mathcal{V}$. Typically – e.g., if $\mathcal{V} = \mathcal{BV}$, the latter defined in the following paragraph – $u \notin \mathcal{V}$, a fact whose repercussions were famously noted by Carnap; for bibliographical references, see the "Strong Claim vs. Weak Claim" discussion in Humberstone (2011, p. 101); this comes up again at note 37 below.) ◀

The converse of the implication (1a) $\Rightarrow$ (1b) in question is in general not forthcoming, as Example 3.3 (*ii*) below illustrates. But in the special case in which $\mathcal{V}$ is the class, $\mathcal{BV}$, of all Boolean valuations,[20] and $\vDash_{\mathcal{BV}}$ is understood specifically as the consequence relation on a language

[19]Numerous pertinent citations are given in the discussion in Humberstone (2011, 3.34). See further, Longer Note G below.

[20]Here "$\mathcal{BV}$" is an unstructured symbol, as in Humberstone (2011). – though there, $\vDash_{\mathcal{BV}}$ is usually referred to as $\vdash_{\mathsf{CL}}$, to suggest classical logic, as $\vdash_{\mathsf{IL}}$ is used similarly for intuitionistic logic.

with a functionally complete stock of Boolean connectives and no non-Boolean connectives – so $\vDash_{\mathcal{BV}}$ is the familiar consequence relation of classical propositional logic – we do have the converse ((1b) ⇒ (1a)) implication.[21] This is due in part, as we shall see, to the structural completeness of this consequence relation.[22]

Similarly, but now picking up on the confusion between properties of binary operations and properties of binary relations, the stronger valuation-by-valuation claim that # is commutative over $\mathcal{V}$ made by saying that

$$\text{For all } A, B: A \# B \vDash_{\mathcal{V}} B \# A \tag{2a}$$

and the potentially weaker $\mathcal{V}$-*en-masse* claim that

$$\text{For all } A, B: \text{ if } \vDash_{\mathcal{V}} A \# B \text{ then } \vDash_{\mathcal{V}} B \# A \tag{2b}$$

are equivalent for any choice of # in the case of $\mathcal{V} = \mathcal{BV}$.[23] Thus we can cite either the consideration that for each $v \in \mathcal{BV}$, the relation $R^{\#}_{v}$ – coinciding with the relation $\overleftarrow{\chi}(v(\cdot), v(\cdot))$ if # is truth-functional over $\mathcal{V}$ – is symmetric, or the consideration that $R^{\#}_{\mathcal{BV}}$ is symmetric, to explain how the commutativity of a connective gets to be referred to as symmetry. Example 3.3 (*i*) shows how these two considerations can in other cases fail to amount to the same thing.

Example 3.3. (i) We illustrate the failure of the implication (2b) ⇒ (2a). Extend the language of $\vDash_{\mathcal{BV}}$ with the 1-ary non-Boolean connective $\Box$ and take $\mathcal{BV}_{\Box}$ to be the class of valuations of the form $v^{\mathcal{M}}_{w}$, where $\mathcal{M}$ is a model (for normal modal logic), w a point in that model, and for all formulas A of the $\Box$-expanded language, $v^{\mathcal{M}}_{w}(A) = T$ iff A is true at w in $\mathcal{M}$, under the usual Kripke semantics for modal logic. (Here models are not taken to have a distinguished element, truth at which is said to constitute truth in the model; the use of such "pointed models" would call for some reformulation.) Note that Remark 3.1 (*i*) applies here in the case of $\Box$, since there is no truth function to "de-characterize", and similarly for the # about to be defined in terms of $\Box$, there is no corresponding **#**. (This comes up again, with intuitionistic implication, in the paragraph following Example 3.8 below.) Take $A \# B$ as $A \wedge \Box B$. Whenever $\vDash_{\mathcal{BV}_{\Box}} A \# B$, we have $\vDash_{\mathcal{BV}_{\Box}} B \# A$, but certainly we do not have the stronger commutativity condition: (for all A, B) $A \# B \vDash_{\mathcal{BV}_{\Box}} B \# A$. (Putting this in terms of the smallest normal modal logic K: the theorems of K are closed under the rule taking us from $\Box A \wedge B$ to $\Box B \wedge A$. See, if required, the index entries under "necessitation" and "denecessitation" in Humberstone (2011, 2016) – either will do – but in general $\Box A \wedge B$ does not provably imply $\Box B \wedge A$ in K: for instance, not if A and B are distinct sentence letters/propositional variables. Indeed,

[21]The functional completeness condition here is convenient but for the sake of the point just made, all we need is that the language of $\vDash_{\mathcal{BV}}$ should contain at least one $\mathcal{BV}$-valid formula – as it would, for example, if we considered the $\{\wedge, \vee, \top\}$-fragment of classical logic. (Suppose that C is a $\mathcal{BV}$-valid formula, and that $A_1, \ldots, A_n \nvDash_{\mathcal{BV}} B$. Exercise 1.25.12 in Humberstone (2011, p. 132). concerns structural completeness in the sense of note 22 below. It describes a standard method for defining, on the basis of a Boolean valuation v with $v(A_1) = \ldots v(A_n) = T$ and $v(B) = F$, a substitution s^{*}_{v}, which renders each of the formulas $s^{*}_{v}(A_i)$, where $i = 1, \ldots, n$, but not the formula $s^{*}_{v}(B)$, $\mathcal{BV}$-valid. Varying the definition so as to replace the formula $p \rightarrow p$ in the proof suggested there, by the formula C, does not affect this familiar argument for the structural completeness of the $\vDash_{\mathcal{BV}}$ for the fragment in question.)

[22]A consequence relation is *structurally complete* if whenever a formula C in its language is not a consequence of a set Γ of formulas, there is a substitution $\mathfrak{s}$ (replacing propositional variables by arbitrary formulas in a uniform manner) with $\mathfrak{s}(A)$ a consequence of $\emptyset$ ("is a $\mathcal{V}$-valid formula"), for each $A \in \Gamma$ but $\mathfrak{s}(C)$ is not; whenever we describe a consequence relation as structurally complete, we shall also assume that it is *substitution-invariant*, meaning that whenever C is a consequence (by that relation) of Γ and $\mathfrak{s}$ is a substitution, then $\mathfrak{s}(C)$ is a consequence of $\{\mathfrak{s}(A) \mid A \in \Gamma\}$. There is also a related though distinct notion of structural completeness arising as a property of proof systems (which may involve, as in the case of natural deduction and also sequent calculi, sequent-to-sequent rules rather than just formula-to-formula rules). Further discussion and references may be found in the Digression on p. 882 *f.* of Humberstone (2011).

[23]Recall that when we write "$\vDash_{\mathcal{BV}}$", the assumption is that only Boolean connectives are present in the language concerned.

the only consistent normal modal logic containing $(\Box p \wedge q) \rightarrow (\Box q \wedge p)$ is the "trivial" system in which $\Box A$ is always provably equivalent to A.)

(ii) We can give a similar counterexample to the implication (1b) ⇒ (1a) by keeping the language the same and now writing $\mathcal{BV}_{\Box 4}$ for the subset of $\mathcal{BV}_{\Box}$ in which the accessibility relations of the models $\mathcal{M}$ (for its induced valuations $v_w^{\mathcal{M}}$) are transitive. Thus we are working with the modal logic K4, though readers more comfortable with S4 (alias KT4) can safely have that in mind instead. Let $A \# B$ be $\Box A \rightarrow B$. It is easy to see that (1b) holds for this choice of # when $\mathcal{V}$ is taken as $\mathcal{BV}_{\Box 4}$, that is, that the K4 (or S4) provability of $\Box A \rightarrow B$ and $\Box B \rightarrow C$ guarantees that of $\Box A \rightarrow C$. To see the failure of (1a), take the case in which A is $\top$ (a truth/verum constant – or any K4-provable formula), B is the sentence letter p, and C is $\Box p$. This turns the relevant case of (1a) into:

$$\Box \top \rightarrow p, \Box p \rightarrow \Box p \vDash_{\mathcal{BV}_{\Box 4}} \Box \top \rightarrow \Box p.$$

We can drop the $\Box\top$ antecedent from the first and third formulas in view of its truth at all points in any model, as well as deleting the second formula on the same grounds. This leaves us with:

$$p \vDash_{\mathcal{BV}_{\Box 4}} \Box p,$$

which is clearly an incorrect claim, since a model's having a transitive accessibility relation does not guarantee that $\Box p$ is true at any p-verifying point in that model. (Less semantically formulated: because $p \rightarrow \Box p$ is not K4-provable – or S4- or even S5-provable.) ◀

In view of the heavy involvement of modal logic with Examples 3.3 (*i*) and (*ii*), it is worth sounding a note of warning about the fact that the usual local/global consequence distinction in modal logic does not coincide exactly with the above use of this terminology for the contrast between preserving, for each valuation in some class $\mathcal{V}$, and preserving the property of being true-on-every-valuation-in- $\mathcal{V}$. The former contrast, between preserving truth at each point in every model in some class, on the one hand, and preserving truth-throughout-the-model for each model in the class is one case of the latter, but another case might take $\mathcal{V}$ as the set of all valuations $v^{\mathcal{M}}$, where $v^{\mathcal{M}}(A) = T$ iff for all w in the universe of $\mathcal{M}$, $v_w^{\mathcal{M}}(A) = T$. Now the "local" preservation characteristic is itself what from a modal point of view would be regarded as global – preservation of truth-throughout-a-model – and the valuations $v^{\mathcal{M}}$ are admittedly not Boolean valuations.[24] For this choice of $\mathcal{V}$ we now have Necessitation in the strong form: $A \vDash_{\mathcal{V}} \Box A$, while for Denecessitation we still have only the weaker form: *if* $\vDash_{\mathcal{V}} \Box A$ *then* $\vDash_{\mathcal{V}} A$.[25]

The discussion will now take an autobiographical turn as we approach the promised illustration of the suggestiveness of taking the truth-relational perspective. Bear with me. We return to the issue of logical relations in the style of Lemmon: as relations $R^{\#}_{\mathcal{BV}}$. (So only Boolean connectives are in play here, and A and B are, for example, contraries, when we take $\neg(A \wedge B)$ in the $A \# B$ role and require that this formula be a classical tautology, contradictories if we trade in this choice of # for exclusive disjunction, and so on.) After introducing these ideas to an

[24]In the more refined terminology of Humberstone (2011, p. 65)., these valuations are $\wedge$-Boolean but not, for instance $\vee$-Boolean or $\neg$-Boolean.

[25]More on this and related matters can be found in Fagin et al. (1992). (erroneously listed in the bibliography of Humberstone (2011) simply as Halpern and Vardi [1992]), as well as (Denec $_0$) on p. 853 of the same work, and other entries under "Denecessitation" in its index.

intermediate level logic class three years ago, I set as an exercise for the students – one among several from which to choose – the following three-part question about these 16 "Lemmon relations". In reproducing it and discussing it here, I will, and in the original exercise did, omit the $\mathcal{BV}$ subscript:

1. For how many of the 16 binary truth functions # is the Lemmon relation $R^{\#}$ transitive?
2. For how many of the 16 binary truth functions # is $R^{\#}$ symmetric?
3. For exactly which of the 16 binary truth functions # is the relation $R^{\#}$ an equivalence relation (i.e., transitive, symmetric and reflexive) ?[26]

We had a follow-up session going through the exercises in class, after the assignments had been marked and returned, explaining the answers. The details on the first and second parts of the current question can be relegated to a footnote here.[27]

As for the final part of the question, I explained, the answer is that the truth functions # for which $R^{\#}$ is an equivalence relation are just the following two: when # is the material biconditional and when # is the constant true binary truth function. Going through the exercises in the follow-up session, we noticed that the awkward part of checking the cases to answer this part of the question, namely transitivity, actually played no role in filtering out candidate equivalence-relational $R^{\#}$ s: we get the same two if we ask instead about which of the 16 truth functions induce Lemmon relations that are reflexive and symmetric: just those induced by the constant true truth function and the commutative truth functions. Thus evidently transitivity follows, in this setting, from reflexivity and symmetry.

After the class was over, I realised that in fact, for the current relations, transitivity followed from reflexivity alone. The relations whose reflexivity and transitivity are at issue here, since we are talking about the Lemmon logical relations, are the global relations $R^{\#}_{\mathcal{BV}}$ for varying choices of #, rather than the local relations $R^{\#}_{v}$ for any such choice of # and (all the) $v \in \mathcal{BV}$. (Here, for clarity, the subscripting on $R^{\#}$ has been restored.) But from the transitivity of each such $R^{\#}_{v}$, the transitivity of $R^{\#}_{\mathcal{BV}}$ follows: this was the (1a) ⇒ (1b) implication above. And in the case of reflexivity, the distinction between the reflexivity of $R^{\#}_{\mathcal{BV}}$ and that of each $R^{\#}_{v}$ for $v \in \mathcal{BV}$ does not even arise, since reflexivity is an *unconditional* condition.

Proposition 3.4. *For any binary Boolean connective #, if $R^{\#}_{\mathcal{BV}}$ is reflexive then $R^{\#}_{\mathcal{BV}}$ is transitive.*

Proof. Suppose that $R^{\#}$ is reflexive: that for all formulas D: $\vDash_{\mathcal{BV}} D \# D$ (i.e., $D \# D$ is a classical tautology). Since it suffices to show that for all A, B, C we have $A \# B, B \# C \vDash_{\mathcal{BV}} A \# C$, let us suppose, for a contradiction, that we have formulas A, B, C and Boolean valuation v assigning the value T to the two formulas on the left and F to that on the right, and for brevity denote $v(A)$,$v(B)$,$v(C)$ by a,b,c. We have, then:

$$①a \# b = T, \qquad ②b \# c = T, \qquad ③a \# c = F.$$

From ① and ③ we infer that $b \neq c$, and similarly from ② and ③ we infer that $a \neq b$. Since $\vdash_{\mathsf{CL}} D \# D$ for all D, we must have for $x \in \{T, F\}$, $x \# x = T$, so ③ tells us that

[26]The students were asked to identify the truth functions in question by means of the alphabetical labels (a)–(p) given in Lemmon (1965, p. 70). to refer to them.

[27]A detailed examination of cases establishes that the answers to the first and second parts of the question are 13 and 8, respectively.

$a \neq c$. But now we have got to the position of concluding that a,b,c are pairwise distinct, contradicting the fact that they were drawn from the two-element set $\{T, F\}$. ■

Proposition 3.4 is in general (i.e., if we vary the logic concerned – represented here by class of valuations picked out by the subscript on "R") potentially weaker than what the proof establishes, namely that for each v in the relevant class, the relation $R^{\#}_{v}$ is transitive, given the hypothesis of reflexivity. That in the classical (or $\mathcal{BV}$) case these two conclusions are equivalent is the result of the interplay of a formal feature – alluded to in Remark 3.2 – shared by conditions like reflexivity, symmetry and transitivity, which is that they are all (metalinguistic) universal Horn conditions on the relations $R^{\#}_{\mathcal{BV}}$, with the structural completeness of the consequence relation $\vDash_{\mathcal{BV}}$. The discussion that follows (from here to the end of Remark 3.6) clarifies the role of these considerations, after which we return to Proposition 3.4 and its proof.

To avoid extraneous distractions, it is actually *strict* universal Horn conditions that we are concerned with. This means those conditions on R expressed by statements of the form (3), where the dotted $\dot{\wedge}, \dot{\vee}$ and $\dot{\rightarrow}$ appear simply to reduce the risk of confusion with similarly notated connectives of whatever object language is under consideration (the language, i.e., constituting the domain of the functions v and from which the connectives # are drawn):

$$\forall x_1 \ldots \forall x_k((\varphi_1 \dot{\wedge} \ldots \dot{\wedge} \phi_m) \dot{\rightarrow} (\psi_1 \dot{\vee} \ldots \dot{\vee} \psi_n)) \tag{3}$$

in which the variables $x_1,\ldots,x_n$ exhaust those occurring in the implication, and the φ_i and ψ_j are atomic formulas in the language with dyadic R as its sole predicate letter, and we require not only that $n \leq 1$ ("Horn") but that $n = 1$ ("strict Horn"); this is because of the correspondence arising in this case with statements about consequence relations – or alternatively (to use terminology from Humberstone (2011).) with sequents in the logical framework SET-FMLA, in which there is a single formula "on the right".[28] Evidently reflexivity, symmetry and transitivity are universal strict Horn conditions on the two-place R, with $k = 1, 2, 3$ respectively, and $m = 0,1,2$ respectively.

In our recent discussion we have been concerned with the case in which the individual variables range over the formulas of some propositional language among whose connectives is the binary #, $\mathcal{V}$ being a class of valuations for that language, and R is taken either as $R^{\#}_{v}$ for $v \in \mathcal{V}$ or else as the global relation $R^{\#}_{\mathcal{V}}$. (1a) and (2a) are then special cases of (3a), and (1b) and (2b) are special cases of (3b), where Φ is of the form (3) with $n = 1$:

$$\text{For each } v \in \mathcal{V}, R^{\#}_{v} \text{ satisfies } \Phi \tag{3a}$$

$$R^{\#}_{\mathcal{V}} \text{ satisfies } \Phi \tag{3b}$$

Then the discussion of (1a,1b) and (2a,2b) above illustrates why (3a) always implies (3b), provided that Φ is a (strict) Horn condition.[29] The "Horn" part here is essential. Allowing $n \geq 2$ in (3) would permit as a simple counterexample (with $m = 0$) classical implicational comparability

[28]Later we broaden the focus and consider generalized ("multiple conclusion") consequence relations – or sequents in the framework SET-SET; cf. the distinction between *rule-like* and *generalized rule-like* conditions at p. 29 of Humberstone (1996). Returning to (3) itself, when $n = 0$ in the non-strict case, the conditional in there is identified with the negation of its antecedent, and when $m = 0$, it is identified with its consequent. (The $n = 1$ incarnation of (3) is a very special kind of rule-like condition, since the premises and conclusion of the corresponding rule in a schematic formulation of the latter all result from applying the connective to suitably many schematic letters. Thus a rule like that taking us from premise $A \wedge B$ to conclusion $B \wedge A$ is among those represented while a rule taking us from premises A and B to conclusion $A \wedge B$ is not.) Often a (strict) Horn sentence is taken to a conjunction of (strict) Horn sentences as defined here. Information on Horn sentences and their distinctive logical properties can be found in Hodges (1993, Chapter 9)., or, for a more computer science oriented overview, in Hodges (1993a); the case of particular interest here – universal Horn sentences – is the focus of McNulty (1977)., *q.v.* also for historical information.

[29]This is a matter of distributing universal quantifiers (over $v \in \mathcal{V}$) across implications, and as already noted in connection with reflexivity, not even this much is involved in the $m = 0$ case.

(see Longer Note E), Φ for this case being $\forall x_1 \forall x_2 (Rx_1x_2 \dot{\vee} Rx_2x_1)$. With this choice of Φ, we have the familiar fact that this instance of (3a) holds: for each $v \in \mathcal{BV}$, $R_v^{\rightarrow}$ satisfies Φ; that is, for all Boolean formulas A, B, and Boolean valuations v, $v(A \rightarrow B) = T$ or $v(B \rightarrow A) = T$. By contrast, the corresponding instance of (3b) for $R_{\mathcal{BV}}^{\rightarrow}$ fails: it is not the case that for arbitrary A, B, at least one of $A \rightarrow B$ or $B \rightarrow A$ is $\mathcal{BV}$-valid (tautologous).

We turn now to the implication (3b) $\Rightarrow$ (3a), and to give the general idea alluded to with the implications from (1b) to (1a) and (2b) to (2a) for the case in which $\vDash_{\mathcal{V}}$ is structurally complete, we will go more slowly through a similar example, the Horn condition "R is Euclidean", familiar especially from the semantics of normal modal logic.

Example 3.5. The condition in question is as follows (and writing x, y, z rather than x_1, x_2, x_3):

$$\forall x \forall y \forall z ((Rxy \dot{\wedge} Rxz) \dot{\rightarrow} Ryz).$$

Now suppose that (instantiating (3b)) for $\mathcal{V}$ for which $\vDash_{\mathcal{V}}$ is (substitution-invariant and) structurally complete, $R_{\mathcal{V}}^{\#}$ satisfies this condition, so we have

$$\text{For all } A, B, C: \text{ if } \vDash_{\mathcal{V}} A \# B \text{ and } \vDash_{\mathcal{V}} A \# C, \text{ then } \vDash_{\mathcal{V}} B \# C \qquad (*)$$

but the corresponding instance of (3a) fails, so there are formulas A', B', C' for which we have

$$A' \# B', A' \# C' \nvDash B' \# C' \qquad (**)$$

Since $\vDash_{\mathcal{V}}$ is substitution-invariant, this must also be the case when A', B', C' are taken as three distinct propositional variables (sentence letters) in $(**)$ – p, q, and r, say – so by the assumed structural completeness of $\vDash_{\mathcal{V}}$, we can find formulas D, E, F to substitute for these variables and for which $\vDash_{\mathcal{V}} D \# E$, $\vDash_{\mathcal{V}} D \# F$, and $\nvDash_{\mathcal{V}} E \# F$: but this would then be a counterexample to $(*)$. ◀

Remark 3.6. At this point one might naturally inquire as to what happens to the (3b) $\Rightarrow$ (3a) direction in the structurally complete case, but when the (3)-style condition is not "strict Horn" – and in particular with $m \geq 1$ and $n \geq 2$. As no pertinent counterexample comes to mind, however, we return to the matter of Proposition 3.4 and its proof. ◀

That proof itself involved a simple appeal to the pigeonhole principle: there is not enough room in our two-pigeonhole loft (with holes T and F) to house the three pigeons $v(A)$, $v(B)$, and $v(C)$ without forcing two of them into the same hole.[30] Still, to the extent that explanation is a matter of reduction to the familiar, there is something more explanatory that might be said here. Explanatory strength then comes to depend, perhaps not inappropriately, in part on what exactly is familiar to the recipient of the explanation. But something anyone who has any experience at all with binary relations – such as drawing or pondering diagrams like those in Figures 1and 2 in Section 2 – will be familiar with is this: a reflexive relation on a two-element set cannot but be transitive.[31] Proposition 3.4, reformulated as at the start of this paragraph to address an arbitrary $R_v^{\#}$ $(v \in \mathcal{BV})$, simply records the upshot of this general fact for the truth-value relation $\overleftarrow{\chi}_{\#}(v(\cdot), v(\cdot))$. All that is being suggested here is that the truth-value relational

[30]More carefully put, it is not $v(A)$,$v(B)$ and $v(C)$ that are three in number, but rather the labels "$v(A)$", "$v(B)$" and "$v(C)$" (or "a", "b", and "c", as it is put in the proof) that there are three of, two of which the argument observes must co-denote.

[31]Elaborating somewhat: with only two elements on the scene, a case in which Rxy and Ryz but not Rxz, must be a case in which $x = z$, and hence a counterexample to reflexivity.

perspective makes matters more evident in directly invoking the fact that any reflexive relation on a two-element set is transitive. If the latter fact is not itself directly evident and a proof is demanded, it is not suggested that any further proof then offered would be very different from the relevant part of the proof of Proposition 3.4itself.[32]

Remark 3.7. That reflexivity implies transitivity when attention is restricted to relations on a two-element set, though not when arbitrary sets are considered, is analogous to facts bearing similarly on the logical behaviour of binary connective facts more conveniently expressed in functional/operational than relational language, such as the fact that any associative binary operation on a two-element set is either commutative or idempotent. One could verify that this is a fact by working through the 16 such operations available and checking them case by case, though, as with a similar strategy in the reflexivity-implies-transitivity case, as well as being time-consuming, this has about it the air of presenting a "brute fact", not supplying any graspable explanation as to why it must be so. For that, one might instead reason as follows.[33] Calling the two elements concerned a,b, and using juxtaposition to indicate the action of some associative binary operation, supposing that the operation is not commutative amounts to supposing that (1) $ab \neq ba$, and supposing that it is not idempotent means that $x \neq xx$ when x is a or b, and we lose no generality in supposing that x is a. So, on that supposition, (2) $aa = b$ since b is the only available value for aa left. Using (2), we can rewrite (1) as $a(aa) \neq (aa)a$. This contradicts the assumed associativity of the operation, so it cannot be both non-commutative and non-idempotent. Of course to complete the illustration, one should also show that in general an associative operation on a set need not be either commutative or idempotent. Since here a simple counterexample suffices, we can rest content with pointing the interested reader in the direction of Budden (1970, p. 371): the operation there denoted by $*$ will do. Budden had set himself the task of finding an associative non-commutative essentially binary operation on the real numbers, but the example he comes up with – on the non-zero reals, as it happens – is also non-idempotent, so it serves our present purposes equally well. Essential binarity – dependence on each of its two argument positions – figures in another example of the present phenomenon: any essentially binary operation on a two-element set satisfying the medial (also called *entropic*) law, that is, $(wx)(yz) = (wy)(xz)$, is commutative. (See Lemma 0.3.3 in Humberstone (1996, p. 26).) Note that properties – associativity, idempotence and mediality – figuring in these examples do not lend themselves readily to representation as simple conditions on the corresponding relations (in the case in which they are predicated of truth functions). It would be interesting to attempt to precisify that idea and to explore the extent of the phenomenon in question. ◀

Let us turn now to a loose end left by one aspect of the foregoing discussion, showing how a shift from classical propositional logic can lead to a situation in which the analogue of Proposition 3.4 fails. For variety, instead of venturing into modal logic again, as with Example 3.3, we turn to intuitionistic (propositional) logic. To have a matching presentation of the case, just as we call the classical consequence relation ($\vdash_{\mathsf{CL}}$) by the semantically based name $\models_{\mathcal{BV}}$, so we here

[32]For example with R a reflexive relation on a two-element set, one might reason that given Rxy and Ryz we cannot but have Rxz, since if not- Rxz, we would have x and y differing in respect of bearing R to z, so $x \neq y$, and y and z differing in respect of having x bear R to them, so $y \neq z$. But $x \neq y$ and $y \neq z$ imply $x = z$, since we are in a two-element set, making not- Rxz contradict the supposed reflexivity of R. But this is essentially a reorganized version of the proof of Proposition 3.4, with the present $x \neq y \,\&\, y \neq z \Rightarrow x = z$ step being an application of the pigeonhole principle.

[33]Not quite as time-consuming as just intimated, in all honesty, since we really need concern ourselves with the 10 isomorphism-types of 2-element groupoids, since nothing about T beyond its distinctness from F matters for present purposes. Still, one would prefer to avoid working through an enumeration of all possible cases.

refer to $\vdash_{\mathsf{IL}}$ as $\vDash_{\mathcal{HV}}$ – "H" for Heyting – where, analogously to the case of $\mathcal{BV}_{\Box}$, $\mathcal{HV}$ comprises those valuations $v_w^{\mathcal{M}}(A) = T$ iff A is true at w in $\mathcal{M}$, where this time we are using the Kripke semantics for intuitionistic propositional logic.[34]

Example 3.8. For a case in which, by contrast with Proposition 3.4, $R^{\#}_{\mathcal{BV}}$ is reflexive without being transitive, we consider a binary "Peircean" connective # for which $A \# B = ((A \to B) \to A) \to A$. Clearly $\vDash_{\mathcal{HV}} A \# A$ for all A, making $R^{\#}_{\mathcal{BV}}$ reflexive. But this relation is not transitive, as we see, for example, by taking p, q as distinct sentence letters and $\top$ as a truth constant (or as $p \to p$, for example, if preferred); in this case we have

$$\vDash_{\mathcal{HV}} p \# \top \quad \textit{and} \quad \vDash_{\mathcal{HV}} \top \# q \quad \textit{whereas} \quad \nvDash_{\mathcal{HV}} p \# q.$$

The first two formulas involved here, in more familiar primitive notation, are

$$((p \to \top) \to p) \to p \quad \text{and} \quad ((\top \to q) \to \top) \to \top,$$

while the third is (the commonest formulation of) Peirce's Law. (This means, incidentally, the smallest extension of intuitionistic logic in which the logical relation induced by the current # is transitive is classical logic. Of additional interest is the fact that in intuitionistic logic formulas A and B stand in this relation to each other – $A \# B \vDash\ \vDash_{\mathcal{HV}} B \# A$ – if and only if for some formula C, they are respectively equivalent to $A_0 \to C$ and $B_0 \to C$ for some choice of A_0, B_0; see Humberstone (2020, pp. 167 & 209), where, as in Humberstone (2011, p. 1319*ff.*)., A, B are said to be "head-linked" when this condition is satisfied.) ◀

Those familiar with the fact that, in contrast to the case of classical propositional logic, the consequence relation $\vDash_{\mathcal{HV}}$ of intuitionistic logic is not structurally complete, may suspect that the demonstration in Example 3.8 of the failure of an intuitionistic analogue of Proposition 3.4 is somehow due to this structural incompleteness. This is not so, however, as the example could have been given entirely with reference to the →-fragment of intuitionistic logic, where the consequence relation concerned *is* structurally complete.[35] The reason nothing like the proof of Proposition 3.4 is available here is rather that unlike the cases of ∧ and ∨, there is no truth function interpreting (or "associated with" – see Remark 3.1 (*i*)) the connective → on the valuations in $\mathcal{HV}$, so the argument involving ①, ②, and ③, which proceeds in terms of the truth function # associated on Boolean valuations with the connective # under discussion (and notated the same way for convenience), does not get started. Naturally, instead of talk of the truth function g (binary, say, for example) interpreting # on the valuation v, when for all A, B, $v(A \# B) = g(v(A), v(B))$, we could equally well be talking about the ("de-characteristic") truth-value relation $\overleftarrow{\chi}(g)$, and write the corresponding condition (subscripting the "g"): for all A, B, $v(A \# B) = T$ iff $\overleftarrow{\chi}_g(v(A), v(B))$: there is in general no such truth-value relation for $v \in \mathcal{HV}$ for # as →. (It is not a matter of a g for which – cf. note 9 – $\overleftarrow{\chi}(g)$ is undefined in the present case, but that there is no g about which even to raise the question of whether or not $\overleftarrow{\chi}(g)$ is defined: no **#** for the current #.)[36] The closest we get to this would be a relation between the local

[34] So, forget any associations the continued use of the variable "w" here may have with possible worlds.

[35] See §2 of Humberstone (2006). for discussion and historical references on these matters.

[36] Not only does → fail to be truth-functional over $\mathcal{HV}$: it is not even "variably" or *pseudo*truth-functional over $\mathcal{HV}$, in the sense of Humberstone (2011, p. 451)., which requires only that each v in this class associate a truth function with the connective – possibly a different truth function as we pass from one v to another. We could equivalently put all this in relational terms, saying that (e.g., *binary*) # is interpreted by $R \subseteq \{T, F\} \times \{T, F\}$ on a valuation v when for all formulas A, B, $v(A \# B) = T$ iff $R(v(A), v(B))$, and is *truth-value*

propositions $[\![A]\!]_w^{\mathcal{M}}$ and $[\![B]\!]_w^{\mathcal{M}}$, recalling that each $v \in \mathcal{HV}$ is $v_w^{\mathcal{M}}$ as explained before Example 3.8, w belongs to the universe U of the model $\mathcal{M}$, and writing $\leq$ for the latter's accessibility relation, for a formula C, $[\![C]\!]_w^{\mathcal{M}}$ is $\{w' \in U \mid w \leq w' \text{ and } v_{w'}^{\mathcal{M}}(C) = T\}$. For the connective $\rightarrow$, of course, the relevant relation is between the local propositions expressed by the antecedent and consequent is the relation $\subseteq$.

Having now introduced the intuitionistic consequence relation $\vDash_{\mathcal{HV}}$, we are in a position to make a couple of observations, both involving disjunction, and most conveniently made with reference to intuitionistic logic. As was mentioned in note 17, while for each $v \in \mathcal{BV}$, the relations $R_v^{\#}$ are all monadically representable, this is by no means so for the relations $R_{\mathcal{BV}}^{\#}$ (# a Boolean connective in both cases). For instance, whereas $R_{\mathcal{BV}}^{\wedge}$ is monadically representable, holding between A and B just in case each is $\mathcal{BV}$-valid, $R_{\mathcal{BV}}^{\vee}$ is not monadically representable. If we shift from $\mathcal{BV}$ to $\mathcal{HV}$ and from classically to intuitionistically definable #, both of these relations become monadically representable in view of the Disjunction Property of intuitionistic logic: $R_{\mathcal{HV}}^{\vee}(A,B)$ just in case A is $\mathcal{HV}$-valid or B is. The second observation is that when non-Horn conditions on the relations $R_{\mathcal{V}}^{\#}$ are at issue, more fine-grained information about $\mathcal{V}$ may be needed than is provided by consideration of the consequence relation $\vDash_{\mathcal{V}}$ (let alone just information as to which formulas are $\mathcal{V}$-valid). This is because we can have $\vDash_{\mathcal{V}} \; = \; \vDash_{\mathcal{V}'}$ even when $\mathcal{V} \neq \mathcal{V}'$. Indeed, we are sitting on the doorstep of a famous illustration of this possibility, in the shape of the contrast between the Kripke semantics and the Beth semantics for intuitionistic propositional logic.[37] $\mathcal{HV}$ was characterized in terms of the former, so let us denote by $\mathcal{HV}*$ the set of valuations arising similarly from the points in Beth models.[38] The key contrast is that the truth of $A \vee B$ at a point w in a Beth model $\mathcal{M}$ does not require that at least one of A, B be true at that point, as long as a certain condition is satisfied involving the Beth-semantical analogues of the local propositions $[\![A]\!]_w^{\mathcal{M}}$ and $[\![B]\!]_w^{\mathcal{M}}$ of the preceding paragraph. Since the consequence relations determined by $\mathcal{HV}$ and $\mathcal{HV}*$ – and in particular the set of valid formulas – coincide, the relations $R_{\mathcal{HV}}^{\vee}$ and $R_{\mathcal{HV}*}^{\vee}$ also coincide and thus we have monadic representability for the global disjunction relation in both cases. But now, contrasting with both the classical and the Kripke-intuitionistic cases of $\mathcal{BV}$ and $\mathcal{HV}$, for the Beth-intuitionistic case we have: with $v \in \mathcal{HV}*$, $R_v^{\vee}$ is not in general a monadically representable relation. (This shows how confusing it might have been in (3), to be recalled presently, to write "$\vee$" rather than "$\dot{\vee}$".)

If one wants to do justice to the such issues as this second observation raises, one needs to pass from the consequence relation to what is variously called the multiple-conclusion or generalized or "Scott" consequence relation determined by a class of valuations $\mathcal{V}$ (as explained in the works cited in note 38), which we may here denote by $\models_{\mathcal{V}}$, defined to relate a set Γ of formulas to (not a formula, but another) set of formulas Δ when every $v \in \mathcal{V}$ verifying all formulas in Γ verifies at least one formula in Δ. (We write "A, B" on the right rather than the more explicit "$\{A, B\}$" etc., extending the conventions mentioned in note 1.) Thus whereas $A \vee B \models_{\mathcal{BV}} A, B$ and $A \vee B \models_{\mathcal{HV}} A, B$, we do not in general have $A \vee B \models_{\mathcal{HV}*} A, B$. When $A = B$, however, we are back with a single formula on the right – recalling that Δ is a set, not a multiset, of formulas – and so we do have $A \vee A \models_{\mathcal{HV}*} A$, as well as the converse, enabling us to exploit the idempotence of disjunction in this setting to re-express the fact that (in general) $A \vee B \not\models_{\mathcal{HV}*} A, B$ in the following terms:

$$A \vee B \not\models_{\mathcal{HV}*} A \vee A, B \vee B$$

relational over $\mathcal{V}$ when there is some R such for all $v \in \mathcal{V}$ # is interpreted by R on each $v \in \mathcal{V}$, with the weakened variable ("pseudo") version obtained by changing this $\exists\forall$ condition to the corresponding $\forall\exists$ formulation.

[37]A similar situation – mentioned in passing in the parenthetical comment at the end of Remark 3.2 above – arises for classical propositional logic, as was first observed by Carnap; discussion and references can found in 1.14 and 6.46 of Humberstone (2011)., and §15.3 of Humberstone (2019).

[38]See Gabbay (1981, Chapter 3). for an explanation of the Beth semantics, as well as Theorems 6 of §1 and 5 of §2 for the fact that what we be calling $\models_{\mathcal{HV}}$ and $\models_{\mathcal{HV}*}$ are the weakest generalized consequence relations that agree with $\vDash_{\mathcal{HV}}$ when exactly one formula appears on the right of the "$\models$". Subsection 6.43 of Humberstone (2011). provides a secondary exposition of the pertinent details.

Thus we see that whereas the following condition on a binary relation, instantiating (3) with $m = 1$, $n = 2$:

$$\forall x \forall y (Rxy \dot{\rightarrow} (Rxx \dot{\vee} Ryy))$$

is not satisfied, for all $v \in \mathcal{HV}*$ by the relation $R_v^{\vee}$, this condition is satisfied by the relation $R_{\mathcal{HV}*}^{\vee}$ – since that is just the Disjunction Property for intuitionistic logic again.[39] Remark 3.6 mentioned the desirability of a counterexample to the (3b) ⇒ (3a) direction for a non-Horn condition, which that case just given may look like. However, a further condition was in place, namely that of structural completeness, which fails for the consequence relation determined by $\mathcal{HV}^*$, as was already recalled for this consequence relation under its description as that determined by $\mathcal{HV}$.

To close, let me address a thought the reader may be entertaining at this point: is this last reference to consequence relations really relevant? We have, after all, just been discussing generalized consequence relations rather than consequence relations proper, so as to match the possibility than $n \geq 2$ in (3) with the possibility that Δ should contain two or more formulas.[40] Taking up this idea, let us note that the definition of structural completeness in note 22 has a natural reformulation so as to apply to generalized consequence relations $\models$, namely: Whenever $\Gamma \not\models \Delta$, there is a substitution $\mathfrak{s}$ for which for each $C \in \Gamma$ we have $\models \mathfrak{s}(C)$, while for each $D \in \Delta$, $\not\models \mathfrak{s}(D)$.[41] It is not hard to check that the classical generalized consequence relation – that determined by $\mathcal{BV}$ in a functionally complete language (or any of numerous fragments thereof) – is structurally complete, and any counterexample to the structural completeness of the usual intuitionistic consequence relation[42] is already a counterexample to the structural completeness of $\models_{\mathcal{HV}*}$. So it does not seem that the envisaged line of thought is as promising as it may have appeared.

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APPENDIX: Longer notes

Longer Note A. (On the "corresponding linguistic move" mentioned before (1.1)–(1.2) in Section 1.) The move in question trades in function/operation symbols for predicate symbols. By way of illustration, consider doing first-order theories in a language without function

[39]The inset condition is a specialization of the condition $\forall x \forall u \forall y \forall z (Rxy \dot{\rightarrow} (Rxz \dot{\vee} Ruy))$, which is a first-order equivalent of the characterization of *or*-representability (in terms of existential quantification over sets) given in note 17, as is easily verified – or alternatively, can be checked by following up the references cited there. An intermediate condition is commonly encountered in the literature on preference and social choice under the name *negative transitivity* (since it amounts to having a transitive complement): $\forall x \forall y \forall z (Rxy \dot{\rightarrow} (Rxz \dot{\vee} Rzy))$.

[40]Since we also allow $n = 0$ – one reason that generalized consequence relation is a better label than multiple-conclusion consequence relation – this also subsumed the "non-strict" Horn condition case.

[41]I do not recall seeing this adaptation of the notion of structural completeness in the literature, but it is implicit in the definition of admissibility for multiple-conclusion rules in Kracht (2007, §7).

[42]For instance, Harrop's example: $\neg p \rightarrow (q \vee r) \not\models_{\mathcal{HV}} (\neg p \rightarrow q) \vee (\neg p \rightarrow r)$. Discussion and references may be found in Humberstone (2011, pp. 878–883).

symbols – say, that of arithmetic, as in Hájek and Pudlák (2016, p. 87)., for instance. Instead of writing $x+y=z$, with the two-place function symbol "+", one uses a three-place predicate symbol, A, say, reading $Axyz$ to say that x, y and z stand, in that order, in the addition relation. (Of course one must also make explicit the functionality of this relation.)

Longer Note B. (A further arity-preserving way of associating a relation with a function, alluded to in note 4.) As well as the example featured in Section 1, forming de-characteristic relations, another (more commonly encountered) way of obtaining a relation from an n-ary function which preserves arity in the sense that the relation it yields is again n-ary, is the following. Where f is the function and $1 \leq m \leq n$, we define

$$R_{f,m,n} = \{\langle x_1, \ldots, x_n \rangle \mid f(x_1, \ldots, x_n) = x_m\}$$

When $n=2$ and $m=1$ (or $m=2$), and f is the fundamental operation of a groupoid with universe S, some such notation as $\leq$ may be used for the derived relation, living up to its usual connotations in being reflexive (on S) when f is idempotent, and transitive when f is associative, and antisymmetric when f is commutative, so that $R_{f,1,2}$ (or equally well, $R_{f,2,2}$) is a partial ordering of S when the original groupoid is a semilattice. We shall not consider this further, though, since one cannot recover the f from the derived relation *in general* (i.e., without special l.u.b./g.l.b. conditions on the relation) – even from the set of such derived relations with $m=1,2,\ldots n$. For example, consider the functions f,g defined using multiplication and addition on the positive integers by

$$f(x,y)=x+y \qquad \text{and} \quad g(x,y)=x+2y.$$

Then $R_{f,1,2} = R_{g,1,2} = \emptyset = R_{f,2,2} = R_{g,2,2}$ although $f \neq g$ (and nor are $\langle Z^+, f\rangle$ and $\langle Z^+, g\rangle$ even isomorphic). A similar point holds for deriving an n-ary relation from an n-ary function by means of quantification, illustrated in the $n=2$ case by the possibility of setting $Rxy := \exists z(f(x,z)=y)$.

Longer Note C. (Terminological remarks pertinent to Section 2.) The more delicate reader may be upset by an asymmetry: if we are talking about *truth functions* shouldn't we also be talking about *truth relations*? On the other hand, if we want to use the phrase "truth-value relations," shouldn't we, for parity, be saying "truth-value functions"? That phrase – mentioned in passing in Kleene (1952, p. 125). as being occasionally seen in that era, and then used (in the title, even) decades later in Segerberg (1983). – is surely too clunky and pedantic: the intended meaning of "truth function" is clear enough to everyone. (Perhaps the intention is to avoid its secondary use, mentioned in the second paragraph of the present section, for linguistic expressions; Segerberg (1983). does not say.) Avoiding it leaves the other suggestion: "truth relations" rather than "truth-value relations". The trouble with this is that the latter phrase has been used already for other things, such as in Humberstone (2004, p. 42). for example, for relations between models, formulas and sometimes further parameters (worlds, or pairs of worlds, for instance), sometimes also called verification or satisfaction relations. So this terminological asymmetry will remain in place, though I am attempting to reduce some asymmetry by writing *truth function* instead of *truth-function* except when quoting others, in view of the lack of a hyphen immediately before "relation" in truth-value relation, restoring the hyphen only for the case of the adjective *truth-functional*.

Longer Note D. (Elaboration of note 17 in Section 3.) The sense in which our truth-value relations are not "genuinely relational" is that they are all *monadically representable* relations. By this is meant relations holding among individuals in virtue of those individuals being elements of sets specifiable independently of other relata; as a special case, we have, for example, *or*-representable binary relations R for which there are sets $S_0, S_1 \subseteq S$ with $\langle a,b\rangle \in R$ iff $a \in S_0$ or $b \in S_1$ (The interested reader is directed to Humberstone (2011); Humberstone (2016);

Humberstone (2013)., and, therein, subsection 5.33, pp. 15 and 178, respectively, where further references are also given. The topic comes up again briefly in the course of Longer Note E below.) For any given Boolean valuation v and Boolean #, the derived relation $R_v^{\#}$ inherits this monadic representability property from the underlying truth-value relation involved – any binary relation on a 2-element set being monadically representable – whereas this is far from the case for the logical relations $R_{\mathcal{V}}^{\#}$ where $\mathcal{V}$ is a set of Boolean valuations, and in particular for the set (baptized after Remark 3.2) $\mathcal{BV}$ of all such valuations for the language under consideration.

Longer Note E. (Elaboration of note 18.) The *coercive* relations are what one gets by closing the Lemmon relations (or their analogues for non-classical logics) under union and intersection – for which the "intersection" part is automatic in the presence of something behaving like a familiar conjunction connective – and the *permissive* logical relations are their complements. (As well as the purely coercive relation of subcontrariety just mentioned, there is also a "mixed" coercive-permissive relation often going by that name: the intersection of $R_{\mathcal{V}}^{\vee}$ with the complement of $R_{\mathcal{V}}^{\mathsf{nand}}$. See Humberstone (2019, Appendix 1). for further discussion and pointers to the literature.) Thus for instance $R_{\mathcal{V}}^{\rightarrow} \cup R_{\mathcal{V}}^{\leftarrow}$ is the coercive relation of implicational comparability for the logic determined by $\mathcal{V}$. (See note 1 for an explanation of "determined by", and the paragraph following Remark 3.2 for the notation "$\mathcal{BV}$" about to be used.) This logical relation is not among the "basic Lemmon relations" of classical logic; one can show, that is, that for no definable binary Boolean #, do we have $R_{\mathcal{BV}}^{\rightarrow} \cup R_{\mathcal{BV}}^{\leftarrow} = R_{\mathcal{BV}}^{\#}$.

Longer Note F. (Lemmon's vs. traditional terminology – more from note 18.) The point just made about unions shows that although the 16 binary Lemmon relation $R_{\mathcal{BV}}^{\#}$ form a Boolean algebra inheriting its structure from the Boolean algebra of the truth functions # themselves, whereas its meets (glb's) are the intersections of the relations concerned, their joins (lub's) are not the corresponding unions. There is a natural exhaustification or "exhaustivity" operation we can apply to these relations (*cf.* Fox (2007)., Spector (2016). and works there cited): letting $\mathbb{R}_{\mathcal{BV}}$ be the set of these relations $R_{\mathcal{BV}}^{\#}$, for $R \in \mathbb{R}_{\mathcal{BV}}$ we define its exhaustification $exh\,(R)$ by saying that for all A, B:

$$exh(R)(A,B) \text{ iff } R(A,B) \text{ and there is no } R' \in \mathbb{R}_{\mathcal{BV}} \text{ with } R' \gneqq R \text{ and } R'(A,B),$$

where $\leq$ is the partial ordering of the Boolean algebra above with universe $\mathbb{R}_{\mathcal{BV}}$, so $\gneqq$ amounts to the restriction of $\subsetneq$ to the relations in $\mathbb{R}_{\mathcal{BV}}$. (Strictly, then, the "*exh*" notation should carry a reference to this algebra, or to the poset concerned.) Thus, A and B stand in the relation $exh\,(R)$ when they stand in the relation R but in no stronger (= more restrictive) relation in $\mathbb{R}_{\mathcal{BV}}$. The relations we get by exhaustifying those in $\mathbb{R}_{\mathcal{BV}}$ will accordingly typically lie outside $\mathbb{R}_{\mathcal{BV}}$, being of the mixed permissive-coercive kind. On Lemmon's usage A and $\neg A$ are contraries, since $R_{\mathcal{BV}}^{\mathsf{nand}}(A,B)$ and $R_{\mathcal{B}}^{\mathsf{nand}}$ is the Lemmon relation of contrariety. But A and $\neg A$ do not stand in the relation $exh\,(R_{\mathcal{BV}}^{\mathsf{nand}})$, since they stand in the more restrictive Lemmon "contradictories"

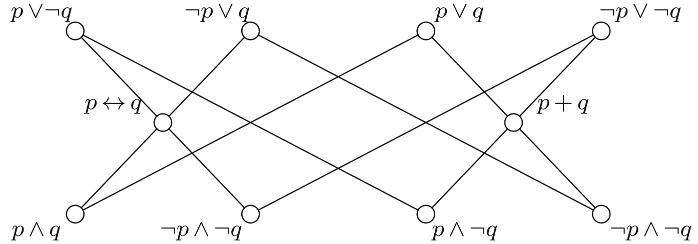

FIGURE A1 The poset of essentially binary truth functions relation $R^{\mathsf{xor}}_{\mathcal{BV}}$. Contraries that are not also contradictories (equivalently: that are not also subcontraries) are called "mere subcontraries" in, for example, Humberstone (2019)., in an attempt to bring the traditional notion of contrariety into play: in the traditional taxonomy, no two statements could stand in two distinct logical relations: the sets of contrary pairs, contradictory pairs, and subcontrary pairs – and superordinate–subordinate pairs – were disjoint (see esp. §3 of Humberstone (2013); "traditional" here means essentially "Aristotle-inspired"). With respect to the current operation of exhaustification; however, *exh* (R^{nand}) does not quite coincide with the relation of mere contrariety. For consider as a counterexample to the r.h.s. of the definition inset above the which chooses as $R' \in \mathbb{R}_{\mathcal{BV}}$ the inessentially binary relation holding between one formula and another when the first formula is false. (The associated truth function thus depends only its first argument and returns as value the negation of that argument.) Here we have $R' \lneq R^{\mathsf{nand}}_{\mathcal{BV}}$ – since whenever $\neg A$ is a tautology, so is $\neg(A \wedge B)$, though not in general conversely – so for example, writing $\#'$ as a connective for this "negation of the first projection" truth function $p\#'q$ (equivalently: $\neg p$) and $p \wedge q$ stand in the relation $R^{\mathsf{nand}}_{\mathcal{BV}}$ but not in the relation $exh(R^{\mathsf{nand}}_{\mathcal{BV}})$. Although for that reason not being "exhaustified contraries", these formulas remain *mere* contraries, in the sense of contraries which are not contradictories: $p\#'q$ are not $R^{\mathsf{xor}}_{\mathcal{BV}}$-related (do not have a tautologous exclusive disjunction).

We would get closer if we exhaustified relative to a smaller class of truth functions. Figure A1 excerpts from the Boolean algebra of all binary truth functions (as in Figure 2 of Humberstone (2013, p. 187).) the poset of essentially binary truth functions (depending on both arguments, that is), in the form of a Hasse diagram of the two-variable formulas representing such functions ("+" for xor), implications running upwards along sequences of edges as usual. But even here, taking $A = B = p \wedge \neg p$, we have $\vDash_{\mathcal{BV}} \neg(A \wedge B)$ making A and B contraries, and indeed, since $\nvDash_{\mathcal{BV}} A + B$, *mere* contraries. But they do not stand in the exhaustified contrariety relation *exh* ($R^{\mathsf{nand}}_{\mathcal{BV}}$) relative to our reduced stock of Lemmon relations, as they do also stand in the more restricted relation $R^{\mathsf{nor}}_{\mathcal{BV}}$, substituting A, B for p, q in the formula $\neg p \wedge \neg q$ in the logically strongest (= lowest) level of Figure A1 – in terms of which figure, the contrariety of A, B, itself is represented by the disjunction with negated disjuncts on the top row.

To align "mere" contrariety with an exhaustification of contrariety, we have to restrict the range of relations w.r.t. the exhaustification is performed, lopping off the bottom row of (those represented in) Figure A1, which amounts to removing all of the original 16 binary relations which are monadically representable in the sense of Longer Note D above. The six remaining relations are, when considered in their exhaustified forms, those in a popular inventory of the "traditional" logical relations: contraries, contradictories, subcontraries, the superordinate–subordinate relation and – typically not registered as an annotation in the square of opposition because its vertices can already be regarded as representing equivalence classes of statements when only one square is considered at a time.[43]Note that *exh* has no effect when applied in the case of contradictories or equivalents, since they are now minimal elements.) Unlike their pre-exhaustified namesakes, when attention is restricted to formulas not taking the same value on all Boolean valuations, no pair of such formulas stand in more than one of these mixed coercive–permissive relations, and if we throw in the maximally permissive relation of independence (as do the writers discussed in Humberstone (2013).), every pair stand in at least one of the (now) seven relations.

Although the concentration has been on contrariety, the same story applies to subcontrariety, mere subcontraries in the sense of "subcontraries which are not contradictories", coinciding with pairs in the exhaustified subcontrariety relation, *exh* ($R^{\vee}_{\mathcal{BV}}$), *exh* understood relative to the reduced version of Figure A1. (Similarly with the remaining pair on the top row,

[43]This qualification is included because one may, for instance, want to regard *No F is G* and *No G is F* as having different "canonical" contraries, their equivalence notwithstanding. (This could be done by means of non-logical axiom schemes for a propositional theory without extending the logical resources beyond the Boolean connectives, or by extending classical propositional logic to incorporate the syllogistic materials, in the style of Łukasiewicz.)

though now *unilateral* would be the more appropriate modifier than *mere*.) Now, the sub-contrariety case may ring a bell with readers following the literature on the role of exhaustification in generating an exclusive implicature for inclusively disjunctive assertions. There are many variations in the details of published proposals as to how this might work, but the broad idea is that assuming the speaker to be following Gricean cooperative principles and making the most relevantly informative contribution on the subject under discussion, says "*A* or *B*", then since "*A* and *B*" may be presumed to bear on that subject and would have been more informative, if the speaker is further presumed to be in an epistemic position to have made the stronger claim, the hearer infers that the stronger claim would not be true, and thus, that only one of A, B is true. So exhaustifying w.r.t. the set $\{A \text{ or } B, A \text{ and } B\}$ of alternatives – here playing the role of $\mathbb{R}$ above – converts the initially inclusive *or* into its exclusive cousin with the "and not both" supplied by the pragmatic enrichment effected by exhaustification. Similarly, at least with the $\mathbb{R}$ of the reduced version of Figure A1, exhaustifying the Lemmon relation $R^{\vee}_{\mathcal{BV}}$ turns it into the relation holding between *A* and *B* when their inclusive disjunction is $\mathcal{BV}$-valid into the relation holding between them when their exclusive disjunction is $\mathcal{BV}$-valid. (Indeed in Humberstone (2013, p. 200)., it is suggested that application of predicates like "contrary" – though the worked example with that application in mind concerned shape predicates, such as *rectangular*, was prone to induce a Gricean implicature – use the most restrictive predicate to hand or you will be taken as implying that it does not apply; the term *minimization* was used rather than *exhaustification*, though.)

Having seen how to modulate, using exhaustification, between a Lemmon-style taxonomy of logical relations and the more traditional style, we should pause to note two respects in which the analogy discussed in the preceding paragraph – though fine for the purpose it was put to there – is not perfect. The first is that the "logical relations" side of the picture places considerable reliance on eliminating the intuitively "less relational" binary relations, in weeding out the monadically representable (including not essentially binary) cases, which plays no role in the use of exhaustification to calculate implicatures in the case of disjunction. (Fox (2007). is concerned as much with explaining "free choice" *or*-constructions as with the strengthening of inclusive to exclusive disjunction.) Thus we should restore the top and bottom elements of the 16-element Boolean algebra, as well as allowing the mid-level nodes that would be labelled $p, q, \neg p, \neg q$ to return to the fold, rejoining their companions $p \leftrightarrow q$ and $p + q$ when considering the case of exhaustifying a statement of the form $A \vee B$, *prima facie* extending the range of candidates strictly stronger than $A \vee B$ from $\{A \wedge B\}$ to $\{A \wedge B, A, B, \bot\}$. This would not be a problem in the absence of the second of the two respects in which the analogy in question is imperfect: for the pragmatic strengthening of $A \vee B$ what was relevant was not validity but truth, or, to locate the issue more precisely in a formal setting, not $\mathcal{BV}$-validity but truth on a given $v \in \mathcal{BV}$.

We can keep the presentation of the issue as close as possible to the above discussion of logical relations by considering the situation with truth-value relations, or rather, to the relations between formulas induced by such plugging in the values on a given valuation. In other words, passing from $R^{\#}_{\mathcal{BV}}$ to $R^{\#}_{v}$ for the various $v \in \mathcal{BV}$, in terms of the contrast drawn at the end of the opening paragraph of Section 3, the latter amounting (Remark 3.1 (*i*)) in the present case to: $\overleftarrow{\boldsymbol{\chi}}\,(v(A), v(B))$, where boldface indicates the corresponding truth function. Since we are considering $\# = \vee$, the relevant $\overleftarrow{\boldsymbol{\chi}}\,(\cdot,\cdot)$ is that depicted first in Figure 1. There is no justification for whittling down the relations worthy of consideration to those that are not monadically representable – the suggested explanation for their non-appearance in traditional discussions of *logical* relations – in this case all of these truth-value relations are, since, as mentioned in note 17 all such relations are monadically representable. Indeed, they are all what we might call monadically representable by means of a single set (instead of the – in general – distinct, S_0, S_1), in the sense that for some S_0 and binary Boolean connective $\dot{\#}$ (here used in the metalanguage, whence the dot, as in (3) and Example 3.5 in Section 3) we have for all a, b iff $\langle a, b\rangle \in R$ iff

$(a \in S_0)\ \dot{\#}\ (b \in S_0)$. For several choices of $\dot{\#}$, Propositions 4, 6(ii), 7(i) and 7(ii) in Humberstone (1995). give equivalent first order characterizations – that is, without the quantification over sets. It would be interesting to see such a characterization of the general case without such quantification: a first order filling for the blank in R is "1-monadically" representable if and only if , and preferably one more informative than is obtained by treating the remaining candidates for $\dot{\#}$ and disjoining the characterizations for all 16 cases.

In fact, if we are thinking of truth-value relations proper, rather than the associated relations $R^{\#}_{v}$, for a given valuation, among the formulas, then not only are monadic representations available using just one set S_0, but that set has only element – the truth-value T. However, let us return to the range of (in general) strictly stronger candidates that $A \vee B$ switching from $\{A \wedge B\}$ to $\{A \wedge B, A, B, \bot\}$. We can conjoin $A \vee B$ with with the negation of $A \vee B$, as in the case of the reduced range, obtaining exclusive disjunction by exhaustification straightforwardly, and nor does negating $\bot$ present a problem, since this gives us a vacuously satisfied condition.[44] The case of adding the disjuncts A, B into the range of candidates whose negations will collectively give a plausible pragmatic enrichment now that we are considering truth (on a Boolean valuation) rather than validity (over $\mathcal{BV}$) is another matter, since, as Fox (2007, p. 96). points out, throwing in both their negations together alongside the disjunction gives us a contradiction. (Fox proposes a principle of "innocent exclusion" according to which exhaustification excludes – in the sense of adding the negation of – only stronger alternatives to what is being exhaustified when trouble of this kind is not on the cards.[45])

For present purposes, the key point is that while a disjunction can be $\mathcal{BV}$-valid without either disjunct being $\mathcal{BV}$-valid, it can be true on $v \in \mathcal{BV}$ without either disjunct being true on v.[46]As noted in the discussion after Example 3.8, in view of the Disjunction Property, despite the difference between the individual valuations in the two determining classes $\mathcal{HV}$ and $\mathcal{HV}^*$ in respect of the issue raised in note 46 for intuitionistic logic there mentioned (Kripke and Beth semantics, resp.), the validity of a disjunction over either class implies the validity of one of the disjuncts over the other, raising a Fox-like issue for exhaustifying subcontrariety in this case: it can never be the strongest thing to say about A and B that their disjunction is intuitionistically valid, so if we wanted to use "mere subcontrariety" for that (relative to the Heyting algebra with two free generators, without making the exclusions that led from the 2-generated free Boolean algebra to the poset of Figure A1), we would be saying that no distinct formulas are mere subcontraries. If instead we meant, as above, by *mere subcontraries*, subcontraries that are not contradictories, we would need to consider what contradictories are in an intuitionistic setting, bearing in mind that A's being equivalent to the negation of B does not in that setting amount to B's being equivalent to the negation of A. Indeed, even the subcontrariety has no (obvious) uniquely natural intuitionistic incarnation, the disjunctive formulation just provisionally employed is distinguished from the validity of $\neg(\neg A \wedge \neg B)$ is cast in Humberstone (2019, p. 332). as the contrast between Priest subcontrariety and Wansing subcontrariety, and there are many further options; more on these issues can be found in Humberstone (2020). The particular case of contradictories is evidently closely related to the question of whether there is some intuitionistically preferred notion of exclusive disjunction (see Exercises 6.12.6, 6.12.7,

[44]Indeed, natural language semanticists would not normally consider $\bot$ as even being a provisional candidate in need of being exhaustifying away. There has, on the other hand, been speculation in view of some empirical data on the use of *or* by young children as to whether the relevant range of candidates in this case depends on whether we are considering adult or child language-users, notably in Singh et al. (2016)., for example, p. 313.

[45]See Fox (2007)., p. 97, the upshot of which, in the present case, is that exclusion of either disjunct, let along both, is not part of the process. Historical references to earlier work on exhaustification, especially by Groenendijk and Stokhof, can also be found in Fox (2007).

[46]This gives the predicate "is $\mathcal{BV}$-valid" the logical properties – give or take the appropriate use/mention adjustments – of the an operator satisfying the conditions on "O" in (47) on p. 93 Fox (2007, p. 93). We should recall, apropos of all this, that the consequence relation of classical propositional logic is determined not only by $\mathcal{BV}$ but by other classes of (still bivalent) valuations, including by classes among whose elements there are v for which $v(A \vee B) = T$ even though $v(A) \neq T$ and $v(B) \neq T$. For further details, see, for example, Humberstone (2019, p. 300); the issue is also raised in note 37 of the present paper.

and the ambient discussion in Humberstone (2011, pp. 785–788).). Nor is the issue in any way specific, among non-classical logics, to intuitionistic logic. For example, there is the well-known contrast in FDE and Anderson–Belnap relevant logics which conservatively extend this $\{\wedge, \vee, \neg\}$ basis with principles governing $\rightarrow$, between the (IL-equivalent) stronger $(\neg p \wedge \neg q) \vee (p \wedge \neg q)$ and weaker $(p \vee q) \wedge (\neg p \vee \neg q)$ exclusive-like disjunctions, discussed in Dale (1982). and Humberstone (2014, p. 22*ff.*). (both cited in Section 2, note 10).

Longer Note G. (Expanding on note 19.) Numerous pertinent citations are given in the discussion in Humberstone (2011). referred to. I first became exercised over the confusions in the text to which note 19 is appended, as a student at the University of York, subjecting my then supervisor Martin (= J. M.) Bell to an essay on the topic airing my grievances on what may have seemed to be this comparatively minor matter. The considerations involved are broadly similar to those on show in Belnap (1975)., and, like them, raise the delicate question of whether what is offered is, on the one hand, the diagnosis of a mistake, or, on the other, the explanation – in the present case, via the de-characterizing transition – of a natural re-purposing of established terminology, not reflecting any confusion at all. The former (adverse) verdict might after all be passed on calling the conjunction connective (of classical logic, for definiteness) commutative. After all, as an operation in the algebra of formulas $\wedge$, mapping any pair of formulas to their conjunction, is not commutative, $A \wedge B$ and $B \wedge A$ being different formulas whenever A and B are. Rather, as, for example, Dunn (1975, p. 183). explains, what is actually commutative in such cases is the corresponding operation in the Lindenbaum algebra – the quotient of the formula algebra under the congruence relation of provable equivalence in the logic in question, and this is what is meant when a connective is said to be commutative according to that logic. (The elements of this algebra, as Dunn suggests, can be thought of as *propositions*, the proposition expressed by a formula being its equivalence class.) More generally one would need to say *complete interreplaceability* for *provable equivalence* here, and take further steps to avoid the identification of logics with (certain – e.g., substitution-closed) sets of formulas.

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